Classification of P-oligomorphic permutation groups

Conjectures of Cameron and Macpherson

Justine Falque
joint work with Nicolas M. Thiéry

Laboratoire de Recherche en Informatique
Université Paris-Sud (Orsay)

SLC, April 17h of 2019

Profile of a permutation group, a finite example

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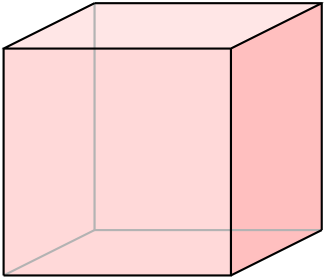

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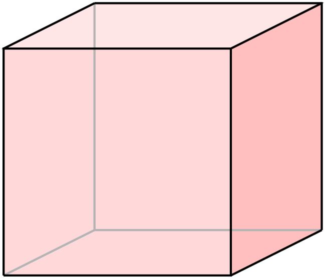

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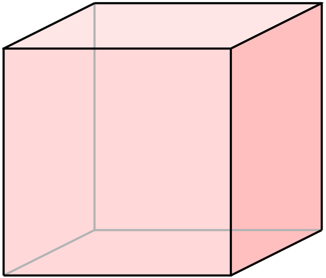

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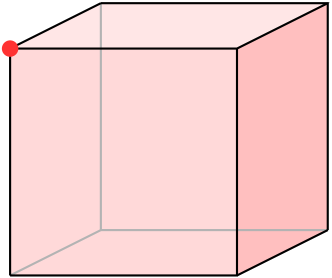

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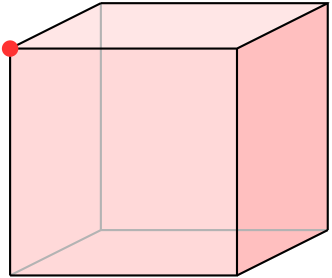

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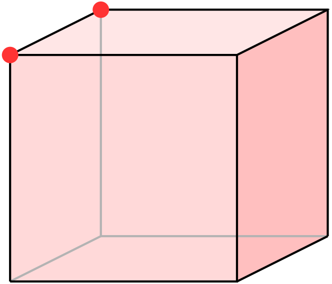

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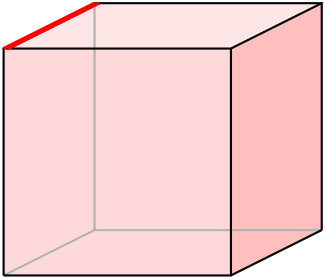

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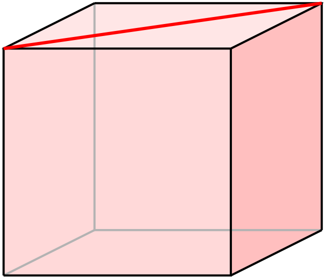

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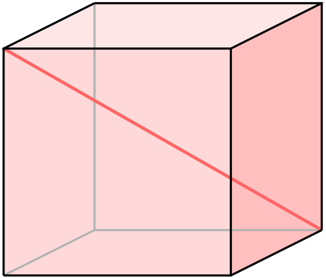

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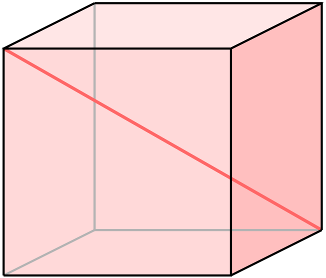

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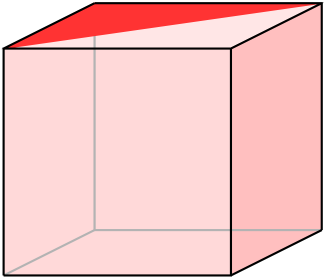

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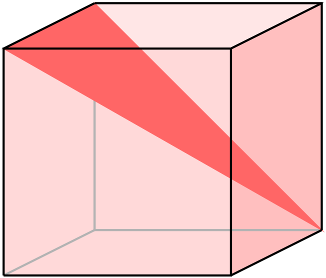

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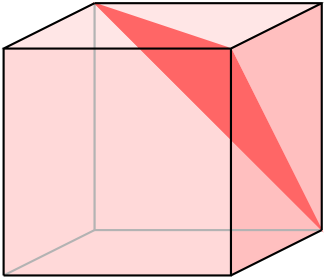

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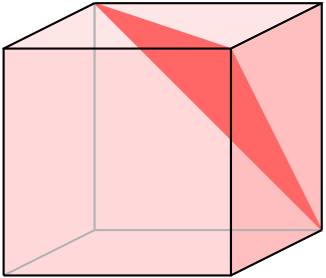

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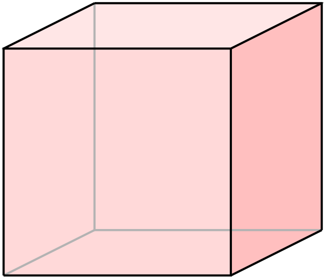

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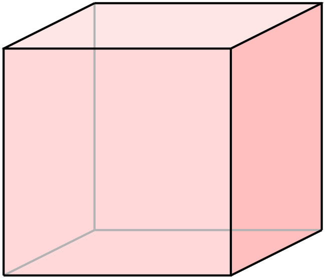

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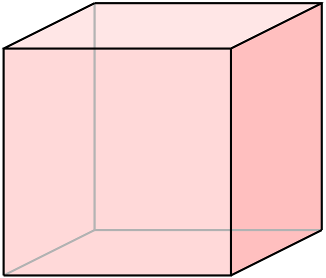

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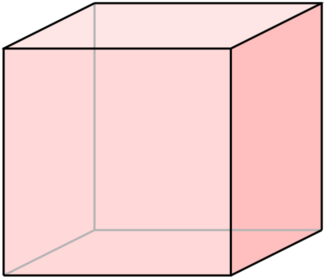

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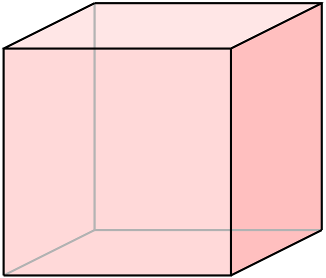

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Conjecture 1 - Cameron, 70's

G P-**oligomorphic** $\Rightarrow \mathcal{H}_G(z) = \frac{N(z)}{\prod_i (1-z^{d_i})}$ with $N(z) \in \mathbb{Z}[z]$

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Conjecture 2 (stronger) - Macpherson, 85

G P-oligomorphic $\Rightarrow$ $\mathcal{A}_G$ is finitely generated

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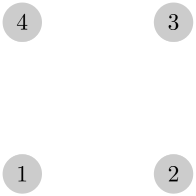
Block systems of C_4



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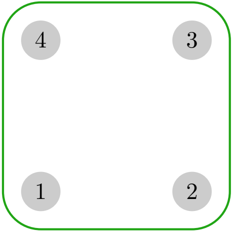

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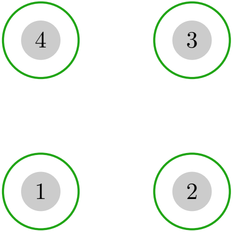

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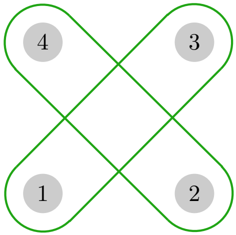

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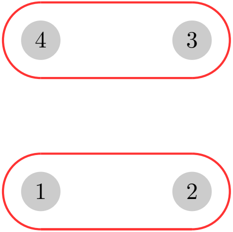
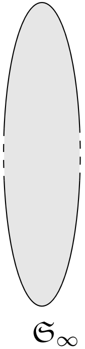


The complete primitive P-oligomorphic groups



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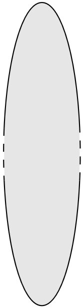
Macpherson:
G P-oligomorphic with no non trivial blocks $\Rightarrow \varphi_G(n) = 1$



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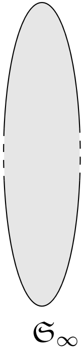

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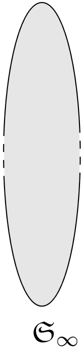

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- $\mathrm{Aut}(\mathbb{Q})$: automorphisms of the rational chain
- $\mathrm{Rev}(\mathbb{Q})$: generated by $\mathrm{Aut}(\mathbb{Q})$ and one reflection
- $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$, preserving the circular order
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- $\mathfrak{S}_\infty$: the symmetric group

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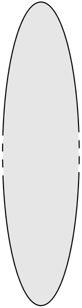
Macpherson:
G P-oligomorphic with no non trivial blocks $\Rightarrow \varphi_G(n) = 1$



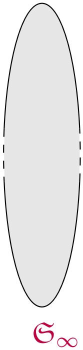
Theorem (Classification, Cameron)

Only 5 complete groups such that $\varphi_G(n) = 1 \ \ \forall n$

- $\mathrm{Aut}(\mathbb{Q})$: automorphisms of the rational chain
- $\mathrm{Rev}(\mathbb{Q})$: generated by $\mathrm{Aut}(\mathbb{Q})$ and one reflection
- $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$, preserving the circular order
- $\mathrm{Rev}(\mathbb{Q}/\mathbb{Z})$: generated by $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$ and a reflection
- $\mathfrak{S}_\infty$: the symmetric group

Well known, nice groups (called *highly homogeneous*).
In particular, their orbit algebra is finitely generated.

An infinite example: $\mathfrak{S}_\infty \wr \mathfrak{S}_3$



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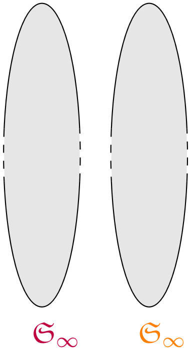

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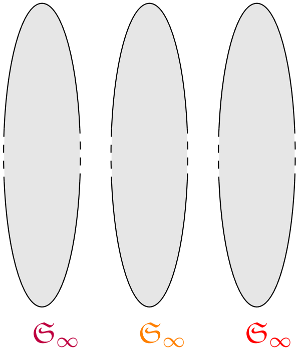

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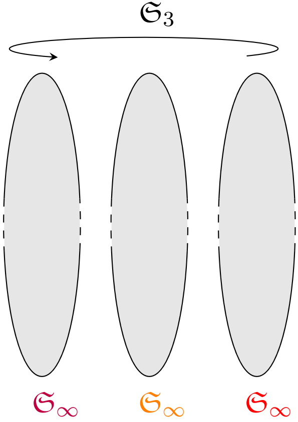

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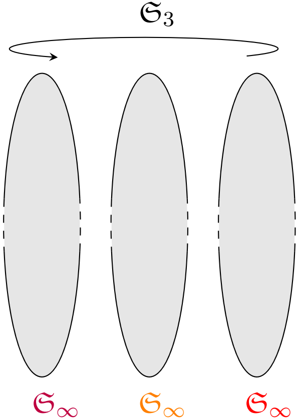

Wreath product

$\mathfrak{S}_\infty \wr \mathfrak{S}_3$

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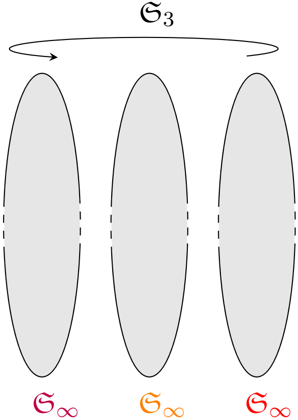

Wreath product

$$\mathfrak{S}_\infty \wr \mathfrak{S}_3 \quad \simeq \quad \mathfrak{S}_\infty^3 \rtimes \mathfrak{S}_3$$

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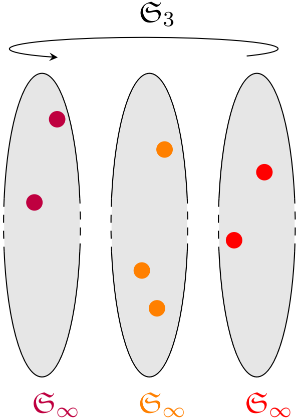
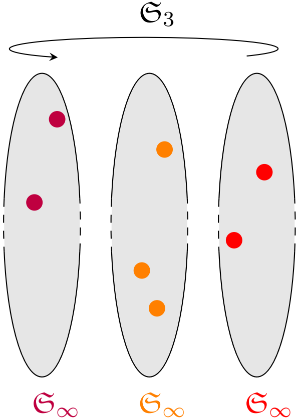


Wreath product

$$\mathfrak{S}_\infty \wr \mathfrak{S}_3 \quad \simeq \quad \mathfrak{S}_\infty^3 \rtimes \mathfrak{S}_3$$

Subset of shape $2, 3, 2 \;\rightarrow\; x_1^2\, x_2^3\, x_3^2$

An infinite example: $\mathfrak{S}_\infty \wr \mathfrak{S}_3$



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Subset of shape 2, 3, 2 $\rightarrow$ $x_1^2 x_2^3 x_3^2$

Orbits of subsets
$\leftrightarrow$ symmetric polynomials in x_1, x_2, x_3

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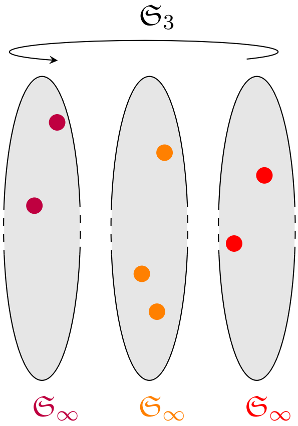

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$\mathfrak{S}_\infty \wr \mathfrak{S}_3 \simeq \mathfrak{S}_\infty^3 \rtimes \mathfrak{S}_3$

Subset of shape $2, 3, 2 \to x_1^2 x_2^3 x_3^2$

Orbits of subsets
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$\mathcal{A}_{\mathfrak{S}_\infty \wr \mathfrak{S}_3} \simeq \mathrm{Sym}_3[X] = \mathbb{Q}[X]^{\mathfrak{S}_3}$

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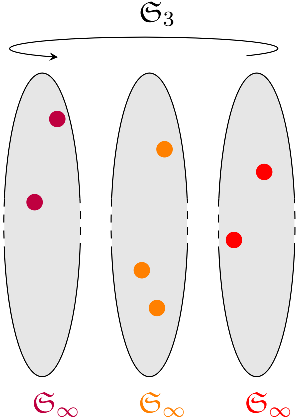

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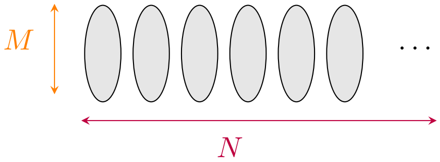
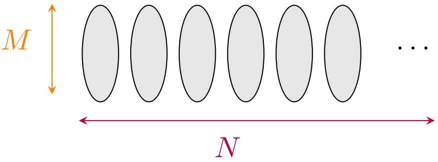
One can obtain functions counting integer partitions, combinations, P-partitions (with optional length and/or hight restrictions) as profiles of wreath products...

Lower bound

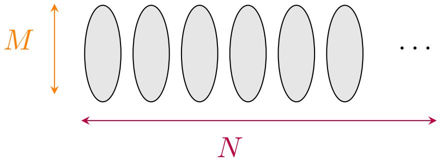


Lower bound



$$\implies G \leq \mathfrak{S}_M \wr \mathfrak{S}_N$$

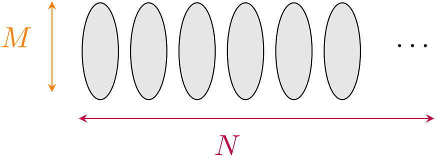
Lower bound



$\implies G \leq \mathfrak{S}_M \wr \mathfrak{S}_N$

Two cases if G is P-oligomorphic :

Lower bound



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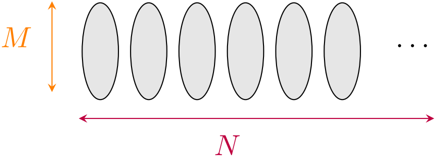
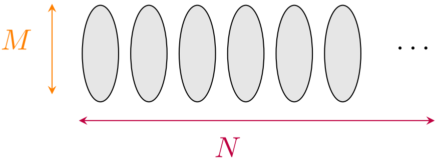


$$\implies G \leq \mathfrak{S}_M \wr \mathfrak{S}_N$$

Two cases if G is P-oligomorphic :

- $M < \infty$

Lower bound

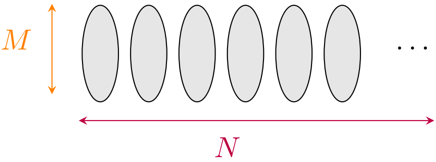


$\Longrightarrow G \le \mathfrak{S}_M \wr \mathfrak{S}_N$

Two cases if G is P-oligomorphic :

- $M < \infty$
- $N < \infty$

Lower bound

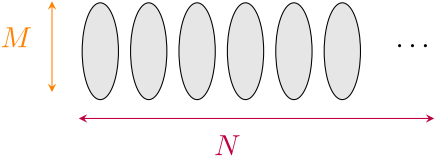


$\Longrightarrow G \leq \mathfrak{S}_M \wr \mathfrak{S}_N$

Two cases if G is P-oligomorphic :

- $M < \infty$ $\longrightarrow \varphi_G(n) \geq O(n^{M-1})$
- $N < \infty$

Lower bound



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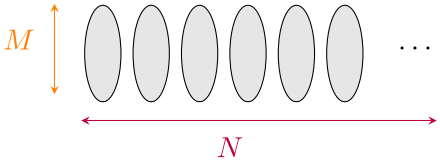

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- $N < \infty$ $\longrightarrow \varphi_G(n) \geq O(n^{N-1})$

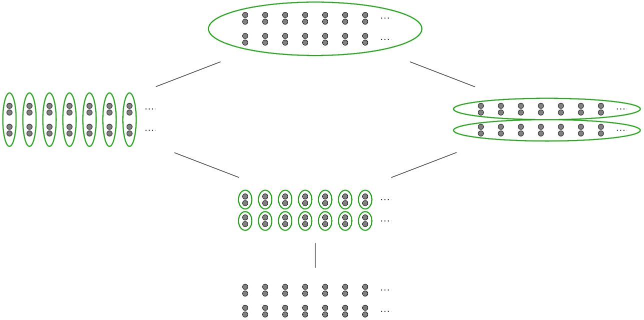
Better have big finite blocks and/or "small" infinite ones...

Lattices of block systems

Lattice of partitions $\rightarrow$ structure of *lattice* on block systems

Lattices of block systems

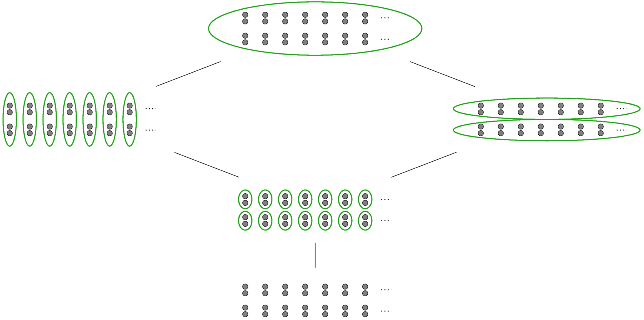
Lattice of partitions $\rightarrow$ structure of *lattice* on block systems



$(\mathcal{C}_2 \times \mathcal{C}_2) \wr \mathfrak{S}_\infty$

Lattices of block systems

Lattice of partitions $\rightarrow$ structure of *lattice* on block systems



$(\mathcal{C}_2 \times \mathcal{C}_2) \wr \mathfrak{S}_\infty$

Non trivial fact

- $\{$Systems with $< \infty$ blocks only$\}$ = sublattice with maximum
- $\{$Systems with ∞ blocks only$\}$ = sublattice with minimum

Lattices of block systems

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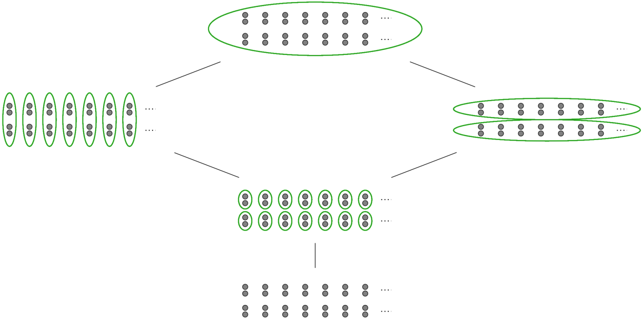

$(\mathcal{C}_2 \times \mathcal{C}_2) \wr \mathfrak{S}_\infty$

Non trivial fact

- $\{$Systems with $< \infty$ blocks only$\}$ = sublattice with maximum
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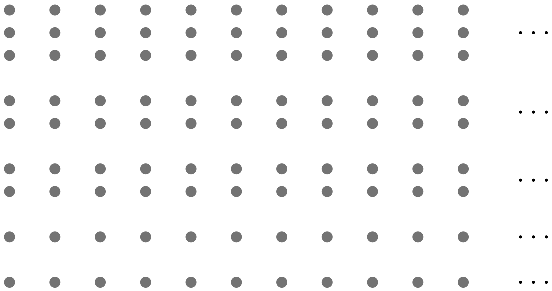
Remark. If G is P-oligomorphic, both of them are actually finite!

The nested block system

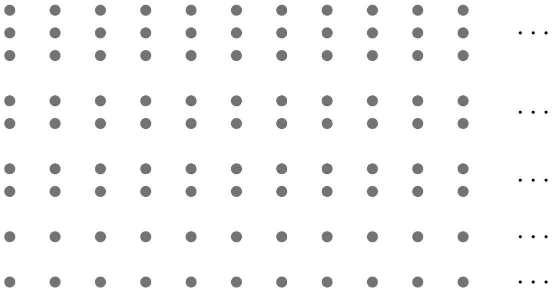
Idea



The nested block system

Idea

1. Take the *maximal* system of finite blocks



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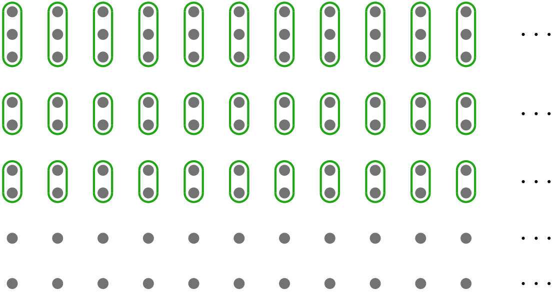

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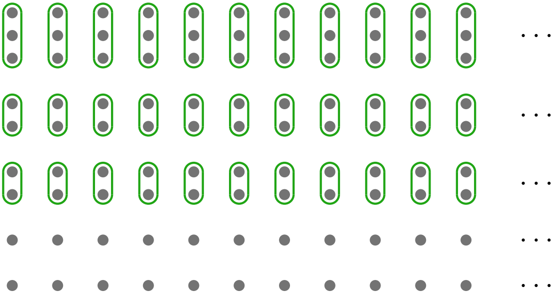

Action on the maximal finite blocks...

The nested block system

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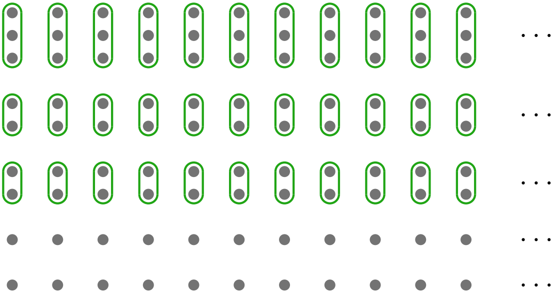
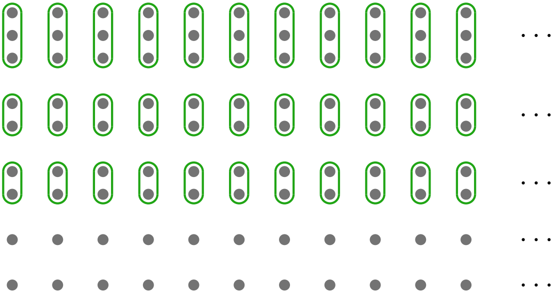


Action on the maximal finite blocks... that has no finite blocks.

The nested block system

Idea

1. Take the *maximal* system of finite blocks
2. Take the *minimal* system of infinite blocks of the action of G on the maximal finite blocks



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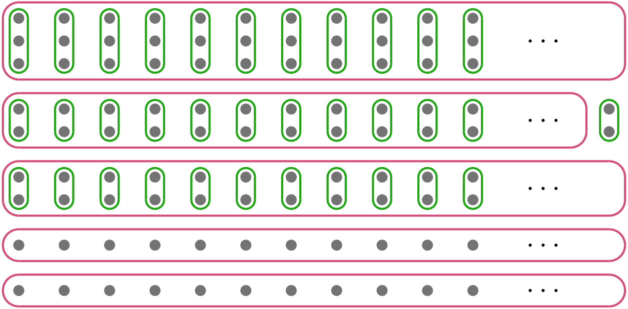

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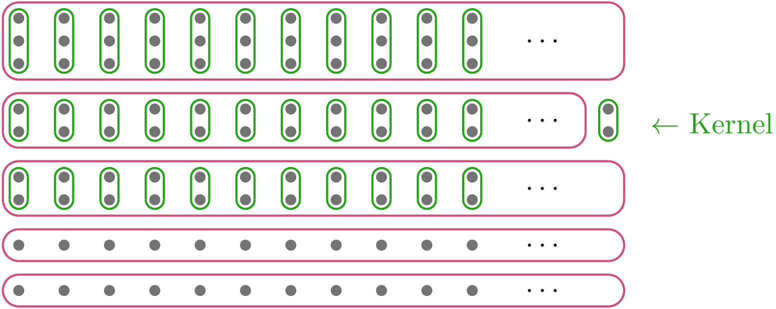
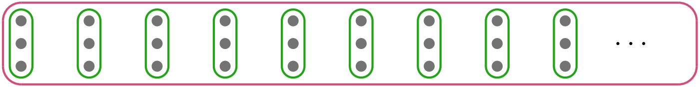


Action on the maximal finite blocks... that has no finite blocks.

One superblock: examples



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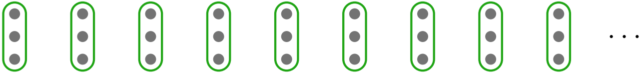

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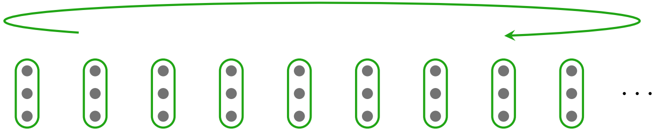

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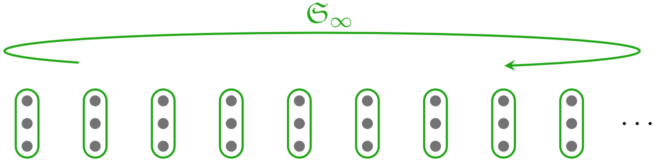

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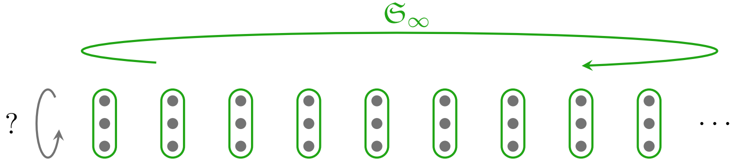

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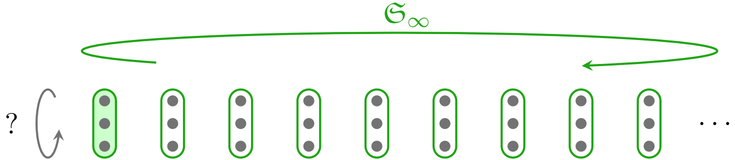

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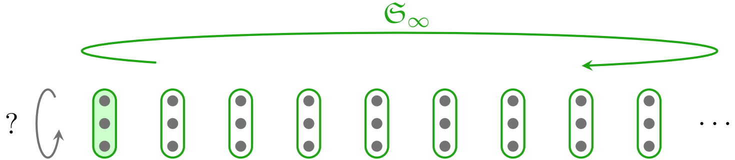
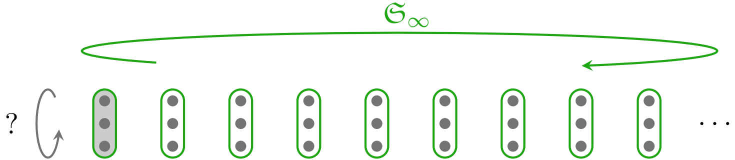


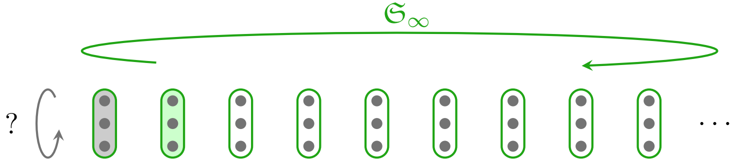
$G_{|B_0} = H_0$

One superblock: examples



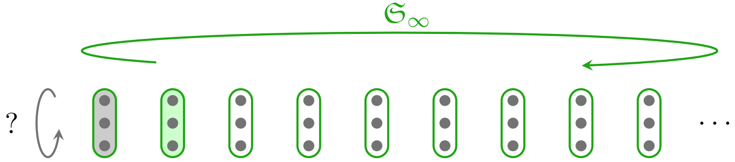
$G_{|B_0} = H_0$, Fix(B_0)

One superblock: examples



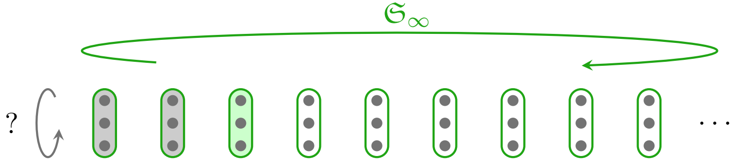
$G_{|B_0} = H_0$, $\mathrm{Fix}(B_0)_{|B_1} = H_1$

One superblock: examples



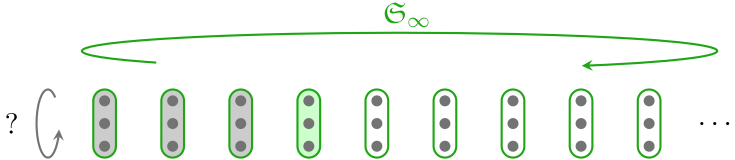
H_0 , H_1

One superblock: examples



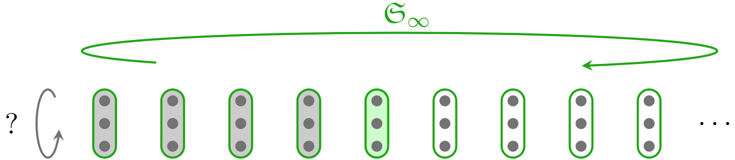
H_0 , H_1 , H_2

One superblock: examples



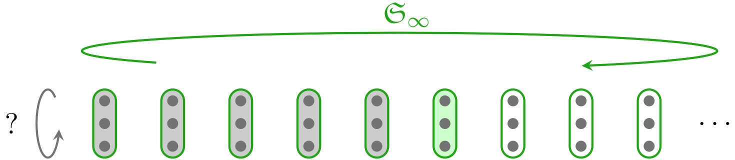
H_0 , H_1 , H_2 , H_3

One superblock: examples



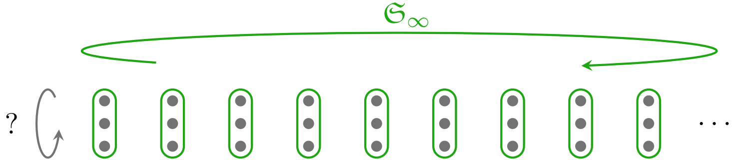
H_0 , H_1 , H_2 , H_3 , H_4

One superblock: examples



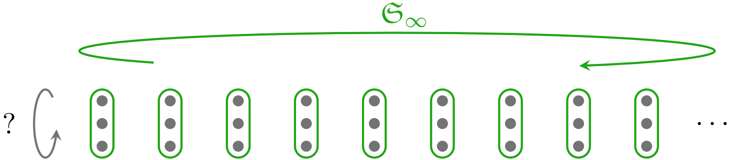
H_0 , H_1 , H_2 , H_3 , H_4 , H_5

One superblock: examples



H_0 , H_1 , H_2 , H_3 , H_4 , H_5 $\cdots$ **Tower** of G

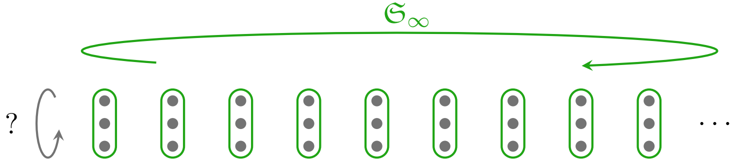
One superblock: examples



H_0 , H_1 , H_2 , H_3 , H_4 , H_5 $\cdots$ **Tower** of G

- $H \wr \mathfrak{S}_\infty$ $\rightarrow$ H , H , H , H , H , H $\cdots$

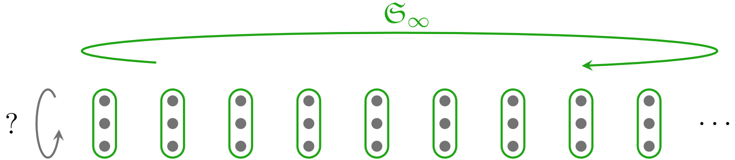
One superblock: examples



H_0 , H_1 , H_2 , H_3 , H_4 , H_5 $\cdots$ **Tower** of G

- $H \wr \mathfrak{S}_\infty$ $\rightarrow$ H , H , H , H , H , H $\cdots$
- "$H_0 \times \mathfrak{S}_\infty$" $\rightarrow$ H_0, Id , Id , Id , Id , Id $\cdots$

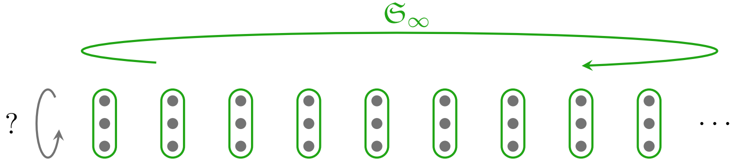
One superblock: examples



H_0 , H_1 , H_2 , H_3 , H_4 , H_5 $\cdots$ **Tower** of G

- $H \wr \mathfrak{S}_\infty$ $\to$ H , H , H , H , H , H $\cdots$
- "$H_0 \times \mathfrak{S}_\infty$" $\to$ H_0, Id , Id , Id , Id , Id $\cdots$
- $<$ "$H_0 \times \mathfrak{S}_\infty$", $H \wr \mathfrak{S}_\infty >$

One superblock: examples



H_0 , H_1 , H_2 , H_3 , H_4 , H_5 $\cdots$ **Tower** of G

- $H \wr \mathfrak{S}_\infty$ $\rightarrow$ H , H , H , H , H , H $\cdots$
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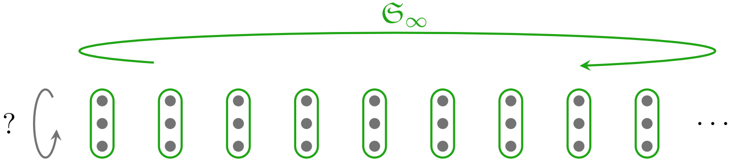
- $<$ "$H_0 \times \mathfrak{S}_\infty$", $H \wr \mathfrak{S}_\infty >$

  $H_0 \triangleright H$ w.l.o.g

One superblock: examples



$H_0\ ,\ \ H_1\ ,\ \ H_2\ ,\ \ H_3\ ,\ \ H_4\ ,\ \ H_5\ \ \cdots$ **Tower** of G

- $H \wr \mathfrak{S}_\infty \quad \rightarrow \quad H\ ,\ H\ ,\ H\ ,\ H\ ,\ H\ ,\ H\ \ \cdots$
- "$H_0 \times \mathfrak{S}_\infty$" $\quad \rightarrow \quad H_0,\ \mathrm{Id}\ ,\ \mathrm{Id}\ ,\ \mathrm{Id}\ ,\ \mathrm{Id}\ ,\ \mathrm{Id}\ \ \cdots$
- $<$ "$H_0 \times \mathfrak{S}_\infty$", $H \wr \mathfrak{S}_\infty >$ $\quad \rightarrow \quad H_0,\ H\ ,\ H\ ,\ H\ ,\ H\ ,\ H\ \ \cdots$

  $H_0 \triangleright H$ w.l.o.g

Notation: $[H_0, H_\infty]$

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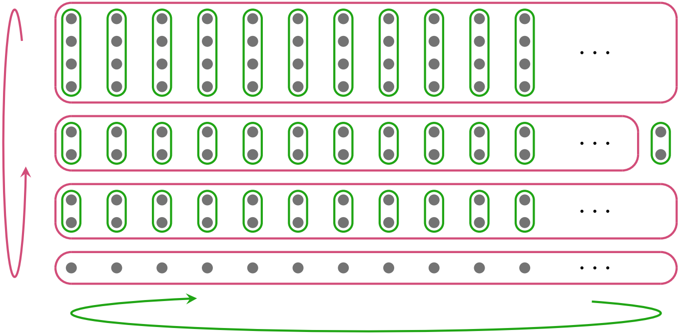
In particular, both conjectures hold.

General case: minimal subgroup of finite index



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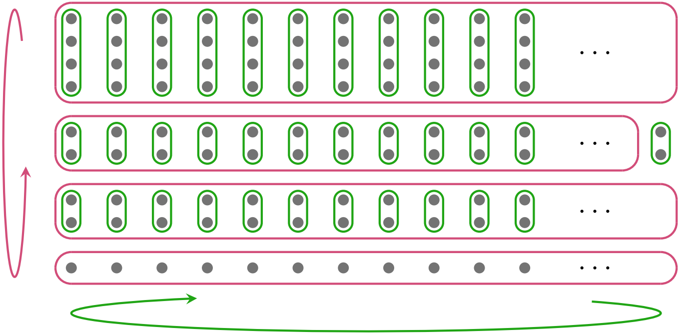

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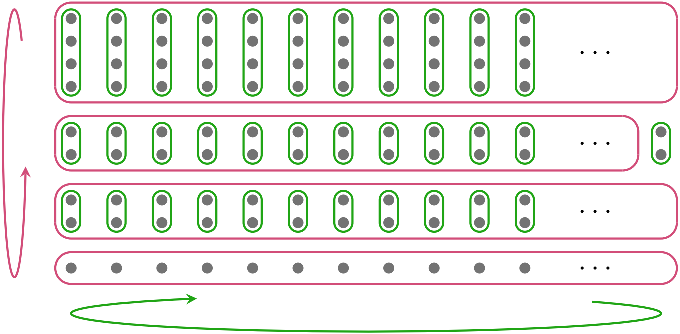

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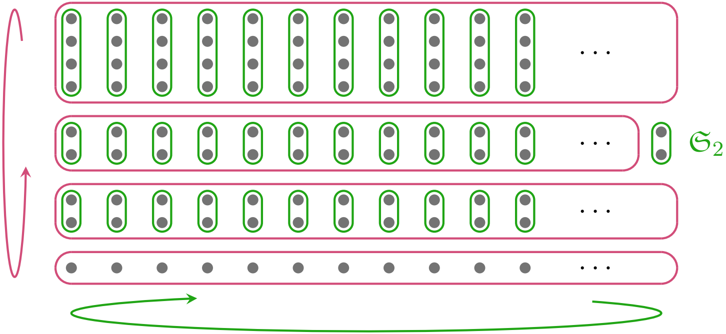

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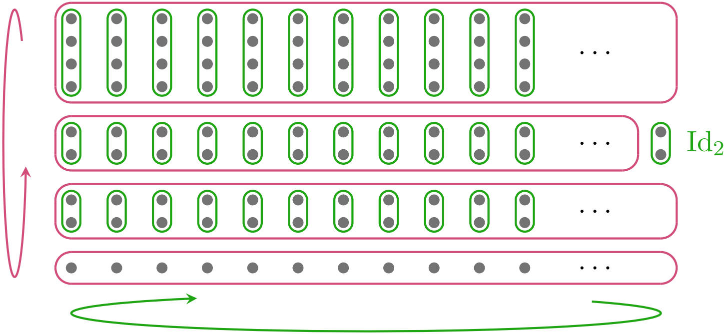

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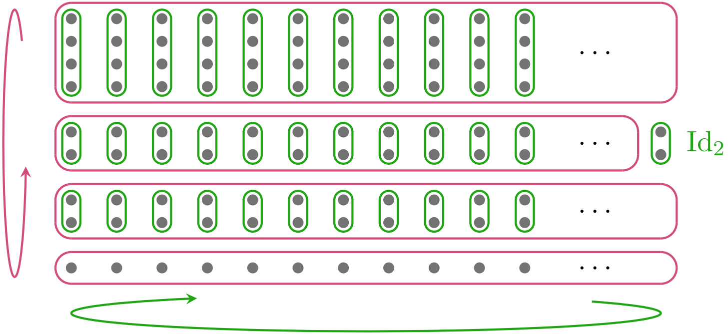

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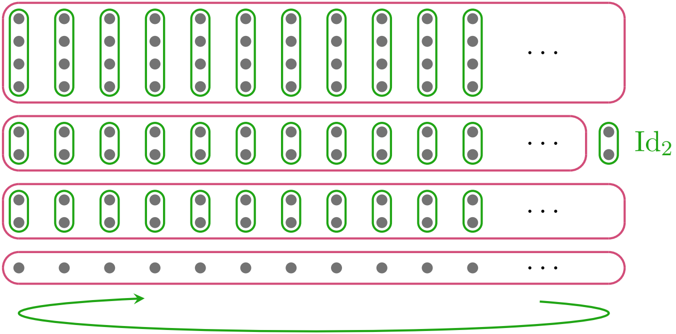

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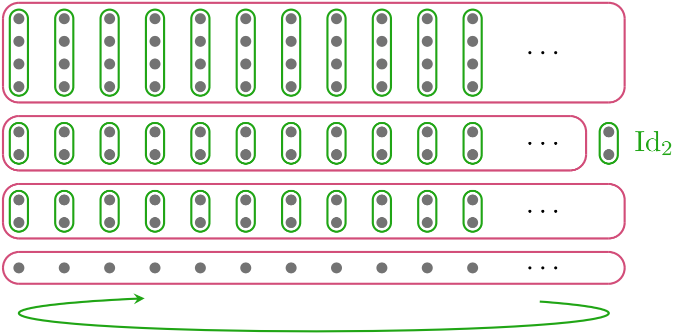

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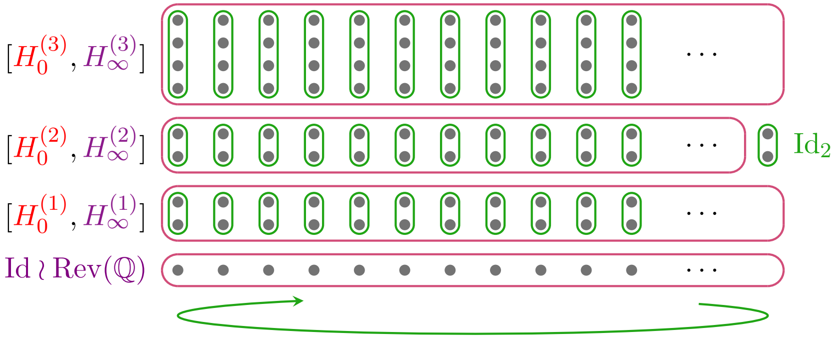

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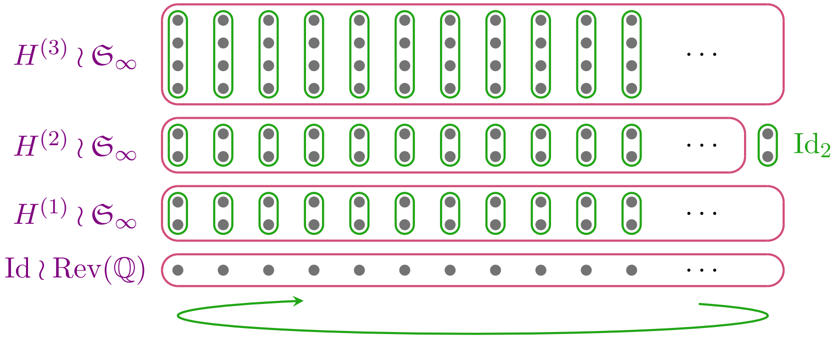

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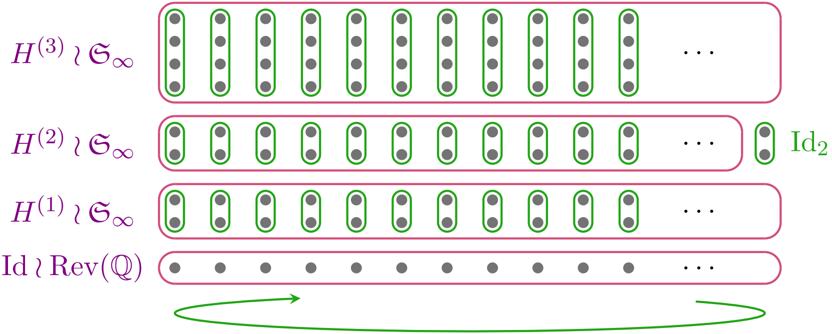

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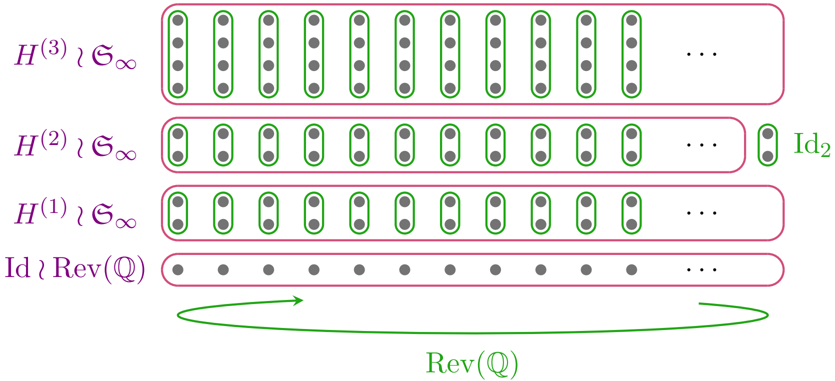

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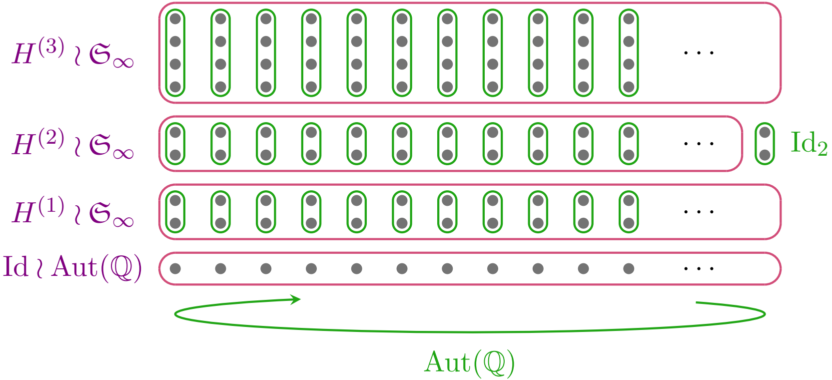

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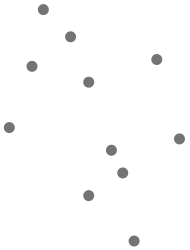
  Which end the proof of the conjectures !

Classification of P-oligomorphic groups

G_0 a finite permutation group



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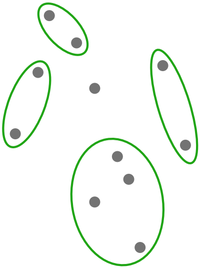

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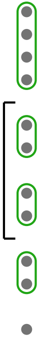

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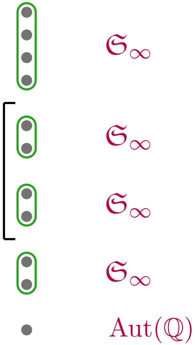

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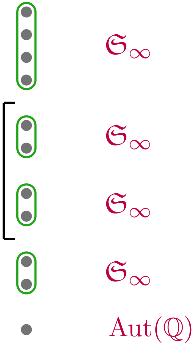

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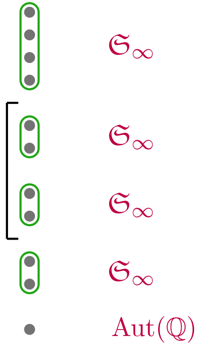

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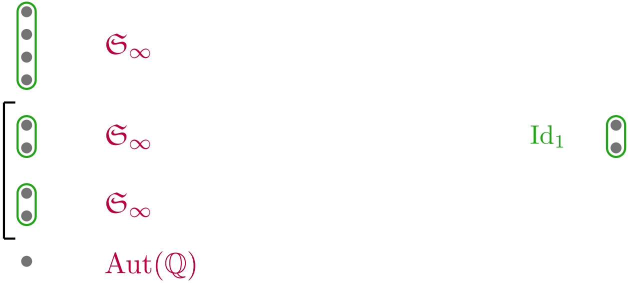

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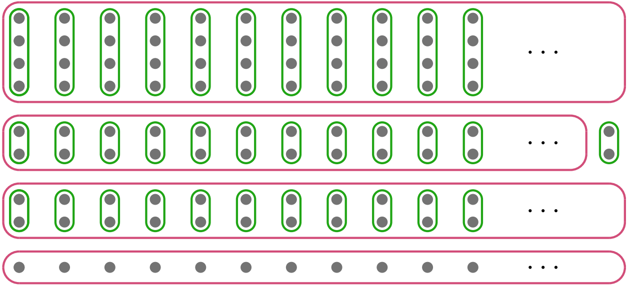

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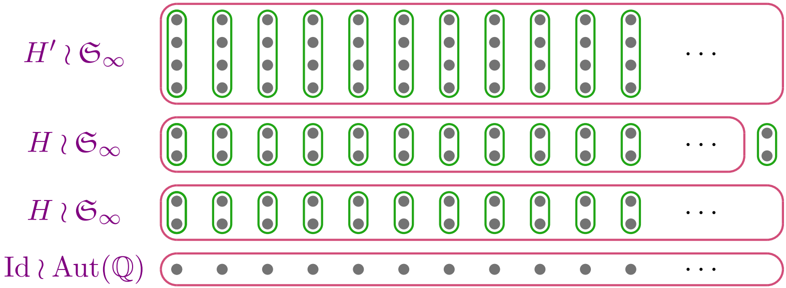

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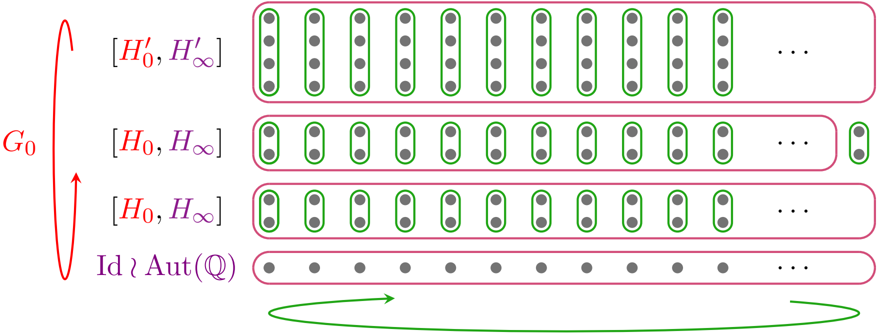


Thank you for your attention !

Context

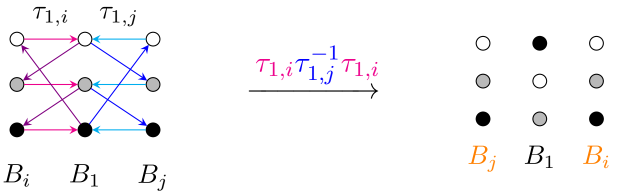
- G permutation group of a countably infinite set E
- Profile φ_G: counts the orbits of finite subsets of E
- Hypothesis: $\varphi_G(n)$ bounded by a polynomial
- Conjecture (Cameron): rational form of the generating series
- Conjecture (Macpherson): finite generation of the orbit algebra

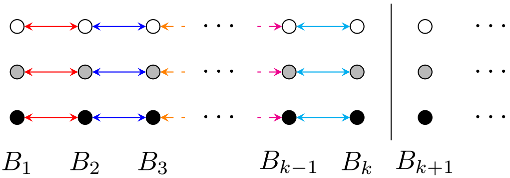
Results

- Both conjectures hold !
- Classification of P-oligomorphic permutation groups
- The orbit algebra is an algebra of invariants (up to some idempotents)

The tower determines the group (1): "straight $\mathfrak{S}_\infty$"

G contains a set of "straight" swaps of blocks





Subdirect product

Subdirect product of G_1 and G_2

- Formalizes the *synchronization* between G_1 and G_2
- Subgroup of $G_1 \times G_2$ (with canonical projections G_1 and G_2)
- $E = E_1 \sqcup E_2$ stable $\Rightarrow G$ subdirect product of $G_{|E_1}$ and $G_{|E_2}$

Synchronization in a subdirect product

Let $N_1 = \mathrm{Fix}_G(E_2)$ and $N_2 = \mathrm{Fix}_G(E_1)$.

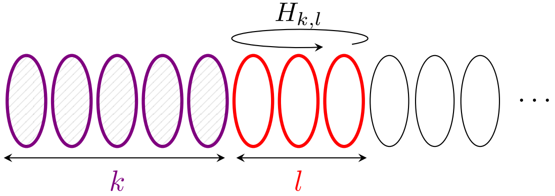
$$\frac{G_1}{N_1} \simeq \frac{G}{N_1 \times N_2} \simeq \frac{G_2}{N_2}$$

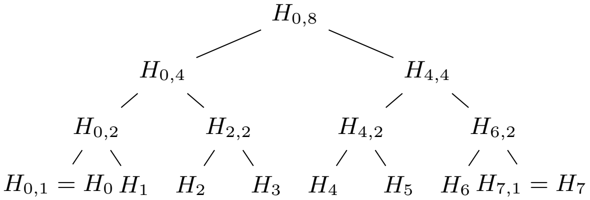
A subdirect product with explicit N_i's is explicit.

Remark. N_1 and N_2 are *normal* in G_1 and G_2, so the possibilities of synchronization of a group is linked to its normal subgroups.

The tower determines the group (2): $\mathrm{Stab}_G(\text{blocks})$

$\mathrm{Stab}_G(\text{blocks})$ = explicit subdirect product of the H_i





← The tower determines $\mathrm{Stab}_G(\text{blocks})$

$G \simeq \mathrm{Stab}_G(\text{blocks}) \rtimes$ "straight $\mathfrak{S}_\infty$" $\rightarrow$ Ok

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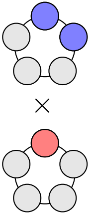

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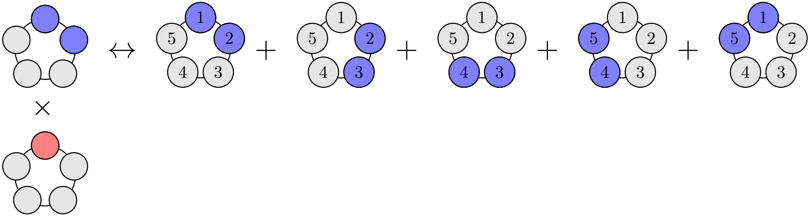

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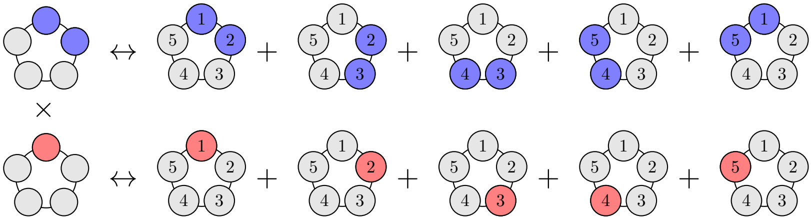

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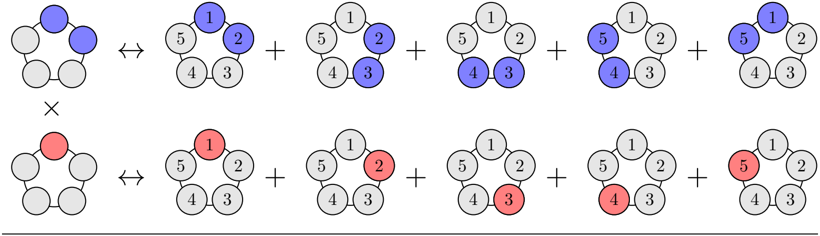

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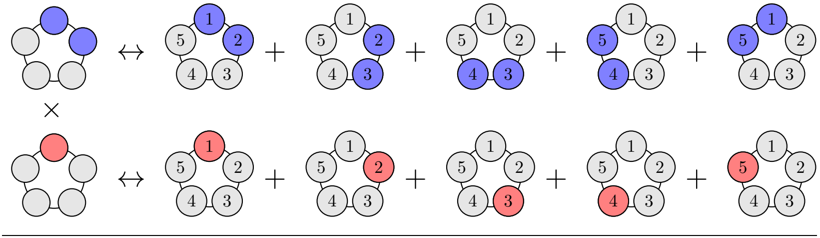

$= \quad 0$

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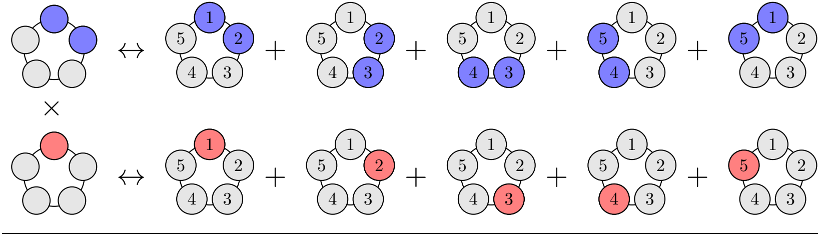

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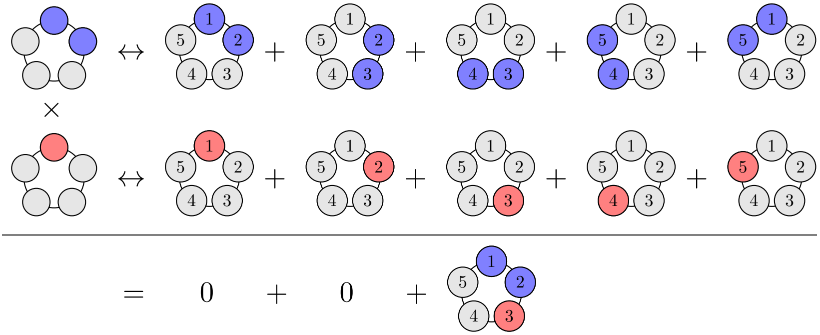

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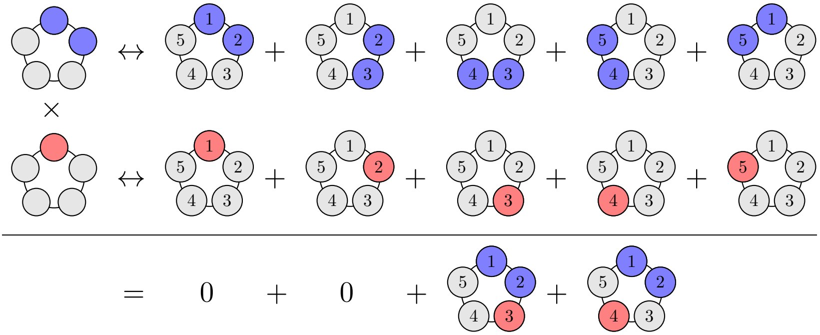

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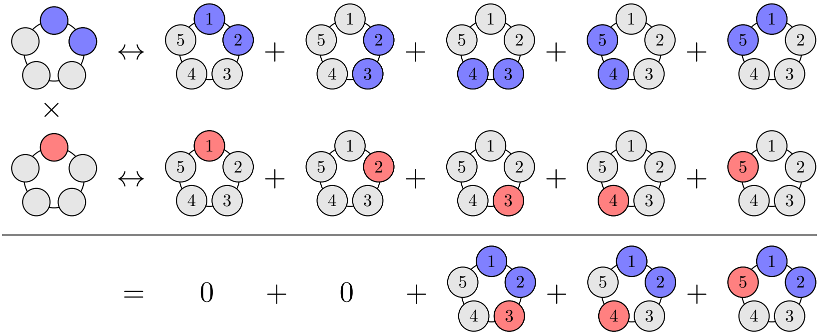

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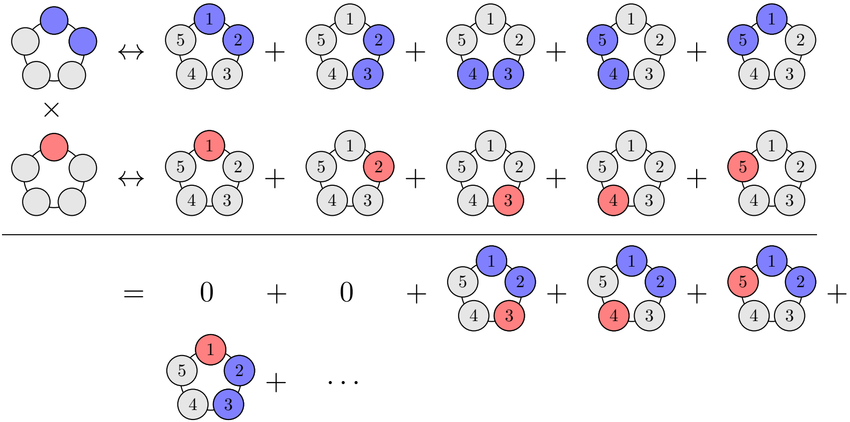

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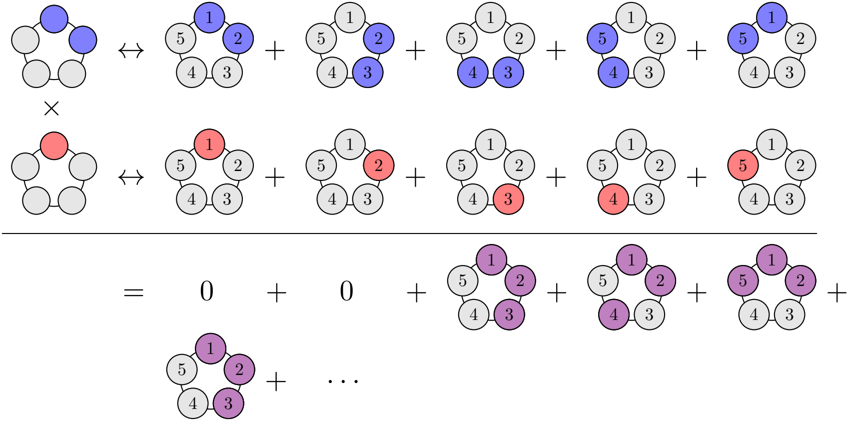

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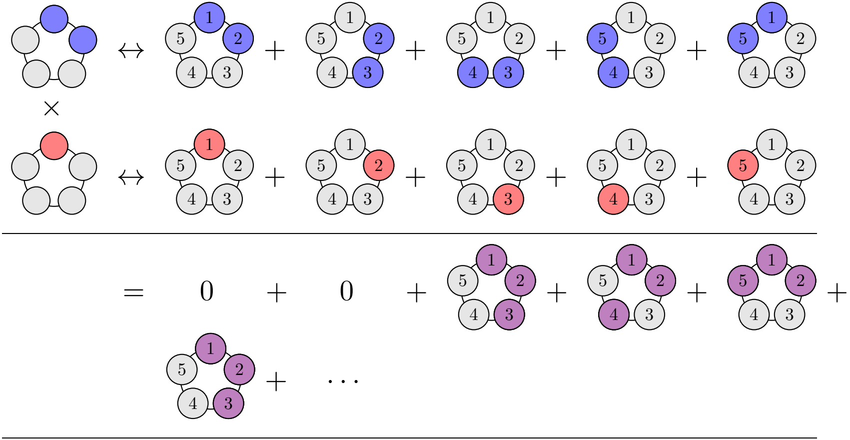

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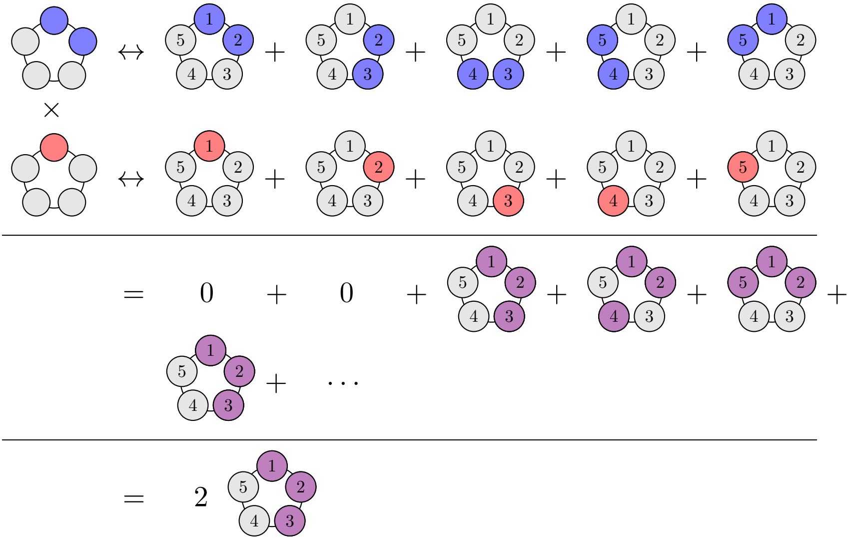

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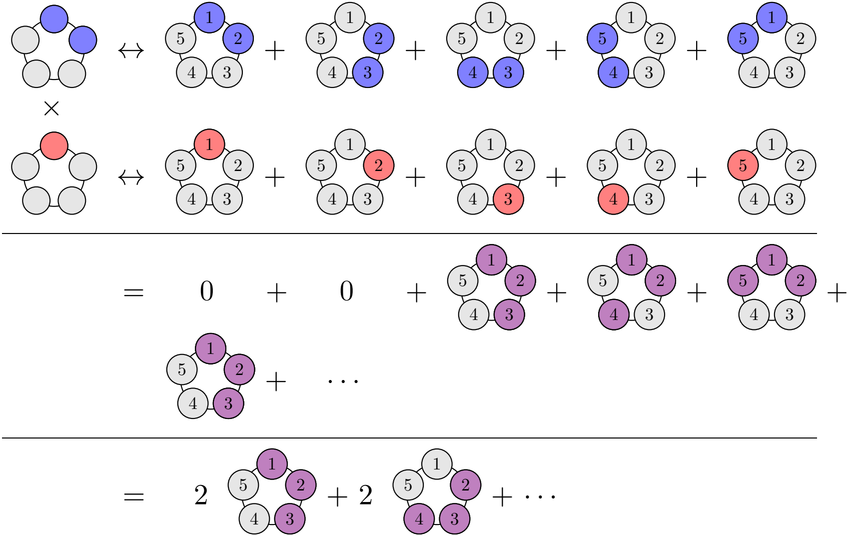

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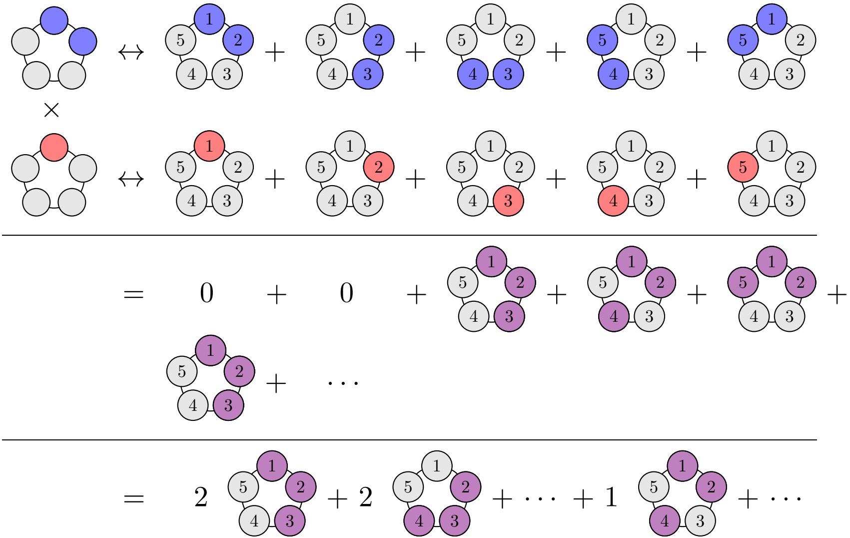
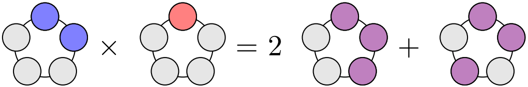


In the end:



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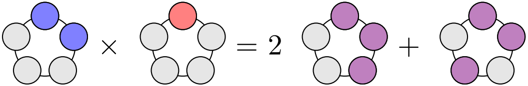
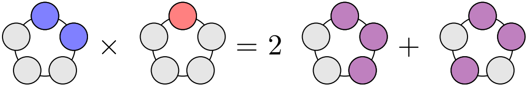


Non trivial fact

Product well defined (and graded) on the space of orbits.

In the end: 

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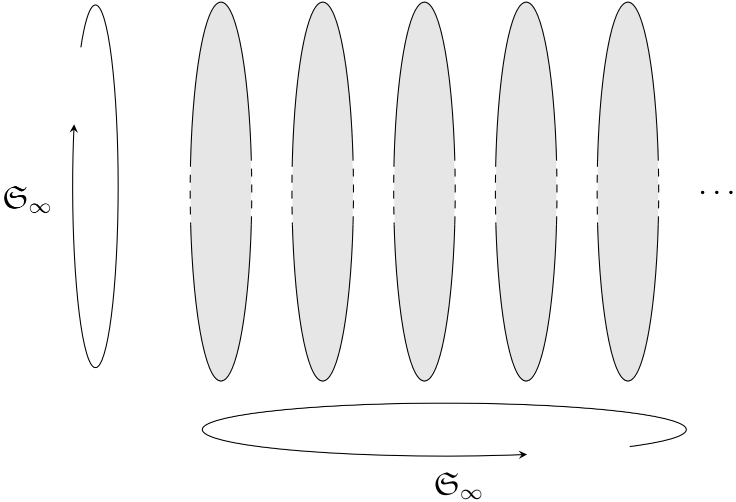
$\longrightarrow$ **The orbit algebra of a permutation group**

Example : $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$

$\varphi_G(n) = \ ?$



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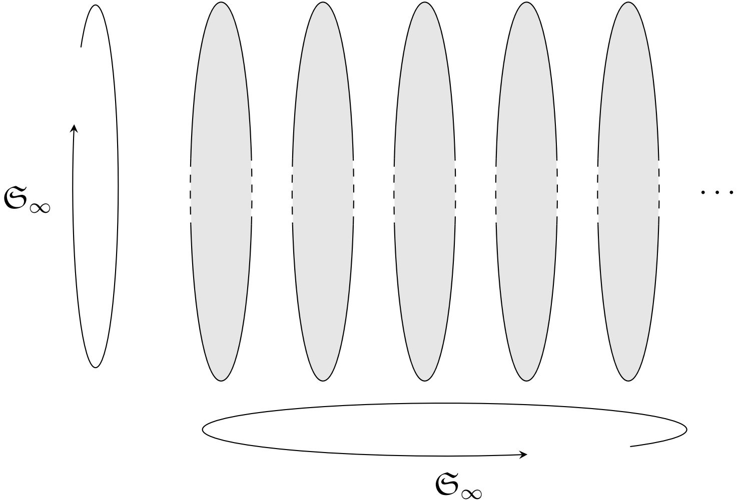
An orbit of degree $n \longleftrightarrow$ a partition of n



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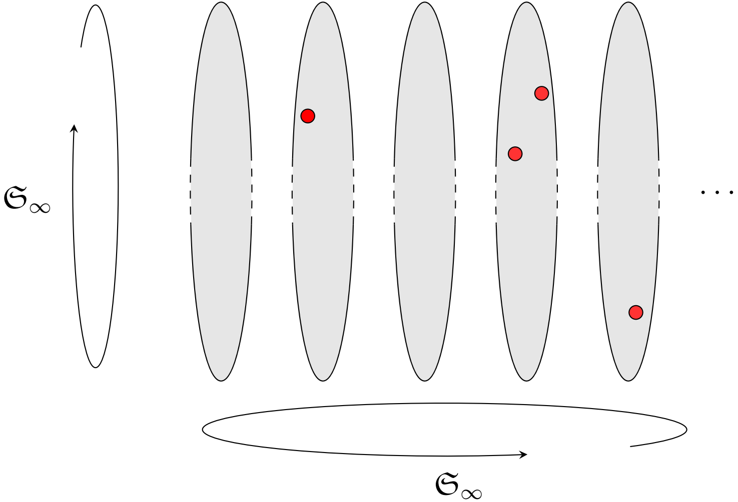

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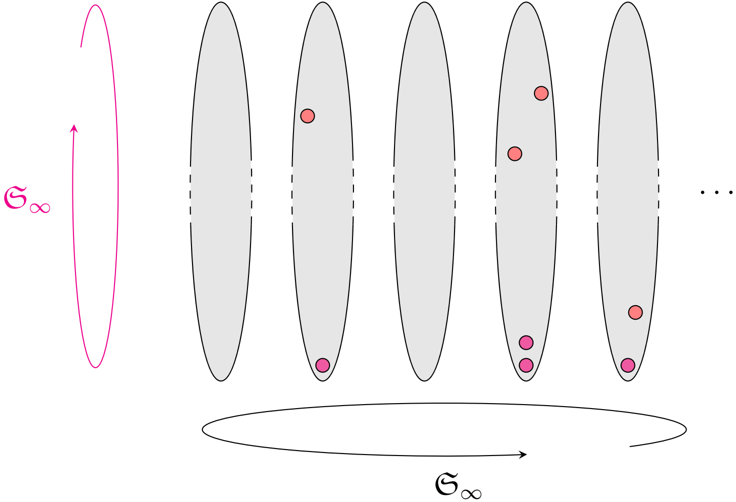

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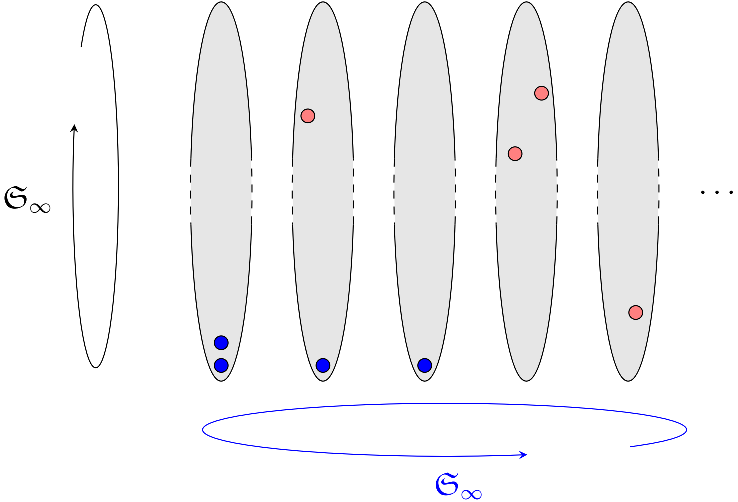
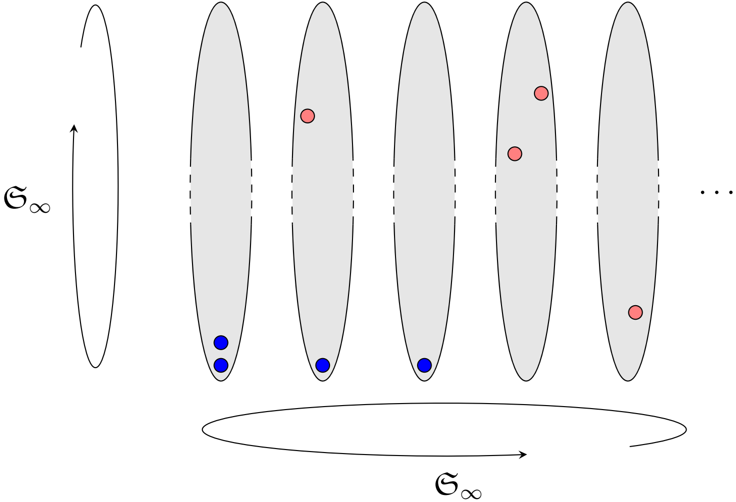


Example : $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$

$\varphi_G(n) = p(n)$

An orbit of degree $n \longleftrightarrow$ a partition of n



Examples of orbit algebras (1)

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If $G = \mathfrak{S}_\infty$, $\varphi_G(n) = 1$ for all n, and $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x]$.

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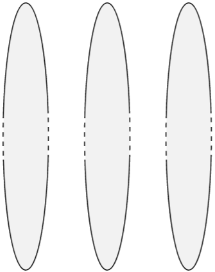
A_n = homogeneous symmetric polynomials of degree n in x_1, x_2, x_3



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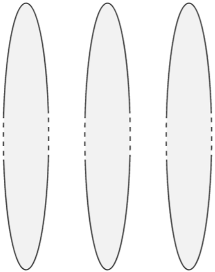


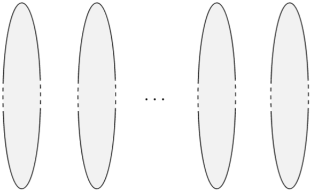
$\rightarrow \mathbb{Q}\mathcal{A}(\mathfrak{S}_\infty \wr \mathfrak{S}_3) = \mathbb{K}[x_1, x_2, x_3]^{\mathfrak{S}_3}$

Examples of orbit algebras (2)

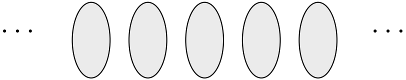
More generally, for H subgroup of $\mathfrak{S}_m$:

- $G = \mathfrak{S}_\infty \wr H$:
  $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x_1, \ldots, x_m]^H$, the algebra of invariants of H

  $\mathbb{Q}\mathcal{A}(G)$ is finitely generated by Hilbert's theorem.

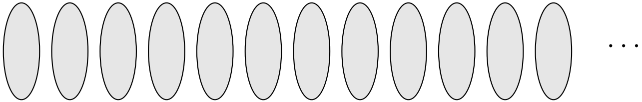


- $G = H \wr \mathfrak{S}_\infty$:
  $\mathbb{Q}\mathcal{A}(G) =$ the free algebra generated by the age of H



Direct product in the case of finite blocks

"Speak, friend..."



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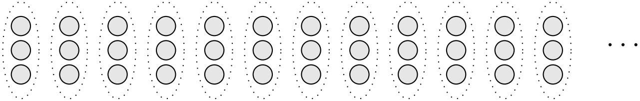
Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



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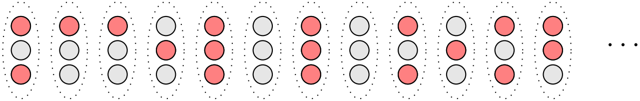

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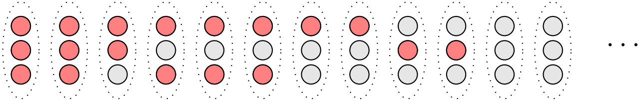

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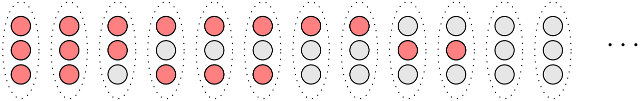

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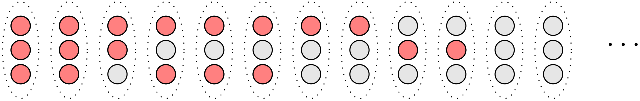

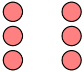

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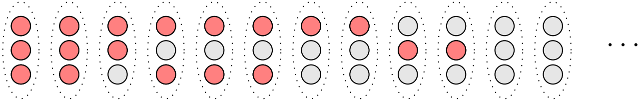

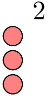

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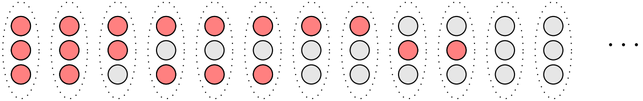

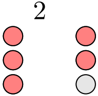

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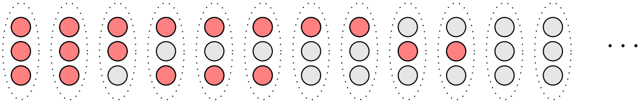

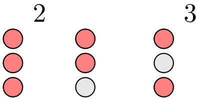

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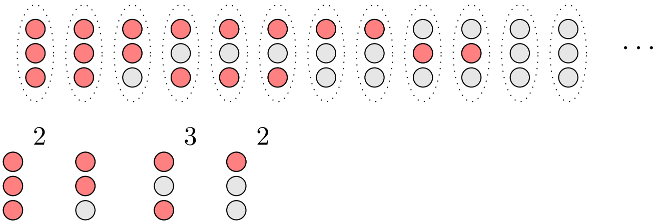

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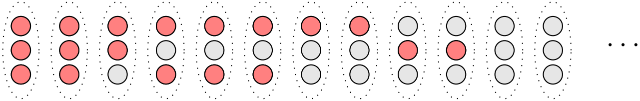

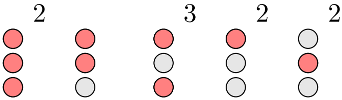

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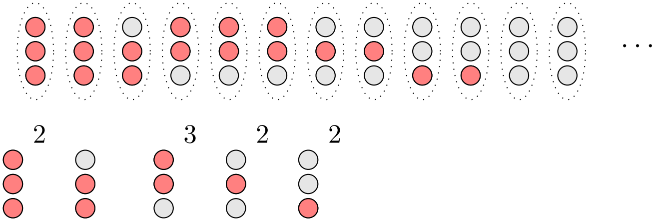

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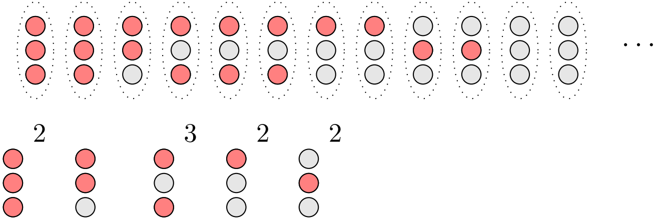

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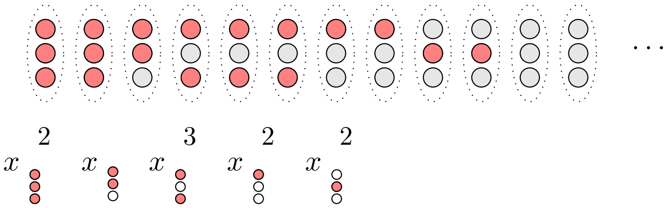

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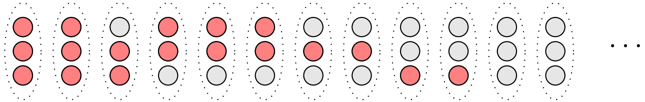

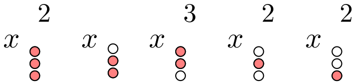

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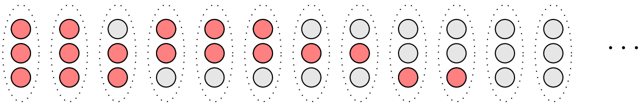

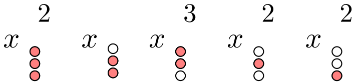

$\rightarrow$ C_3 acts on monomials

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"Speak, friend..."

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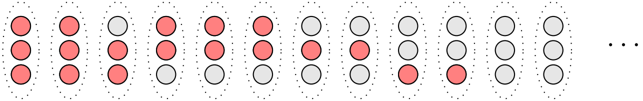

$G' = C_3$ *acting on (non empty) subsets*

$$\mathbb{K}[\ x\ ]^{G'} \quad \longleftrightarrow \quad \text{Orbit algebra of } C_3 \times \mathfrak{S}_\infty\ \ ?$$

Direct product in the case of finite blocks

"Speak, friend..."

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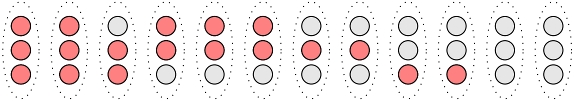
$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



...

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x 

x 

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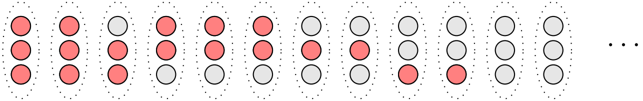

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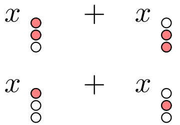

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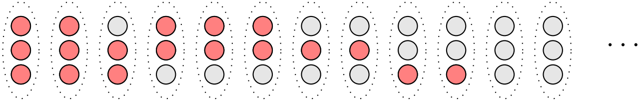

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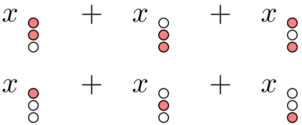

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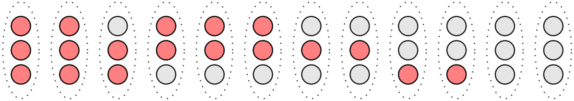

$\dots$

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O(x )

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"Speak, friend..."

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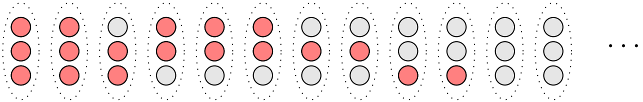

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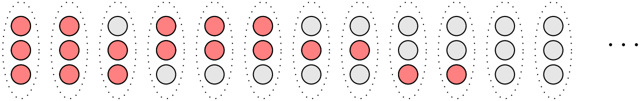
Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



$G' = C_3$ *acting on (non empty) subsets*

$\mathbb{K}[\ x\ ]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?

O(x ).O(x ) = O(x  x )

Direct product in the case of finite blocks

"Speak, friend..."

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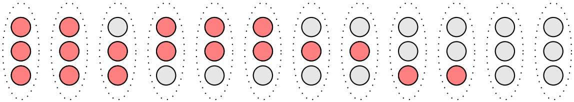

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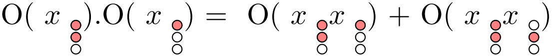

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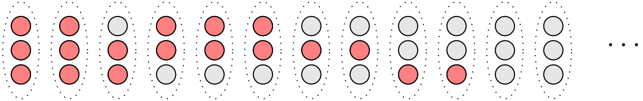

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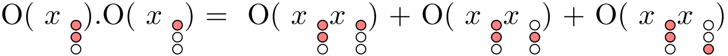

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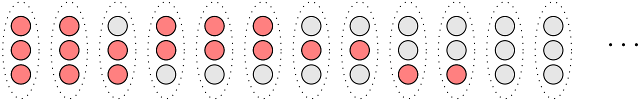

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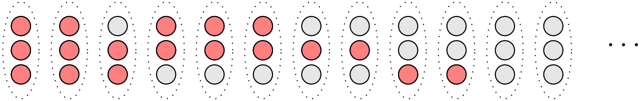

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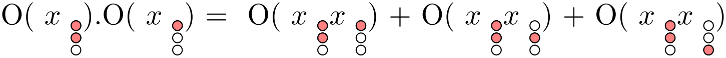

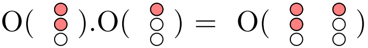

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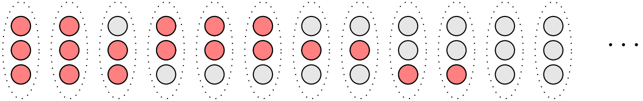

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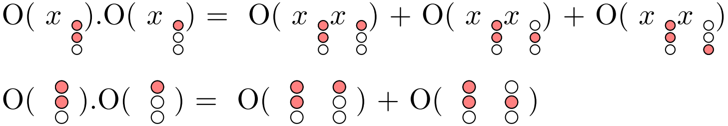

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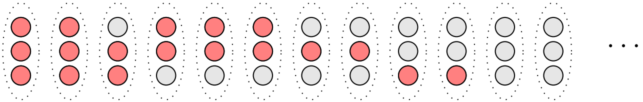

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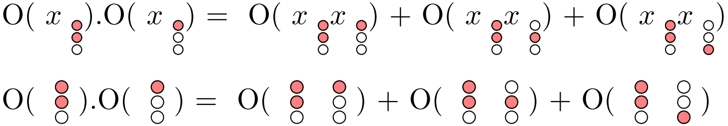

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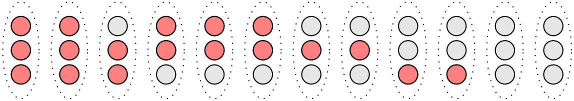

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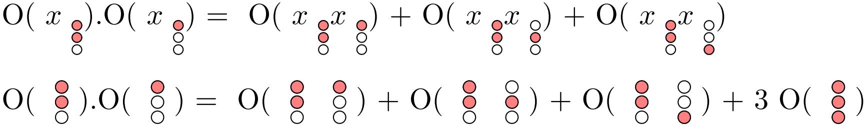

The tower has shape $H_0, \ H, \ H, \ H \ \cdots$

Lemma to prove

G has tower $H_0 \ H_1 \ H_2 \ H_3 \Rightarrow H_1 = H_2$

Proof.

An element $s \in G$ stabilizing the blocks $\leftrightarrow$ a quadruple
$g \in H_1 \quad \rightarrow \quad \exists \ (1, g, h, k), \ \ h, k \in H_1$.
Let σ be an element of G that permutes "straightforwardly" the first two blocks and fixes the other two.
Conjugation of x by σ in $G \quad \rightarrow \quad y = (g, 1, h, k)$
Then: $x^{-1}y = (g, g^{-1}, 1, 1)$
By arguing that the tower does not depend on the ordering of the blocks, g^{-1} and therefore g are in H_2.

In the infinite case, apply to each restriction to four consecutive blocks of the fixator of the previous ones in G.