Which geometry is predicted by electromagnetic waves?

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Electromagnetism and geometry

The relation between GR and geometry is well known. Does there exist some relation between electromagnetism and the geometry of basic spacetime?

Example — Electromagnetic field in vacuum

$$\nabla \cdot \mathbf{H} = 0\,, \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = 0\,, \tag{1}$$

$$\nabla \cdot \mathbf{E} = 0\,, \qquad \nabla \times \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} = 0\,. \tag{2}$$

Characteristic equation *(Courant-Hilbert "Methods of math. physics, vol. 2)*

$$w^2(w^2 - k_1^2 - k_2^2 - k_3^2)^2 = 0$$

represents:

1. dispersion relation for the electromagnetic waves and

2. metric of the Lorentzian geometry.

Note: The signs in (1-2) are due to the Dufay law (repulsion of like charges and attraction of opposite ones), and to the Lenz rule (the pulling of a ferromag. core into a solenoid independently on the direction of the current).

Thus the Lorentz signature of the metric of spacetime is electromagnetic in origin. *(Y.I. and F.-W. Hehl, Ann. Phys. 312 (2004) 60-83)*.

Plain:

To generalize this result we need answer the questions:

1. What are a most general electromagnetic field equations?

2. How must be generalized the notion of the electromagnetic waves?

3. What is a general characteristic equation?

4. Which covariant dispersion relation it yields?

5. Which geometry corresponds to this dispersion relation?

Differential Maxwell equations

Premetric description of electromagnetism — F. W. Hehl, Yu. N. Obukhov, *Foundations of Classical Electrodyn. (2003).*

The **field strength** F (even 2-form), the **excitation** $\mathcal{H}$ (odd 2-form) are defined. For charge-free regions, they are related by the equations: in forms —

$$dF = 0\,, \qquad d\mathcal{H} = 0\,, \tag{3}$$

in tensors —

$$\epsilon^{ijkl} F_{jk,l} = 0\,, \qquad \mathcal{H}^{ij}{}_{,j} = 0\,, \tag{4}$$

where F_{ij} – tensor, $\mathcal{H}^{ij}$ – tensor density,

$$F = \frac{1}{2} F_{ij} dx^i \wedge dx^j\,, \qquad \mathcal{H} = \frac{1}{2} \mathcal{H}^{ij} \epsilon_{ijmn}\, dx^m \wedge dx^n\,, \tag{5}$$

All the equations are invariant under **general smooth transformations** of the coordinates even the ordinary derivatives are used.

Integral Maxwell equations

In a lot of physical situations, the differential Maxwell equations are not well posed. For instance in the case of two media of different electromagnetic characteristics, the fields F and H are singular on the hypersurface between them.

The integral Maxwell equations do not have this problem.
Replace

$$dF = 0 \qquad \longmapsto \qquad \int_C F = 0\,. \tag{6}$$

$$d\mathcal{H} = 0 \qquad \longmapsto \qquad \int_C \mathcal{H} = 0\,. \tag{7}$$

C is a 2-dim. boundary of some connected 3-dim. domain Σ.

$$C = \partial\Sigma\,, \qquad \partial C = 0\,. \tag{8}$$

Jump conditions

Theorem *(Y.I., Ann. of Phys., 327, p. 359-375.)*
Let field F satisfies the equation $\int_C F = 0$. *For an arbitrary smooth hypersurface* $\varphi(x^i) = 0$, *holds:*

$$[F] \wedge d\varphi = 0$$

Proof

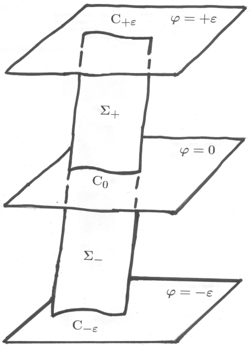


Fig. 1. Here Σ is a part of a 3-dimensional surface, which is transversal to the boundary $\varphi(x^i) = 0$ and bounded by the 3-surfaces $\varphi(x^i) = +\epsilon$ and $\varphi(x^i) = -\epsilon$. The boundary $\varphi(x^i) = 0$ divides Σ into two connected disjoint pieces Σ_+ and Σ_-. The parts of the boundary $\partial\Sigma$ lying on the surfaces $\varphi(x^i) = \pm\epsilon$ are denoted by $C_{+\epsilon}$ and $C_{-\epsilon}$ correspondingly. $\widetilde{C}$ denotes the remaining part of $\partial\Sigma$.

Proof

The 2-form F is assumed to be smooth in the domains Σ_+ and Σ_-. In the whole domain Σ, it is discontinuous, but finite (bounded).
Due to (6), we have

$$\int_{C_{+\epsilon}} {}^{(+)}F + \int_{C_{-\epsilon}} {}^{(-)}F + \int_{\widetilde{C}} F = 0\,. \tag{9}$$

When the limit $\epsilon \to 0$ is taken, the third integral goes to zero, since the domain of integration $\tilde{C}$ approaches zero. The domains $C_{+\epsilon}$ and $C_{-\epsilon}$ approach the same domain C_0 lying in the 3-dimensional surface $\varphi = 0$. Due to the opposite orientations of the domains $C_{+\epsilon}$ and $C_{-\epsilon}$, we are left with

$$\int_{C_0} \left[F\right] = 0\,. \tag{10}$$

Write $\left[F\right] = A \wedge d\phi + B$. Since C_0 is an arbitrary domain embedded in the surface with a constant value of the function φ,

$$\int_{C_0} \left[F\right] = \int_{C_0} A \wedge d\phi + \int_{C_0} B = 0 + \int_{C_0} B = 0\,. \tag{11}$$

Thus $B = 0$, i.e., $\left[F\right] = A \wedge d\phi$. Finally, $\left[F\right] \wedge d\varphi = 0\,.$ □

Eikonal ansatz

Let the fields be expressed as

$$F_{ij} = f_{ij} e^{\varphi} \,, \qquad \mathcal{H}^{ij} = h^{ij} e^{\varphi} \,. \tag{12}$$

Here the eikonal φ is a scalar function of a spacetime point. The tensors h^{ij} and f_{ij} are assumed to be slow functions of a point. The derivatives of φ give the main contributions to the field equations. Define the **wave covector**

$$q_i = \frac{\partial \varphi}{\partial x^i} \,. \tag{13}$$

Substituting into the field equations

$$\epsilon^{ijkl} F_{jk,l} = 0; \qquad \mathcal{H}^{ij}{}_{,j} = 0 \tag{14}$$

and removing the derivatives of the amplitudes relative to the derivative of the eikonal function, we come to the algebraic system

$$\epsilon^{ijkl} q_j f_{kl} = 0 \,, \qquad q_j h^{ij} = 0 \,. \tag{15}$$

Jump conditions → waves

On an arbitrary hypersurface we have jump conditions

$$[F] \wedge d\varphi = 0\,, \qquad [\mathcal{H}] \wedge d\varphi = 0\,. \tag{16}$$

These equations are metric-free and hold for an arbitrary hypersurface. So they generalize the standard boundary conditions between two different media, the initial conditions for Cauchy problem, and the conditions on the wave front surface.
Definition: Generalized "wave solutions" of Maxwell system are non-trivial F and $\mathcal{H}$ that satisfy on some hypresurface $\varphi(x^i) = 0$ the conditions

$$F \wedge d\varphi = 0\,, \qquad \mathcal{H} \wedge d\varphi = 0\,. \tag{17}$$

In tensorial form ($q_i = \partial\varphi/\partial x^i$ – *wave covector*)

$$\epsilon^{ijkl} F_{jk} q_{,l} = 0\,, \qquad H^{ij} q_{,j} = 0\,. \tag{18}$$

For a somewhat similar definition, see *A. Lichnerowicz, (1994) Magneto-hydrodynamics waves and shock waves.*

Constitutive relation

Local linear homogeneous **constitutive relation**

$$\mathcal{H}^{ij} = \frac{1}{2}\chi^{ijkl}F_{kl} \qquad (19)$$

with the **constitutive pseudo-tensor** χ^{ijkl}. Due to the Young diagrams analysis such a tensor is irreducibly decomposed under the group $GL(4,\mathbb{R})$

$$\chi^{ijkl} = {}^{(1)}\chi^{ijkl} + {}^{(2)}\chi^{ijkl} + {}^{(3)}\chi^{ijkl}\,. \qquad (20)$$

axion part	${}^{(3)}\chi^{ijkl} = \chi^{[ijkl]}$	– 1 component
skewon part	${}^{(2)}\chi^{ijkl} = \frac{1}{2}\left(\chi^{ijkl} - \chi^{klij}\right)$	– 15 components
principal part	${}^{(1)}\chi^{ijkl}$ – the reminder	– 20 components

χ^{ijkl} is assumed to represent all electromagnetic properties of the media.
For a curved space-time of GR,

$$\chi^{ijmn} = \sqrt{-g}\left(g^{im}g^{jn} - g^{in}g^{jm}\right)$$

The constitutive tensor of a dielectric media ,

$$\chi^{ijkl} = \begin{pmatrix} \varepsilon^{\alpha\beta} & \gamma^{\alpha}{}_{\beta} \\ \tilde{\gamma}_{\alpha}{}^{\beta} & \mu^{-1}_{\alpha\beta} \end{pmatrix}$$

Characteristic system

We relate to the integral system of source-free Maxwell equations an algebraic **characteristic system** ($q_i = \partial\varphi/\partial x^i$)

$$\epsilon^{ijkl} f_{kl} q_j = 0; \qquad \chi^{ijkl} f_{kl} q_j = 0 \tag{21}$$

Proposition 1. *A most general solution of the first Maxwell equation is*

$$\epsilon^{ijkl} q_j f_{kl} = 0 \quad \Longrightarrow \quad f_{kl} = \frac{1}{2}\left(a_k q_l - a_l q_k\right)\,, \tag{22}$$

where a_k is an arbitrary covector.
The second Maxwell equation reads

$$\chi^{ijkl} q_l q_j a_k = 0\,. \tag{23}$$

Define the **characteristic matrix**

$$M^{ik} = \chi^{ijkl} q_l q_j\,. \tag{24}$$

It is an analog of Christoffel's matrix from elasticity theory.
We remain with the **characteristic equation**

$$M^{ik} a_k = 0\,. \tag{25}$$

Proposition 3. The characteristic matrix M^{ik} is irreducibly decomposed as

$$M^{ik} = M^{(ik)}\left({}^{(1)}\chi\right) + M^{[ik]}\left({}^{(2)}\chi\right)\,. \qquad (26)$$

The axion part ${}^{(3)}\chi^{ijkl}$ does not contribute to M^{ik}.

Gauge invariant condition:

$$M^{ik}q_k = 0 \qquad \Longrightarrow \qquad RowRank(M) = 3 \qquad (27)$$

It is a linear relation between the rows of the matrix M^{ik}. Every solution of (**??**) is defined only up to an unphysical addition $a_i \sim q_i$ – does not contribute to F.

Charge conservation condition:

$$M^{ik}q_i = 0 \qquad \Longrightarrow \qquad ColRank(M) = 3\,. \qquad (28)$$

It is a linear relations between the columns of the matrix M^{ik}. When a full equation is used it yields $j^i q_i = 0$ — momentum representation of $J^i{}_{,i} = 0$.

Gauge – charge duality: Charge conservation and the gauge invariance hold simultaneously. Indeed, for any matrix, $RowRank(M) = ColRank(M)$.

Existence of a wave-type solution

We must have a solution of

$$M^{ik} a_k = 0\,. \qquad (29)$$

When this equation has a non-trivial solution?

- The solution $a_l = Cq_l$ does not contribute to the electromagnetic field strength so it is unphysical. Hence, the system can have a physical meaning, only if it has an **additional linearly independent solution**.
- Consequently (29) must have at least **two linearly independent solutions** – one for gauge and one for physics.
- Linear algebra: a linear system has two (or more) linearly independent solutions if and only if the rank of the matrix M^{ij} is of 2 (or less).
- For a matrix of a rank 2 (or less), the adjoint matrix (constructed from the cofactors of M^{ij}) is equal to zero, $A_{ij} = 0$.

Theorem 3. The Maxwell system with a general linear constitutive relation has a non-trivial wave-type solution if and only if its adjoint matrix is equal to zero.

Scalar condition

Thus we have a tensorial condition $A_{ij} = Adj(M^{ij}) = 0$. But we need a scalar condition – one dispersion relation.

Proposition 4. *Let for an n-covector $q_i \neq 0$, an $n \times n$ matrix M^{ij} satisfies*

$$M^{ij} q_i = 0\,, \qquad M^{ij} q_j = 0\,. \tag{30}$$

Then the adjoint matrix A_{ij} is proportional to the tensor product of q_i,

$$A_{ij} = \lambda(q) q_i q_j\,. \tag{31}$$

where $\lambda(q)$ is a 4-th order polynomial of q.

Compare to the Maxwell vacuum form *(Courant-Hilbert)*

$$(w^2 - k_1^2 - k_2^2 - k_3^2)^2 w^2 = 0$$

Covariant dispersion relation is a scalar equation

$$\lambda(q) = 0\,. \tag{32}$$

Some basic properties of the dispersion relation

Without an explicit form, only from the fact that $\lambda(q)$ is a factor of the adjoint matrix, we derive:

Corollary 4. $\lambda(q)$ is a homogeneous 4-th order polynomial of the wave covector q_i, i.e.,

$$\lambda(q) = \mathcal{G}^{ijkl} q_i q_j q_k q_l \,, \tag{33}$$

Corollary 5. $\lambda(q)$ is a homogeneous 3-rd order polynomial of the constitutive pseudotensor χ^{ijkl}.

$$\mathcal{G} \sim \chi \cdot \chi \cdot \chi$$

Corollary 6. The equation $\lambda(q) = 0$ defines a (complex) algebraic cone.

$$q - \text{solution} \qquad \Longrightarrow \qquad kq - \text{solution}$$

Corollary 7. For a given wave covector q, the cone is separated to a future cone - $\mathcal{C}_+ = \{kq | k > 0\}$ and a past cone - $\mathcal{C}_- = \{kq | k < 0\}$.

Corollary 8. The axion part of the constitutive tensor does not contribute to the dispersion relation.

Corollary 9. The skewon part alone does not emerge in a non-trivial dispersion relation.

$$\lambda \left({}^{(2)}\chi \right) = 0 \tag{34}$$

Corollary 10. A non-trivial (non-zero) dispersion relation emerges only if the principal part of the constitutive tensor is non-zero ${}^{(1)}\chi \neq 0$.

Explicit forms

Problem: How to find $\lambda(q)$ from $A_{ij} = \lambda(q) q_i q_j$?

Theorem 10. For the Maxwell system with a general local linear constitutive relation, the dispersion relation is given by

$$\lambda(q) = \frac{1}{72} \frac{\partial^2 A_{ij}}{\partial q_i \partial q_j} = 0\,. \qquad (35)$$

Proof: Euler's theorem for homogeneous functions used twice.

Theorem 11. In term of the characteristic matrix,

$$\lambda(q) = \frac{1}{144} \epsilon_{ii_1 i_2 i_3} \epsilon_{jj_1 j_2 j_3} \left(\frac{\partial^2 M^{i_1 j_1}}{\partial q_i \partial q_j} M^{i_2 j_2} + 2 \frac{\partial M^{i_1 j_1}}{\partial q_i} \frac{\partial M^{i_2 j_2}}{\partial q_j} \right) M^{i_3 j_3}\,. \qquad (36)$$

Explicit forms

Different equivalent forms of the λ-function are available:

Hehl-Obukhov-Rubilar, 2002

$$\lambda(q) = \frac{1}{4!}\epsilon_{ii_1i_2i_3}\epsilon_{jj_1j_2j_3}\chi^{ii_1ja}\chi^{bi_2j_1c}\chi^{di_3j_2j_3}q_aq_bq_cq_d = 0\,. \tag{37}$$

Y.I. J.Phys.A42:475402,2009.

$$\lambda(q) = \frac{1}{4!}\epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1aij_1}\chi^{i_2bjj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d = 0\,. \tag{38}$$

ibid

$$\lambda(q) = \frac{1}{4!}\epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{i_1iaj_1}\chi^{i_2jbj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d = 0\,. \tag{39}$$

ibid

$$\lambda(q) = \frac{1}{4!}\epsilon_{ii_1i_2i_3}\,\epsilon_{jj_1j_2j_3}\,\chi^{ii_1jj_1}\chi^{i_2abj_2}\,\chi^{i_3cdj_3}\,q_aq_bq_cq_d = 0\,. \tag{40}$$

Lorentz, Riemann, and Finsler geometry

When the constitutive tensor is chosen as ($\eta^{ij} = diag(1, -1, -1, -1)$)

$$\chi^{ijmn} = \left(\eta^{im}\eta^{jn} - \eta^{in}\eta^{jm}\right)$$

we get **Lorentz metric** $\lambda(q) = \left((q_0)^2 - (q_1)^2 - (q_2)^2 - (q_3)^2\right)^2$
For

$$\chi^{ijmn} = \sqrt{-g}\left(g^{im}g^{jn} - g^{in}g^{jm}\right)$$

we have **pseudo-Riemann metric** $\lambda(q) = \left(g^{ij}q_iq_j\right)^2$
In general case, we have the **Finsler form** $\lambda(q) = \mathcal{G}^{ijkl}q_iq_jq_kq_l$

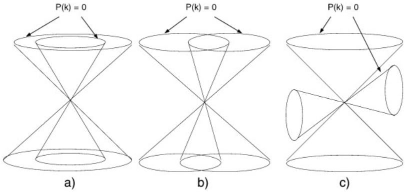


Fig. from D. Raetzel, S. Rivera and F. P. Schuller, Phys. Rev. D **83**, 044047 (2011)

Axion modification of electrodynamics – CFJ-model

The axion modification of electrodynamics has a goal to examine the possibility of **Lorentz and PCT symmetries violations** *F. Wilczek; PRL* **58**, *1799 (1987)*.
The model predicts the rotation of the plane of **polarization of radiation** from distance galaxies . *S. M. Carroll, G. B. Field and R. Jackiw PRD* **41**, *1231 (1990)*
The **first field equation** is postulated to be the same as in the ordinary electrodynamics, while the **second Maxwell equation** is **modified**

$$\partial_a F^{ab} + v_a {}^*F^{ab} = J^b \,,$$

where v_a is a pseudo-covector. To preserve $SO(3)$ space invariance, v_a is taken in the form

$$v_a = (\mu, 0, 0, 0)\,, \qquad \mu = \text{constant}.$$

Gauge invariance, charge and energy conservation are preserved.
The wave propagation is described by a **dispersion relation**

$$w = \sqrt{|\mathbf{k}|^2 \pm \mu|\mathbf{k}|} \approx |\mathbf{k}| \pm \frac{\mu}{2}$$

This birefringence of the vacuum generates a Faraday-like rotation on polarized light.

Astronomical measurements impose an upper bound on μ.

CFJ-model in the premetric approach

Axion field naturally emerges in the premetric electrodynamics *Post, Hehl and Obukhov*. But in this construction, the axion **is not modify the light propagation.** In fact, the CFJ-axion is in fact a special type of the premetric axion. *Y. I., PRD* **70**, *025012 (2004)*
Assume the ordinary Maxwell equations

$$\epsilon^{ijkl}F_{jk,l}=0\,,\qquad \mathcal{H}^{ij}{}_{,j}=0\,, \tag{41}$$

with the **axion modified constitutive tensor**

$$\chi^{ijkl}=\left(g^{ik}g^{jl}-g^{il}g^{jk}\right)\sqrt{-g}+\psi\epsilon^{ijkl}\,.$$

Substitute potential into the second field equation

$$\chi^{ijkl}A_{k,lj}+\chi^{ijkl}{}_{,j}A_{k,l}=0$$

We apply the following approximation

$$\chi^{ijkl}{}_{,j}=\psi_{,j}\epsilon^{ijkl}$$

In other words, we restrict to the spacetime regions where the gravitational field varies slowly as compared to the change of the pseudoscalar field.

Wave ansatz

Consider an ansatz

$$A_k = \mathrm{Re}\left(a_k e^{iq_m x^m}\right)$$

Substituting in the field equation we have

$$M^{ij} = \left(g^{ij}q^2 - q^i q^j\right)\sqrt{-g} + i\psi_{,k}q_l\epsilon^{ijkl}$$

The linear system $M^{ij}a_j = 0$ has two linear independent solutions (one for gauge and one for physics) if and only if its matrix M^{ij} is of rank 2 or less, i.e, the adjoint matrix $A_{ij} = 0$. Algebraic calculations yield

$$A_{ij} = -\sqrt{-g}\left[q^4 + \left(\psi_{,m}\psi^{,m}\right)q^2 - \left(\psi_{,m}q^m\right)^2\right]q_i q_j\,.$$

The **axion modified dispersion relation**

$$\boxed{q^4 + \left(\psi_{,m}\psi^{,m}\right)q^2 - \left(\psi_{,m}q^m\right)^2 = 0\,.}$$

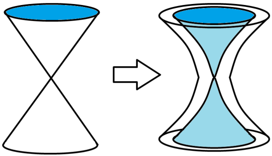
In the $(1+3)$ notation, $q = (\omega, \mathbf{k})$ and $\psi_{,i} = (\mu, \mathbf{m})$

$$(\omega^2 - k^2)^2 + \left(\mu^2 - m^2\right)(\omega^2 - k^2) - (\omega\mu + mk\cos\alpha)^2 = 0\,.$$

Axion field with a timelike derivative $\psi_{,i} = (\mu, 0, 0, 0)$

The dispersion relation is

$$w^2 = k^2 \pm \mu k\,.$$

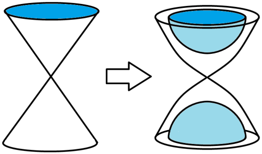


The first hypersurface is topologically equivalent to the ordinary light cone. The second hypersurface is topologically different. It consists of two pieces which are joined by a sphere ($k = \mu$, $w = 0$) and by a point ($k = 0$, $w = 0$). Consequently the interior region cannot be separated into two disjoint parts even if the origin is removed. It means that the future (the upper region) and the past (the downer region) cannot be disjoint. The causality on the second branch is violated.

Axion field with a spacelike derivative $\psi_{,i} = (0, m, 0, 0)$

The dispersion relation

$$w^2 = k^2 + \frac{m^2}{2} \pm \sqrt{\frac{m^4}{4} + m^2 k^2 \cos^2 \alpha}$$



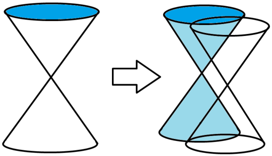
The first hypersurface is topologically different from the standard one. Two branches are represented by two disjoint surfaces. In other words there is not, for this lightlike hypersurfaces, a way from the past to the future. The second hypersurface is topologically equivalent to the ordinary one.

w is real for every values of parameters. No runaway solutions.

Axion field with a lightlike derivative $\psi_{,i} = (m, m, 0, 0)$

The dispersion relation

$$w^2 = k^2 \pm m(w + k_1)$$



This topological structure is also different from the standard light cone structure. The future and the past surfaces are contacted at points

$$w = \pm \frac{m}{2}\,, \quad k_1 = \pm \frac{m}{2}\,, \quad k_2 = 0\,, \quad k_3 = 0\,.$$

The **birefringence effect** emerges for arbitrary varying null axion fields.

Problems:

- What is a precise math. and physical sense of the characteristic polynomial $\lambda(q)$? How much of the e-m system it characterizes?
- How $\lambda(q)$ can be directly derived from the field equation?
- Under which conditions on the constitutive pseudotensor χ^{ijkl} we have some meaningful physical situation:

  a) birefringence/ non-birefringence

  b) time-symmetry

  c) hyperbolicity

  d) ...
- Extended characteristic polynomial

$$\lambda(\chi, \partial\chi)$$