Lagrangian Points

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July 5, 2017

Abstract

In this text we shortly address some questions about the Lagrangian points. We show that there are five such points and then consider the stability of two of them. In finding their approximate positions we demonstrate a problem in perturbation theory and give a method to deal with a certain class of problems in approximations. To show stability we consider a powerful method that reduces the problem to calculating eigenvalues of a matrix.

The three-body problem is not exactly solvable. If however the mass of the third object is really much smaller than the other two masses, we can ignore the backreaction of the third mass on the other two masses. In this setting the Lagrangian points are discussed: We have a configuration of two masses rotating in circular orbits around their center of mass. The Lagrangian points are those co-rotating points where an infinitesimal mass would experience no force. In these points one could leave a space-ship, without having to burn fuel to stay there.

Let M_1 and M_2 be the masses of the two larger objects. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors, where the origin is taken to be the center of mass. We thus have

$$M_1\mathbf{r}_1 + M_2\mathbf{r}_2 = 0\,.$$

We assume that the masses rotate in circular orbits. The angular velocity ω can be calculated using Kepler's law, or by considering the centripetal force and the gravitational force. In any case, one finds

$$\omega^2 = \frac{G(M_1 + M_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} = \frac{GM_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2|\mathbf{r}_1|}\,,$$

where G is Newton's gravitational constant and where the last equality uses $M_1\mathbf{r}_1 + M_2\mathbf{r}_2 = 0$.

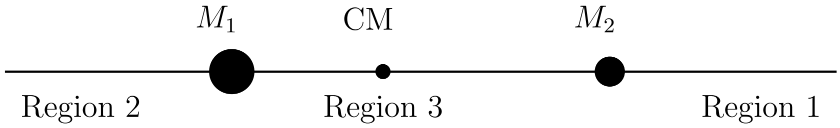
In addition to the choice of the origin, we make another choice: We consider a coordinate system that is co-rotating, thus one in which the masses are at fixed positions. Since this is a non-inertial frame, two additional forces show up: the centrifugal force and the Coriolis force. The forces $\mathbf{F}_m$ on a mass m at position $\mathbf{r}$ are now

$$\mathbf{F}_m = -\frac{GM_1 m}{|\mathbf{r}-\mathbf{r}_1|^3}(\mathbf{r}-\mathbf{r}_1) - \frac{GM_2 m}{|\mathbf{r}-\mathbf{r}_2|^3}(\mathbf{r}-\mathbf{r}_2) + m\omega^2\mathbf{r} - 2m\omega \times \dot{\mathbf{r}}\,, \tag{1}$$

where ω is the angular velocity vector; ω^2 is as above the square of its norm. For an explanation where the Coriolis force and the centrifugal forces come from, one can consult any classical mechanics textbook, or the hand-out

`http://www.mat.univie.ac.at/~westra/coriolis.pdf`.

For stationary points the Coriolis force is irrelevant, as we want points with zero velocity $\dot{\mathbf{r}}$. We first consider those points that lie on the line through the masses M_1 and M_2. We divide this line into three parts: Region 1 is to the right of M_2, region 3 is between M_1 and M_2 and region 2 is to the left of M_1 – see the sketch below, where CM stands for center of mass. We will in general not be able to solve exactly for the positons where $\mathbf{F}_m = 0$, but we will find good approximations if $M_1 >> M_2$.



Region 1 and region 2 have the same equation for $\mathbf{F}_m = 0$, but with M_1 and M_2 interchanged. However, if we assume $M_2 << M_1$ the approximate equations are different.

Region 1

Coordinates on the line through M_1 and M_2 are given as follows: The center of mass ist at $x = 0$, mass M_1 is at position $x = -r_1$, mass M_2 is at position $x = r_2$. We thus have $|\mathbf{r}_1 - \mathbf{r}_2| = r_1 + r_2$. Our task is to find a position x such that all forces are balanced at this point, that is $\mathbf{F}_m = 0$.

Balance of forces in the third region is determined by the following equation

$$\frac{GM_1}{(x+r_1)^2} + \frac{GM_2}{(x-r_2)^2} = \frac{G(M_1+M_2)}{(r_1+r_2)^3}x\,.$$

This is a quintic equation, so that solving it by hand seems hopeless. We introduce the new parameter $\alpha = \frac{M_2}{M_1}$ and $z = \frac{x}{r_2}$ and obtain the following equation

$$(z-1)^2 + \alpha(z+\alpha)^2 = \frac{1}{(1+\alpha)^2} z(z-1)^2(z+\alpha)^2\,. \tag{2}$$

We now want to use perturbation theory. The perturbation parameter is α. We would like to find a perturbative expansion in α for a solution. The zeroth order problem is obtained by setting $\alpha = 0$. For $\alpha = 0$ eqn.(2) reduces to

$$(z-1)^2 = z^3(z-1)^2$$

which is solved for by $z = 1$. Naive perturbation theory now would proceed as follows: expand the original equation to first order α and consider a solution of the form $z = 1 + \lambda\alpha$ and put it into the equation, discard all higher order terms α^2, α^3 and so on and then determine λ. We now show this does not work, why it does not work and how to proceed in this and similar cases.

If we use the expansion

$$\frac{1}{(1+\alpha)^2} = 1 - 2\alpha + 3\alpha^2 - 4\alpha^3 - \ldots$$

we find that the function

$$P(z) = (z-1)^2 + \alpha(z+\alpha)^2 - \frac{1}{(1+\alpha)^2} z(z-1)^2(z+\alpha)^2$$

is to first in α given by

$$P(z) = (z-1)^2(1-z^3) + \alpha z^2(1 + 2(z-1)^3) + O(\alpha^2)\,.$$

To find an approximate zero, we write $z = 1 + \alpha\lambda$ and find

$$0 = \alpha + O(\alpha^2)\,.$$

The symbol $O(\alpha^2)$ stands for terms that are at least quadratic in α. The variable λ has dropped out of the equation!

The reason for this bad bevavior is that $z = 1$ is not an ordinary zero of $P(z)$, but a third order zero, indeed $P(z) = -(z-1)^3(1 + z + z^2) + O(\alpha)$. To study this kind of problems, we consider the following problem: Let $P(x;\epsilon)$ be a polynomial in x that is given as an expansion in ϵ:

$$P(x;\epsilon) = P_0(x) + \epsilon P_1(x) + \epsilon^2 P_2(x) + \ldots$$

where all the $P_i(x)$ are polynomials – which is not really a necessary requierement yet. We assume $P_0(0) = 0$ and want a perturbative expansion for this zero in ϵ. We thus want an expression $x(\epsilon) = \epsilon x_1 + \epsilon^2 x_2 + \ldots$ such that $P(x(\epsilon);\epsilon) = 0$ for all orders in ϵ. Using the first order Taylor expansion $P_0(\epsilon x_1) = \epsilon x_1 P'(0)$, we find that the first order equation is

$$\epsilon x_1 P_0'(0) + \epsilon P_1(0) = 0\,.$$

Now we clearly see the structure; this equation can only be solved for x_1 if $x = 0$ is an ordinary zero of P_0. If it is a double zero, then $P_0'(0) = 0$ and one cannot solve for x_1.

Let us assume that $P_0(0) = P_0'(0) = P_0''(0)$ but $P_0'''(0) \neq 0$. Then $P_0(x) = \frac{1}{6}P_0'''(0)x^3+$ higher order terms. This makes clear that x^3 should be of order ϵ, and thus we can expand in powers of $\epsilon^{1/3}$, thus $x = \epsilon^{1/3}a + \epsilon^{2/3}b + \ldots$.

We thus consider the problem to find approximate solutions for the zero of $P(x;\epsilon) = x^3Q(x) + \epsilon R(x) + \ldots$, where $Q(0) \neq 0$ and the ellipsis contains terms of higher order in ϵ. Putting in $x(\epsilon) = a\epsilon^{1/3} + b\epsilon^{2/3} + \ldots$ one finds

$$P(x(\epsilon);\epsilon) = \epsilon(a^3Q(0) + R(0)) + \epsilon^{4/3}(3a^2bQ(0) + a^3Q'(0) + aR'(0)) + \ldots\,.$$

Requiring $P(x(\epsilon);\epsilon) = 0$ to all orders in ϵ thus results in $a = -\sqrt[3]{\frac{R(0)}{Q(0)}}$.

Now we return to our original problem and consider the transformation $z = 1 + y$, so that the zero lies at $y = 0$. We then find

$$P(y) = -y^3(3+3y+y^2) + \alpha(1+\alpha)^2(1+2y^3) + \alpha^2(1+y)(2-y^2)(2+6y+3y^2) + \ldots.$$

Identifying $Q(y) = -(3+3y+y^2)$ and $R(y) = (1+y)^2(1+2y^2)$, we find $Q(0) = -3$ and $R(0) = 1$. Thus we have

$$y(\epsilon) = \epsilon^{1/3}\sqrt[3]{\frac{1}{3}} + \ldots = \sqrt[3]{\frac{\epsilon}{3}} + \ldots\,.$$

Retracing back the definitions we find

$$x = r_2\Big(1 + \sqrt[3]{\frac{M_2}{3M_1}}\Big)\,.$$

In the literature this point is called L_2.

Now let us plug in some numbers. For the system earth-sun we have $M_1 = M_{sun} \approx 2\cdot 10^{30}$ kg, $M_2 = M_{earth} \approx 6 \cdot 10^{24}$ kg and $r_2 \approx 150\cdot 10^6$ km. Thus $x \approx r_2 + 1.5\cdot 10^6$ km. That is, L_2 lies at 1.5 Million km from the earth, away from the sun.

Region 2

We let x be the distance between the test mass and the center of mass. Mass M_1 is r_1 left from the center of mass, M_2 is r_2 right from the center of mass. Balance of forces in region 3 is then given by the equation

$$\omega^2 x = \frac{GM_1}{(x-r_1)^2} + \frac{GM_2}{(x+r_2)^2}\,. \tag{3}$$

Again, a general solution is hopeless to find, so we assume $M_1 >> M_2$, and thus $r_1 << r_2$. We put $x = zr_2$, $M_2 = \epsilon M_1$ and thus $r_1 = \epsilon r_2$. Together with $\omega^2 = \frac{G(M_1+M_2)}{(r_1+r_2)^3}$ this turns eqn.(3) into

$$\frac{1}{(1+\epsilon)^2} z(z-\epsilon)^2(z+1)^2 = (z+1)^2 + \epsilon(z-\epsilon)^2\,. \tag{4}$$

For convenience we multiply through by $(1+\epsilon)^2$. Then, to first order in ϵ eqn.(4) reduces to

$$0 = (z^3-1)(z+1)^2 - \epsilon(2z^4+4z^3+5z^2+4z+2)\,. \tag{5}$$

The relevant solution for $\epsilon = 0$ is $z = 1$. Since $z^3 - 1 = (z-1)(1+z+z^2)$ we recast eqn.(5) in the form

$$0 = (z-1)Q(z) + \epsilon R(z)\,.$$

Now $z = 1$ is an ordinary zero and we thus can savely take $z(\epsilon) = 1 + \lambda\epsilon$ and find to first order $\lambda = \frac{R(1)}{Q(1)}$. In our case $Q(1) = 12$ and $R(1) = 17$.

If we now retrace the definitions we find $x = r_2(1 + \frac{17}{12}\frac{M_2}{M_1})$; the distance between this point and the mass M_1 is thus

$$x - r_1 = r_2\Big(1 + \frac{5}{12}\frac{M_2}{M_1}\Big)$$

whereas the distance between M_1 and M_2 is $r_1 + r_2 = r_2(1 + \frac{M_2}{M_1})$. This Lagrangian point, called L_3 in the literature, is thus closer to M_1 than M_2 is to M_1. For the sun-earth system we find that L_3 is just some 600 km outside the orbit of the earth.

Region 3

Balance of forces is now given by

$$\frac{GM_1}{(x+r_1)^2} - \frac{GM_2}{(x-r_2)^2} = \frac{G(M_1+M_2)}{(r_1+r_2)^3}x\,. \tag{6}$$

We put $r_1 + r_2 = a$, $z = \frac{x}{a}$, $s = \frac{r_2}{r_1+r_2}$, $t = \frac{r_1}{r_1+r_2} = 1 - s$, so that $M_1 = s(M_1 + M_2)$ and $M_2 = t(M_1 + M_2)$. This reduces eqn.(6) to

$$s(z-s)^2 - t(z+t)^2 = z(z+t)^2(z-s)^2\,.$$

If $s = t = \frac{1}{2}$ then indeed one finds a solution $z = 0$, i.e., precisely in between the two masses. If $M_1 >> M_2$, then s tends to 1 and t tends to 0. We thus put $s = 1 - \beta$ and $t = \beta$ and find the following equation for z up to first order in β

$$0 = (z - 1)^3(1 + z + z^2) + \beta(4z^4 - 6z^3 + 4z^2 - 4z + 3)\,.$$

This equation is solved perturbatively by $z = 1 + \lambda\beta^{1/3}$ and one finds $\lambda = -\sqrt[3]{\frac{1}{3}}$. Hence we have

$$x = a\Big(1 - \sqrt[3]{\frac{M_2}{3M_1}}\Big)\,,$$

so that for the system sun-earth we find that this Lagrangian points lies approximately 1.5 Mio km from the earth in the direction of the sun. In the literature this point is called L_1.

Other Lagrangian points

Are there other Lagrangian points? It turns out, yes, but only two more, making thus a total of five. Balance of forces leads to an equation of the form

$$\alpha(\mathbf{r} - \mathbf{r}_1) + \beta(\mathbf{r} - \mathbf{r}_2) = \mathbf{r}\,,$$

where α and β depend on $\mathbf{r}$ through $|\mathbf{r} - \mathbf{r}_1|$ and $|\mathbf{r} - \mathbf{r}_2|$. Rearranging one concludes that $\alpha\mathbf{r}_1 + \beta\mathbf{r}_2$ has to be parallel to $\mathbf{r}$. Since $\mathbf{r}_1$ and $\mathbf{r}_2$ lie on the line through M_1 and M_2 a solution is only possible if m is on the line through M_1 and M_2, or the coefficients are chosen such that $\alpha\mathbf{r}_1 + \beta\mathbf{r}_2 = 0$. The center of mass equation $M_1\mathbf{r}_1 + M_2\mathbf{r}_2 = 0$ gives the ratio between the lengths of $\mathbf{r}_1$ and $\mathbf{r}_2$ and thus $\alpha\mathbf{r}_1 + \beta\mathbf{r}_2 = 0$ if and only if $\alpha : \beta = M_1 : M_2$. Looking at the precise form of α and β,

$$\alpha = \frac{GM_1}{\omega^2|\mathbf{r} - \mathbf{r}_1|}\,, \qquad \beta = \frac{GM_2}{\omega^2|\mathbf{r} - \mathbf{r}_2|}\,,$$

we see that $\alpha : \beta = M_1 : M_2$is only satisfied if $|\mathbf{r} - \mathbf{r}_1| = |\mathbf{r} - \mathbf{r}_2|$. Let us consider this case and define $s = |\mathbf{r} - \mathbf{r}_1| = |\mathbf{r} - \mathbf{r}_2|$. Balance of forces is then given by

$$\frac{M_1}{s^3}(\mathbf{r} - \mathbf{r}_1) + \frac{M_2}{s^3}(\mathbf{r} - \mathbf{r}_2) = \frac{M_1 + M_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}\mathbf{r}\,.$$

It follows straightforwardly that this equation is only satisfied if $|\mathbf{r}_1 - \mathbf{r}_2| = s$. Thus the final two Lagrangian points are those two points in the plane of rotation that make the three masses lie on the vertices of an equilateral triangle. In the literature these are called L_4 and L_5.

Stability of L_4 and L_5

The first free Lagrangian points are unstable. This means that if a space-ship is parked there a small deviation from the precise position will let the space-ship wander off from the Lagrangian point. The speed with which this happens differs from point to point, but a typical time-scale is of the order of years. A good discussion on these issues is found in

`https://map.gsfc.nasa.gov/ContentMedia/lagrange.pdf`,

a nice document by Neil Cornish. In fact, we will discuss the stability of L_4 and L_5 by the methods outlined in this document, but with a slightly worse notation perhaps.

In order to study stability we slightly perturb the equations of motion around the solution. Since Newton's equations are second order equations, it is convenient to turn them into a system of first-order differential equations. This is done as follows: We first introduce the six-dimensional vector $Z = (\mathbf{r}, \dot{\mathbf{r}})$, we then have

$$\dot{Z} = (z_4, z_5, z_6, F_1, F_2, F_3)\,.$$

Since the forces depend on positions and velocities the right-hand side is a function of components of Z.

In the case at hand a few simplifications are possible; from physical grounds we know that the mass m will always be atracted to the plane of rotation and the Coriolis force does not work in this direction. We may thus simplify and consider motion in the plane of rotation. We thus take $Z = (x, y, v_x, v_y)$. We have a solution $Z_0 = (\frac{1}{2}a + x_1, \frac{\sqrt{3}}{2}a, 0, 0)$ for L_4, say. Of course $a = |\mathbf{r}_1 - \mathbf{r}_2|$ is the length of the sides of the equilateral triangle on whose corners the three masses are.

The equations of motion are $\dot{Z} = (v_x, v_y, a_x, a_y) = G(Z)$, where $G(Z)$ is some vector whose components depend on those of Z, and we consider solutions of the form $Z = Z_0 + \delta Z$, where $\delta Z = (\delta x, \delta y, \delta v_x, \delta v_y)$ represents a small deviation from Z_0. Since $G(Z_0) = 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta Z = G(Z_0 + \delta Z)\,.$$

The trick is now to expand $G(Z_0 + \delta Z)$ to first order in δZ. The resulting equation will then be of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta Z = M\delta Z\,, \tag{7}$$

where M is some matrix. Stability is then guaranteed if all the eigenvalues of M have non-positive real part. Indeed, if $\lambda = a + ib$ is some eigenvalue for an eigenvector

W and $a > 0$, then the vector $We^{(a+ib)t}$ solves equation (7). Clearly this solution wanders away from the solution Z_0 on time-scales $1/a$. Below we will perform these calculations for $Z_0 = (\frac{1}{2}a + x_1, \frac{\sqrt{3}}{2}a, 0, 0)$.

We take M_1 to be at position $(x_1, y_1) = (-\frac{M_2 a}{M_1+M_2}, 0)$ and M_2 to be at $(x_2, y_2) = (\frac{M_1 a}{M_1+M_2}, 0)$. The accelerations are given by

$$a_x = -\frac{GM_1}{((x-x_1)^2+y^2)^{3/2}}(x-x_1) - \frac{GM_2}{((x-x_2)^2+y^2)^{3/2}}(x-x_2) + \omega^2 x + 2\omega v_y$$

and

$$a_y = -\frac{GM_1}{((x-x_1)^2+y^2)^{3/2}}(y-y_1) - \frac{GM_2}{((x-x_2)^2+y^2)^{3/2}}(y-y_2) + \omega^2 y - 2\omega v_x \,.$$

We now insert $x = \frac{1}{2}a + x_1 + \delta x$, $y = \frac{\sqrt{3}}{2}a + \delta y$, $v_x = \delta v_x$, $v_y = \delta v_y$, and keep only terms linear in δx, δy, δv_x and δv_y. A useful expansion is

$$\frac{1}{|\mathbf{r}-\mathbf{r}_1|^3} = \frac{1}{|\mathbf{r}_0-\mathbf{r}_1|^3} - \frac{3(\mathbf{r}_0-\mathbf{r}_1)\cdot(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_1|^5} + O(|\mathbf{r}-\mathbf{r}_0|^2)\,.$$

We however stipulate an even easier way. We write the accelerations as

$$a_x = \frac{\partial}{\partial x}U(x,y) + 2\omega v_y\,, \quad a_y = \frac{\partial}{\partial y}U(x,y) - 2\omega v_x\,,$$

with $U(x,y) = \frac{GM_1}{|\mathbf{r}-\mathbf{r}_1|} + \frac{GN_2}{|\mathbf{r}-\mathbf{r}_2|} + \frac{1}{2}\omega^2 r^2$, which plays the role of the negative of a potential. At the point Z_0 the gradient of $U(x,y)$ vanishes. Writing $U_{xx} = \frac{\partial^2}{\partial x^2}U(x,y)$, $U_{xy} = \frac{\partial^2}{\partial x \partial y}U(x,y)$, $U_{yy} = \frac{\partial^2}{\partial y^2}U(x,y)$ evaluated at the point $(x,y) = (\frac{1}{2}a + x_1, \frac{\sqrt{3}}{2}a)$ the matrix M is found to be

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2\omega \\ U_{xy} & U_{yy} & -2\omega & 0 \end{pmatrix}.$$

Thus we only have to calculate the derivatives of U, evaluate at the solution point, and then calculate the eigenvalues of M.

One finds that

$$U_{xx} = \frac{3}{4}\omega^2\,, \quad U_{xy} = \frac{3}{4}\tau^2\,, \quad U_{yy} = \frac{9}{4}\omega^2\,,$$

where $\tau^2 = \frac{G(M_1-M_2)}{a^3}\sqrt{3}$, where we already anticipated that $M_1 > M_2$. The equation $\det(M - \lambda 1) = 0$ then becomes

$$\lambda^4 + \omega^2\lambda^2 + \frac{27}{16}\omega^4 - \frac{9}{16}\tau^4\,,$$

which is quite remarkable, since no odd powers of λ appear. For us, this is convenient, as we can now solve exactly by hand. First we remark that

$$\frac{27}{16}\omega^4 - \frac{9}{16}\tau^4 = \frac{27}{4}\frac{G^2 M_1 M_2}{a^6}\,.$$

Then we calculate the discriminant as $D = \frac{G^2}{a^6}(M_1^2 + M_2^2 - 25 M_1 M_2)$, which is smaller than ω^4. We have for λ^2 the expression

$$\lambda^2 = -\frac{1}{2}\omega^2 \pm \frac{1}{2}\sqrt{D}\,.$$

If $D > 0$, then $\lambda^2 < 0$ since $\sqrt{D} < \omega^2$ and we thus have a total of four imaginary eigenvalues λ, which implies stability. If $D = 0$, then we have two imaginary eigenvalues and thus also stability. If however $D < 0$, then λ^2 aquires an imaginary part. Thus $\lambda^2 = \rho e^{\pm i\varphi}$ with $\frac{\pi}{2} < \varphi < \pi$. Thus $\lambda = \pm\sqrt{\rho}e^{\pm i\varphi/2}$ and at least one of them has a positive real part and in this case we thus find an unstable direction.

Since the zeros of $t^2 - 25t + 1$ are at $t_1 = \frac{25}{2}(1 + \sqrt{1 - \frac{4}{625}})$ and at $t_2 = 1/t_1$ we conclude that L_4 and L_5 are stable if the larger mass M_1 is at least t_1 times as large as the smaller mass M_2. Indeed, we have

$$D = \frac{G^2 M_2^2}{a^6}\Big(\Big(\frac{M_1}{M_2}\Big)^2 + 1 - 25\frac{M_1}{M_2}\Big)$$

which is positive if $\frac{M_1}{M_2} > t_1$ or $\frac{M_1}{M_2} < t_2$. Since $t_2 < 1$ the last inequality cannot be satisfied if M_1 is larger than M_2, hence we need the first, and we have thus proven stability for large enough M_1, i.e.

$$M_1 > \frac{25 M_2}{2}\Big(1 + \sqrt{1 - \frac{4}{625}}\Big) \approx 24.96 M_2\,.$$

For the earth-sun system this is satisfied, and even for the system earth-moon it holds. The stability of the Trojan asteroids in between Mars and Jupiter is explained this way: At the points L_4 and L_5 in the system sun-jupiter little rocks that fly around our solar system accumulate. In these points they can stay for a long time and small perturbations will let them tend back to their position.