From presheaves to Hopf algebras

82nd Seminaire Lotharigiene de Combinatoire, Curia

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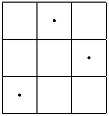
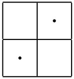
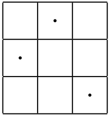
Slides can be found in `http://user.math.uzh.ch/penaguiao/`

Baby steps

Permutations as a square configuration:

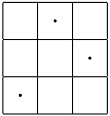
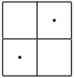
$\pi =$  $= 132$

$\sigma =$  $= 12\,,\ \tau =$  $= 231$

One-line notation: read left-to-right the height of each element.
Family of permutations with finite points - $\mathcal{G}(\texttt{Per})$.

Counting occurences of a pattern

Let π be a permutation and I a set of columns of the square configuration of π. The **restriction to** I is a permutation $\pi|_I$, called a **pattern** of π, and I is its **occurence** in π.
If $\pi = 132$ as above,

$\pi|_{\{1,3\}} =$  $\Big|_{\{1,3\}} =$ 

In fact, there are 2 occurences of the pattern 12 in π. We write

$$\mathbf{p}_{12}(132) = 2,\ \mathbf{p}_{123}(123456) = 20,\ \mathbf{p}_{2413}(762341895) = 0\,.$$

Permutation pattern algebra

Pattern function p_τ are in the space of functions $\mathcal{F}(\mathcal{G}(\mathtt{Per}), \mathbb{R})$
The linear span of all pattern functions - $\mathcal{A}(\mathtt{Per})$.
Products on $\mathcal{G}(\mathtt{Per})$

$$\pi \oplus \tau = \begin{array}{|c|c|}\hline & \tau \\ \hline \pi & \\ \hline \end{array} \qquad \pi \ominus \tau = \begin{array}{|c|c|}\hline \pi & \\ \hline & \tau \\ \hline \end{array}$$

By the magic properties of dualizing functions, we have a coproduct on $\mathcal{A}(\mathtt{Per})$:

$$\Delta \, \mathbf{p}_\pi = \sum_{\pi = \tau_1 \oplus \tau_2} \mathbf{p}_{\tau_1} \otimes \mathbf{p}_{\tau_2} \, ,$$

so that we have a Hopf algebra

$$\mathbf{p}_\pi(\sigma_1 \oplus \sigma_2) = \Delta \, \mathbf{p}_\pi(\sigma_1 \otimes \sigma_2) \, .$$

Permutation pattern algebra

Proposition (Linear independence)

The set $\{\mathbf{p}_\pi \mid \pi \in \uplus_{n\geq 0} S_n\}$ *is linearly independent - Triangularity argument*

Proposition (Product formula)

Let $\binom{\sigma}{\pi, \tau}$ *count the number of covers of* σ *with permutations* π, τ.

$$\mathbf{p}_\pi \cdot \mathbf{p}_\tau = \sum_\sigma \binom{\sigma}{\pi, \tau} \mathbf{p}_\sigma \,,$$

where σ *runs over equivalence classes of pairs of orders.*

Theorem (Vargas, 2014)

The Hopf algebra $\mathcal{A}(\texttt{Per})$ *is free comutative.* *what is free?*

Outline of the talk

Pattern algebra

What do we need to have a pattern Hopf algebra?

1. Assignment $S \mapsto h[S] = \{\text{structures over } S\}$ + notion of *relabelling*.
2. For any inclusion $V \hookrightarrow W$, a restriction map $h[W] \to h[V]$.
3. An associative monoid operation $*$ with unit, in $\mathcal{G}(\mathtt{h})$ that is compatible with restrictions.
4. A unique element of size zero.

A structure with 1 and 2 - *combinatorial presheaf*.
If in addition it has a structure as in 3 - *monoid in combinatorial presheaves*.
A combinatorial presheaf that satisfies 4 - *connected presheaf*.

Category theory formulation

Observation: The product structure on $\mathcal{A}(\mathtt{h})$ depends only on the combinatorial presheaf structure, and not on the monoid structure $*$, so the same product structure may be compatible with several coproducts.

Examples with several products: the presheaves of **marked graphs** or **permutations**.

We have a functor $\mathcal{A}$ that sends

$$\mathcal{A} : \mathtt{CPSh} \to \mathrm{GAlg}_{\mathbb{R}} \, ,$$

and restricts $\mathcal{A} : \mathrm{Mon}(\mathtt{CPSh}) \to \mathrm{GHopf}_{\mathbb{R}}$.

A presheaf on graphs

- For each set V we are given the set $\mathtt{G}[V]$ of graphs with vertex set V., and for any bijection $\phi : V \to W$ gives us a relabelling of graphs $\mathtt{G}[W] \to \mathtt{G}[V]$.
- Induced subgraphs endow graphs with the structure of restrictions.
- The disjoint union of graphs is an associative monoid structure. It is also **commutative**.
- The empty graph fortunately exists!

Theorem (P - 2019+)

If $\mathtt{h}$ is a connected commutative presheaf, then $\mathcal{A}(\mathtt{h})$ is free. The free generators are the indecomposable objects with respect to the commutative product.

Connected commutative combinatorial presheafs

Proof (by example):
Graphs, with a disjoint union, form a commutative presheaf. Every graph has a unique factorisation into **indecomposables** $\mathcal{I}$.

$$\mathcal{A}(\mathtt{G}) \text{ is free commutative} \Leftrightarrow \{\prod_{l \in L} \mathbf{p}_l \,|\, L \subseteq \mathcal{I} \text{ multiset } \} \text{ is lin. ind.}$$

$$\Leftrightarrow_{\text{triangularity argument}} \prod_{l \in L} \mathbf{p}_l = \mathbf{p}_\alpha + \sum_{\beta \leq \alpha} c_\beta \, \mathbf{p}_\beta \text{ for some order } \leq .$$

where $\alpha = \biguplus_{l \in L} l$.
Highly important: **We have a unique factorisation theorem.**
Moral of the story: If we have a unique factorisation theorem up to commutativity of factors, we have a nice order to go with it.

Unique factorisation theorem on permutations

Vargas used the $\oplus$ product on permutations to obtain a unique factorisation theorem on permutations.

$$\pi = \tau_1 \oplus \cdots \oplus \tau_k = \begin{array}{|c|c|c|} \hline & & \tau_k \\ \hline & \iddots & \\ \hline \tau_1 & & \\ \hline \end{array}$$

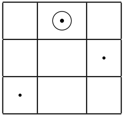
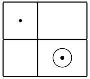
- The factorisation is **not** unique up to order of factors.

Enlarge the set $\mathcal{I}$ to $\mathcal{L}$ with **Lyndon permutations**, by adding some decomposable elements. Choose between $\pi_1 \oplus \pi_2$ and $\pi_2 \oplus \pi_1$, and between more factors.

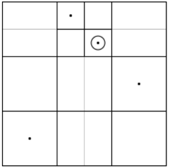
Lyndon words - used to prove the freeness of the shuffle algebra on $\mathbb{K}\langle A \rangle$.

The inflation product - Marked permutations

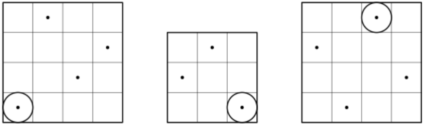
In marked permutations - use the inflation product.

$\pi =$  , $\sigma_1 =$ 

Inflation of $\pi * \sigma$ is



Examples of indecomposable marked permutations (in $\mathcal{I}$):



Unique factorisation theorem on marked permutations

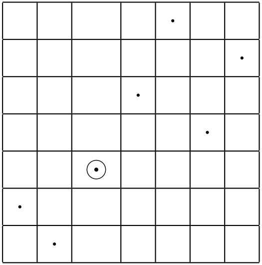
- The factorisation is **not** unique up to order of factors.
- The order of the factors does matter **only to some extent.**

The inflation map is a morphism of monoids $* : \mathcal{W}(\mathcal{I}) \to \mathcal{A}(\texttt{MPer})$.
If τ_1, τ_2 $\oplus$-indecomposable.

$$(\bar{1} \oplus \tau_1) * (\tau_2 \oplus \bar{1}) = (\tau_2 \oplus \bar{1}) * (\bar{1} \oplus \tau_1) = \tau_2 \oplus \bar{1} \oplus \tau_1 \, .$$

For $\tau_1 = 2413$ and $\tau_2 = 21$ we have

$$(\bar{1} \oplus \tau_1) * (\tau_2 \oplus \bar{1}) = 21 \oplus \bar{1} \oplus 2413 =$$



Unique factorisation theorem on marked permutations

Monoid morphism $* : \mathcal{W}(\mathcal{I}) \to \mathcal{A}(\mathtt{MPer})$

$\oplus$- relations : $(\bar{1} \oplus \tau_1) * (\tau_2 \oplus \bar{1}) = (\tau_2 \oplus \bar{1}) * (\bar{1} \oplus \tau_1) = \tau_2 \oplus \bar{1} \oplus \tau_1$.

Theorem (P - 2019+)

The equivalence relation $\ker *$ *is spanned by relations as the one above and their* $\ominus$ *equivalent.*

Further questions - MEASURE THEORY

Permutons P - A doubly stochastic probability measure in the square $[0,1] \times [0,1]$. Intuition: the limit of a sequence of permutations. Notion of patterns of π can be extended to a permuton P: $\mathbf{p}_\pi(P) = \mathbb{E}[\text{ something}(\pi)]$.

Conjecture

Let $\mathcal{L}_q = \{\mathbf{p}_l \,|l$ is a Lyndon permutation with size $\geq q\}$ be the set of free generators of $\mathcal{A}(\mathtt{Per})$. The image of the map

$$\prod_{l \in \mathcal{L}_q} \mathbf{p}_l : \{\textit{Permutons}\} \to \mathbb{R}^{\#\mathcal{L}_q} \,,$$

is full dimensional.

Partial results for the map $\prod_{l \in \mathcal{I}} \mathbf{p}_l$ by Kenyon, Krall et al.

Further questions - ALGEBRA

- *Character Theory:* characters with ”compact support” are constructed. In particular, all characters of the form

$$\zeta_a(\mathbf{p}_b) = \mathbf{p}_b(a)\,,$$

  and all its convolutions. Can we describe all characters? Are these all ”compactly supported characters” of a free pattern algebra?
- *Freeness:* Are pattern algebras free in general? Other examples include set compositions, etc.

Biblio

- Aguiar, M., & Mahajan, S. A. (2010). Monoidal functors, species and Hopf algebras (Vol. 29). *Providence, RI: American Mathematical Society*.
- Vargas, Y. (2014). Hopf algebra of permutation pattern functions. In Discrete Mathematics and Theoretical Computer Science (pp. 839-850). *Discrete Mathematics and Theoretical Computer Science*.
- Kenyon, R., Kral, D., Radin, C., & Winkler, P. (2015). Permutations with fixed pattern densities. *arXiv preprint* arXiv:1506.02340.

Thank you