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Generating Functions of Some Families of Directed Uniform Hypergraphs

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Abstract. In this paper, we count acyclic and strongly connected uniform directed labeled hypergraphs. For these combinatorial structures, we introduce a specific generating function allowing us to recover and generalize some results on the number of directed acyclic graphs and the number of strongly connected directed graphs.

Résumé. Dans cet article, nous comptons les composantes acycliques et fortement connexes des hypergraphes uniformes dirigés étiquetés. Pour ces structures combinatoires, nous introduisons une classe spécifique de fonctions génératrices qui nous permet de retrouver et de généraliser les résultats sur le nombre de graphes dirigés acycliques et le nombre de digraphes fortement connexes.

Keywords: Directed labeled hypergraphs, generating functions, acyclic digraphs

1 Introduction

A *directed graph* or *digraph* consists of a finite node set $\mathcal{V}$ with a subset $\mathcal{E}$ of $\mathcal{V} \times \mathcal{V}$ (the arcs) and we do not allow neither loops nor multiple arcs.

In the seventies, several researchers including Liskovets [16, 17], Robinson [26, 23], Stanley [27] or Wright [28] studied enumerative aspects of important families of digraphs including *Directed Acyclic Graphs* (DAGs) or *strongly connected digraphs*.

A hypergraph is a generalization of a graph in which an (hyper)edge can join any number of nodes. Hypergraphs have been extensively studied [4, 5] as they are very useful to model concepts and structures in various aspects of computer science (combinatorial optimization, algorithmic game theory, machine learning, constraint satisfaction problem, data mining and indexing, and more).

In this paper, we deal with directed hypergraphs or simply *dihypergraphs* (also known as *And/Or graphs* [19, 15, 9]). As far as we know, these objects have been introduced in the Computer Science literature by Boley as a representation language [6]. For detailed surveys on directed hypergraphs, algorithms and applications, we refer the reader to

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the papers of Gallo, Longo, Pallotino and Nguyen [9] and of Ausiello and Luigi [2]. Following the recent enumerative results on digraphs of de Panafieu and Dovgal [7], Archer, Gessel, Graves and Liang [1], our aim in this article is to study enumerative aspects of some families of dihypergraphs.

2 Definitions

Terminology for dihypergraphs is established in the book of Harary, Norman, and Cartwright [13] or in the paper of Gallo, Longo, Pallottino, and Nguyen [9].

A directed (labeled) hypergraph (or simply dihypergraph) $\mathcal{H}$ is a pair $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a non-empty finite set of nodes and $\mathcal{E}$ is a set of ordered pairs of non-empty subsets of $\mathcal{V}$ called *directed hyperedges* (or *hyperarcs*). That means a hyperarc e is an ordered pair $(T(e), H(e))$, of disjoint subsets of $\mathcal{V}$ such that $T(e) \neq \emptyset$, $H(e) \neq \emptyset$. $T(e)$ is called the *tail* of the hyperarc e while $H(e)$ is its *head*.

A dihypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is called *b-uniform* iff for any $e \in \mathcal{E}$, $|T(e)| + |H(e)| = b$ (that is all hyperarcs are built with the same number of nodes). Clearly, the 2-uniform dihypergraph is the standard digraph. The dihypergraph $(\emptyset, \emptyset)$ is called the empty dihypergraph.

A *directed path* (or *path*) P_{st} of length ℓ in a dihypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, is a sequence of nodes and hyperarcs $P_{st} = \big(v_1 = s, e_{i_1}, v_2, \ldots, v_\ell, e_{i_\ell}, v_{\ell+1} = t\big)$ where:

$$s \in T(e_{i_1}), \quad t \in T(e_{i_\ell}) \text{ and } v_j \in T(e_{i_{j-1}}) \cap H(e_{i_j}) \text{ for } j = 2..\ell.$$

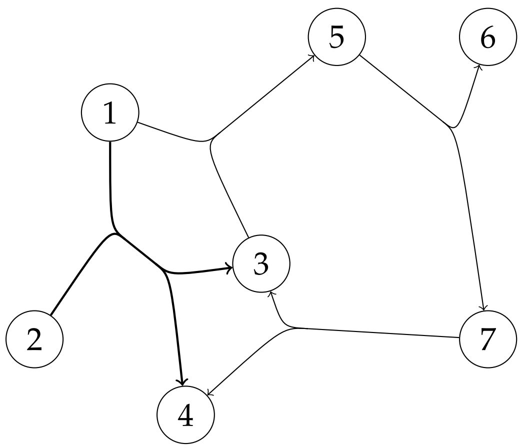
Nodes s and t are respectively the *origin* and the *destination* of the path P_{st} and we say that t is connected to s. The path P_{st} is said *simple* if all nodes on the path are distincts except possibly the origin s and the destination t. A *directed cycle* (or simply *cycle*) in a dihypergraph is a path where the origin and the destination coincide. A dihypergraph is said *acyclic* iff it has no cycle.

Given a dihypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, we define the relation $\mathcal{R}$ on $\mathcal{V}$ by $u\, \mathcal{R}\, v$ if there is a (directed) path from u to v in $\mathcal{H}$ and vice versa. It is easy to show that $\mathcal{R}$ is an equivalence relation on $\mathcal{V}$. The equivalence classes are called the *strongly connected components* of $\mathcal{H}$. A dihypergraph is *strongly connected* (or simply *strong*) if it has a unique strong component.

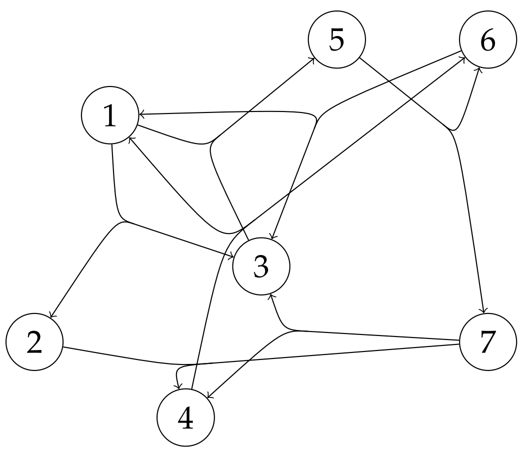
According to Robinson [26, 23] an *out-component* of a digraph is a strong component which cannot be reached from any other strong component. Such a component is called *source strong component* by Gessel [1] and *source-like strong connected component* by de Panafieu and Dovgal [7]. A source (strong) component is called simply a *source* if it contains exactly one node.

Obviously, we have the following Lemma.

Lemma 2.1. *Every non-empty dihypergraph has at least a source strong component.*



On the left, a general directed hypergraph with nodes $\{1, 2, 3, 4, 5, 6, 7\}$ built with 4 hyperarcs $\{1,3\} \to \{5\}$, $\{5\} \to \{6, 7\}$, $\{7\} \to \{3, 4\}$ and $\{1, 2\} \to \{3, 4\}$. The subset of nodes $\{3, 5, 7\}$ forms a directed cycle.



From the structure drawn above, by removing the hyperarc $\{1, 2\} \to \{3, 4\}$, we obtain a 3-uniform directed hypergraph. Then, by adding the hyperarcs $\{6\} \to \{1,3\}$, $\{1\} \to \{2, 3\}$, $\{4\} \to \{1, 6\}$ and $\{2, 7\} \to 4$, we get a strongly connected 3-uniform dihypergraph.

Throughout the rest of this paper, a dihypergraph is a b-uniform directed hypergraph. Similarly a hyperarc with b nodes is called simply a hyperarc. Graphs, digraphs or dihypergraphs are labeled.

3 Hypergraphic genenerating functions

We introduce a new type of generating function called *hypergraphic generating function* defined as follow. The variables x and y are reserved to mark nodes and hyperarcs.

Definition 3.1. The hypergraphic generating function (or simply HGF) for the sequence $(f_n(y))_{n\geq 0}$ is defined by

$$F(x,y) := \sum_{n=0}^{\infty} \frac{f_n(y)}{(1+y)^{\binom{n}{b}}} \frac{x^n}{n!}, \tag{3.1}$$

where $b \geq 2$.

Our hypergraphic generating function is a generalization of the *graphic generating function* (GGF) introduced by Read [22] and Robinson [23]. In particular, the *special generating function* of Robinson [23] corresponds to the case $b = 2$ and $y = 1$ and the graphical generating function corresponds to the case $b = 2$. Graphic generating functions are very useful as shown by the results of Bender, Richmond, Robinson, and Wormald [3], of Gessel [10], of Gessel and Sagan [12], and very recently of Archer, Gessel, Graves, and Liang [1] and de Panafieu and Dovgal [7].

For convenience, given a family of dihypergraphs $\mathcal{F}$ enumerated by the sequence $(f_n(y))_{n\geq 0}$, the *exponential generating function* (EGF) will be denoted by

$$f(x,y) := \sum_{n=0}^{\infty} f_n(y)\frac{x^n}{n!}, \tag{3.2}$$

and its HGF by (3.1). As some additionnal variables may be added for specific parameters, we often use *multivariate generating functions* (see Flajolet and Sedgewick [8, Definition III.4] for *multi-index convention*).

Definition 3.2. The exponential multivariate generating function of a family $\mathcal{F}$ will be denoted by

$$f(x,y,u) = \sum_{n,p} f_{n,p}(y)u^p\frac{x^n}{n!},$$

and the corresponding multivariate hypergraphic generating function is

$$F(x,y,u) = \sum_{n,p} \frac{f_{n,p}(y)}{(1+y)^{\binom{n}{b}}}u^p\frac{x^n}{n!},$$

where u the variable for some source component. Throughout this paper, the quantities $f(x,y,1)$ and $F(x,y,1)$ coincide with $f(x,y)$ and $F(x,y)$ respectively.

We observe that the HGF is obtained by dividing the coefficient of $n!\,x^n$ in the EGF by $(1+y)^{\binom{n}{b}}$. This linear operation is named by Robinson [23] as Δ for the case $b = 2$ and $y = 1$. We can use similar notation to convert an EGF to a HGF of family of dihypergraphs $\mathcal{F}$.

Definition 3.3. Let $\mathcal{F}$ be a family of dihypergraphs with EGF f and HGF F. We define $\Delta_{y,b}$ as the linear operator on generating functions which transform f into F :

$$F(x,y) = \Delta_{y,b}\left(f(x,y)\right) . \tag{3.3}$$

Let us remark that the operator $\Delta_{y,b}$ acts only w.r.t. the variable x. As an example of using $\Delta_{y,b}$, consider all sets of empty dihypergraphs (dihypergraph that contains no hyperarc). The EGF of such graphs is $\sum_{n\geq 0} x^n/n!$ and then the associated HGF is

$$\begin{aligned}\theta_b(x,y) &:= \Delta_{y,b}\left(\sum_{n\geq 0}\frac{x^n}{n!}\right), \\ &= \sum_{n\geq 0}^{\infty}\frac{1}{(1+y)^{\binom{n}{b}}}\frac{x^n}{n!}.\end{aligned} \tag{3.4}$$

Observe that de Panafieu and Dovgal [7] used the *exponential Hadamard product* to convert an EGF to a graphic generating function when working on digraphs. Such operation is simply defined below.

Definition 3.4. The exponential Hadamard product of $f(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}$ and $g(x) = \sum_{n\geq 0} g_n \frac{x^n}{n!}$ is the exponential generating functions of the sequence $(f_n g_n)_{n\geq 0}$. It is denoted $f(x)\odot g(x)$ and we have

$$f(x)\odot g(x) = \sum_{n\geq 0} f_n g_n \frac{x^n}{n!}.$$

Then, given a family of dihypergraphs $\mathcal{F}$ with EGF f and HGF F, the linear operator $\Delta_{y,b}$ and the exponential Hadamard product are linked by the equation

$$F(x,y) = \theta_b(x,y)\odot f(x,y) = \Delta_{y,b}\left(f(x,y)\right),$$

where $\theta_b(x,y)$ is the HGF defined by (3.4).

Now, we introduce the *arrow product* which already appears in [23, 25, 11]. The definition of the arrow product of two families of digraphs $\mathcal{A}$ and $\mathcal{B}$ viewed as symbolic methods is defined explicitly in [7]. Such definition is extended here to dihypergraphs.

Definition 3.5. The arrow product $\mathcal{C}$ of two families of dihypergraphs $\mathcal{A}$ and $\mathcal{B}$ is the family that consists in pairs (A,B) with $A\in\mathcal{A}$ and $B\in\mathcal{B}$ relabeled so that objects A and B have disjoint labels and where an arbitrary number of hyperarcs have their tails belonging to A while their heads belong to B.

The following lemmas extend on dihypergraphs some results on symbolic methods of EGFs (cf. Flajolet and Sedgewick [8]) and symbolic methods of GGFs as introduced by de Panafieu and Dovgal [7].

Lemma 3.6. *Given two families $\mathcal{F}$ and $\mathcal{G}$ of dihypergraphs with HGFs $F(x,y)$ and $G(x,y)$, the HGF of the disjoint union $\mathcal{F}+\mathcal{G}$ is*

$$F(x,y)+G(x,y),$$

where x and y mark respectively nodes and hyperarcs.

Lemma 3.7. *Given a family of dihypergraphs $\mathcal{F}$ with HGF F, if a variable u marks the number of some family of source components in the HGF $F(x,y,u)$ the HGF for the elements of $\mathcal{F}$ which have a distinguished subset of source components is $F(x,y,u+1)$ where x and y mark respectively nodes and hyperarcs.*

The proofs of Lemmas 3.6 and 3.7 are elementary by means of symbolic methods on EGFs and Definition 3.3 and Definition 3.5.

As an example of using the parameter u for a family of dihypergraphs $\mathcal{F}$, we may use u to mark the number of sources in the HGF $F(x,y,u)$. Then, $F(x,y,1)$ is the HGF of the whole family $\mathcal{F}$ without distinguishing if a node is a source or not and $F(x,y,0)$ is the HGF of dihypergraphs in $\mathcal{F}$ without any source.

Remark 3.8. The substitution of u by $u+1$ means that items are marked or left unmarked. Conversely, replacing u with $u-1$ corresponds to an inclusion-exclusion principle.

Lemma 3.9. *Let $F(x,y)$ and $G(x,y)$ be the HGFs of two families of dihypergraphs $\mathcal{F}$ and $\mathcal{G}$. The HGF of the arrow product (cf. Definition 3.5) of the families $\mathcal{F}$ and $\mathcal{G}$ is equal to $F(x,y)\,G(x,y)$.*

Proof. Let $(f_n(y))$ and $(g_n(y))$ be the associated sequences of the two families $\mathcal{F}$ and $\mathcal{G}$. Then, the sequence associated to the HGFs $F(x,y)G(x,y)$ is

$$\begin{aligned} c_n(y) &= (1+y)^{\binom{n}{b}} n![x^n]\left(\sum_{k\geq 0}\frac{f_k(y)}{(1+y)^{\binom{k}{b}}}\frac{x^k}{k!}\right)\left(\sum_{\ell\geq 0}\frac{g_\ell(y)}{(1+y)^{\binom{\ell}{b}}}\frac{x^\ell}{\ell!}\right), \\ &= \sum_{k=0}^{n}\binom{n}{k}(1+y)^{\binom{n}{b}-\binom{k}{b}-\binom{n-k}{b}} f_k(y) g_{n-k}(y). \end{aligned}$$

□

There is a direct combinatorial explanation for the exponent $\binom{n}{b}-\binom{k}{b}-\binom{n-k}{b}$. Consider two dihypergraphs F and G of sizes k and $n-k$, and their arrow product H (of size n). F and G are combined and relabeled. Any of the $\binom{n}{b}$ possible sets of nodes can become a hyperarc from the arrow product, except the $\binom{k}{b}$ sets that contain only nodes from F, and the $\binom{n-k}{b}$ sets that contain only nodes from G. We can also use the Vandermonde's identity for any nonnegative integers b,m,n:

$$\binom{m+n}{b} = \sum_{k=0}^{b}\binom{m}{k}\binom{n}{b-k},$$

to show that

$$\binom{n}{b}-\binom{k}{b}-\binom{n-k}{b} = \sum_{i+j=b,i,j>0}\binom{k}{i}\binom{n-k}{j}.$$

Lemma 3.10. *The total number of hyperarcs on n nodes is equal to*

$$(2^b - 2)\binom{n}{b}, \quad \text{for } b \geq 2.$$

Proof. The number of hyperarcs with exactly k tails ($0 < k < b$) and $b - k$ heads is equal to

$$\binom{n}{k}\binom{n-k}{b-k}.$$

Summing over k, we have

$$\sum_{k=1}^{b-1}\binom{n}{k}\binom{n-k}{b-k} = (2^b - 2)\binom{n}{b}.$$

□

Lemma 3.11. *The EGF of all dihypergraphs $h(x, y)$ is*

$$h(x, y) = \sum_{n \geq 0}^{\infty} (1 + y)^{(2^b - 2)\binom{n}{b}} \frac{x^n}{n!}. \tag{3.5}$$

The HGF of all dihypergraphs $H(x, y)$ is

$$H(x, y) = \sum_{n \geq 0}^{\infty} (1 + y)^{(2^b - 3)\binom{n}{b}} \frac{x^n}{n!}. \tag{3.6}$$

Proof. The proof is obvious from the definition of the HGFs and by Lemma 3.10. □

4 Acyclic or strong dihypergraphs

In this Section, we give exact enumerations of acyclic or strongly dihypergraphs. Our results extend those in [27, 23, 12, 24, 7] on enumeration of these families in digraphs to dihypergraphs. We notice also that a different approach has been given by Ostroff [20] to count strong digraphs

Let us recall that Robinson [23, Corollary 1] showed that the counting sequence $\alpha_n(y)$ of acyclic digraphs on n nodes satisfies

$$\sum_{n=0}^{\infty} \frac{\alpha_n(y)}{(1 + y)^{\binom{n}{2}}} \frac{x^n}{n!} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(1 + y)^{\binom{n}{2}}} \frac{x^n}{n!} \right)^{-1}.$$

Theorem 4.1 generalizes this identity for dihypergraphs. Let us define the HGF of the sequence $((-1)^n)_{n \geq 0}$ denoted $\phi(x, y)$. We have

$$\phi(x, y) := \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\,(1 + y)^{\binom{n}{b}}}. \tag{4.1}$$

Theorem 4.1. *Let $a_n(y) = \sum_{q=0}^{\binom{n}{b}} a_{n,q} y^q$ be the counting sequence of acyclic dihypergraphs where $a_{n,q}$ denotes the number of acyclic dihypergraphs with n nodes and q hyperarcs, and $A(x,y) = \sum_{n=0}^{\infty} \frac{a_n(y)x^n}{n!(1+y)^{\binom{n}{b}}}$ be its associated HGF. $A(x,\, y)$ satisfies*

$$A(x,y) = \phi(x,y)^{-1}, \tag{4.2}$$

where ϕ is defined by (4.1).

Proof. Let u be the variable marking the number of sources in the EGF or HGF of all acyclic dihypergraphs $A(x,y,u)$. By the Lemma 3.7, the HGF for the dihypergraphs where each source node is either marked, or left unmarked by the variable u is $A(x,y,u+1)$. Next, the EGF of a set of isolated nodes is $\exp(ux)$ (dihypergraph without any hyperarc) and so the associated HGF is $\Delta_{y,b}(\exp(ux))$. We observe that an acyclic dihypergraph with some marked sources can be viewed as an arrow product of a set of nodes (the marked sources) with an acyclic dihypergraph. This decomposition implies

$$A(x,y,u+1) = \Delta_{y,b}(\exp(ux)) \times A(x,y)\,.$$

Substituting u by -1 leads to $A(x,y,0) = 1$ (the only acycic dihypergraph without a source is the empty dihypergraph). Since $\Delta_{y,b}(\exp(ux)) = \phi(x,y)$ where ϕ is given by (4.1), we get the result. □

Remark 4.2. A similar proof can be obtained using first the inclusion-exclusion principle to get $a_0(y) = 1$ and for $n \geq 1$

$$a_n(y) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (1+y)^{\binom{n}{b}-\binom{k}{b}-\binom{n-k}{b}} a_{n-k}(y)\,, \tag{4.3}$$

which can be rewritten as

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (1+y)^{\binom{n}{b}-\binom{k}{b}-\binom{n-k}{b}} a_k = \delta_{n0}, \tag{4.4}$$

where δ_{n0} is Kronecker's symbol, and then by checking that $A(x,y)\phi(x,y) = 1$. In terms of n, an explicit expression of $a_n(y)$ can be obtained from the identity $A(x,y)\phi(x,y) = 1$:

$$a_n(y) = \sum_{j\geq 0} (-1)^j \sum_{n_1+\cdots+n_j=n} \binom{n}{n_1,\ldots,n_j} (1+y)^{\binom{n}{b}-\sum_{i=1}^{j}\binom{n_i}{b}}$$

Theorem 4.3. *Let S be the set of all strongly connected dihypergraphs, if s is the associated EGF, then the HGF of all dihypergraphs defined by (3.6) and the EGF $s(x,y)$ verify*

$$H(x,y) = \left(\Delta_{y,b}\left(\exp\left(-s(x,y)\right)\right)\right)^{-1}. \tag{4.5}$$

Proof. Let u be a variable marking the number of strongly connected components which are source components (see Lemma 2.1) in the EGF or in the HGF $H(x,y,u)$ of all dihypergraphs. By the Lemma 3.7, the HGF for the dihypergraphs where each source strong component is either marked, or left unmarked by the variable u is $H(x,y,u+1)$. Next, the EGF of the set of strongly connected components is $\exp(u\, s(x,y))$ and so the associated HGF is $\Delta_{y,b}\left(\exp(u\, s(x,y))\right)$. We observe that a dihypergraph with some marked source components can be viewed as an arrow product of a set of strong dihypergraphs (the marked source components) with a dihypergraph. This decomposition implies

$$H(x,y,u+1) = \Delta_{y,b}\left(\exp(u\, s(x,y))\right) \times H(x,y)\,.$$

Then replacing u with -1 gives the result since $H(x,y,0) = 1$ (the only dihypergraph without a source component is the empty dihypergraph). □

Remark 4.4. Notice also that Theorem 4.3 leads to a recursive relation satisfied by $(s_n(y))$ where $s_n(y) = n![x^n]s(x,y)$ with $s(x,y)$ is the EGF of all strongly connected dihypergraphs. Following the same techniques using by Robinson in [23, Section 4.], we can easily show that $s_0(y) = 1$ and

$$s_n(y) = \lambda_n(y) + \sum_{t=1}^{n-1} \binom{n-1}{t} s_{n-t}(y)\lambda_t(y)\,,$$

with

$$\lambda_n(y) = (1+y)^{(2^b-2)\binom{n}{b}} - \sum_{t=1}^{n-1} \binom{n}{t} (1+y)^{(2^b-2)\binom{t}{b}} \lambda_{n-1}(y)\,.$$

5 Conclusion

Our paper deals with directed uniform hypergraphs by introducing a specific type of generating functions to obtain generating functions of acyclic and strong dihypergraphs. We think that many families of dihypergraphs can be enumerated using the same methods. More generally, what is the most general model of graph-like objects where DAGs and strongly connected components can be defined and counted using the same techniques?

In future works, it would be interesting to compute the asymptotic number of these combinatorial structures (as in [3] for dense digraphs and in [21] for sparse random digraphs) and to study the appearance of strongly connected components (as in [14, 18]) during some random dihypergraphs processes. For example, when enriching the structures by adding hyperarc one by one, how many hyperarcs are needed to have asymptotically almost surely structures containing complex strong components?

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