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Uniformization by Wignerization

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Contents of the talk

Scope of the talk

to explain by a concrete example how to uniformize a two-phase WKB function using the Wigner transform

Geometric optics-WKB method

Consider for example the Helmholtz equation

$$\Delta u(\mathbf{x},\kappa)+\kappa^2\eta^2(\mathbf{x})u(\mathbf{x},\kappa)=f(\mathbf{x})\ , \quad \mathbf{x}\in M\subset\mathbb{R}^n_{\mathbf{x}}$$

$\eta(\mathbf{x})=c_0/c(\mathbf{x})$ smooth refraction index
($c_0 =$ some reference velocity, $c(\mathbf{x}) =$ phase velocity at $\mathbf{x}$)
$f\in C_c^\infty(\mathbb{R}^n_{\mathbf{x}})$, source of waves
$\kappa=\omega/c_0 =$ wavenumber ($\omega =$ circular frequency)

WKB method aims to the construction of asymptotic solutions for high frequencies ($\kappa\to\infty$)

Geometrical optics - WKB method

Single-phase optics (Bensoussan, Lions, Papanicolaou-1979)

$$
\begin{aligned}
u(\mathbf{x},\kappa) &= A(\mathbf{x},\kappa)\ e^{i\kappa S(\mathbf{x})} \\
A(\mathbf{x},\kappa) &= A_0(\mathbf{x}) + \frac{1}{i\kappa}A_1(\mathbf{x}) + \dots
\end{aligned}
$$

- $S(\mathbf{x})$ solution of the eikonal equation

$$
\begin{cases}
(\nabla S(\mathbf{x}))^2 = \eta^2(\mathbf{x}) \\
S(\mathbf{x}) = s_0(\mathbf{x})\,, \quad \mathbf{x} = \mathbf{x}_0(\theta) \in \Lambda_0 \\
\Lambda_0 := \{\mathbf{x} = \mathbf{x}_0(\theta)\ ,\ \theta = (\theta_1, \dots, \theta_{n-1}) \in U_0 \subset \mathbb{R}^{n-1}\}
\end{cases}
$$

- $A_0(\mathbf{x})$ solution of the transport equation

$$
\begin{cases}
2\nabla S(\mathbf{x}) \cdot \nabla A_0(\mathbf{x}) + A_0(\mathbf{x})\Delta S(\mathbf{x}) = 0 \\
A_0(\mathbf{x}) = \alpha_0(\mathbf{x}_0(\theta))
\end{cases}
$$

WKB method-Caustics

The eikonal is solved by integration along the Rays $\{\mathbf{x} : \mathbf{x} = \mathbf{x}(t;\theta)\} \equiv$ trajectories of the Hamiltonian system

$$\begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{k} \ , & \mathbf{x}(0) = \mathbf{x}_0(\theta) \\ \frac{d\mathbf{k}}{dt} = \eta \cdot \nabla\eta \ , & \mathbf{k}(0) = \nabla s_0(\mathbf{x}_0(\theta)) \end{cases}$$

Integration of the transport on a ray tube gives the amplitude

$$A_0(\mathbf{x}) = \frac{\alpha_0(\theta)}{\sqrt{J(t,\theta)}} \quad , \qquad J(t,\theta) = \det \frac{\partial \mathbf{x}(t,\theta)}{\partial(t,\theta)}$$

Caustic $\{\mathbf{x} = \mathbf{x}(t;\theta) : J(t,\theta) = 0\} \quad \Rightarrow A_0(\mathbf{x})$ becomes infinite

Modified geometric optics

Uniform solutions (finite amplitude on the caustics)

- Fourier integral operators
  - Canonical integrals (Kravtsov-1968, Ludwig-1960) ▸ Kravtsov
  - Maslov's canonical operator(Maslov-1965)
- Boundary layer techniques (Fock & Leontovich-1940, Buchal & Keller-1960, book by Babich & Kirpichnikova-1979)
- Phase space equations using the Wigner transformWigner (Filippas & Makrakis-2003: exploiting the pioneering approximation constructed by Berry-1977)

Geometric optics for the Airy equation-1

Semiclassical Airy equation with point source ▸ Airy

$$\epsilon^2 \frac{d^2}{dx^2} u^\epsilon(x) + x u^\epsilon(x) = \sigma(\epsilon)\delta(x - x_0)\,, \quad x \in \mathbb{R}_x \quad \epsilon = 1/\kappa$$

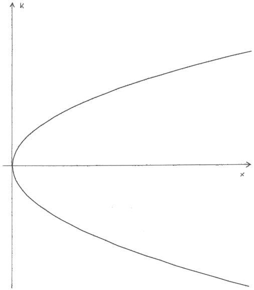


Figure: Lagrangian "manifold" $\Lambda = \{(x,k) : x = k^2\}$

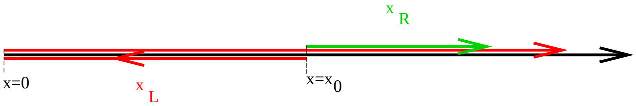


Figure: Configuration space $\mathbb{R}_x$

caustic : $x = 0$ (turning point)

Geometric optics for the Airy equation-3

$$
\begin{aligned}
u^{\epsilon}_{WKB}(x) &= \frac{1}{2} e^{-1/4} x_0^{-1/2} e^{i\frac{1}{\epsilon}\frac{2}{3}x_0^{3/2}} \left(-i x^{-1/4} e^{i\frac{1}{\epsilon}\frac{2}{3}x^{3/2}} + x^{-1/4} e^{-i\frac{1}{\epsilon}\frac{2}{3}x^{3/2}} \right) \\
u^{\epsilon}_{KL}(x) &= \pi^{1/2} e^{-i\pi/2} \left(x_0^{-1/4} e^{i\frac{1}{\epsilon}\frac{2}{3}x_0^{3/2}} \right) \epsilon^{-1/6} Ai\left(-\frac{x}{\epsilon^{2/3}} \right) , \quad x > 0
\end{aligned}
$$

Remark: Because of the simplicity of the equation $u^{\epsilon}_{KL}(x) = u^{\epsilon}(x)$, $u^{\epsilon}(x)$ being the fundamental solution of the Airy function

Scope: Use the Wigner transform to uniformize $u^{\epsilon}_{WKB}(x)$

Wigner transform

The scaled Wigner transform of a complex-valued function ψ^ϵ (let it $\in \mathcal{S}(\mathbb{R})$), is defined by

$$W^\epsilon[\psi^\epsilon] = W^\epsilon(x,k) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iky}\psi^\epsilon\left(x+\frac{\epsilon y}{2}\right)\overline{\psi^\epsilon\left(x-\frac{\epsilon y}{2}\right)}\,dy$$

(Lions & Paul-1993, Gerard et al-1997)

- $W^\epsilon[\psi^\epsilon] = W^\epsilon(x,k)$ is a function defined on phase space $\mathbb{R}^2_{x,k}$

Basic properties

1. $W^\epsilon(x,k) \in \mathbb{R}$
2. $\int_{\mathbb{R}} W^\epsilon(x,k)\,dk = |\psi^\epsilon(x)|^2$, amplitude of ψ^ϵ
3. $\int_{\mathbb{R}} kW^\epsilon(x,k)\,dk = \epsilon Im\left(\overline{\psi}^\epsilon \partial_x \psi^\epsilon\right)$, energy flux

Wigner transform of WKB functions

using appropriate weak limits (Lions & Paul-1993)

- For single-phase WKB function $\psi^\epsilon(x) = A(x)\ e^{iS(x)/\epsilon}$,

$$W^\epsilon(x,k) \xrightarrow{\epsilon\to 0} |A(x)|^2 \delta(k - S'(x))$$

- For two-phase WKB function
$\psi^\epsilon(x) = A_+(x)\ e^{iS_+(x)/\epsilon} + A_-(x)\ e^{iS_-(x)/\epsilon}$,

$$W^\epsilon(x,k) \xrightarrow{\epsilon\to 0} |\ A_+(x)\ |^2 \delta\left(k - S'_+(x)\right) + |\ A_-(x)\ |^2 \delta\left(k - S'_-(x)\right)\ ,$$

Semiclassical Wigner function-1

Let a single-phase WKB function $u^{\epsilon}(x) = A(x)\ e^{iS(x)/\epsilon}$ where $A(x)$, $S(x)$ smooth real-valued functions, and $S'(x)$ globally convex

Berry's pioneering paper: Construction of uniform asymptotic expansion of $W^{\epsilon}[u^{\epsilon}]$ in phase space (Phil. Trans. Roy. Soc., 1977)

Transforming by Wigner $u^{\epsilon}(x)$ we get the Fourier-type integral

$$W^{\epsilon}[u^{\epsilon}] = W^{\epsilon}(x,k) = \frac{1}{\pi\epsilon}\int_{\mathbb{R}} D(\sigma,x)\ e^{i\frac{1}{\epsilon}F(\sigma,x,k)}\,d\sigma$$

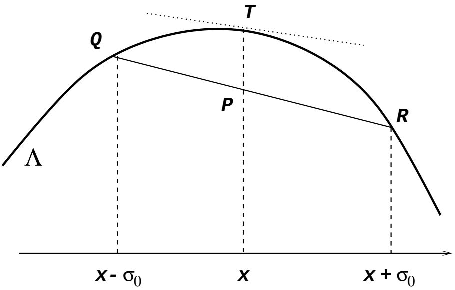
where

$D(\sigma,x) = A(x+\sigma)A(x-\sigma)$ (amplitude)

$F(\sigma,x,k) = S(x+\sigma) - S(x-\sigma) - 2k\sigma$ (Wigner phase)

Semiclassical Wigner function-2

Critical points of Wigner phase:
$F_\sigma(\sigma; x, k) = S'(x+\sigma) + S'(x-\sigma) - 2k = 0$



Berry's chord construction: There is a pair of symmetric roots $\pm\sigma_0(x, k)$, such that the point $P = (x, k)$ is the middle of the chord QR with end points on the Lagrangian manifold $\Lambda = \{k = S'(x)\}$.

Semiclassical Wigner function-3

Choosing $\alpha = \alpha(x,k) := k - S'(x)$ as the parameter controlling the uniformity of stationary-phase approximation, we get

$$W^\epsilon(x,k) \approx \widetilde{W}^\epsilon(x,k) :=$$
$$= \frac{2^{2/3}}{\epsilon^{2/3}} \left(\frac{2}{|S'''(x)|} \right)^{1/3} D(\sigma_o(x,k),x)\ Ai\left(-\frac{2^{2/3}}{\epsilon^{2/3}} \left(\frac{2}{|S'''(x)|} \right)^{1/3} (k - S'(x)) \right)$$

we refer to $\widetilde{W}^\epsilon(x,k)$ as the Semiclassical Wigner function

Airy equation with point source

$$\epsilon^2 \frac{d^2}{dx^2} u^\epsilon(x) + x u^\epsilon(x) = \sigma(\epsilon)\delta(x - x_0)$$

Fundamental solution

$$u^\epsilon(x) = \pi^{1/2} e^{-i\pi/2} \left(x_0^{-1/4} e^{i\frac{1}{\epsilon}\frac{2}{3}x_0^{3/2}} \right) \epsilon^{-1/6} Ai\left(-\epsilon^{-2/3} x\right) , \quad x_0 >> \epsilon$$

(Exact) Wigner function of $u^\epsilon(x)$

$$W^\epsilon_{Ai}(x,k) := W^\epsilon[u^\epsilon](x,k) = \frac{1}{2\sqrt{x_0}} \left(\frac{2}{\epsilon}\right)^{2/3} Ai\left(\left(\frac{2}{\epsilon}\right)^{2/3} (k^2 - x) \right)$$

Remark about the stationary Wigner equation

For the homogeneous semiclassical Helmholtz equation

$$\epsilon^2 u^{\epsilon\prime\prime}(x) + \eta^2(x) u^{\epsilon}(x) = 0, \quad x \in R$$

the stationary Wigner equation is

$$k\partial_x f^{\epsilon} + 1/2\big(\eta^2(x)\big)'\partial_k f^{\epsilon} = -1/2 \sum_{m=1}^{\infty} \alpha_m \epsilon^{2m} \big(\eta^2(x)\big)^{(2m+1)}(x)\partial_k^{2m+1} f^{\epsilon}(x,k)$$

For the Airy equation ($\eta^2 = x$) the Wigner equation simplifies to

$$k\partial_x f^{\epsilon} + 1/2\partial_k f^{\epsilon} = 0$$

which implies that

$$f^{\epsilon}(x,k) = F^{\epsilon}(k^2 - x)$$

Note that $W^{\epsilon}_{Ai}(x,k)$ has the required form and it satisfies the stationary Wigner equation exactly, because we have "moved the source to infinity" ($x_0 >> \epsilon$)

Wigner distribution of the fundamental solution

Observe that

Wigner distribution $\equiv$ weak limit of the Wigner function (fixed x)

$$W^{\epsilon}_{Ai}(x,k) \xrightarrow{\epsilon\to 0} W^0(x,k) = \frac{1}{2x_0^{1/2}}\delta\left(k^2 - x\right)$$

Illuminated zone $x > 0$

$$W^0(x,k) = \frac{1}{4x_0^{1/2}x^{1/2}}\left(\delta\left(k - x^{\frac{1}{2}}\right) + \delta\left(k + x^{\frac{1}{2}}\right)\right)$$

Shadow zone $x < 0$

$$W^0(x,k) = 0 \quad (\textit{as distribution in } \mathrm{k}, \ \textit{fixed } \mathrm{x})$$

Two-phase WKB solution of the Airy equation

Now we depart from the two-phase WKB solution

$$u^{\epsilon}_{WKB}(x) = A_+(x)\ e^{\frac{i}{\epsilon}S_+(x)} + A_-(x)\ e^{\frac{i}{\epsilon}S_-(x)}$$

where

$$A_+(x) = (-i)\frac{1}{2}x^{-1/4}e^{-i\pi/4}{x_0}^{-1/4}$$

$$A_-(x) = \frac{1}{2}x^{-1/4}e^{-i\pi/4}{x_0}^{-1/4}$$

and

$$S_\pm(x) = \pm\frac{2}{3}x^{3/2} + \frac{2}{3}{x_0}^{3/2}$$

Wigner distribution for the two-phase WKB solution

and we recall that the two-phase WKB solution

$$u^{\epsilon}_{WKB}(x) \quad = \quad A_+(x)e^{\frac{i}{\epsilon}S_+(x)} + A_-(x)e^{\frac{i}{\epsilon}S_-(x)}$$

converges weakly to the Wigner distribution

$$W^{\epsilon}[u^{\epsilon}_{WKB}](x,k) \xrightarrow{\epsilon\to 0} \mid A_+(x)\mid^2 \delta\left(k - S'_+(x)\right) + \mid A_-(x)\mid^2 \delta\left(k - S'_-(x)\right)$$

It follows that

$$W^{\epsilon}[u^{\epsilon}_{WKB}](x,k) \xrightarrow{W^0} (x,k) = \frac{1}{2x_0^{1/2}}\delta\left(k^2 - x\right) \ ,$$

that is, we derive again the Wigner distribution of the fundamental solution in the illuminated zone $x > 0$

The uniformization procedure

We proceed now to the main

Question: Can we depart from the Wigner function of the two-phase WKB solution (which lives in the illuminated region $x > 0$) to construct a reasonable approximation of the exact Wigner function, by appropriate "asymptotic surgery"?

(From here the title: Uniformization by Wignerization)

Asymptotic surgery of the Wigner function

Plugging the two-phase WKB solution into the Wigner transform, we get

$$W^{\epsilon}_{WKB}(x,k) = \frac{1}{\pi\epsilon}\sum_{\ell=1}^{4}\int_{\mathbb{R}} D_{\ell}(\sigma;x)\; e^{\frac{i}{\epsilon}F_{\ell}(\sigma;x,k)}d\sigma = \sum_{\ell=1}^{4} W^{\epsilon}_{\ell}(x,k)$$

where, the amplitudes are

$$D_1(\sigma;x) = A_{+}(x+\sigma)\overline{A}_{+}(x-\sigma)$$
$$D_2(\sigma;x) = A_{-}(x+\sigma)\overline{A}_{-}(x-\sigma)$$
$$D_3(\sigma;x) = A_{+}(x+\sigma)\overline{A}_{-}(x-\sigma)$$
$$D_4(\sigma;x) = A_{-}(x+\sigma)\overline{A}_{+}(x-\sigma)$$

and the phases are

$$F_1(\sigma;x,k) = S_{+}(x+\sigma) - S_{+}(x-\sigma) - 2k\sigma$$
$$F_2(\sigma;x,k) = S_{-}(x+\sigma) - S_{-}(x-\sigma) - 2k\sigma$$
$$F_3(\sigma;x,k) = S_{+}(x+\sigma) - S_{-}(x-\sigma) - 2k\sigma$$
$$F_4(\sigma;x,k) = S_{-}(x+\sigma) - S_{+}(x-\sigma) - 2k\sigma$$

Asymptotic approximation of the diagonal terms-1

The stationary points of the phase $F_1(x,k)$

k, stationary points	real	complex
$(-\infty, \sqrt{x})$	$\nexists$	$\nexists$
$[\sqrt{x/2}, \sqrt{x})$	$\pm\sigma_0$ simple	$\nexists$
$k \to \sqrt{x}$	$k = \sqrt{x} - 0\,,\ \sigma_0 = 0$ double	$k = \sqrt{x} + 0\,,\ \sigma_0 = 0$ double
$(\sqrt{x}, +\infty)$	$\nexists$	$\pm i\sigma_0$ simple

where

$$\sigma_0 = \sigma_0(x,k) := 2|k||x - k^2|^{1/2}$$

Asymptotic approximation of the diagonal terms-2

Approximation of the diagonal term $W_1^\epsilon(x,k)$

k	$W_1^\epsilon(x,k)$	approximation technique
$(-\infty, \sqrt{x/2})$	asymptotically negligible	integration by parts
$[\sqrt{x/2}, \sqrt{x})$	$\approx \widetilde{W_1^\epsilon}$	uniform stat. ph.
$k = \sqrt{x} - 0$	$\approx \widetilde{W_1^\epsilon}$	uniform stat. ph.
$k = \sqrt{x} + 0$	$\approx \widetilde{W_1^\epsilon}$	uniform stat. ph.
$(\sqrt{x}, +\infty)$	approx. of same formula	standard stat. ph.

where

$$\widetilde{W_1^\epsilon} = 2^{-1} {x_0}^{-1/2} (2/\epsilon)^{2/3} Ai((2/\epsilon)^{2/3}(k^2 - x))$$

is the semiclassical Wigner function corresponding to $k = S'_+(x)$ (upper branch of the Lagrangian manifold)

Asymptotic approximation of the off diagonal terms -1

The stationary points of the phase $F_3(x, k)$

k, stationary points	real	complex
$(-\infty, -\sqrt{x})$	$\nexists$	$\pm i\sigma_0$ simple
$[-\sqrt{x}, -\sqrt{x/2})$	$\nexists$	$\nexists$
$[-\sqrt{x/2}, 0)$	$-\sigma_0$ simple	$\nexists$
$(0, \sqrt{x/2}]$	$+\sigma_0$ simple	$\nexists$
$(\sqrt{x/2}, \sqrt{x}]$	$\nexists$	$\nexists$
$(\sqrt{x}, +\infty)$	$\nexists$ simple	$\pm i\sigma_0$

Here $\sigma_0 = \sigma_0(x, k) := 2|k||x - k^2|^{1/2}$

Asymptotic approximation of the off diagonal terms -2

Approximation of the off-diagonal term $W_3^\epsilon(x,k)$

k	$W_3^\epsilon(x,k)$	approximation technique
$(-\infty, -\sqrt{x})$	$\approx \widehat{W_3}^\epsilon(x,k)$	steepest descents
$[-\sqrt{x}, -\sqrt{x/2})$	asymptotically negligible	integration by parts
$[-\sqrt{x/2}, \sqrt{x/2}]$	$\approx \widehat{\widehat{W_3}}^\epsilon(x,k)$	standard stat. ph.
$(\sqrt{x/2}, \sqrt{x}]$	asymptotically negligible	integration by parts
$(\sqrt{x}, +\infty)$	$\approx \widehat{W_3}^\epsilon(x,k)$	steepest descents

$$\widehat{W_3}^\epsilon(x,k) := \frac{-i^{1/2}{x_0}^{-1/2}}{2^{5/2}\pi^{1/2}\epsilon^{1/2}}(x^2+{\sigma_0}^2)^{-1/4}\frac{1}{\mid F_{3\sigma\sigma}(i\sigma_0)\mid^{1/2}}\cdot$$

$$\cdot\left[e^{\frac{i}{\epsilon}F_3(i\sigma_0)+i\pi/2}+e^{\frac{i}{\epsilon}F_3(i\sigma_0)+i3\pi/2}+e^{\frac{i}{\epsilon}F_3(-i\sigma_0)+i\pi/2}+e^{\frac{i}{\epsilon}F_3(-i\sigma_0)+i3\pi/2}\right]$$

$$\widehat{\widehat{W_3}}^\epsilon(x,k) := -\frac{i{x_0}^{-1/2}}{2^{3/2}\pi^{1/2}\epsilon^{1/2}}(x-k^2)^{-1/4}e^{i\pi/4}e^{i\frac{4}{3\epsilon}(x-k^2)^{3/2}}$$

Remark 1 : In the region $x < k^2$,

$$\widehat{W_3}^{\epsilon}(x,k) := \frac{-i^{1/2}{x_0}^{-1/2}}{2^{5/2}\pi^{1/2}\epsilon^{1/2}}(x^2+{\sigma_0}^2)^{-1/4}\frac{1}{\mid F_{3\sigma\sigma}(i\sigma_0)\mid^{1/2}}\cdot$$
$$\cdot\left[e^{\frac{i}{\epsilon}F_3(i\sigma_0)+i\pi/2}+e^{\frac{i}{\epsilon}F_3(i\sigma_0)+i3\pi/2}+e^{\frac{i}{\epsilon}F_3(-i\sigma_0)+i\pi/2}+e^{\frac{i}{\epsilon}F_3(-i\sigma_0)+i3\pi/2}\right]$$

$$\widehat{W_4}^{\epsilon}(x,k) := \frac{i^{1/2}{x_0}^{-1/2}}{2^{5/2}\pi^{1/2}\epsilon^{1/2}}(x^2+{\sigma_0}^2)^{-1/4}\frac{1}{\mid F_{4\sigma\sigma}(i\sigma_0)\mid^{1/2}}\cdot$$
$$\cdot\left[e^{\frac{i}{\epsilon}F_4(i\sigma_0)+i\pi/2}+e^{\frac{i}{\epsilon}F_4(i\sigma_0)+i3\pi/2}+e^{\frac{i}{\epsilon}F_4(-i\sigma_0)+i\pi/2}+e^{\frac{i}{\epsilon}F_4(-i\sigma_0)+i3\pi/2}\right]$$

thus

$$W_3^{\epsilon}(x,k)+W_4^{\epsilon}(x,k)\approx\ \widehat{W_3}^{\epsilon}(x,k)+\widehat{W_4}^{\epsilon}(x,k)=0$$

Asymptotic surgery of the Wigner function (wrapping-2)

Remark 2 : In the region $x > 2k^2$,

$$\widehat{W_3}^{\epsilon}(x,k) := -\frac{i{x_0}^{-1/2}}{2^{3/2}\pi^{1/2}\epsilon^{1/2}}(x-k^2)^{-1/4}e^{i\pi/4}e^{i\frac{4}{3\epsilon}(x-k^2)^{3/2}}$$

$$\widehat{W_4}^{\epsilon}(x,k) := \frac{i{x_0}^{-1/2}}{2^{3/2}\pi^{1/2}\epsilon^{1/2}}(x-k^2)^{-1/4}e^{-i\pi/4}e^{-i\frac{4}{3\epsilon}(x-k^2)^{3/2}}$$

thus

$$\begin{aligned} {W_3}^{\epsilon}(x,k) + {W_4}^{\epsilon}(x,k) &\approx \widehat{W_3}^{\epsilon}(x,k) + \widehat{W_4}^{\epsilon}(x,k) \\ &\approx \frac{1}{2\sqrt{x_0}}\Big(\frac{2}{\epsilon}\Big)^{2/3} Ai\Big(\Big(\frac{2}{\epsilon}\Big)^{2/3}(k^2-x)\Big) \end{aligned}$$

Asymptotic surgery of the Wigner function (wrapping-3)

Combining the asymptotics from the various regions, we get the following approximation of the Wigner transform of the WKB expansion in the region $(x > 0, k \in \mathbb{R})$

$$W^{\epsilon}_{WKB}(x,k) \approx \widetilde{W}^{\epsilon}_{WKB}(x,k) := \frac{1}{2\sqrt{x_0}}\left(\frac{2}{\epsilon}\right)^{2/3} Ai\left(\left(\frac{2}{\epsilon}\right)^{2/3}(k^2 - x)\right)$$

Observe now that

$$\widetilde{W}^{\epsilon}_{WKB}(x,k) \equiv W^{\epsilon}_{Ai}(x,k) \quad !!!!!!!$$

that is, the approximation $\widetilde{W}^{\epsilon}_{WKB}$ of W^{ϵ}_{WKB} coincides with the exact Wigner function W^{ϵ}_{Ai} of the fundamental solution of the Wigner equation

Wave amplitude by $dk-$ integration

Using the identity

$$\int_{\mathbb{R}} Ai(r_1 k^2 + r_2 k + r_3)\, dk = \frac{2\pi}{\sqrt{r_1}} \frac{1}{2^{1/3}}\, Ai^2\Big(-\frac{r_2^2 - 4r_1 r_3}{4^{4/3} r_1}\Big), \quad r_1 > 0$$

with $r_1 = (2/\epsilon)^{2/3}$, $r_2 = 0$, $r_3 = -(2/\epsilon)^{2/3} x$, we assert that

$$\begin{aligned}
\int_{\mathbb{R}} W^{\epsilon}_{WKB}(x,k)\, dk &\approx \int_{\mathbb{R}} \widetilde{W}^{\epsilon}_{WKB}(x,k)\, dk \equiv \int_{\mathbb{R}} W^{\epsilon}_{Ai}(x,k)\, dk \\
&= \int_{\mathbb{R}} \frac{1}{2\sqrt{x_0}} \Big(\frac{2}{\epsilon}\Big)^{2/3} Ai\Big(\Big(\frac{2}{\epsilon}\Big)^{2/3}(k^2 - x)\Big)\, dk \\
&= \frac{\pi}{\sqrt{x_0}\,\epsilon^{1/3}}\, Ai^2(-\epsilon^{-2/3} x) \\
&= |u^{\epsilon}(x)|^2
\end{aligned}$$

Therefore, the approximation of W^{ϵ}_{WKB} provides the correct finite amplitude of the wave function, even on the turning point

Conclusions

1. Appropriate surgery of the semiclassical approximation of the Wigner transform of the two-phase WKB solution, in different regions of phase space, uniformizes the wave field near the caustics (turning point)

2. The interaction of the two phases plays a crucial role in putting together the pieces from different regions of phase space

3. The computation can be applied to any fold caustic by introducing local coordinates, but the computation becomes extremely complicated

4. The computation suggests that the Wigner transform of a two-phase WKB solution would be the correct local solution of the Wigner equation

End

Thank you for your attention!!

Kravtsov-Ludwig technique -1

Assume that

$$u(\mathbf{x},\kappa) = \left(\frac{i\kappa}{2\pi}\right)^{\frac{1}{2}} \int_{\Xi} e^{i\kappa S(\mathbf{x},\xi)} A(\mathbf{x},\xi)\, d\xi \ , \quad \mathbf{x} \in M \subset \mathbb{R}^n_{\mathbf{x}} \ , \quad \xi \in \Xi \subset \mathbb{R}_{\xi}$$

where $S(\mathbf{x},\xi)$, $A(\mathbf{x},\xi)$ satisfy eikonal and transport equations, resp., for any ξ

Near smooth caustic (fold)

$$S(\mathbf{x},\xi) = \phi(\mathbf{x}) + \xi\rho(\mathbf{x}) - \frac{\xi^3}{3}$$

$$A(\mathbf{x},\xi) = g_0(\mathbf{x}) + \xi g_1(\mathbf{x}) + h(\mathbf{x},\xi)\partial_{\xi} S(\mathbf{x},\xi)$$

where $h(\mathbf{x},\xi)$ smooth function

Kravtsov-Ludwig technique-2

- we substitute the expressions for the phase and the amplitude in the integral
- The first two terms lead to the integral representation for Ai and Ai'.
- By stationary phase the third term is asymptotically negligible

Therefore we get

$$
\begin{aligned}
u(\mathbf{x}) &= \sqrt{2\pi}\kappa^{\frac{1}{6}} e^{\frac{i\pi}{4}} e^{i\kappa\phi(\mathbf{x})} \Bigg(g_0(\mathbf{x}) Ai\left(-\kappa^{\frac{2}{3}}\rho(\mathbf{x})\right) \\
&+ i\kappa^{-\frac{1}{3}} g_1(\mathbf{x}) Ai'\left(-\kappa^{\frac{2}{3}}\rho(\mathbf{x})\right) \Bigg) + O(\kappa^{-1}) \ , \quad \kappa \to \infty
\end{aligned}
$$

where Ai denotes the Airy function

Substituting the asymptotic expansions of Ai and Ai' (negative argument–illuminated region) we get

$$\begin{aligned} u(\mathbf{x},\kappa) &= \frac{1}{\sqrt{2}}\Bigg(\left(g_0(\mathbf{x}) + g_1(\mathbf{x})\sqrt{\rho(\mathbf{x})}\right)\rho^{-\frac{1}{4}}(\mathbf{x})\; e^{i\kappa\Phi_+(\mathbf{x})} \\ &+ \left(g_0(\mathbf{x}) - g_1(\mathbf{x})\sqrt{\rho(\mathbf{x})}\right)\rho^{-\frac{1}{4}}(\mathbf{x})\; e^{i\kappa\Phi_-(\mathbf{x})+\frac{i\pi}{2}}\Bigg), \quad \kappa \to \infty \end{aligned}$$

where $\Phi_\pm(\mathbf{x}) = \phi(\mathbf{x}) \pm \frac{2}{3}\rho^{\frac{3}{2}}(\mathbf{x})$.

Kravtsov-Ludwig technique-4

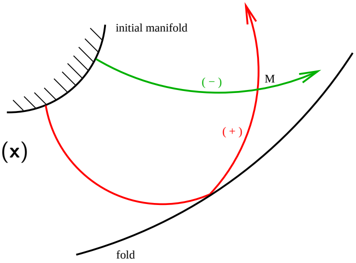
Kravtsov-Ludwig asymptotic expansion far from the fold

$$u_{KL}(\mathbf{x},\kappa) = \frac{1}{\sqrt{2}}\Bigg(\Big(g_0(\mathbf{x}) + g_1(\mathbf{x})\sqrt{\rho(\mathbf{x})}\Big)\,\rho^{-\frac{1}{4}}(\mathbf{x})\;e^{i\kappa\Phi_+(\mathbf{x})}$$

$$+\;\Big(g_0(\mathbf{x}) - g_1(\mathbf{x})\sqrt{\rho(\mathbf{x})}\Big)\,\rho^{-\frac{1}{4}}(\mathbf{x})\;e^{i\kappa\Phi_-(\mathbf{x})+\frac{i\pi}{2}}\Bigg),\quad \kappa\to\infty$$

should match with the standard two-phase WKB expansion

$$u_{WKB}(\mathbf{x},\kappa) = A_+(\mathbf{x})\,e^{i\kappa S_+(\mathbf{x})} + A_-(\mathbf{x})\,e^{i\kappa S_-(\mathbf{x})}$$



$$u_{KL}(\mathbf{x},\kappa) \equiv u_{WKB}(\mathbf{x},\kappa)$$

Kravtsov-Ludwig technique-5

From this matching we get the amplitudes

$$
\begin{aligned}
g_0(\mathbf{x}) &= \frac{\rho^{\frac{1}{4}}(\mathbf{x})}{\sqrt{2}}\Big(A_+(\mathbf{x}) - iA_-(\mathbf{x})\Big) \\
g_1(\mathbf{x}) &= \frac{\rho^{-\frac{1}{4}}(\mathbf{x})}{\sqrt{2}}\Big(A_+(\mathbf{x}) + iA_-(\mathbf{x})\Big)
\end{aligned}
$$

and the functions paramatrizing the phase

$$
\begin{aligned}
\phi(\mathbf{x}) &= \frac{1}{2}\Big(S_+(\mathbf{x}) + S_-(\mathbf{x})\Big) \\
\rho(\mathbf{x}) &= \left[\frac{3}{4}\Big(S_+(\mathbf{x}) - S_-(\mathbf{x})\Big)\right]^{2/3}
\end{aligned}
$$

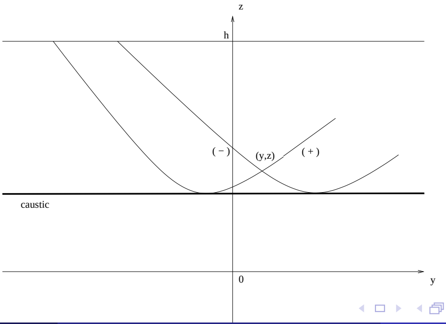
Kravtsovback

Example: Plane wave in linear layer-1

Example

$$\Delta U(y,z) + \kappa_0^2 \eta^2(z) U(y,z) = 0$$
$$\eta^2(z) = \mu_0 + \mu_1 z\,, \quad \mu_1 > 0$$
$$\kappa_0 := \kappa\eta_0\,, \quad {\eta_0}^2 = \mu_0 + \mu_1 h$$

We consider a " linearly stratifies" medium filling the layer $0 < z < h$, and a plane wave entering the face $z = h$ at angle ψ with respect to $z-$ axis:



Example: Plane wave in linear layer-2

Dirichlet problem in the half-plane $z < h$

- Separation of variables $U(y,z) = u(z)\, e^{iy\kappa_0 \sin\psi}$

$$\Rightarrow \quad \frac{d^2}{dz^2} u(z) + \kappa_0^2 \big(\mu_0 - \mu_1 z - \sin^2\psi\big) u(z) = 0$$

- Change of vertical coordinate $Z = \mu_0 - \mu_1 z - \sin^2\psi$?? $x = Z\epsilon^{2/3}$ and $\epsilon = 1/\kappa_0$

$$\Rightarrow \quad \epsilon^2 \frac{d^2}{dx^2} u^\epsilon(x) + x u^\epsilon(x) = 0\,, \quad x \in \mathbb{R} \quad \textit{Airy equation}$$

We are interested in th high-frequency regime, and therefore we study the semiclassical Airy equation as the simplest model of geometrical optics with caustic (actually turning point) ▸ Airyback