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A POSITIVE-DEFINITE INNER PRODUCT FOR VECTOR-VALUED MACDONALD POLYNOMIALS

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Abstract. In a previous paper J.-G. Luque and the author (Sem. Loth. Combin. 2011) developed the theory of nonsymmetric Macdonald polynomials taking values in an irreducible module of the Hecke algebra of the symmetric group $\mathcal{S}_N$. The polynomials are parametrized by (q,t) and are simultaneous eigenfunctions of a commuting set of Cherednik operators, which were studied by Baker and Forrester (IMRN 1997). In the Dunkl–Luque paper there is a construction of a pairing between $\left(q^{-1},t^{-1}\right)$-polynomials and (q,t)-polynomials, and for which the Macdonald polynomials form a biorthogonal set. The present work is a sequel with the purpose of constructing a symmetric bilinear form for which the Macdonald polynomials form an orthogonal basis and of determining the region of (q,t)-values for which the form is positive-definite. Irreducible representations of the Hecke algebra are characterized by partitions of N. The positivity region depends only on the maximum hook-length of the Ferrers diagram of the partition.

1. Introduction

The theory of nonsymmetric Jack polynomials was generalized by Griffeth [4] to polynomials on the complex reflection groups of type $G(n,p,N)$ taking values in irreducible modules of the groups. This theory simplifies somewhat for the group $G(1,1,N)$, the symmetric group of N objects, where any irreducible module is spanned by standard Young tableaux all of the same shape, corresponding to a partition of N. Luque and the author [3] developed an analogous theory for vector-valued Macdonald polynomials taking values in irreducible modules of the Hecke algebra of a symmetric group. The structure has parameters (q,t) and depends on a commuting set of Cherednik operators whose simultaneous eigenfunctions are the aforementioned Macdonald polynomials. The paper showed how to construct the polynomials by means of a Yang–Baxter graph (see [5]). Also a bilinear form was

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defined which paired polynomials for the parameters (q^{-1}, t^{-1}) with those parametrized by (q, t) and resulted in biorthogonality relations for the Macdonald polynomials. The present paper is a sequel whose aim is to define a symmetric bilinear form for which these polynomials are mutually orthogonal. Some other natural conditions are imposed on the form to force uniqueness. The form is positive-definite for a (q, t)-region determined by the specific module.

For purposes of illustration the form is first defined for the scalar case, and leads to expressions only slightly different from the well-known hook-product formulas. For the trivial representation of the Hecke algebra, corresponding to the one-part partition, the vector-valued polynomials specialize to the scalar polynomials. Section 3 contains a short outline of representation theory of the Hecke algebra, the Yang–Baxter graph of vector-valued Macdonald polynomials and the process leading to the definition of the symmetric bilinear form, followed by the characterization of (q, t)-values yielding positivity of the form. The details of the construction of the polynomials and related operators along with the proofs of their properties are found in [3].

1.1. **Notation.** Let $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$. The elements of $\mathbb{N}_0^N$ are called *compositions,* and for $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$ let $|\alpha| := \sum_{i=1}^N \alpha_i$. Let $\mathbb{N}_0^{N,+}$ denote the set of partitions $\left\{\lambda \in \mathbb{N}_0^N : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N\right\}$, and let α^+ denote the nonincreasing rearrangement of α; for example, if $\alpha = (1, 2, 1, 4)$, then $\alpha^+ = (4, 2, 1, 1)$. There are two partial orders on compositions used in this work: for $\alpha, \beta \in \mathbb{N}_0^N$ the relation $\alpha \succ \beta$ means $\alpha \neq \beta$ and $\sum_{i=1}^j (\alpha_i - \beta_i) \geq 0$ for $1 \leq j \leq N$ (the dominance order), and $\alpha \rhd \beta$ means $|\alpha| = |\beta|$ and $\alpha^+ \succ \beta^+$, or $\alpha^+ = \beta^+$ and $\alpha \succ \beta$. The rank function for $\alpha \in \mathbb{N}_0^N$ is

(1.1)
$$r_\alpha(i) := \#\{j : \alpha_j > \alpha_i\} + \#\{j : 1 \leq j \leq i, \alpha_j = \alpha_i\}, \ 1 \leq i \leq N.$$

We have $\alpha = \alpha^+$ if and only if $r_\alpha(i) = i$ for all i.

The symmetric group $\mathcal{S}_N$ is generated by the adjacent transpositions $s_i := (i, i+1)$ for $1 \leq i < N$, where s_i acts on an N-tuple $a = (a_1, \ldots, a_N)$ by $a.s_i = (\ldots, a_{i+1}, a_i, \ldots)$, interchanging entries $\#i$ and $\#(i+1)$. For a composition $\alpha \in \mathbb{N}_0^N$ the inversion number is $\operatorname{inv}(\alpha) := \#\{(i, j) : 1 \leq i < j \leq N, \alpha_i < \alpha_j\}$. If $\alpha_i < \alpha_{i+1}$ then $\operatorname{inv}(\alpha.s_i) = \operatorname{inv}(\alpha) - 1$.

The space of polynomials is $\mathcal{P} := \mathbb{K}[x_1, x_2, \ldots, x_N]$, where $\mathbb{K} := \mathbb{Q}(q, t)$ and q, t are transcendental or generic, that is, complex numbers satisfying $q \neq 1$ and $q^a t^b \neq 1$ for $a, b \in \mathbb{Z}$ and $-N \leq b \leq N$. For $\alpha \in$

$\mathbb{N}_0^N$ we write x^α for the *monomial* $\prod_{i=1}^{N} x_i^{\alpha_i}$. The space of homogeneous polynomials of degree n is defined as $\mathcal{P}_n := \operatorname{span}_{\mathbb{K}}\{x^\alpha : |\alpha| = n\}$ for $n = 0, 1, 2, \ldots$. The group $\mathcal{S}_N$ acts on polynomials by permutation of coordinates, $p(x) \to (ps_i)(x) := p(x.s_i)$.

The Hecke algebra $\mathcal{H}_N(t)$ is the associative algebra generated by $\{T_1, T_2, \ldots, T_{N-1}\}$ subject to the relations

$$\begin{aligned} &(T_i + 1)(T_i - t) = 0, \\ &T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad 1 \le i \le N-2, \\ &T_i T_j = T_j T_i, \quad 1 \le i < j-1 \le N-2. \end{aligned} \tag{1.2}$$

The quadratic relation implies $T_i^{-1} = \frac{1}{t}(T_i + 1 - t) \in \mathcal{H}_N(t)$. For generic t there is a linear isomorphism $\mathbb{K}\mathcal{S}_N \to \mathcal{H}_N(t)$ generated by $s_i \to T_i$.

For $p \in \mathcal{P}$ and $1 \le i < N$ define

$$p(x) T_i := (1-t) x_{i+1} \frac{p(x) - p(x.s_i)}{x_i - x_{i+1}} + tp(x.s_i). \tag{1.3}$$

It can be shown straightforwardly that these operators satisfy the defining relations of $\mathcal{H}_N(t)$. Also $ps_i = p$ (symmetry in (x_i, x_{i+1})) if and only if $pT_i = tp$ (because $pT_i - tp = \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(p - ps_i)$), and $pT_i = -p$ if and only if $p(x) = (tx_i - x_{i+1}) p_0(x)$ where $p_0 \in \mathcal{P}$ and $p_0 s_i = p_0$. Also $x_i T_i = x_{i+1}$ and $1T_i = t$.

2. Scalar nonsymmetric Macdonald polynomials

For $f \in \mathcal{P}$ define shift, Cherednik and Dunkl operators by (see [1], and also [3])

$$\begin{aligned} fw(x) &:= f(qx_N, x_1, x_2, \ldots, x_{N-1}), \\ f\xi_i &:= t^{i-1} f T_{i-1}^{-1} T_{i-2}^{-1} \cdots T_1^{-1} w T_{N-1} T_{N-2} \cdots T_i, \\ f\mathcal{D}_N &:= (f - f\xi_N)/x_N, \ f\mathcal{D}_i := \frac{1}{t} f T_i \mathcal{D}_{i+1} T_i. \end{aligned} \tag{2.1}$$

Note that $\xi_i = \frac{1}{t} T_i \xi_{i+1} T_i$. It is a nontrivial result that D_i maps $\mathcal{P}_n$ to $\mathcal{P}_{n-1}$. The operators ξ_i commute with each other and there is a basis of simultaneous eigenfunctions, the nonsymmetric Macdonald polynomials M_α, labeled by $\alpha \in \mathbb{N}_0^N$ with $\rhd$-leading term $q^a t^b x^\alpha$ with $\alpha, \beta \in \mathbb{N}_0$ such that

$$\begin{aligned} M_\alpha(x) &= q^a t^b x^\alpha + \sum_{\alpha \rhd \beta} A_{\alpha\beta}(q,t) x^\beta \\ M_\alpha \xi_i &= q^{\alpha_i} t^{N - r_\alpha(i)} M_\alpha, 1 \le i \le N; \end{aligned} \tag{2.2}$$

where the coefficients $A_{\alpha\beta}(q,t)$ are rational functions of (q,t) whose denominators are of the form $\left(1-q^at^b\right)$. The *spectral vector* is $\zeta_\alpha(i) = q^{\alpha_i}t^{N-r_\alpha(i)}$, $1\leq i\leq N$. There is a simple relation between M_α and $M_{\alpha.s_i}$ when $\alpha_i<\alpha_{i+1}$ and $\rho=\zeta_\alpha(i+1)/\zeta_\alpha(i)=q^{\alpha_{i+1}-\alpha_i}t^{r_\alpha(i)-r_\alpha(i+1)}$, namely

$$M_\alpha T_i = M_{\alpha s_i} - \frac{1-t}{1-\rho}M_\alpha, \tag{2.3}$$

$$M_{\alpha.s_i}T_i = \frac{(1-\rho t)(t-\rho)}{(1-\rho)^2}M_\alpha + \frac{\rho(1-t)}{(1-\rho)}M_{\alpha.s_i}, \tag{2.4}$$

and $\zeta_{\alpha.s_i}=\zeta_\alpha.s_i$. If $\alpha_i=\alpha_{i+1}$ then

$$M_\alpha T_i = tM_\alpha. \tag{2.5}$$

The other step needed to construct any M_α starting from 1 is the affine step

$$\begin{aligned} M_{\alpha\Phi} &= x_N(M_\alpha w), \\ \alpha\Phi &:= (\alpha_2,\alpha_3,\ldots,\alpha_N,\alpha_1+1), \\ \zeta_{\alpha\Phi} &= (\zeta_\alpha(2),\ldots,\zeta_\alpha(N),q\zeta_\alpha(1)). \end{aligned} \tag{2.6}$$

Formulas (2.3) and (2.6) can be interpreted as edges of a Yang–Baxter graph for generating the polynomials (see [5, Sec. 9]). This graph has the root $\left(\mathbf{0},\left[t^{N-i}\right]_{i=1}^N,1\right)$ and nodes $(\alpha,\zeta_\alpha,M_\alpha)$. There are steps $(\alpha,\zeta_\alpha,M_\alpha)\overset{s_i}{\to}(\alpha.s_i,\zeta_\alpha.s_i,M_{\alpha.s_i})$ for $\alpha_{i+1}>\alpha_i$ given by

$$M_{\alpha s_i} = M_\alpha T_i + \frac{1-t}{1-\zeta_\alpha(i+1)/\zeta_\alpha(i)}M_\alpha,$$

and affine steps $(\alpha,\zeta_\alpha,M_\alpha)\overset{\Phi}{\to}(\alpha\Phi,\zeta_{\alpha\Phi},M_{\alpha\Phi})$ (given by (2.6)).

There is a short proof using Macdonald polynomials that $\mathcal{D}_N$ maps $\mathcal{P}_N$ to $\mathcal{P}_{N-1}$: when $\alpha_N=0$ then $r_\alpha(N)=N$, $\zeta_\alpha(N)=1$ and $M_\alpha\xi_N = M_\alpha$, thus $M_\alpha\mathcal{D}_N=0$; if $\alpha_N\geq 1$ then the raising (affine) formula is $M_\alpha(x)=x_NM_\beta w(x)$ where $\beta=(\alpha_N-1,\alpha_1,\alpha_2,\ldots,\alpha_{N-1})$, thus

$$M_\alpha(1-\xi_N) = (1-\zeta_\alpha(N))M_\alpha,$$

which is divisible by x_N.

Our logical outline is to first state a number of hypotheses to be satisfied by the inner product, then deduce consequences leading to a formula which is used as a definition. To finish one has to show that the hypotheses are satisfied. The presentation is fairly sketchy for the scalar case which is mostly intended as illustration. The material for vector-valued Macdonald polynomials is more detailed.

The **hypotheses** (BF1) for the symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathcal{P}$, with $w^{*}:=T_{N-1}^{-1}\cdots T_{1}^{-1}wT_{N-1}\cdots T_{1}$, are (for $f,g\in\mathcal{P}$):

$$\langle 1,1\rangle=1, \tag{2.7}$$

$$\langle fT_{i},g\rangle=\langle f,gT_{i}\rangle,\ 1\leq i<N, \tag{2.8}$$

$$\langle f\xi_{N},g\rangle=\langle f,g\xi_{N}\rangle, \tag{2.9}$$

$$\langle f\mathcal{D}_{N},g\rangle=(1-q)\,\langle f,x_{N}\,(gw^{*}w)\rangle. \tag{2.10}$$

From the definition of $w^{*}=T_{N-1}^{-1}\cdots T_{1}^{-1}\xi_{1}$ it follows that

$$\begin{aligned}\langle f,gw^{*}\rangle&=\langle f,gT_{N-1}^{-1}\cdots T_{1}^{-1}\xi_{1}\rangle=\langle f\xi_{1},gT_{N-1}^{-1}\cdots T_{1}^{-1}\rangle\\&=\langle f\xi_{1}T_{1}^{-1}\cdots T_{N-1}^{-1},g\rangle=\langle fw,g\rangle.\end{aligned} \tag{2.11}$$

(It is a trivial exercise to show $\langle fT_{i}^{-1},g\rangle=\langle f,gT_{i}^{-1}\rangle$.) Here w^{*} is taken as a symbolic name without claiming that it is the adjoint. Since it is possible that there is a subspace $\mathcal{N}$ of $\mathcal{P}$ such that $\langle f,h\rangle=0$ for all $f\in\mathcal{P}$ and $h\in\mathcal{N}$, the adjoint of an operator is only defined modulo $\mathcal{N}$. It follows from (2.8), (2.9) and $\xi_{i}=\frac{1}{t}T_{i}\xi_{i+1}T_{i}$ that $\langle f\xi_{i},g\rangle=\langle f,g\xi_{i}\rangle$ for all $f,g\in\mathcal{P}$ and all i. This implies the mutual orthogonality of $\left\{M_{\alpha}:\alpha\in\mathbb{N}_{0}^{N}\right\}$ because the spectral vector ζ_{α} determines α. Implicitly $t\in\mathbb{R}$ since the eigenvalues of T_{i} are $t,-1$. If $\deg f\neq\deg g$ then $\langle f,g\rangle=0$ because the Macdonald polynomials form a homogeneous basis. For convenience denote $\langle f,f\rangle=\|f\|^{2}$ (no claim is being made about positivity).

Definition 1. *For $z\in\mathbb{K}$ let*

$$u\,(z):=\frac{(t-z)\,(1-zt)}{(1-z)^{2}}. \tag{2.12}$$

Note that $u\,(z^{-1})=u\,(z)$.

Proposition 1. *Suppose* (BF1) *holds, $a_{i}<\alpha_{i+1}$ and*

$$\rho=q^{\alpha_{i+1}-\alpha_{i}}t^{r_{\alpha}(i)-r_{\alpha}(i+1)}.$$

Then

$$\|M_{\alpha.s_{i}}\|^{2}=u\,(\rho)\,\|M_{\alpha}\|^{2}.$$

Proof. From equation (2.3) we infer $\langle M_{\alpha}T_{i},M_{\alpha.s_{i}}\rangle=\|M_{\alpha.s_{i}}\|^{2}$ (by hypothesis $\langle M_{\alpha},M_{\alpha.s_{i}}\rangle=0$), and by equation (2.4) we have

$$\begin{aligned}\langle M_{\alpha}T_{i},M_{\alpha.s_{i}}\rangle&=\langle M_{\alpha},M_{\alpha.s_{i}}T_{i}\rangle\\&=\frac{(1-\rho t)\,(t-\rho)}{(1-\rho)^{2}}\,\|M_{\alpha}\|^{2}=u\,(\rho)\,\|M_{\alpha}\|^{2}.\end{aligned}$$

□

Definition 2. *For $\alpha \in \mathbb{N}_0^N$ let*

$$\mathcal{E}\left(\alpha\right) := \prod_{1\leq i<j\leq N, \alpha_i<\alpha_j} u\left(q^{\alpha_j-\alpha_i} t^{r_\alpha(i)-r_\alpha(j)}\right). \tag{2.13}$$

Proposition 2. *Suppose* (BF1) *holds and $\alpha \in \mathbb{N}_0^N$. Then $\left\|M_{\alpha^+}\right\|^2 = \mathcal{E}\left(\alpha\right)\left\|M_\alpha\right\|^2$.*

Proof. Arguing by induction on $\operatorname{inv}\left(\alpha\right)$ it suffices to show that $\alpha_i < \alpha_{i+1}$ implies $\mathcal{E}\left(\alpha\right)/\mathcal{E}\left(\alpha.s_i\right) = u\left(q^{\alpha_{i+1}-\alpha_i} t^{r_\alpha(i)-r_\alpha(i+1)}\right)$. The factors corresponding to pairs (l,j) with $l,j \neq i, i+1$ are the same in the products, and the pairs with just one of $i, i+1$ are interchanged in $\mathcal{E}\left(\alpha\right), \mathcal{E}\left(\alpha.s_i\right)$. There is only one factor in $\mathcal{E}\left(\alpha\right)$ that is not in $\mathcal{E}\left(\alpha.s_i\right)$, namely $u\left(q^{\alpha_{i+1}-\alpha_i} t^{r_\alpha(i)-r_\alpha(i+1)}\right)$ coming from $(i, i+1)$. Thus

$$\mathcal{E}\left(\alpha\right)\left\|M_\alpha\right\|^2 = \mathcal{E}\left(\alpha.s_i\right)\left\|M_{\alpha.s_i}\right\|^2. \qquad \square$$

Lemma 1. *Suppose $\alpha \in \mathbb{N}_0^N$. Then $M_{\alpha\Phi}\mathcal{D}_N = \left(1 - q\zeta_\alpha\left(1\right)\right) M_\alpha w$.*

Proof. By definition we have

$$\begin{aligned} M_{\alpha\Phi}\mathcal{D}_N &= \left(1/x_N\right) M_{\alpha\Phi}\left(1-\xi_N\right) \\ &= \left(1/x_N\right)\left(1-\zeta_{\alpha\Phi}\left(N\right)\right) M_{\alpha\Phi} = \left(1 - q\zeta_\alpha\left(1\right)\right) M_\alpha w. \qquad \square \end{aligned}$$

Remark 1. It is incompatible with (2.7), (2.8) and (2.9) to require either $\langle f\mathcal{D}_N, g\rangle = c\langle f, x_N g\rangle$ with some constant c, or $\langle x_N f, x_N g\rangle = \langle f, g\rangle$. Let $f = M_{\alpha\Phi}$ and $g = M_\beta w$ with $|\alpha| = |\beta|$; then $\langle f, x_N g\rangle = \langle M_{\alpha\Phi}, M_{\beta\Phi}\rangle$ while

$$\langle f\mathcal{D}_N, g\rangle = \left(1 - q\zeta_\alpha\left(1\right)\right)\langle M_\alpha w, M_\beta w\rangle = \left(1 - q\zeta_\alpha\left(1\right)\right)\langle M_\alpha, M_\beta w w^*\rangle.$$

If $\alpha \neq \beta$ then $\langle f, x_N g\rangle = 0$ but in general $\langle M_\alpha, M_\beta w w^*\rangle \neq 0$; for example $\alpha = (1,0,0,0)$ and $\beta = (0,1,0,0)$. For the second part let $f = M_\alpha w$ so that $\langle x_N f, x_N g\rangle = \langle M_{\alpha\Phi}, M_{\beta\Phi}\rangle$, while $\langle f, g\rangle = \langle M_\alpha, M_\beta w w^*\rangle$.

Proposition 3. *Suppose* (BF1) *holds and $\alpha \in \mathbb{N}_0^N$. Then $\left\|M_{\alpha\Phi}\right\|^2 = \frac{1-q\zeta_\alpha(1)}{1-q}\left\|M_\alpha\right\|^2$.*

Proof. Let $g \in \mathcal{P}$ with $\deg g = |\alpha|$. Then by the previous lemma

$$\langle M_{\alpha\Phi}\mathcal{D}_N, g\rangle = \left(1 - q\zeta_\alpha\left(1\right)\right)\langle M_\alpha w, g\rangle = \left(1 - q\zeta_\alpha\left(1\right)\right)\langle M_\alpha, g w^*\rangle.$$

Specialize to $gw^* = M_\alpha$ to obtain

$$\langle M_{\alpha\Phi}\mathcal{D}_N, M_\alpha\left(w^*\right)^{-1}\rangle = \left(1 - q\zeta_\alpha\left(1\right)\right)\langle M_\alpha, M_\alpha\rangle.$$

By (2.10) we have

$$\begin{aligned} \langle M_{\alpha\Phi}\mathcal{D}_N, M_\alpha\left(w^*\right)^{-1}\rangle &= \left(1-q\right)\langle M_{\alpha\Phi}, x_N\left(M_\alpha\left(w^*\right)^{-1} w^* w\right)\rangle \\ &= \left(1-q\right)\langle M_{\alpha\Phi}, M_{\alpha\Phi}\rangle. \end{aligned}$$

This completes the proof. $\square$

Next we use (BF1) to derive a formula for $\|M_\lambda\|^2$ for any $\lambda \in \mathbb{N}_0^{N,+}$. Suppose $\lambda_m \geq 1$ and $\lambda_i = 0$ for $i > m$. Let

$$\alpha = (\lambda_1, \ldots, \lambda_{m-1}, 0, \ldots 0, \lambda_m),$$
$$\beta = (\lambda_m - 1, \lambda_1, \ldots, \lambda_{m-1}, 0, \ldots),$$

so that $\alpha = \beta\Phi$, and $\gamma = \beta^+ = (\lambda_1, \ldots, \lambda_{m-1}, \lambda_m - 1, 0, \ldots)$. Then $\|M_\lambda\|^2 = \mathcal{E}(\alpha)\|M_\alpha\|^2$, $\|M_\alpha\|^2 = \frac{1-q\zeta_\beta(1)}{1-q}\|M_\beta\|^2$ and $\|M_\gamma\|^2 = \mathcal{E}(\beta)\|M_\beta\|^2$. The rank vectors for α, β are $(\ldots, m+1, \ldots, N, m)$ and $(m, 1, 2, \ldots, m-1, m+1\ldots)$ respectively. Then

(2.14)
$$\mathcal{E}(\alpha) = \prod_{i=m+1}^{N} u\left(q^{\lambda_m} t^{i-m}\right) = t^{N-m} \frac{\left(1-q^{\lambda_m}\right)\left(1-q^{\lambda_m} t^{N-m+1}\right)}{\left(1-q^{\lambda_m} t\right)\left(1-q^{\lambda_m} t^{N-m}\right)},$$
$$\zeta_\beta(1) = q^{\lambda_m - 1} t^{N-m},$$
$$\mathcal{E}(\beta) = \prod_{i=1}^{m-1} u\left(q^{\lambda_i - \lambda_m + 1} t^{m-i}\right)$$

and

$$\|M_\lambda\|^2 = \frac{1-q^{\lambda_m} t^{N-m}}{1-q} \frac{\mathcal{E}(\alpha)}{\mathcal{E}(\beta)} \|M_\gamma\|^2.$$

(The product $\mathcal{E}(\alpha)$ telescopes. If $m = N$ then $\mathcal{E}(\alpha) = 1$.) This is the key ingredient for an inductive argument. Denote the transpose of (the Ferrers diagram) $\lambda \in \mathbb{N}_0^{N,+}$ by λ', so that $\operatorname{arm}(\lambda; i, j) = \lambda_i - j$ and $\operatorname{leg}(\lambda; i, j) = \lambda'_j - i$, and define the hook product

$$h_{q,t}(\lambda; z) := \prod_{(i,j)\in\lambda} \left(1 - z q^{\operatorname{arm}(i,j)} t^{\operatorname{leg}(i,j)}\right). \tag{2.15}$$

The changes in the hook product going from λ to γ come from the hooks at $\{(i, \lambda_m) : 1 \leq i \leq m-1\}$ and $\{(m, j) : 1 \leq j \leq \lambda_m\}$. Thus

$$\frac{h_{q,t}(\lambda; z)}{h_{q,t}(\gamma; z)} = \prod_{i=1}^{m-1} \frac{1 - z q^{\lambda_i - \lambda_m} t^{m-i}}{1 - z q^{\lambda_i - \lambda_m} t^{m-i-1}} \left(1 - z q^{\lambda_m - 1}\right), \tag{2.16}$$

because $\prod\limits_{j=1}^{\lambda_m - 1} \frac{1 - z q^{\lambda_m - j}}{1 - z q^{\lambda_m - j - 1}} (1 - z) = 1 - z q^{\lambda_m - 1}$ by telescoping (this telescoping property is unique to the scalar case and the norm formulas for the vector-valued case look quite different). Furthermore

$$\frac{h_{q,t}(\lambda; tz)}{h_{q,t}(\gamma; tz)} \frac{h_{q,t}(\gamma; z)}{h_{q,t}(\lambda; z)} = t^{1-m} \prod_{i=1}^{m-1} u\left(z q^{\lambda_i - \lambda_m} t^{m-i}\right) \frac{1 - z t q^{\lambda_m - 1}}{1 - z q^{\lambda_m - 1}}.$$

Set $z=q$ to obtain

$$\mathcal{E}(\beta)=t^{m-1}\left(\frac{1-q^{\lambda_m}}{1-tq^{\lambda_m}}\right)\frac{h_{q,t}(\lambda;tq)}{h_{q,t}(\gamma;tq)}\frac{h_{q,t}(\gamma;q)}{h_{q,t}(\lambda;q)}$$

and

(2.17)

$$\begin{aligned}\frac{\|M_\lambda\|^2}{\|M_\gamma\|^2}&=t^{1-m}\left(\frac{1-q^{\lambda_m}t^{N-m}}{1-q}\right)\left(\frac{1-q^{\lambda_m}t}{1-q^{\lambda_m}}\right)\\&\quad\times\frac{h_{q,t}(\lambda;q)}{h_{q,t}(\gamma;q)}\frac{h_{q,t}(\gamma;tq)}{h_{q,t}(\lambda;tq)}t^{N-m}\frac{\left(1-q^{\lambda_m}\right)\left(1-q^{\lambda_m}t^{N-m+1}\right)}{\left(1-q^{\lambda_m}t\right)\left(1-q^{\lambda_m}t^{N-m}\right)}\\&=t^{N-2m+1}\left(\frac{1-q^{\lambda_m}t^{N-m+1}}{1-q}\right)\frac{h_{q,t}(\lambda;q)}{h_{q,t}(\gamma;q)}\frac{h_{q,t}(\gamma;tq)}{h_{q,t}(\lambda;tq)}.\end{aligned}$$

Define the generalized q,t factorial for $\lambda\in\mathbb{N}_0^{N,+}$ by $(z;q,t)_\lambda=\prod_{i=1}^{N}\left(zt^{1-i};q\right)_{\lambda_i}$, where $(z;q)_n:=\prod_{i=1}^{N}\left(1-zq^{i-1}\right)$.

Theorem 1. *Suppose* (BF1) *holds and* $\lambda\in\mathbb{N}_0^{N,+}$. *Then*

$$\|M_\lambda\|^2=t^{k(\lambda)}\frac{h_{q,t}(\lambda;q)}{h_{q,t}(\lambda;qt)}\frac{\left(qt^{N-1};q\right)_\lambda}{(1-q)^{|\lambda|}}, \tag{2.18}$$

$$k(\lambda):=\sum_{i=1}^{N}(N-2i+1)\lambda_i.$$

Proof. The formula gives the trivial result $\|1\|^2=1$, where $M_{\mathbf{0}}=1$. One needs only check

$$\frac{\left(qt^{N-1};q,t\right)_\lambda}{\left(qt^{N-1};q,t\right)_\gamma}=\frac{\left(qt^{N-m};q\right)_{\lambda_m}}{\left(qt^{N-m};q\right)_{\lambda_m-1}=1-q^{\lambda_m}t^{N-m}}$$

and $k(\lambda)-k(\gamma)=N-2m+1$. □

Note that $k(\lambda)=\sum_{i=1}^{\lfloor N/2\rfloor}(\lambda_i-\lambda_{N+1-i})(N-2i+1)\geq 0$. We now use formula (2.18), together with $\langle M_\alpha,M_\beta\rangle=0$ for $\alpha\neq\beta$ and $\|M_\alpha\|^2=\mathcal{E}(\alpha)^{-1}\|M_{\alpha^+}\|^2$, as definition of the form. It is straightforward to check properties (2.7), (2.8) and (2.9). For (2.10) we need to show $\|M_{\alpha\Phi}\|^2=\frac{1-q\zeta_\alpha(1)}{1-q}\|M_\alpha\|^2$ (detailed argument in Section 3) and the formula $\langle f\mathcal{D}_N,g\rangle=(1-q)\langle f,x_N(gw^*w)\rangle$. It suffices to prove this for $f=M_\gamma$ and $gw^*=M_\beta$ with $|\gamma|=|\beta|+1$; indeed $\langle M_\gamma\mathcal{D}_N,M_\beta(w^*)^{-1}\rangle=\langle M_\gamma\mathcal{D}_Nw^{-1},M_\beta\rangle$ and $\langle M_\gamma,x_NM_\beta w\rangle=\langle M_\gamma,M_{\beta\Phi}\rangle$. If $\gamma=\alpha\Phi$ for some α then both terms vanish for $\alpha\neq\beta$, otherwise the equation

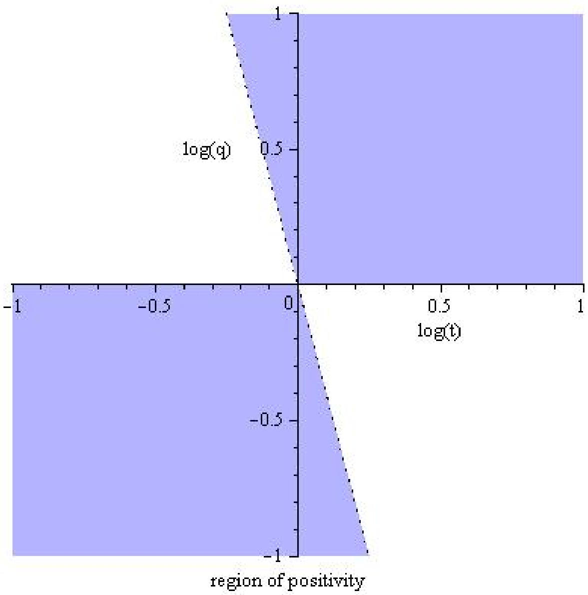

FIGURE 1. Logarithmic coordinates, N = 4

$\|M_{\alpha\Phi}\|^2 = \frac{1-q\zeta_\alpha(1)}{1-q}\|M_\alpha\|^2$ holds. If $\gamma_N = 0$ then $M_\gamma \mathcal{D}_N = 0$ and $\langle M_\gamma, M_{\beta\Phi}\rangle = 0$ (since $\gamma \neq \beta\Phi$).

The last of our concerns here is to determine the (q,t) region of positivity of $\langle\cdot,\cdot\rangle$. Inspection of the norm formula shows that there is an even number of factors of the form $1-q^a t^b$ where $a \geq 1$ and $0 \leq b \leq N$. There are two possibilities: either each such factor is positive or each is negative. Always assume $q,t > 0$ and $q \neq 1$. If each is positive then $0 < q < 1$ and $q^a t^b \leq qt^b$. If $0 < t \leq 1$ then $qt^b \leq q < 1$, or if $t > 1$ then $qt^b \leq qt^N < 1$, that is $q < t^{-N}$. If each factor is negative then $q > 1$: if $t \geq 1$ then $q^a t^b \geq q > 1$, or if $0 < t \leq 1$ then $q^a t^b \geq qt^b \geq qt^N > 1$, that is, $q > t^{-N}$.

Proposition 4. *The inner product $\langle\cdot,\cdot\rangle$ is positive-definite, that is, $\langle M_\alpha, M_\alpha\rangle > 0$ for all $\alpha \in \mathbb{N}_0^N$ provided $q,t > 0$, and $q < \min\left(1, t^{-N}\right)$ or $q > \max\left(1, t^{-N}\right)$.*

Figure 1 is an illustration of the positivity region with $N = 4$ using logarithmic coordinates.

3. Vector-valued Macdonald polynomials.

These are polynomials whose values lie in an irreducible $\mathcal{H}_N(t)$-module. The generating relations for the Hecke algebra $\mathcal{H}_N(t)$ are stated in (1.2). For the purpose of constructing a positive symmetric bilinear form we make the restriction $t > 0$. Also throughout $q, t \neq 0, 1$.

3.1. Representations of the Hecke algebra.

The irreducible modules of $\mathcal{H}_N(t)$ correspond to partitions of N and are constructed in terms of Young tableaux (see [2]).

Let τ be a partition of N, that is, $\tau \in \mathbb{N}_0^{N,+}$ and $|\tau| = N$. Thus $\tau = (\tau_1, \tau_2, \ldots)$ and often the trailing zero entries are dropped when writing τ. The length of τ is $\ell(\tau) = \max\{i : \tau_i > 0\}$. There is a Ferrers diagram of shape τ (given the same label), with boxes at points (i, j) with $1 \leq i \leq \ell(\tau)$ and $1 \leq j \leq \tau_i$. A *tableau* of shape τ is a filling of the boxes with numbers, and a *reverse standard Young tableau* (RSYT) is a filling with the numbers $\{1, 2, \ldots, N\}$ so that the entries decrease in each row and each column. Denote the set of RSYT's of shape τ by $\mathcal{Y}(\tau)$ and let $V_\tau = \mathrm{span}_{\mathbb{K}}\{S : S \in \mathcal{Y}(\tau)\}$ with orthogonal basis $\mathcal{Y}(\tau)$ (recall $\mathbb{K} = \mathbb{Q}(q, t)$). Set $n_\tau := \dim V_\tau = \#\mathcal{Y}(\tau)$. The formula for the dimension is a hook-length product. For $1 \leq i \leq N$ and $S \in \mathcal{Y}(\tau)$ the entry i is at coordinates $(\mathrm{row}(i, S), \mathrm{col}(i, S))$ and the *content* of the entry is $c(i, S) := \mathrm{col}(i, S) - \mathrm{row}(i, S)$. Each $S \in \mathcal{Y}(\tau)$ is uniquely determined by its *content vector* $[c(i, S)]_{i=1}^N$. For example let $\tau = (4, 3)$ and $S = \begin{array}{cccc} 4 & 3 & 1 & \\ 7 & 6 & 5 & 2 \end{array}$. Then the content vector is $[1, 3, 0, -1, 2, 1, 0]$. There is a representation of $\mathcal{H}_N(t)$ on V_τ, also denoted by τ (slight abuse of notation). The description will be given in terms of the actions of $\{T_i\}$ on the basis elements.

Definition 3. *The representation τ of $\mathcal{H}_N(t)$ is defined by the action of the generators specified as follows: for $1 \leq i < N$ and $S \in \mathcal{Y}(\tau)$,*

(1) *if* $\mathrm{row}(i, S) = \mathrm{row}(i+1, S)$ *(implying* $\mathrm{col}(i, S) = \mathrm{col}(i+1, S) + 1$ *and* $c(i, S) - c(i+1, S) = 1$*) then*
$$S\tau(T_i) = tS;$$

(2) *if* $\mathrm{col}(i, S) = \mathrm{col}(i+1, S)$ *(implying* $\mathrm{row}(i, S) = \mathrm{row}(i+1, S) + 1$ *and* $c(i, S) - c(i+1, S) = -1$*) then*
$$S\tau(T_i) = -S;$$

(3) *if* $\mathrm{row}(i, S) < \mathrm{row}(i+1, S)$ *and* $\mathrm{col}(i, S) > \mathrm{col}(i+1, S)$ *then* $c(i, S) - c(i+1, S) \geq 2$*; the tableau* $S^{(i)}$ *obtained from* S *by*

exchanging i and $i+1$, is an element of $\mathcal{Y}(\tau)$ and

$$S\tau(T_i) = S^{(i)} + \frac{t-1}{1-t^{c(i+1,S)-c(i,S)}} S;$$

(4) *if* $c(i,S) - c(i+1,S) \leq -2$, *thus* $\operatorname{row}(i,S) > \operatorname{row}(i+1,S)$ *and* $\operatorname{col}(i,S) < \operatorname{col}(i+1,S)$, *then with* $b = c(i,S) - c(i+1,S)$,

$$S\tau(T_i) = \frac{t\left(t^{b+1}-1\right)\left(t^{b-1}-1\right)}{\left(t^b-1\right)^2} S^{(i)} + \frac{t^b(t-1)}{t^b-1} S.$$

The formulas in (4) are consequences of those in (3) by interchanging S and $S^{(i)}$ and applying the relations $(\tau(T_i) + I)(\tau(T_i) - tI) = 0$ (where I denotes the identity operator on V_τ). There is a partial order on $\mathcal{Y}(\tau)$ related to the inversion number, namely

$$\operatorname{inv}(S) := \#\{(i,j) : 1 \leq i < j \leq N, c(i,S) \geq c(j,S) + 2\}, \tag{3.1}$$

so $\operatorname{inv}\left(S^{(i)}\right) = \operatorname{inv}(S) - 1$ in (3) above. The *inv*-maximal element S_0 of $\mathcal{Y}(\tau)$ has the numbers $N, N-1, \ldots, 1$ entered column-by-column, and the *inv*-minimal element S_1 of $\mathcal{Y}(\tau)$ has the numbers $N, N-1, \ldots, 1$ entered row-by-row. The set $\mathcal{Y}(\tau)$ can be given the structure of a Yang–Baxter graph, with root S_0, sink S_1 with arrows labeled by T_i joining S to $S^{(i)}$ as in (3). Some properties can be proved by induction on the inversion number. Recall $u(z) = \frac{(t-z)(1-tz)}{(1-z)^2} = u\left(z^{-1}\right)$.

Definition 4. *The bilinear symmetric form $\langle \cdot, \cdot \rangle_0$ on V_τ is defined to be the linear extension of*

$$\langle S, S' \rangle_0 = \delta_{S,S'} \prod_{\substack{i<j \\ c(j,S)-c(i,S) \geq 2}} u\left(t^{c(i,S)-c(j,S)}\right). \tag{3.2}$$

Proposition 5. *Suppose $f, g \in V_\tau$. Then $\langle f\tau(T_i), g \rangle_0 = \langle f, g\tau(T_i) \rangle_0$ for $1 \leq i < N$. If $c(i,S) - c(i+1,S) \geq 2$ for some i, S then $\langle S^{(i)}, S^{(i)} \rangle_0 = u\left(t^{c(i,S)-c(i+1,S)}\right) \langle S, S \rangle_0$.*

Proof. If $\operatorname{row}(i,S) = \operatorname{row}(i+1,S)$ or $\operatorname{col}(i,S) = \operatorname{col}(i+1,S)$ then

$$\langle S\tau(T_i), S \rangle_0 = t\langle S, S \rangle_0 = \langle S, S\tau(T_i) \rangle_0$$

or

$$\langle S\tau(T_i), S \rangle_0 = -\langle S, S \rangle_0 = \langle S, S\tau(T_i) \rangle_0$$

respectively. If $c(i,S) - c(i+1,S) \geq 2$ and $b = c(i+1,S) - c(i,S)$ then $\langle S^{(i)}, S^{(i)} \rangle_0 / \langle S, S \rangle_0 = u\left(t^{-b}\right)$ (in the product the only difference is the term $(i,i+1)$, appearing in $\langle S^{(i)},S^{(i)}\rangle_0)$. Then

$$\langle S\tau(T_i),S^{(i)}\rangle_0=\langle S^{(i)},S^{(i)}\rangle_0+\frac{t-1}{1-t^b}\langle S^{(i)},S\rangle_0=\langle S^{(i)},S^{(i)}\rangle_0,$$

$$\langle S^{(i)}\tau(T_i),S\rangle_0=\frac{t\left(t^{b+1}-1\right)\left(t^{b-1}-1\right)}{\left(t^b-1\right)^2}\langle S,S\rangle_0+\frac{t^b(t-1)}{t^b-1}\langle S^{(i)},S\rangle_0$$
$$=\frac{t\left(t^{b+1}-1\right)\left(t^{b-1}-1\right)}{\left(t^b-1\right)^2}\langle S,S\rangle_0=u\left(t^b\right)\langle S,S\rangle_0,$$

thus $\langle S\tau(T_i),S^{(i)}\rangle_0=\langle S^{(i)}\tau(T_i),S\rangle_0$. These statements imply that $\langle f\tau(T_i),g\rangle_0=\langle f,g\tau(T_i)\rangle_0$ for $f,g\in V_\tau$. ☐

Furthermore if $t>0$ then $\langle S,S\rangle_0\geq 0$; each term is of the form $\frac{(t-t^m)\left(1-t^{m+1}\right)}{(1-t^m)^2}$ with $m\geq 2$; either all parts are positive or all are negative depending on $0<t<1$ or $t>1$ respectively (the limit as $t\to 1$ is $\frac{m^2-1}{m^2}>0$). Denote $\langle f,f\rangle_0=\|f\|_0^2$ for $f\in V_\tau$.

There is a commutative set of Jucys–Murphy elements in $\mathcal{H}_N(t)$ which are diagonalized with respect to the basis $\mathcal{Y}(\tau)$.

Definition 5. *Set* $\phi_N:=1$ *and* $\phi_i:=\frac{1}{t}T_i\phi_{i+1}T_i$ *for* $1\leq i<N$.

Proposition 6. *Suppose* $1\leq i\leq N$ *and* $S\in\mathcal{Y}(\tau)$. *Then* $S\tau(\phi_i)=t^{c(i,S)}S$.

Proof. Arguing inductively suppose that $S\tau(\phi_{i+1})=t^{c(i+1,S)}S$ for all $S\in\mathcal{Y}(\tau)$; this is trivially true for $i=N-1$ since $c(N,S)=0$ and $\phi_N=1$. If $\operatorname{row}(i,S)=\operatorname{row}(i+1,S)$ then $S\tau(\phi_i)=\frac{1}{t}S\tau(T_i)\tau(\phi_{i+1})\tau(T_i)=t^{c(i+1,S)+1}S$ and $c(i,S)=c(i+1,S)+1$. If $\operatorname{col}(i,S)=\operatorname{col}(i+1,S)$ then $S\tau(\phi_i)=\frac{1}{t}S\tau(T_i)\tau(\phi_{i+1})\tau(T_i)=\frac{1}{t}t^{c(i+1,S)}S$ and $c(i,S)=c(i+1,S)-1$ (since $S\tau(T_i)=-S$). Suppose $c(i,S)-c(i+1,S)\geq 2$. Then the matrices $\mathcal{T},\Phi$ of $\tau(T_i),\tau(\phi_{i+1})$ respectively with respect to the basis $\left[S,S^{(i)}\right]$ are

$$\mathcal{T}=\begin{bmatrix}-\frac{1-t}{1-\rho} & 1\\ \frac{(1-\rho t)(t-\rho)}{(1-\rho)^2} & \frac{\rho(1-t)}{1-\rho}\end{bmatrix},\ \Phi=\begin{bmatrix}t^{c(i+1,S)} & 0\\ 0 & t^{c(i,S)}\end{bmatrix},$$

where $\rho=t^{c(i+1,S)-c(i,S)}$. A simple calculation shows

$$\frac{1}{t}\mathcal{T}\Phi\mathcal{T}=\begin{bmatrix}t^{c(i,S)} & 0\\ 0 & t^{c(i+1,S)}\end{bmatrix}.$$ ☐

3.2. **Polynomials and operators.** Let $\mathcal{P}_\tau:=\mathcal{P}\otimes V_\tau$. The equation $\deg(f)=n$ means $f\in\mathcal{P}_n\otimes V_\tau$. The action of $\mathcal{H}_N(t)$ and the operators are defined as follows: with $p \in \mathcal{P}, S \in \mathcal{Y}(\tau)$ and $1 \leq i < N$,

$$(p(x) \otimes S)\boldsymbol{T}_i := (1-t)\,x_{i+1}\frac{p(x) - p(x.s_i)}{x_i - x_{i+1}} \otimes S + p(x.s_i) \otimes S\tau(T_i), \tag{3.3}$$

$$\omega := T_1 T_2 \cdots T_{N-1}, \tag{3.4}$$

$$(p(x) \otimes S)\boldsymbol{w} := p(qx_N, x_1, \ldots, x_{N-1}) \otimes S\tau(\omega), \tag{3.5}$$

$$\xi_i := t^{i-N}\boldsymbol{T}_{i-1}^{-1} \cdots \boldsymbol{T}_1^{-1}\boldsymbol{w}\boldsymbol{T}_{N-1} \cdots \boldsymbol{T}_i, \tag{3.6}$$

$$\mathcal{D}_N := (1-\xi_N)/x_N,\ \mathcal{D}_i := \frac{1}{t}\boldsymbol{T}_i\mathcal{D}_{i+1}\boldsymbol{T}_i. \tag{3.7}$$

By the braid relations we have

$$\begin{aligned} T_{i+1}\omega &= T_1 \cdots T_{i+1}T_iT_{i+1}T_{i+2} \cdots T_{N-1} \\ &= T_1 \cdots T_iT_{i+1}T_iT_{i+2} \cdots T_{N-1} = \omega T_i, \end{aligned}$$

for $1 \leq i < N-1$. It follows that $\boldsymbol{T}_{i+1}\boldsymbol{w} = \boldsymbol{w}\boldsymbol{T}_i$ acting on $\mathcal{P}_\tau$. The operators $\{\xi_i\}$ mutually commute and the simultaneous polynomial eigenfunctions are the vector-valued (nonsymmetric) Macdonald polynomials. The factor t^{i-N} in ξ_i appears to differ from the scalar case, but if $\tau = (N)$, the trivial representation, then $S\tau(T_i) = tS$ (the unique RSYT of shape (N)) and $S\tau(\omega) = t^{N-1}S$, and thus ξ_i coincides with (2.1). The operator ξ_i acting on constants coincides with $I \otimes \tau(\phi_i)$:

$$\begin{aligned} (1 \otimes S)\,\xi_i &= t^{i-N} \otimes S\tau\left(T_{i-1}^{-1} \cdots T_1^{-1}T_1T_2 \cdots T_{N-1}T_{N-1} \cdots T_i\right) \\ &= t^{i-N} \otimes S\tau(T_i \cdots T_{N-1}T_{N-1} \cdots T_i) \\ &= 1 \otimes S\tau(\phi_i) = t^{c(i,S)}(1 \otimes S). \end{aligned}$$

For each $\alpha \in \mathbb{N}_0^N$ and $S \in \mathcal{Y}(\tau)$ there is an $\{\xi_i\}$ eigenfunction

$$M_{\alpha,S}(x) = \eta(\alpha,S)\,x^\alpha \otimes S\tau(R_\alpha) + \sum_{\alpha \rhd \beta} x^\beta \otimes B_{\alpha,\beta,S}(q,t), \tag{3.8}$$

where $\eta(\alpha,S) = q^a t^b$ with $a,b \in \mathbb{N}_0$, $R_\alpha \in \mathcal{H}_N(t)$, $B_{\alpha,\beta,S}(q,t) \in V_\tau$. Furthermore R_α is an analog of r_α (see [3, p. 9]); if $\alpha \in \mathbb{N}_0^{N,+}$ then $R_\alpha = I$, and if $\alpha_i < \alpha_{i+1}$ then $R_{\alpha.s_i} = R_\alpha T_i$ (there is a definition of R_α below). Furthermore

$$\begin{aligned} M_{\alpha,S}\xi_i &= \zeta_{\alpha,S}(i)\,M_{\alpha,S}, 1 \leq i \leq N, \\ \zeta_{\alpha,S}(i) &= q^{\alpha_i}t^{c(r_\alpha(i),S)}. \end{aligned} \tag{3.9}$$

These polynomials are produced with the Yang–Baxter graph. The typical node (labeled by (α, S)) is

$$(\alpha, S, \zeta_{\alpha,S}, R_\alpha, M_{\alpha,S})$$

and the root is $\left(\mathbf{0}, S_0, \left[t^{c(i,S_0)}\right]_{i=1}^N, I, 1 \otimes S_0\right)$.

There are steps:

- if $\alpha_i < \alpha_{i+1}$ there is a step labeled s_i

$$(\alpha, S, \zeta_{\alpha,S}, R_\alpha, M_{\alpha,S}) \to (\alpha.s_i, S, \zeta_{\alpha.s_i,S}, R_{\alpha.s_i}, M_{\alpha.s_i,S}),$$

$$M_{\alpha.s_i,S} = M_{\alpha,S}\boldsymbol{T}_i + \frac{t-1}{\zeta_{\alpha,S}(i+1)/\zeta_{\alpha,S}(i) - 1} M_{\alpha,S}, \tag{3.10}$$

$$R_{\alpha.s_i} = R_\alpha T_i,\ \eta(\alpha.s_i, S) = \eta(\alpha, S)$$

(note that $\left(x_i^{\alpha_i} x_{i+1}^{\alpha_{i+1}} \otimes S R_\alpha\right) \boldsymbol{T}_i = x_i^{\alpha_{i+1}} x_{i+1}^{\alpha_i} \otimes S\tau(R_\alpha T_i) + \cdots$);

- if $\alpha_i = \alpha_{i+1}$, $j = r_\alpha(i)$ (thus $j+1 = r_\alpha(i+1)$, and $R_\alpha T_i = T_j R_\alpha$; see [3, Lemma 2.14]) and $c(j,S) - c(j+1,S) \geq 2$ there is a step

$$(\alpha, S, \zeta_{\alpha,S}, R_\alpha, M_{\alpha,S}) \to \left(\alpha, S^{(j)}, (\zeta_{\alpha,S}).s_i, R_\alpha, M_{\alpha,S^{(j)}}\right),$$

$$M_{\alpha,S^{(j)}} = M_{\alpha,S}\boldsymbol{T}_i + \frac{t-1}{\zeta_{\alpha,S}(i+1)/\zeta_{\alpha,S}(i) - 1} M_{\alpha,S}, \tag{3.11}$$

$$\frac{\zeta_{\alpha,S}(i+1)}{\zeta_{\alpha,S}(i)} = t^{c(j+1,S)-c(j,S)}, \eta\left(\alpha, S^{(j)}\right) = \eta(\alpha, S);$$

For these formulas to be valid it is required that the denominators $\zeta_{\alpha,S}(i+1)/\zeta_{\alpha,S}(i) - 1 \neq 0$, that is, $q^{\alpha_{i+1}-\alpha_i} t^{c(r_\alpha(i+1),S)-c(r_\alpha(i),S)} \neq 1$. From the bound $|c(j,S) - c(j',S)| \leq \tau_1 + \ell(\tau) - 2$ we obtain the necessary condition $q^q t^b \neq 1$ for $a \geq 0$ and $|b| \leq \tau_1 + \ell(\tau) - 2$. These conditions are satisfied in the region of positivity described in Proposition 11.

The other possibilities for the action of $\boldsymbol{T}_i$ are:

- if $\alpha_i > \alpha_{i+1}$ set $\rho := \zeta_{\alpha,S}(i)/\zeta_{\alpha,S}(i+1)$ then

$$M_{\alpha,S}\boldsymbol{T}_i = \frac{(1-t\rho)(t-\rho)}{(1-\rho)^2} M_{\alpha.s_i,S} + \frac{\rho(1-t)}{(1-\rho)} M_{\alpha,S}; \tag{3.12}$$

- if $\alpha_i = \alpha_{i+1}$ and $j = r_\alpha(i)$, $c(j,S) - c(j+1,S) \leq 2$, $\rho = t^{c(j,S)-c(j+1,S)}$ then

$$M_{\alpha,S}\boldsymbol{T}_i = \frac{(1-t\rho)(t-\rho)}{(1-\rho)^2} M_{\alpha,S^{(j)}} + \frac{\rho(1-t)}{(1-\rho)} M_{\alpha,S}; \tag{3.13}$$

- if $\alpha_i = \alpha_{i+1}$ and $j = r_\alpha(i)$, $\mathrm{row}(j,S) = \mathrm{row}(j+1,S)$ then $M_{\alpha,S}\boldsymbol{T}_i = tM_{\alpha,S}$;

- if $\alpha_i = \alpha_{i+1}$ and $j = r_\alpha(i), \mathrm{col}(j,S) = \mathrm{col}(j+1,S)$ then $M_{\alpha,S}\boldsymbol{T}_i = -M_{\alpha,S}$.

The degree-raising operation, namely, the affine step, takes α to $\alpha\Phi := (\alpha_2, \alpha_3, \ldots, \alpha_N, \alpha_1 + 1)$:

$$(3.14)\qquad \begin{aligned} (\alpha, S, \zeta_{\alpha,S}, R_\alpha, M_{\alpha,S}) &\to (\alpha\Phi, S, \zeta_{\alpha\Phi,S}, R_{\alpha\Phi}, M_{\alpha\Phi,S}), \\ M_{\alpha\Phi,S} &= x_N (M_{\alpha,S}\boldsymbol{w}), \\ \alpha\Phi &= (\alpha_2, \alpha_3, \ldots, \alpha_N, \alpha_1 + 1), \\ \zeta_{\alpha\Phi,S} &= (\zeta_{\alpha,S}(2), \ldots, \zeta_{\alpha,S}(N), q\zeta_{\alpha,S}(1)). \end{aligned}$$

The inversion number $\mathrm{inv}(\alpha)$ of $\alpha \in \mathbb{N}_0^N$ is the length of the shortest product $g = s_{i_1}s_{i_2}\cdots s_{i_m}$ such that $\alpha.g = \alpha^+$. From this and the Yang–Baxter graph we deduce that the series of steps $s_{i_1}, s_{i_2}, \cdots, s_{i_m}$ leads from $M_{\alpha,S}$ to $M_{\alpha^+,S}$ and $R_\alpha T_{i_i}T_{i_2}\cdots T_{i_m} = R_{\alpha^+} = I$.

Definition 6. *Suppose* $\alpha \in \mathbb{N}_0^N$. *Then* $R_\alpha := (T_{i_i}T_{i_2}\cdots T_{i_m})^{-1}$ *where* $\alpha.s_{i_1}s_{i_2}\cdots s_{i_m} = \alpha^+$ *and* $m = \mathrm{inv}(\alpha)$.

There may be different products $\alpha.s_{j_1}s_{j_2}\cdots s_{ij} = \alpha^+$ of length $\mathrm{inv}(\alpha)$ but they all give the same value of R_α by the braid relations. It is shown in [3, p. 10, Eq. (2.15)] that $R_\alpha\omega = t^{N-m}\phi_m R_{\alpha\Phi}$ with $m = r_\alpha(1)$.

3.3. **The bilinear symmetric form.** We will define a symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathcal{P}_\tau$ satisfying certain postulates, using the same logical outline as in Section 2; first we derive consequences from these, then state the definition and show that the desired properties apply.

The **hypotheses** (BF2) for the symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathcal{P}_\tau$, with $\boldsymbol{w}^* := \boldsymbol{T}_{N-1}^{-1}\cdots\boldsymbol{T}_1^{-1}\boldsymbol{w}\boldsymbol{T}_{N-1}\cdots\boldsymbol{T}_1$, are (for $f, g \in \mathcal{P}_\tau,\ S, S' \in \mathcal{Y}(\tau),\ 1 \le i < N$):

$$(3.15a)\qquad \langle 1\otimes S, 1\otimes S'\rangle = \langle S, S'\rangle_0,$$

$$(3.15b)\qquad \langle f\boldsymbol{T}_i, g\rangle = \langle f, g\boldsymbol{T}_i\rangle,$$

$$(3.15c)\qquad \langle f\xi_N, g\rangle = \langle f, g\xi_N\rangle,$$

$$(3.15d)\qquad \langle f\mathcal{D}_N, g\rangle = (1-q)\langle f, x_N(g\boldsymbol{w}^*\boldsymbol{w})\rangle.$$

Properties (3.15b) and (3.15c) imply $\langle f\xi_i, g\rangle = \langle f, g\xi_i\rangle$ for each i and thus the $M_{\alpha,S}$'s are mutually orthogonal. As in the scalar case $\langle f\boldsymbol{w}, g\rangle = \langle f, g\boldsymbol{w}^*\rangle$. As before denote $\langle f, f\rangle = \|f\|^2$. First we will show that these hypotheses determine the form uniquely when $q, t \neq 0, 1$ without recourse to the Macdonald polynomials. We use the commutation relationships $(x_{i+1}f)\boldsymbol{T}_i = x_i(f\boldsymbol{T}_i) + (t-1)x_{i+1}f$ and $(x_jf)\boldsymbol{T}_i = x_j(f\boldsymbol{T}_i)$ for $f \in \mathcal{P}_\tau$ and $j \neq i, i+1$ (a simple direct computation).

Proposition 7. *Suppose* (BF2) *holds. Then for* $1 \leq i \leq j \leq N$ *and* $q, t \neq 0, 1$ *there are operators* $A_{i,j}, B_{i,j}$*, on* $\mathcal{P}_\tau$ *preserving degree of homogeneity such that* $A_{i,i}$ *and* $B_{i,i}$ *are invertible and for* $f, g \in \mathcal{P}_\tau$

$$\langle f\mathcal{D}_i, g\rangle = \sum_{j=i}^{N} \langle f, x_j\,(gA_{i,j})\rangle,$$

$$\langle f, x_i g\rangle = \sum_{j=i}^{N} \langle f\mathcal{D}_j, gB_{i,j}\rangle.$$

Proof. Suppose $i = N$. Then $A_{N,N} = (1-q)\,\boldsymbol{w}^*\boldsymbol{w}$ and $B_{N,N} = A_{N,N}^{-1}$ by (3.15d). Arguing by induction suppose the statement is true for $k+1 \leq i \leq N$. Then for any $f, g \in \mathcal{P}_\tau$

$$\begin{aligned}\langle f\mathcal{D}_k, g\rangle &= \frac{1}{t}\langle f\boldsymbol{T}_k\mathcal{D}_{k+1}\boldsymbol{T}_k, g\rangle = \frac{1}{t}\langle f\boldsymbol{T}_k\mathcal{D}_{k+1}, g\boldsymbol{T}_k\rangle \\ &= \frac{1}{t}\sum_{j=k+1}^{N} \langle f\boldsymbol{T}_k, x_j\,(g\boldsymbol{T}_k A_{k+1,j})\rangle \\ &= \frac{1}{t}\sum_{j=k+1}^{N} \langle f, \{x_j\,(g\boldsymbol{T}_k A_{k+1,j})\}\,\boldsymbol{T}_k\rangle.\end{aligned}$$

Hence

$$\begin{aligned}\{x_{k+1}\,(g\boldsymbol{T}_k A_{k+1,k+1})\}\,\boldsymbol{T}_k &= x_k\,(g\boldsymbol{T}_k A_{k+1,k+1}\boldsymbol{T}_k) \\ &\quad + (t-1)\,x_{k+1}\,(g\boldsymbol{T}_k A_{k+1,k+1})\,, \\ \{x_j\,(g\boldsymbol{T}_k A_{k+1,j})\}\,\boldsymbol{T}_k &= x_j\,(g\boldsymbol{T}_k A_{k+1,j}\boldsymbol{T}_k)\,.\end{aligned}$$

Thus set $A_{k,k} := \frac{1}{t}\boldsymbol{T}_k A_{k+1,k+1}\boldsymbol{T}_k$, $A_{k,k+1} := \frac{t-1}{t}\boldsymbol{T}_k A_{k+1,k+1}$ and $A_{k,j} := \frac{1}{t}\boldsymbol{T}_k A_{k+1,j}\boldsymbol{T}_k$ for $j > k+1$. Next

$$\langle f, x_k\,(gA_{k,k})\rangle = \langle f\mathcal{D}_k, g\rangle - \sum_{j=k+1}^{N} \langle f, x_j\,(gA_{k,j})\rangle.$$

Replace g by $gA_{k,k}^{-1}$ and use the inductive hypothesis to get

$$\langle f, x_k g\rangle = \langle f\mathcal{D}_k, gA_{k,k}^{-1}\rangle + \sum_{m=k+1}^{N} \langle f\mathcal{D}_m, gB_{k,m}\rangle,$$

$$B_{k,m} := -\sum_{j=k+1}^{m} A_{k,k}^{-1}A_{k,j}B_{j,m}, B_{k,k} := A_{k,k}^{-1}.$$

This completes the induction. $\square$

Corollary 1. *The symmetric bilinear form is uniquely determined by the hypotheses* (BF2)*. If* $\deg(f) < \deg(g)$ *then* $\langle f, g\rangle = 0$.

Proof. If $\deg(g) = n \geq 1$ then g can be expressed as a sum $g = \sum_{i=1}^{N} x_i g_i$ with $\deg(g_i) = n-1$ for i such that $g_i \neq 0$. This shows that if $f = 1 \otimes S$ and $\deg(g) \geq 1$ then $\langle f, g\rangle = 0$ because $f\mathcal{D}_i = 0$ for all i. Arguing inductively suppose the stated orthogonality property holds for all h with $\deg(h) \leq k$ and let $\deg(f) = k+1, \deg(g) > k$. Then $\langle f, x_i g\rangle = \sum_{j=i}^{N} \langle f\mathcal{D}_j, gB_{i,j}\rangle = 0$ because $\deg(f\mathcal{D}_j) = k < \deg(gB_{i,j})$. Thus the orthogonality property holds for $k+1$. The form is uniquely defined for $\mathcal{P}_0 \otimes V_\tau$ and a similar inductive argument shows that $\langle f, g\rangle$ is uniquely determined when $\deg(f) = \deg(g) > 0$. □

However the result does not prove existence. A closer look at the formulas shows that $(1-q)^{|\alpha|}\langle x^\alpha \otimes S, x^\beta \otimes S'\rangle$ is a Laurent polynomial in q, t (a sum of $q^a t^b$ with $a, b \in \mathbb{Z}$) for any $\alpha, \beta \in \mathbb{N}_0^N$, $S, S' \in \mathcal{Y}(\tau)$.

Recall $u(z) := \frac{(1-tz)(t-z)}{(1-z)^2}$.

Lemma 2. *Suppose* (BF2) *holds and suppose* (α, S) *satisfies* $\alpha_i < \alpha_{i+1}$*. Then with* $\rho = \zeta_{\alpha,S}(i+1)/\zeta_{\alpha,S}(i)$ *we have*

$$\|M_{\alpha.s_i,S}\|^2 = u(\rho)\|M_{\alpha,S}\|^2.$$

Proof. From (3.10) and (3.12) we have

$$M_{\alpha,S}\boldsymbol{T}_i = -\frac{1-t}{1-\rho}M_{\alpha,S} + M_{\alpha.s_i,S},$$
$$M_{\alpha.s_i,S}\boldsymbol{T}_i = \frac{(1-t\rho)(t-\rho)}{(1-\rho)^2}M_{\alpha,S} + \frac{\rho(1-t)}{(1-\rho)}M_{\alpha.s_i,S}.$$

Take the inner product of the first equation with $M_{\alpha.s_i,S}$ and use $\langle M_{\alpha,S}, M_{\alpha.s_i,S}\rangle = 0$, then take the inner product of the second equation with $M_{\alpha,S}$ and again use $\langle M_{\alpha,S}, M_{\alpha.s_i,S}\rangle = 0$ to obtain

$$\langle M_{\alpha,S}\boldsymbol{T}_i, M_{\alpha.s_i,S}\rangle = \|M_{\alpha.s_i,S}\|^2,$$
$$\langle M_{\alpha,S}, M_{\alpha.s_i,S}\boldsymbol{T}_i\rangle = \frac{(1-t\rho)(t-\rho)}{(1-\rho)^2}\|M_{\alpha,S}\|^2.$$

The hypothesis $\langle M_{\alpha,S}\boldsymbol{T}_i, M_{\alpha.s_i,S}\rangle = \langle M_{\alpha,S}, M_{\alpha.s_i,S}\boldsymbol{T}_i\rangle$ completes the proof. □

Lemma 3. *Suppose* (BF2) *holds and suppose* (α, S) *satisfies* $\alpha_i = \alpha_{i+1}$*,* $j = r_\alpha(i)$*,* $c(j,S) - c(j+1,S) \geq 2$*. Then with*

$$\rho = \zeta_{\alpha,S}(i+1)/\zeta_{\alpha,S}(i) = t^{c(j+1,S)-c(j,S)}$$

we have

$$\left\|M_{\alpha,S^{(j)}}\right\|^{2}=u\left(\rho\right)\left\|M_{\alpha,S}\right\|^{2}=\frac{\left\|S^{(j)}\right\|_{0}^{2}}{\left\|S\right\|_{0}^{2}}\left\|M_{\alpha,S}\right\|^{2}.$$

Proof. Using the same argument as in the previous lemma on formulas (3.11) and (3.13), one shows $\left\|M_{\alpha,S^{(j)}}\right\|^{2}=u\left(\rho\right)\left\|M_{\alpha,S}\right\|^{2}$. Proposition 5 asserted that $u\left(\rho\right)=\left\|S^{(j)}\right\|_{0}^{2}/\left\|S\right\|_{0}^{2}$. □

Definition 7. *For $\alpha\in\mathbb{N}_{0}^{N},S\in\mathcal{Y}\left(\tau\right)$ let*

$$\mathcal{E}\left(\alpha,S\right):=\prod_{\substack{1\leq i<j\leq N\\ \alpha_{i}<\alpha_{j}}}u\left(q^{\alpha_{j}-\alpha_{i}}t^{c(r_{\alpha}(j),S)-c(r_{\alpha}(i),S)}\right). \tag{3.16}$$

There are $\operatorname{inv}\left(\alpha\right)$ terms in $\mathcal{E}\left(\alpha,S\right)$.

Lemma 4. *Suppose $\alpha\in\mathbb{N}_{0}^{N},S\in\mathcal{Y}\left(\tau\right)$. Then*

$$M_{\alpha\Phi,S}\mathcal{D}_{N}=\left(1-q\zeta_{\alpha,S}\left(1\right)\right)M_{\alpha,S}\boldsymbol{w}.$$

Proof. By definition we have

$$\begin{aligned}M_{\alpha\Phi,S}\mathcal{D}_{N}&=\left(1/x_{N}\right)M_{\alpha\Phi,S}\left(I-\xi_{N}\right)=\left(1/x_{N}\right)\left(1-\zeta_{\alpha\Phi,S}\left(N\right)\right)M_{\alpha\Phi,S}\\&=\left(1-q\zeta_{\alpha,S}\left(1\right)\right)M_{\alpha,S}\boldsymbol{w}.\end{aligned}$$

□

The following is proved exactly like Propositions 2 and 3.

Proposition 8. *Suppose* (BF2) *holds and $\alpha\in\mathbb{N}_{0}^{N}$, $S\in\mathcal{Y}\left(\tau\right)$. Then*

$$\begin{aligned}\left\|M_{\alpha^{+},S}\right\|^{2}&=\mathcal{E}\left(\alpha,S\right)\left\|M_{\alpha,S}\right\|^{2},\\ \left\|M_{\alpha,\Phi,S}\right\|^{2}&=\frac{1-q^{\alpha_{1}+1}t^{c(r_{\alpha}(1),S)}}{1-q}\left\|M_{\alpha,S}\right\|^{2}.\end{aligned}$$

The intention here is to find the explicit formula for $\left\|M_{\alpha,S}\right\|^{2}$ implied by (BF2) and then prove that, as a definition, it satisfies (BF2). We use the same inductive scheme as in Section 2.

Suppose (BF2) holds and $\lambda\in\mathbb{N}_{0}^{N,+},S\in\mathcal{Y}\left(\tau\right)$ and $\lambda_{m}>0=\lambda_{m+1}$. Then set

$$\begin{aligned}\alpha&:=\left(\lambda_{1},\ldots,\lambda_{m-1},0,\ldots0,\lambda_{m}\right),\\ r_{\alpha}&=\left(1,\ldots,m-1,m+1,\ldots,N,m\right),\\ \beta&:=\left(\lambda_{m}-1,\lambda_{1},\ldots,\lambda_{m-1},0,\ldots\right),\\ r_{\beta}&=\left(m,1,\ldots,m-1,m+1,\ldots,N\right),\\ \gamma&:=\left(\lambda_{1},\ldots,\lambda_{m-1},\lambda_{m}-1,0,\ldots\right)=\beta^{+}.\end{aligned} \tag{3.17}$$

Thus $\|M_{\lambda,S}\|^2 = \mathcal{E}(\alpha,S)\|M_{\alpha,S}\|^2$ and $\|M_{\beta,S}\|^2 = \mathcal{E}(\beta,S)^{-1}\|M_{\gamma,S}\|^2$; by Proposition 8 we have $\|M_{\alpha,S}\|^2 = \frac{1-q^{\lambda_m}t^{c(m,S)}}{1-q}\|M_{\beta,S}\|^2$. Also $\alpha.(s_{N-1}s_{N-2}\cdots s_m) = \lambda$ and $\beta.(s_1s_2\cdots s_{m-1}) = \gamma$ thus $R_\alpha = T_m^{-1}\cdots T_{N-1}^{-1}$ and $R_\beta = T_{m-1}^{-1}\cdots T_1^{-1}$. The leading term of $M_{\beta,S}$ is $\eta(\beta,S)x^\beta\otimes S\tau(R_\beta)$, so the leading term of $M_{\gamma,S}$ is $\eta(\beta,S)x^\gamma\otimes S$ (and $\eta(\gamma,S)=\eta(\beta,S)$).

Apply $\mathbf{w}$ to $M_{\beta,S}$. Then we have

(3.18)
$$\begin{aligned} x_N\left(\left(x^\beta\right)w\right)S\tau\left(R_\beta\omega\right) &= q^{\beta_1}x^\alpha\otimes S\tau\left(\left(T_{m-1}^{-1}\cdots T_1^{-1}\right)T_1\cdots T_{N-1}\right)\\ &= q^{\beta_1}x^\alpha\otimes S\tau\left(T_m\cdots T_{N-1}\right), \end{aligned}$$

and

$$\begin{aligned} S\tau\left(T_m\cdots T_{N-1}\right) &= S\tau\left(\left(T_m\cdots T_{N-1}\right)\left(T_{N-1}\cdots T_m\right)R_\alpha\right)\\ &= t^{N-m}S\tau\left(\phi_m R_\alpha\right) = t^{N-m+c(m,S)}S\tau\left(R_\alpha\right). \end{aligned} \tag{3.19}$$

Thus

$$\begin{aligned} \eta(\alpha,S) &= q^{\lambda_m-1}t^{N-m+c(m,S)}\eta(\beta,S),\\ \eta(\lambda,S) = \eta(\alpha,S) &= q^{\lambda_m-1}t^{N-m+c(m,S)}\eta(\gamma,S). \end{aligned} \tag{3.20}$$

Compute

$$\begin{aligned} \mathcal{E}(\alpha,S) &= \prod_{j=m+1}^{N} u\left(q^{\lambda_m}t^{c(m,S)-c(j,S)}\right),\\ \mathcal{E}(\beta,S) &= \prod_{i=1}^{m-1} u\left(q^{\lambda_i-\lambda_m+1}t^{c(i,S)-c(m,S)}\right). \end{aligned} \tag{3.21}$$

The argument also shows that $\eta(\lambda,S) = q^{\Sigma_1(\lambda)}t^{\Sigma_2(\lambda,S)}$ where $\Sigma_1(\lambda) := \frac{1}{2}\sum_{i=1}^N \lambda_i(\lambda_i-1)$ and $\Sigma_2(\lambda,S) = \sum_{i=1}^N \lambda_i(N-i+c(i,S))$. Recall $k(\lambda) = \sum_{i=1}^N (N-2i+1)\lambda_i$ for $\lambda\in\mathbb{N}_0^{N,+}$.

Theorem 2. *Suppose* (BF2) *holds,* $\lambda\in\mathbb{N}_0^{N,+}$ *and* $S\in\mathcal{Y}(\tau)$. *Then*

$$\begin{aligned} \|M_{\lambda,S}\|^2 &= t^{k(\lambda)}\|S\|_0^2\,(1-q)^{-|\lambda|}\prod_{i=1}^N\left(qt^{c(i,S)};q\right)_{\lambda_i}\\ &\times\prod_{1\le i<j\le N}\frac{\left(qt^{c(i,S)-c(j,S)-1};q\right)_{\lambda_i-\lambda_j}\left(qt^{c(i,S)-c(j,S)+1};q\right)_{\lambda_i-\lambda_j}}{\left(qt^{c(i,S)-c(j,S)};q\right)^2_{\lambda_i-\lambda_j}}. \end{aligned}$$

Proof. Denote the (i,j)-product by Π_λ. Suppose $\lambda_m > 0 = \lambda_{m+1}$ and $\gamma = (\lambda_1, \ldots, \lambda_{m-1}, \lambda_m - 1, 0, \ldots)$.with α, β as in (3.17). Then

(3.22)

$$\frac{\Pi_\lambda}{\Pi_\gamma} = \prod_{i=1}^{m-1} \frac{\left(1 - q^{\lambda_i - \lambda_m + 1} t^{c(i,S) - c(m,S)}\right)^2}{\left(1 - q^{\lambda_i - \lambda_m + 1} t^{c(i,S) - c(m,S) - 1}\right)\left(1 - q^{\lambda_i - \lambda_m + 1} t^{c(i,S) - c(m,S) + 1}\right)}$$

$$\times \prod_{j=m+1}^{N} \frac{\left(1 - q^{\lambda_m} t^{c(m,S) - c(j,S) - 1}\right)\left(1 - q^{\lambda_m} t^{c(m,S) - c(j,S) + 1}\right)}{\left(1 - q^{\lambda_m} t^{c(m,S) - c(j,S)}\right)^2}$$

$$= t^{2m-1-N} \prod_{i=1}^{m-1} u\left(q^{\lambda_i - \lambda_m + 1} t^{c(i,S) - c(m,S)}\right)^{-1} \prod_{j=m+1}^{N} u\left(q^{\lambda_m} t^{c(m,S) - c(j,S)}\right)$$

$$= t^{2m-1-N} \mathcal{E}(\alpha, S) / \mathcal{E}(\beta, S).$$

Also $\prod_{i=1}^{N} \left(q t^{c(i,S)}; q\right)_{\lambda_i} / \prod_{i=1}^{N} \left(q t^{c(i,S)}; q\right)_{\gamma_i} = 1 - q^{\lambda_m} t^{c(m,S)}$. The formula satisfies the relation $\|M_{\lambda,S}\|^2 = \frac{1 - q^{\lambda_m} t^{c(m,S)}}{1-q} \frac{\mathcal{E}(\alpha,S)}{\mathcal{E}(\beta,S)} \|M_{\gamma,S}\|^2$ and is valid at $\lambda = \mathbf{0}$ since $M_{\mathbf{0},S} = 1 \otimes S$ and $\|1 \otimes S\|^2 = \|S\|_0^2$. $\square$

Definition 8. *The symmetric bilinear form is given by* (3.22) *for* $\lambda \in \mathbb{N}_0^{N,+}, S \in \mathcal{Y}(\tau)$, *by* $\|M_{\alpha,S}\|^2 = \mathcal{E}(\alpha, S)^{-1} \|M_{\alpha^+,S}\|^2$ *for* $\alpha \in \mathbb{N}_0^N$ *and by* $\langle M_{\alpha,S}, M_{\beta,S'} \rangle = 0$ *for* $(\alpha, S) \neq (\beta, S')$.

Next we show that the definition satisfies the hypotheses (BF2).

The step s_i with $\alpha_i < \alpha_{i+1}$ satisfies (3.15b) because of the value $\frac{\mathcal{E}(\alpha.s_i,S)}{\mathcal{E}(\alpha,S)}$. It remains to check the step with $\alpha_i = \alpha_{i+1}$ and the affine step. The (i,j)-product in (3.22) can be written as (note $t^{-1} u(z) = \frac{(1-z/t)(1-tz)}{(1-z)^2}$)

$$\prod_{1 \le i < j \le N} t^{\lambda_j - \lambda_i} \prod_{l=1}^{\lambda_i - \lambda_j} u\left(q^l t^{c(i,S) - c(j,S)}\right).$$

Suppose $\alpha \in \mathbb{N}_0^N$ and $\lambda := \alpha^+$; in the formula for $\mathcal{E}(\alpha, S)$ the condition $(i < j) \,\&\, (\alpha_i < \alpha_j)$ is equivalent to $(i < j) \,\&\, (r_\alpha(i) > r_\alpha(j))$. Let $v_\alpha = r_\alpha^{-1}$ so that $\lambda_i = \alpha_{v_\alpha(i)}$. Then the product can be indexed by $(v_\alpha(i') < v_\alpha(j')) \,\&\, (i' > j')$ (where $i' = r_\alpha(i)$, $j' = r_\alpha(j)$). Thus

$$\mathcal{E}(\alpha, S) = \prod_{1 \le j' < i' \le N, v_\alpha(i') < v_\alpha(j')} u\left(q^{\lambda_{i'} - \lambda_{j'}} t^{c(i',S) - c(j',S)}\right).$$

Proposition 9. *Suppose* $\alpha_i = \alpha_{i+1}$, $j = r_\alpha(i)$ *and* $m = c(j,S) - c(j+1,S) \ge 2$. *Then* $\left\|M_{\alpha,S^{(j)}}\right\|^2 = \frac{\left(1-t^{1-m}\right)\left(t-t^{-m}\right)}{\left(1-t^{-m}\right)^2} \|M_{\alpha,S}\|^2$.

Proof. By hypothesis $\zeta_{\alpha,S}(i) = q^{\alpha_i}t^{c(j,S)}$ and $\zeta_{\alpha,S}(i+1) = q^{\alpha_i}t^{c(j+1,S)}$ so that $\zeta_{\alpha,S}(i+1)/\zeta_{\alpha,S}(i) = t^{-m}$. Also by Proposition 5 we have $\left\|S^{(j)}\right\|_0^2 = u(t^{-m})\|S\|_0^2$. Suppose first that $\alpha \in \mathbb{N}_0^{N,+}$. Then $j = i$. In the formula for $\|M_{\alpha,S}\|^2$ the first product does not change when S is replaced by $S^{(j)}$; the factors $\left(qt^{c(i,S)};q\right)_{\lambda_i}, \left(qt^{c(i+1,S)};q\right)_{\lambda_i}$ trade places. By a similar argument the (i,j)-product also does not change, and $\left\|M_{\alpha,S^{(j)}}\right\|^2/\left\|S^{(j)}\right\|_0^2 = \|M_{\alpha,S}\|^2/\|S\|_0^2$. Otherwise $\alpha \neq \alpha^+$ and

$$\begin{aligned}(3.23)\qquad \frac{\left\|M_{\alpha,S^{(j)}}\right\|^2}{\|S^{(j)}\|_0^2} &= \frac{\left\|M_{\alpha^+,S^{(j)}}\right\|^2}{\mathcal{E}(\alpha,S^{(j)})\|S^{(j)}\|_0^2} = \frac{\|M_{\alpha^+,S}\|^2}{\mathcal{E}(\alpha,S^{(j)})\|S\|_0^2}\\ &= \frac{\mathcal{E}(\alpha,S)}{\mathcal{E}(\alpha,S^{(j)})}\frac{\|M_{\alpha,S}\|^2}{\|S\|_0^2}.\end{aligned}$$

Recall $\mathcal{E}(\alpha,S) = \prod_{1\le l<n\le N,\,\alpha_l<\alpha_n} u\left(q^{\alpha_n-\alpha_l}t^{c(r_\alpha(n),S)-c(r_\alpha(l),S)}\right)$ and the product does not change when S is replaced by $S^{(j)}$ (the factors involving $l=i$ or $n=i$ are interchanged with those involving $l=i+1$ or $n=i+1$). Thus $\mathcal{E}\left(\alpha,S^{(j)}\right) = \mathcal{E}(\alpha,S)$. ∎

Proposition 10. *Suppose* $\alpha \in \mathbb{N}_0^N, S \in \mathcal{Y}(\tau)$. *Then*

$$\|M_{\alpha\Phi,S}\|^2 = \frac{1-q^{\alpha_1+1}t^{c(r_\alpha(1),S)}}{1-q}\|M_{\alpha,S}\|^2.$$

Proof. We need to compute various ratios of $\mathcal{E}(\alpha,S)$, $\|M_{\alpha^+,S}\|^2$, $\mathcal{E}(\alpha\Phi,S)$, $\left\|M_{(\alpha\Phi)^+,S}\right\|^2$. Also $r_\alpha(i+1) = r_{\alpha\Phi}(i)$ for $1 \le i < N$, $r_\alpha(1) = r_{\alpha\Phi}(N)$. Let $\lambda := \alpha^+$. Then $\lambda_{r_\alpha(i)} = \alpha_i$ for all i. Let $m := r_\alpha(1)$. This implies $\#\{i : \alpha_i > \alpha_1\} = m-1$, thus $\lambda_{m-1} > \lambda_m$ and $(\alpha\Phi)_m^+ = \lambda_m + 1$. Also $k\left((\alpha\Phi)^+\right) - k(\lambda) = N-2m+1$. This implies

$$\begin{aligned}\frac{\left\|M_{(\alpha\Phi)^+,S}\right\|^2}{\|M_{\lambda,S}\|^2} = t^{N-2m+1}&\frac{1-q^{\lambda_m+1}t^{c(m,S)}}{1-q}\\ &\times t^{m-1}\prod_{i=1}^{m-1}u\left(q^{\lambda_i-\lambda_m}t^{c(i,S)-c(m,S)}\right)^{-1}\\ &\times t^{m-N}\prod_{j=m+1}^{N}u\left(q^{\lambda_m+1-\lambda_i}t^{c(m,S)-c(j,S)}\right).\end{aligned}$$

Let $\mu := (\alpha\Phi)^{+}$. Then

$$\mathcal{E}(\alpha\Phi,S)=\prod_{\substack{i<j\\ v_{\Phi}(i)>v_{\alpha\Phi}(j)}} u\left(q^{\mu_i-\mu_j}t^{c(i,S)-c(j,S)}\right)$$

and

$$\mathcal{E}(\alpha,S)=\prod_{\substack{i<j\\ v_{\alpha}(i)>v_{\alpha}(j)}} u\left(q^{\lambda_i-\lambda_j}t^{c(i,S)-c(j,S)}\right).$$

From $v_\alpha(i)=v_{\alpha\Phi}(i)+1$ except $v_\alpha(m)=1$, $v_{\alpha\Phi}(m)=N$ the inversions $\{(i,j):i<j,v_\alpha(i)>v_\alpha\ (j)\}$ occur in both the products provided that $j\neq m$ in which case the pairs $\{(i,m):1\leq i\leq m-1\}$ do not occur in $\mathcal{E}(\alpha\Phi,S)$, or if $i=m$ and the pairs $\{(m,j):m<j\leq N\}$ do not occur in $\mathcal{E}(\alpha,S)$. Also $\mu_i=\lambda_i$ for all i except $\mu_m=\lambda_m+1$. Thus

$$\frac{\mathcal{E}(\alpha,S)}{\mathcal{E}(\alpha\Phi,S)}=\prod_{i=1}^{m-1}u\left(q^{\lambda_i-\lambda_m}t^{c(i,S)-c(m,S)}\right)\times\prod_{j=m+1}^{N}u\left(q^{\lambda_m+1-\lambda_j}t^{c(m,S)-c(j,S)}\right)^{-1},$$

and

$$\frac{\|M_{\alpha\Phi,S}\|^2}{\|M_{\alpha,S}\|^2}=\frac{\left\|M_{(\alpha\Phi)^+,S}\right\|^2}{\|M_{\lambda,S}\|^2}\frac{\mathcal{E}(\alpha,S)}{\mathcal{E}(\alpha\Phi,S)}=\frac{1-q^{\lambda_m+1}t^{c(m,S)}}{1-q}.$$

Finally $\zeta_{\alpha,S}(1)=q^{\alpha_1}t^{c(r_\alpha(1),S)}=q^{\lambda_m}t^{c(m,S)}$. □

Corollary 2. *The bilinear form satisfies* (3.15d)*.*

Proof. By Lemma 4 we have

$$\begin{aligned}\left\langle M_{\alpha\Phi,S}\mathcal{D}_N,M_{\alpha,S}(\boldsymbol{w}^*)^{-1}\right\rangle&=(1-q\zeta_{\alpha,S}(1))\left\langle M_{\alpha,S}\boldsymbol{w},M_{\alpha,S}(\boldsymbol{w}^*)^{-1}\right\rangle\\&=(1-q\zeta_{\alpha,S}(1))\|M_{\alpha,S}\|^2\end{aligned}$$

and

$$\begin{aligned}\left\langle M_{\alpha\Phi,S},x_N\left(M_{\alpha,S}(\boldsymbol{w}^*)^{-1}\right)\boldsymbol{w}^*\boldsymbol{w}\right\rangle&=\langle M_{\alpha\Phi,S},M_{\alpha\Phi,S}\rangle\\&=\frac{1-q\zeta_{\alpha,S}(1)}{1-q}\|M_{\alpha,S}\|^2\end{aligned}$$

by the proposition, thus $(1-q)\langle M_{\alpha\Phi,S},x_Ng\boldsymbol{w}^*\boldsymbol{w}\rangle=\langle M_{\alpha\Phi,S}\mathcal{D}_N,g\rangle$ when $g=M_{\alpha,S}(\boldsymbol{w}^*)^{-1}$. It suffices to prove

$$\langle f\mathcal{D}_N,g\rangle=(1-q)\langle f,x_N(g\boldsymbol{w}^*\boldsymbol{w})\rangle$$

for $f = M_{\gamma,S}$ and $g\boldsymbol{w}^* = M_{\beta,S'}$ with $|\gamma| = |\beta| + 1$. If $\gamma_N = 0$ then $M_{\gamma,S}\mathcal{D}_N = 0$ and $\langle M_{\gamma,S}\mathcal{D}_N, M_{\beta.S'}(\boldsymbol{w}^*)^{-1}\rangle = 0$ while

$$\langle M_{\gamma,S}, x_N (M_{\beta,S'}\boldsymbol{w})\rangle = \langle M_{\gamma,S}, M_{\beta\Phi,S'}\rangle = 0$$

because $\gamma \neq \beta\Phi$. If $\gamma = \alpha\Phi$ for some α with $(\alpha, S) \neq (\beta, S')$ then

$$\langle M_{\alpha\Phi,S}, x_N (M_{\beta,S'}\boldsymbol{w})\rangle = \langle M_{\alpha\Phi,S}, M_{\beta\Phi,S'}\rangle = 0$$

and

$$\begin{aligned}\langle M_{\alpha\Phi,S}\mathcal{D}_N, M_{\beta.S'}(\boldsymbol{w}^*)^{-1}\rangle &= (1 - q\zeta_\alpha(1))\langle M_{\alpha,S}\boldsymbol{w}, M_{\beta.S'}(\boldsymbol{w}^*)^{-1}\rangle \\ &= (1 - q\zeta_\alpha(1))\langle M_{\alpha,S}, M_{\beta.S'}\rangle = 0.\end{aligned}$$

The case $(\alpha, S) = (\beta, S')$ is already done. □

Proposition 11. *Suppose* $\dim V_\tau \geq 2$, $q, t > 0$ *and* $q \neq 1$. *Then the form* $\langle\cdot,\cdot\rangle$ *is positive-definite provided* $0 < q < \min(t^{-h_\tau}, t^{h_\tau})$ *or* $q > \max(t^{-h_\tau}, t^{h_\tau})$, *that is,* $\min(q^{-1/h_\tau}, q^{1/h_\tau}) < t < \max(q^{-1/h_\tau}, q^{1/h_\tau})$.

Proof. In the definition of $\langle M_{\alpha,S}, M_{\alpha,S}\rangle$ there is an even number of factors of the form $1 - q^a t^b$ where $a = 1, 2, 3, \ldots$ and b is one of $c(i, S)$, $c(i, S) - c(j, S)$, or $c(i, S) - c(j, S) \pm 1$. The $c(i, S)$ values lie in $[1 - \ell(\tau), \tau_1 - 1]$; thus $-h_\tau \leq b \leq h_\tau$ where $h_\tau = \tau_1 + \ell(\tau) - 1$, the maximum hook length in the Ferrers diagram λ. Consider the four cases

(1) $0 < q < 1$, $0 < t < 1$. Then $q^a t^b \leq qt^{-h_\tau} < 1$ provided $q < t^{h_\tau}$.
(2) $0 < q < 1$, $t \geq 1$. Then $q^a t^b \leq qt^{h_\tau} < 1$ provided $q < t^{-h_\tau}$.
(3) $q > 1$, $0 < t < 1$. Then $q^a t^b \geq qt^{h_\tau} > 1$ provided $q > t^{-h_\tau}$.
(4) $q > 1$, $t \geq 1$. Then $q^q t^b \geq qt^{-h_\gamma} > 1$ provided $q > t^{h_\tau}$.

Thus $\|M_{\alpha,S}\|^2 > 0$ if $\min(q^{-1/h_\tau}, q^{1/h_\tau}) < t < \max(q^{-1/h_\tau}, q^{1/h_\tau})$. □

There is an illustration in Figure 2 with $h_\tau = 3$ (for $\tau = (2, 1)$ or $\tau = (2, 2)$).

From a similar argument it follows that the transformation formulas for Macdonald polynomials have no poles when $\min(q^{-1/k}, q^{1/k}) < t < \max(q^{-1/k}, q^{1/k})$ with $k = h_\tau - 1$.

3.4. **Singular polynomials.** A singular polynomial $f \in \mathcal{P}_\tau$ is one which satisfies $f\mathcal{D}_i = 0$ for all i when (q, t) are specialized to some specific relation of the form $q^a t^b = 1$. By Proposition 7 the polynomial f satisfies $\langle f, g\rangle = 0$ for all $g \in \mathcal{P}_\tau$, and in particular $\langle f, f\rangle = 0$. Thus the singular polynomial phenomenon can not occur in the (q, t)-region of positivity. The boundary of the region does allow singular polynomials. There are Macdonald polynomials which are singular when specialized to $q = t^{h_\tau}$ or $q = t^{-h_\tau}$. These values do not produce poles in the polynomial coefficients as remarked above, since $\frac{1}{h_\tau} < \frac{1}{h_\tau - 1}$.

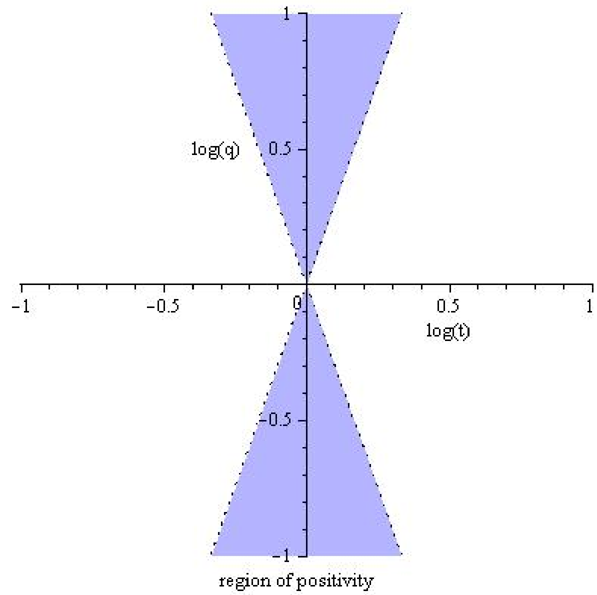

FIGURE 2. Logarithmic coordinates, $h = 3$

Proposition 12. *Suppose $\alpha \in \mathbb{N}_0^N$, $S \in \mathcal{Y}(\tau)$ and $\alpha_i = 0$ for $m < i \leq N$. Then $M_{\alpha,S}\mathcal{D}_j = 0$ for $m < j \leq N$.*

Proof. Arguing by induction the start is

$$M_{\alpha,S}\mathcal{D}_N = \frac{1}{x_N} M_{\alpha,S}(1 - \xi_N) = \frac{1}{x_N}(1 - \zeta_{\alpha,S}(N)) M_{\alpha,S} = 0,$$

since $\zeta_{\alpha,S}(N) = 1$. Suppose now that $\beta_i = 0$ for $i \geq k + 1$ implies $M_{\beta,S'}\mathcal{D}_j = 0$ for $j \geq k + 1$ and any (β, S'). Suppose $\alpha_i = 0$ for $i \geq k$. Then $r_\alpha(i) = i$ and $\zeta_{\alpha,S}(i) = t^{r(i,S)}$ for $i \geq k$ and $M_{\alpha,S}\boldsymbol{T}_k$ is one of $tM_{\alpha,S}$, $-M_{\alpha,S}$, $M_{\alpha,S^{(k)}} - \frac{t-1}{\rho-1}M_{\alpha,S}$, $\frac{(1-t\rho)(t-\rho)}{(1-\rho)^2}M_{\alpha,S^{(k)}} - \frac{t-1}{(1-\rho)}M_{\alpha,S}$ depending on $c(k+1, S) - c(k, S) = 1, = -1, \geq 2, \leq -2$ respectively and $\rho = t^{c(k+1,S)-c(k,S)}$. Then $M_{\alpha,S}\mathcal{D}_k = \frac{1}{t}(M_{\alpha,S}\boldsymbol{T}_k)\mathcal{D}_{k+1}\boldsymbol{T}_k$ and $(M_{\alpha,S}\boldsymbol{T}_k)\mathcal{D}_{k+1} = 0$ by the inductive hypothesis. □

Lemma 5. *Suppose $\alpha = (\alpha_1, \ldots, \alpha_{m-1}, 1, 0, \ldots)$ with $\alpha_i \geq 1$ for $i \leq m$ and $S \in \mathcal{Y}(\tau)$. Then*

$$M_{\alpha,S}\mathcal{D}_m = t^{m-N} \prod_{j=m}^{N-1} u\left(qt^{c(m,S)-c(j+1,S)}\right) M_{\alpha^{(N)},S}\mathcal{D}_N \boldsymbol{T}_{N-1} \cdots \boldsymbol{T}_m,$$

where $\alpha^{(N)} = (\alpha_1, \ldots, \alpha_{m-1}, 0, 0, \ldots, 1)$.

Proof. For $m \leq j \leq N$ let $\alpha^{(j)} = \left(\alpha_1, \ldots, \alpha_{m-1}, 0, \ldots, \overset{j}{1}, 0 \ldots\right)$ so that $\alpha_i^{(j)} = \alpha_i$ except $\alpha_j^{(j)} = 1$ and $\alpha_m^{(j)} = 0$ (when $j \neq m$). Then $\zeta_{\alpha^{(j)},S}(j) = qt^{c(m,S)}$ and $\zeta_{\alpha^{(j)},S}(j+1) = t^{c(j+1,S)}$ (since $r_{\alpha^{(m)}}(j) = m$) and

$$M_{\alpha^{(j)},S}\boldsymbol{T}_j = \frac{(1-t\rho)(t-\rho)}{(1-\rho)^2} M_{\alpha^{(j+1)},S} + \frac{\rho(1-t)}{(1-\rho)} M_{\alpha^{(j)},S}$$

from (3.12) with $\rho = qt^{c(m,S)-c(j+1,S)}$. Thus

$$\begin{aligned} M_{\alpha^{(j)},S}\mathcal{D}_j &= \frac{1}{t} M_{\alpha^{(j)},S}\boldsymbol{T}_j \mathcal{D}_{j+1}\boldsymbol{T}_j \\ &= \frac{1}{t} u\left(qt^{c(m,S)-c(j+1,S)}\right) M_{\alpha^{(j+1)},S}\mathcal{D}_{j+1}\boldsymbol{T}_j, \end{aligned}$$

because $M_{\alpha^{(j)},S}\mathcal{D}_{j+1} = 0$. Iterate this formula starting with $j = m$ and $\alpha^{(m)} = \alpha$, ending with $j = N-1$ to obtain the stated formula. ∎

Recall that S_1 is the *inv*-minimal RSYT with the numbers $N, N-1, N-2, \ldots, 1$ entered row-by-row and let $l = \ell(\tau)$, $\alpha = \left(1^{\tau_l}, 0^{N-\tau_l}\right)$. Thus the entry at $(l,1)$ is τ_l and $c(\tau_l, S_1) = 1-l$. The entry at $(1, \tau_1)$ is $N - \tau_1 + 1$ and $c(N-\tau_1+1, S_1) = \tau_1 - 1$.

Proposition 13. *M_{α,S_1} is singular for $q = t^{h_\tau}$.*

Proof. By the lemma with $m = \tau_l$, we have

$$M_{\alpha,S_1}\mathcal{D}_{\tau_l} = t^{\tau_l - N} \prod_{j=\tau_l}^{N-1} u\left(qt^{1-l-c(j+1,S_1)}\right) M_{\alpha^{(N)},S_1}\mathcal{D}_N \boldsymbol{T}_{N-1}\cdots\boldsymbol{T}_{\tau_l}.$$

The factors in the denominator of the product are of the form $1 - qt^{1-l-c(j+1,S_1)}$ with $c(j+1,S_1) \leq \tau_1 - 1$ so that $1 - l - c(j+1,S_1) \geq 2 - l - \tau_1 = 1 - h_\tau > h_\tau$. Furthermore the numerator factor at $j = N - \tau_1$ is $\left(t - qt^{2-l-\tau_1}\right)\left(1 - qt^{3-l-\tau_1}\right)$ which vanishes at $qt^{-h_\tau} = 1$. By Proposition 12 we have $M_{\alpha,S_1}\mathcal{D}_i = 0$ for $i > \tau_l$. If $1 \leq i < \tau_l$ then $M_{\alpha,S_1}\boldsymbol{T}_i = tM_{\alpha,S_1}$ (because $i, i+1$ are in the same row of S_1), thus

$$\begin{aligned} M_{\alpha,S_1}\mathcal{D}_i &= t^{i-\tau_l} M_{\alpha,S_i}\boldsymbol{T}_i\boldsymbol{T}_{i+1}\cdots\boldsymbol{T}_{\tau_l-1}\mathcal{D}_{\tau_l}\boldsymbol{T}_{\tau_l-1}\cdots\boldsymbol{T}_i \\ &= M_{\alpha,S_i}\mathcal{D}_{\tau_l}\boldsymbol{T}_{\tau_l-1}\cdots\boldsymbol{T}_i = 0 \end{aligned}$$

when $q = t^{h_\tau}$. ∎

We apply the same argument to S_0 where the numbers $N, N-1, \ldots, 1$ are entered column-by-column. Let $m = \tau'_{\tau_1}$, that is, the length of the last column of τ. Then the entry at $(\tau_1, 1)$ is m and $c(m, S_0) = \tau_1 - 1$. Also the entry at $(l, 1)$ is $N - l + 1$ and $c(N-l+1) = 1 - l$.

Proposition 14. *Set* $\alpha = \left(1^m, 0^{N-m}\right)$*. Then* M_{α,S_0} *is singular for* $q = t^{-h_\tau}$*.*

Proof. By Lemma 5 we have

$$M_{\alpha,S_0}\mathcal{D}_m = t^{m-N}\prod_{j=m}^{N-1} u\left(qt^{\tau_1-1-c(j+1,S_1)}\right) M_{\alpha^{(N)},S_0}\mathcal{D}_N \boldsymbol{T}_{N-1}\cdots\boldsymbol{T}_m.$$

The factors in the denominator of the product are of the form $1 - qt^{\tau_1-1-c(j+1,S_1)}$ with $c(j+1,S_1) \geq 1-l$ so that $\tau_1 - 1 - c(j+1,S_1) \leq \tau_1 + l - 2 < h_\tau$. Furthermore the numerator factor at $j = N-l$ is $\left(t - qt^{\tau_1+l-2}\right)\left(1 - qt^{\tau_1+l-1}\right)$ which vanishes at $qt^{h_\tau} = 1$. The rest of the argument is as in the previous proposition with the difference that $M_{\alpha,S_0}\boldsymbol{T}_i = -M_{\alpha,S_0}$ for $1 \leq i < m$. □

In conclusion we have constructed a symmetric bilinear form on $\mathcal{P}_\tau$ for which the operators $\boldsymbol{T}_i$ and ξ_i are self-adjoint, the Macdonald polynomials $M_{\alpha,S}$ are mutually orthogonal, and the form is positive-definite for $q > 0, q \neq 1$ and $\min\left(q^{-1/h_\tau}, q^{1/h_\tau}\right) < t < \max\left(q^{-1/h_\tau}, q^{1/h_\tau}\right))$ where $h_\tau = \tau_1 + \ell(\tau) - 1$. The bound is sharp, as demonstrated by the existence of singular polynomials for $q = t^{\pm h_\tau}$.

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