A two-component nonlinear Schrödinger system with linear coupling

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System of two coupled nonlinear Schrödinger equations with an external driven field

Let us consider:

$$
\begin{aligned}
i\partial_t\psi_1 &= -\frac{1}{2}\Delta\psi_1 + \frac{\gamma^2}{2}|x|^2\psi_1 + \beta_{11}|\psi_1|^2\psi_1 + \beta_{12}|\psi_2|^2\psi_1 + \lambda\psi_2 \\
i\partial_t\psi_2 &= -\frac{1}{2}\Delta\psi_2 + \frac{\gamma^2}{2}|x|^2\psi_2 + \beta_{12}|\psi_1|^2\psi_2 + \beta_{22}|\psi_2|^2\psi_2 + \lambda\psi_1 \\
&\psi_1(x,0) = \varphi_1(x), \quad \psi_2(x,0) = \varphi_2(x) \qquad (1)
\end{aligned}
$$

with $x \in \mathbb{R}^N$ in $N \leq 3$

- $\beta_{jj}, \beta_{12} \in \mathbb{R}$ intraspecific and interspecific scattering lengths, respectively
- $\lambda \in \mathbb{R}$ external driven field constant

Physical Experiments

- First experiment concerning with the binary Bose-Einstein condensate (BEC) was performed in JILA with $|F = 2, m_f = 2 >$ and $|1, -1 >$ spin states of ^{87}Rb. (C. J. Myatt et al.,Phys. Rev. Lett., 78 (1997))
- When $\lambda = 0$, the above system models a mixture of Bose-Einstein condensates consisting of two different hyperfine states of Rubidium atoms confined in the same harmonic trap. By applying a weak magnetic (driven) field with the Rabi frequency λ, the two components are coupled in the overlap region. This coupling realizes a Josephson-type junction and gives rise to nonlinear oscillations in the relative populations.(J.Williams et. al, Phys.Rev.A,59(1999))

Physical literature

- question: will two-component BEC with one repulsive and one attractive component collapse or may it reach a stable state?
- a stabilization method for the single Bose-Einstein condensate $\rightarrow$ controle the scattering length using the Feshbach-resonance;
  Phys. Rev. A **67** *(2003), 013605;*
  Phys. Rev. Lett **90** *(2003) 040403.*
- for the two-component BEC Saito et. al. proposed to use the Rabi oscillations in order to achieve oscillations of the scattering lengths and consequently stabilize the Bose-Einstein condensate.
  Phys. Rev. A **76** *(2007) 053619*

Motivation

- Does two-component NLS-system with focusing and defocusing nonlinearities in the presence of the Rabi frequency blow-up or exist globally?
- Does the Rabi term influence the long time behavior of the system, thus can it avoid blow-up?
- Numerical experiments suggest the fact that the Rabi frequency may affect the long time behavior of the system *Math. Models Methods Appl. Sci. (2013)* A.Jüngel, R.W.
- It will make sense to deal with the case of great Rabi frequencies $\Rightarrow$ we are interested in asymptotics when $|\lambda| \to \infty$.

Outline

Some Definitions I

Definition

For functions $\Psi = (\psi_1, \psi_2)^T : \mathbb{R}^N \times [0, T] \to \mathbb{C}^2$, we define the norms, where $\|\psi_j(t)\|_{L^p(\mathbb{R}^N)}$ is the standard L^p-norm:

$$\|\Psi(t)\|_p = \begin{cases} \left(\sum_{j=1}^2 \|\psi_j(t)\|^p_{L^p(\mathbb{R}^N)}\right)^{1/p} & \text{for } 1 \leq p < \infty \\ \sum_{j=1}^2 \|\psi_j(t)\|_{L^\infty(\mathbb{R}^N)} & \end{cases}$$

$$\|\Psi\|_{q,p} = \left\| \|\Psi(t)\|_{L^p(\mathbb{R}^N)} \right\|_{L^q(0,T)}$$

with the corresponding Banach spaces $L^p(\mathbb{R}^N)$ and $L^q((0,T), L^p(\mathbb{R}^N))$.

Some Definitions II

We introduce the energy-type space

$$\Sigma(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : |xu| \in L^2(\mathbb{R}^N)\}.$$

We remind the definition of an admissible pair (q, r):

$$\frac{2}{q} = N\left(\frac{1}{2} - \frac{1}{r}\right).$$

with $2 \leq r \leq \frac{2N}{N-2}$ ($2 \leq r \leq \infty$ if $N = 1$, and $2 \leq r < \infty$ if $N = 2$)
(q', r') denote the Hölder dual exponents of an admissible pair.

Local Existence

Let $\varphi := (\varphi_1, \varphi_2) \in \Sigma(\mathbb{R}^N)$, then there exists a unique, maximal solution $\Psi \in \mathcal{C}([0, T_{max}), \Sigma(\mathbb{R}^N))$ of (1).
The blow-up alternative holds true, i.e. $T_{max} < \infty$ if and only if

$$\|\Psi(t)\|_{H^1} \to \infty$$

as $t \to T_{max}^-$. Moreover for any admissible pair (q, r), we have

$$\Psi, \nabla\Psi, |\cdot|\Psi \in L^q((0, T_{max}); L^r(\mathbb{R}^N)).$$

Conserved quantities

Total mass:

$$M(t) = M_1(t) + M_2(t) = \int_{\mathbb{R}^N} |\psi_1(x,t)|^2 dx + \int_{\mathbb{R}^N} |\psi_2(x,t)|^2 dx$$

Total energy:

$$E(t) = \int_{\mathbb{R}^N} \Big[\sum_{j=1}^{2} \Big(\frac{1}{2}|\nabla \psi_j|^2 + \frac{\gamma^2}{2}|x|^2|\psi_j|^2 + \frac{\beta_{jj}}{2}|\psi_j|^4 \Big) + \beta_{12}|\psi_1|^2|\psi_2|^2 + 2\lambda \Re(\psi_1^* \psi_2) \Big] (x,t) dx,$$

$M(t) = M(0)$ and $E(t) = E(0)$ for all $t \geq 0$.

Global Existence I

Theorem

Let $N \leq 3$ and set $\beta = \max\{(-\beta_{11})^+, (-\beta_{22})^+\}$. Then there exists a global-in-time solution to in the following cases:

- *all $\beta_{ij} \geq 0$ with $i, j = 1, 2$*
- *at least one $\beta_{ij} < 0$*
  1. $\beta_{11}, \beta_{22} > 0$ *and* $\beta_{12}^2 < \beta_{11}\beta_{22}$
  2. $N = 1$
  3. $N = 2$ *and*
     - $M(0) < 2/(C_2|\beta_{12}|)$, *if* $\beta_{12} < 0$
     - $M(0) < 1/(C_2\beta)$, *if* $\min\{\beta_{11}, \beta_{22}\} < 0$
     - $M(0) < 4/(C_2(2\beta + |\beta_{12}|))$, *if* $\min\{\beta_{11}, \beta_{22}\} < 0$ *and* $\beta_{12} < 0$

Global Existence II

4. $N = 3$, $\|\nabla\Psi(0)\|_2^2 \leq 2(E(0) + |\lambda| M(0))$, *and*
    - $M(0)(E(0) + |\lambda| M(0)) < \frac{8}{27 C_3^2 \beta_{12}^2}$, *if* $\beta_{12} < 0$
    - $M(0)(E(0) + |\lambda| M(0)) < \frac{2}{27 C_3^2 \beta^2}$, *if* $\min\{\beta_{11}, \beta_{22}\} < 0$
    - $M(0)(E(0) + |\lambda| M(0)) < \frac{8}{27 C_3^2 (2\beta + |\beta_{12}|)^2}$, *if* $\min\{\beta_{11}, \beta_{22}\} < 0$ *and* $\beta_{12} < 0$

where C_N *is the best constant in the Gagliardo-Nirenberg inequality:*

$$\|\Psi\|_4^4 \leq C_N \|\nabla\Psi\|_2^N \|\Psi\|_2^{4-N} \quad \Psi \in H^1(\mathbb{R}^N)$$

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Blow-up of the system I

Theorem

Let $\varphi := (\varphi_1, \varphi_2) \in \Sigma(\mathbb{R}^N)$ *and denote by* $I(t) := \int_{\mathbb{R}^N} |x|^2(|\psi_1|^2 + |\psi_2|^2)dx$. *If one of the conditions*

$$E(0) + |\lambda| M(0) < \frac{\gamma^2}{2} I(0), \quad \textit{or}$$

$$I'(0) < 0, \quad E(0) + |\lambda| M(0) < -\frac{\gamma}{2} I'(0)$$

is satisfied, the solution $\Psi = (\psi_1, \psi_2)$ *to the system blows up at time* $t^* \leq \pi/(2\gamma)$ *or* $t^* \leq \pi/(4\gamma)$, *respectively, i.e.*

$$\lim_{t \to t^*} \|\nabla \Psi\|_2 = +\infty,$$

Blow-up of the system II

if the additional conditions on N are fulfilled- in the (mass) critical or super critical case:

1. $N = 2$ *and at least one* $\beta_{ij} < 0$, *with* $i, j = 1, 2$
2. $N = 3$ $\beta_{11} < 0, \beta_{22} < 0$; *if* $\beta_{12} > 0$ *we should have additionally* $\beta_{12} \leq \sqrt{|\beta_{11}\beta_{22}|}$

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Sharp threshold for $N = 2$

In *Phys. Lett. A 374 (2010) 2133–2136*, Zhongxue and Zuhan showed that for $\beta_{ij} < 0$ for $i, j = 1, 2$ and for $|\beta_{12}| < \sqrt{|\beta_{11}\beta_{22}|}$ the system:

$$\Delta v_1 - v_1 - (\beta_{11}|v_1|^2 + \beta_{12}|v_2|^2)v_1 = 0$$
$$\Delta v_2 - v_2 - (\beta_{12}|v_1|^2 + \beta_{22}|v_2|^2)v_2 = 0$$

has a ground state solution $V := (v_1, v_2)$. All v_i , $i = 1, 2$ must be positive, radially symmetric and strictly decreasing.

Sharp threshold for $N = 2$

If $(\varphi_1, \varphi_2) \in \Sigma(\mathbb{R}^2)$ (remember $M(0) = \|\varphi_1\|_2^2 + \|\varphi_2\|_2^2$) and

$$M(0) < \frac{1}{2}\|V\|_2^2$$

then the corresponding solution $\Psi = (\psi_1, \psi_2)$ exists **globally in time**.

At the same time, for arbitrary positive μ and complex c satisfying $|c| \geq \sqrt{\frac{1+\lambda^2}{2}}$ if we take initial data $\varphi_1 = c\mu v_1(\mu x)$ and $\varphi_2 = c\mu v_2(\mu x)$, then

$$M(0) \geq \frac{1}{2}\|V\|_2^2,$$

and the corresponding solution $\Psi = (\psi_1, \psi_2)$ **blows up in finite time**.

In *East Asian J. Appl. Math. 1, no.1 (2011), 49-81* W. Bao and Y. Cai showed existence of the ground state $(\phi_1^g, \phi_2^g)^T$ if at least one of the conditions holds:

- $N = 1$
- $N = 2$ and $\beta_{11} \geq -1/C_2$, $\beta_{22} \geq -1/C_2$, $\beta_{12} \geq -1/C_2 - \sqrt{1/C_2 + \beta_{11}}\sqrt{1/C_2 + \beta_{22}}$
- $N = 3$ either all $\beta_{ij} \geq 0$, or $\beta_{11} \geq 0$ and $\beta_{12}^2 \leq \beta_{11}\beta_{22}$

In addition $(e^{i\theta_1}|\phi_1^g|, e^{i\theta_2}|\phi_2^g|)$, with $\theta_1 - \theta_2 = \pi$ for $\lambda > 0$ and $\theta_1 - \theta_2 = 0$ for $\lambda < 0$, respectively. Furthermore if $\beta_{11} \geq 0$ and $\beta_{12}^2 \leq \beta_{11}\beta_{22}$, and one of the parameters λ, γ are nonzero, then the ground state is $(|\phi_1^g|, -sign(\lambda)|\phi_2^g|)^T$ is unique.

Example: one focusing, one defocusing nonlinearity

Let $\beta_{11} < 0$ and $\beta_{22}, \beta_{12} \geq 0$ and $N = 2$. If the initial mass is not smaller than the critical mass $M(0) < 1/(C_2|\beta_{11}|)$, and the sufficient condition for blow-up $E(0) + |\lambda| M(0) < \frac{\gamma^2}{2} I(0)$ is not satisfied, we cannot say anything on the long time behavior of the system

$$\begin{aligned} i\partial_t \psi_1 &= -\frac{1}{2}\Delta\psi_1 + \frac{\gamma^2}{2}|x|^2\psi_1 - |\psi_1|^2\psi_1 + \lambda\psi_2 \\ i\partial_t \psi_2 &= -\frac{1}{2}\Delta\psi_2 + \frac{\gamma^2}{2}|x|^2\psi_2 + |\psi_2|^2\psi_2 + \lambda\psi_1 \\ &\psi_1(x,0) = \varphi_1(x), \quad \psi_2(x,0) = \varphi_2(x) \end{aligned}$$

Numerical simulations suggests that the system may blow-up or "exist globally" depending on λ.

The effect of the external driven field

Remember:

$$M_1(t) = \int_{\mathbb{R}^N} |\psi_1(x,t)|^2 dx, \quad M_2(t) = \int_{\mathbb{R}^N} |\psi_2(x,t)|^2 dx.$$

The total mass equals $M = M_1 + M_2$ is conserved. We also define

$$M_{12}(t) = \Im \int_{\mathbb{R}^N} \psi_1(x,t)\psi_2^*(x,t) dx,$$

Lemma

M_2 and M_{12} satisfy the following differential equations:

$$\partial_t M_2 = -2\lambda M_{12}, \quad \partial_t M_{12} = \lambda M(0) - 2\lambda M_2 - Q(t), \quad t > 0, \textit{ where}$$

$$Q(t) = \Re \int_{\mathbb{R}^N} \psi_1\psi_2^*\left(\beta_{11}|\psi_1|^2 - \beta_{22}|\psi_2|^2 - \beta_{12}(|\psi_1|^2 - |\psi_2|^2)\right)(x,t) dx.$$

The functions $M_2(t)$ and $M_{12}(t)$ can be computed explicitly from the ODE system. Then $M_1(t) = -M_2(t) + M(0)$. The solution reads as

$$\begin{aligned} M_1(t) &= -\sin(2\lambda t)M_{12}(0) + \cos(2\lambda t)M_1(0) + \tfrac{1}{2}(1-\cos(2\lambda t))M(0) \\ &\quad + \int_0^t \sin(2\lambda(t-s))Q(s)ds, \\ M_2(t) &= \sin(2\lambda t)M_{12}(0) + \cos(2\lambda t)M_2(0) + \tfrac{1}{2}(1-\cos(2\lambda t))M(0) \\ &\quad - \int_0^t \sin(2\lambda(t-s))Q(s)ds. \end{aligned}$$

$\to$ the components exchange their mass periodically.
In the special case $\beta_{11} = \beta_{22} = \beta_{12}$, this exchange occurs actually with the frequency 2λ.

The Transformed System

We first perform the following transformation:

$$\begin{aligned} \phi_1(x,t) &= \frac{\exp(i\lambda t)}{\sqrt{2}}(\psi_1(x,t) + \psi_2(x,t)) \\ \phi_2(x,t) &= \frac{\exp(-i\lambda t)}{\sqrt{2}}(\psi_1(x,t) - \psi_2(x,t)) \end{aligned}$$

Let us denote by $H := -\frac{1}{2}\Delta + \frac{\gamma^2}{2}|x|^2$

Nonautonomous System

We obtain the non-autonomous system:

$$\begin{aligned} i\partial_t\phi_1 &= H\phi_1 + \sigma_1|\phi_1|^2\phi_1 + \sigma_2|\phi_2|^2\phi_1 + \sigma_3(\lambda t)|\phi_1|^2\phi_2 \\ &\quad +\sigma_4(\lambda t)|\phi_2|^2\phi_2 + \sigma_5(\lambda t)\phi_1^*\phi_2^2 + \sigma_6(\lambda t)\phi_1^2\phi_2^* \\ i\partial_t\phi_2 &= H\phi_2 + \sigma_1|\phi_2|^2\phi_2 + \sigma_2|\phi_1|^2\phi_2 + \sigma_3^*(\lambda t)|\phi_2|^2\phi_1 \\ &\quad +\sigma_4^*(\lambda t)|\phi_1|^2\phi_1 + \sigma_5^*(\lambda t)\phi_2^*\phi_1^2 + \sigma_6^*(\lambda t)\phi_2^2\phi_1^* \end{aligned} \tag{2}$$

$$\phi_1(x,0) = \varphi_1(x) + \varphi_2(x); \phi_2(x,0) = \varphi_1(x) - \varphi_2(x).$$

For the single nonlinear Schrödinger equation with a periodic coefficient there is a rigorous result by Cazenave and Scialom *Revista Matématica Complutense, 23, 2(2010), 321–339*

With the coefficients:

$$\sigma_1 = \frac{\beta_{11} + 2\beta_{12} + \beta_{22}}{4};$$
$$\sigma_2 = \frac{\beta_{11} + \beta_{22}}{2};$$
$$\sigma_3(\lambda t) = \frac{\beta_{11} - \beta_{22}}{2} \exp(2i\lambda t);$$
$$\sigma_4(\lambda t) = \frac{\beta_{11} - \beta_{22}}{4} \exp(2i\lambda t);$$
$$\sigma_5(\lambda t) = \frac{\beta_{11} - 2\beta_{12} + \beta_{22}}{4} \exp(4i\lambda t);$$
$$\sigma_6(\lambda t) = \frac{\beta_{11} - \beta_{22}}{4} \exp(-2i\lambda t).$$

Remark Note that for $\beta_{11} = \beta_{22} = \beta_{12} = \beta$ the system (2) does not depend on λ:

$$\begin{aligned} i\partial_t \phi_1 &= -\frac{1}{2}\Delta\phi_1 + \frac{\gamma^2}{2}|x|^2\phi_1 + \beta|\phi_1|^2\phi_1 + \beta|\phi_2|^2\phi_1 \\ i\partial_t \phi_2 &= -\frac{1}{2}\Delta\phi_2 + \frac{\gamma^2}{2}|x|^2\phi_2 + \beta|\phi_2|^2\phi_2 + \beta|\phi_1|^2\phi_2 \end{aligned}$$

Formal Limit

We expect the coefficients of the nonlinearities to go to their average in time:

$$\bar{\sigma}_j = \frac{1}{2\pi}\int_0^{2\pi} \sigma_j(t)dt = 0 \quad \text{for } j = 3, 4\ldots 6.$$

and the solution (ϕ_1, ϕ_2) to converges locally in time for $|\lambda| \to \infty$ to the solution $U = (u_1, u_2)$ of:

$$\begin{aligned} i\partial_t u_1 &= -\tfrac{1}{2}\Delta u_1 + \tfrac{\gamma^2}{2}|x|^2 u_1 + \sigma_1|u_1|^2 u_1 + \sigma_2|u_2|^2 u_1 \\ i\partial_t u_2 &= -\tfrac{1}{2}\Delta u_2 + \tfrac{\gamma^2}{2}|x|^2 u_2 + \sigma_1|u_2|^2 u_2 + \sigma_2|u_1|^2 u_2 \\ & u_1(x,0) = \varphi_1(x) + \varphi_2(x); \quad u_2(x,0) = \varphi_1(x) - \varphi_2(x) \end{aligned}$$

Main Result I

Theorem

Let $\varphi := (\varphi_1, \varphi_2) \in \Sigma(\mathbb{R}^N)$ *be a fixed initial value. Given* $\lambda \in \mathbb{R}$*, let* Φ^λ *denote the maximal solution of* (2)*. Let* U *be the maximal solution of*

$$\begin{aligned} i\partial_t u_1 &= -\tfrac{1}{2}\Delta u_1 + \tfrac{\gamma^2}{2}|x|^2 u_1 + \sigma_1|u_1|^2 u_1 + \sigma_2|u_2|^2 u_1 \\ i\partial_t u_2 &= -\tfrac{1}{2}\Delta u_2 + \tfrac{\gamma^2}{2}|x|^2 u_2 + \sigma_1|u_2|^2 u_2 + \sigma_2|u_1|^2 u_2 \\ & u_1(x,0) = \varphi_1(x) + \varphi_2(x); \quad u_2(x,0) = \varphi_1(x) - \varphi_2(x) \end{aligned}$$

defined on the maximal interval $[0, S_{max})$*.*

Main Result II

- *Given any* $0 < T < S_{max}$ *the solution* Φ^λ *exists on* $[0, T]$ *provided that* $|\lambda|$ *is sufficiently large.*
- *And we have convergence* $\begin{pmatrix} \Phi^\lambda \\ \nabla\Phi^\lambda \\ |\cdot|\Phi^\lambda \end{pmatrix} \to \begin{pmatrix} U \\ \nabla U \\ |\cdot|U \end{pmatrix}$ *in* $L^q((0,T), L^r(\mathbb{R}^N))$ *as* $|\lambda| \to \infty$ *, for all admissible pairs* (q, r) *and all* $0 < T < S_{max}$*. In particular, we have*

$$\Phi^\lambda \to U \text{ in } \mathcal{C}([0,T]; H^1(\mathbb{R}^N)) \quad \forall 0 < T < S_{max}.$$

Main Result III

Where

$$\sigma_1 = \frac{\beta_{11} + 2\beta_{12} + \beta_{22}}{4},$$

$$\sigma_2 = \frac{\beta_{11} + \beta_{22}}{2}.$$

- With the standard techniques it follows that the Cauchy problems for Ψ, Φ^λ, U are locally well-posed.
- The same result stated in the above Theorem for solution Ψ holds true also for solutions Φ^λ of the non-autonomous system.
- We can easily check that $|\psi_1|^2 + |\psi_2|^2 = |\phi_1|^2 + |\phi_2|^2$ and consequently also

$$\|\Psi(t)\|_{\Sigma(\mathbb{R}^N)} = \|\Phi^\lambda(t)\|_{\Sigma(\mathbb{R}^N)}$$

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Preliminary Results

We first need uniform in λ bounds on the H^1-norm of the solution:

Proposition

Given $M > 0$, there exists a $\delta = \delta(M) > 0$ such that for any $\varphi := (\varphi_1, \varphi_2) \in \Sigma(\mathbb{R}^N)$, with $\|\varphi\|_{\Sigma(\mathbb{R}^N)} \leq M$, there exists a unique solution $\Psi \in \mathcal{C}((0,\delta); \Sigma(\mathbb{R}^N))$ for the system. In addition,

$$\|\Psi\|_{L^\infty((0,\delta);\Sigma(\mathbb{R}^N))} \leq 2\|\phi\|_{\Sigma(\mathbb{R}^N)}.$$

Lemma

For any $\varphi \in \Sigma(\mathbb{R}^N)$ let Φ^λ be the maximal solution of the non-autonomous system. Let U be the maximal solution of the limiting system, defined on $[0, S_{max})$. Let $0 < l < S_{max}$ and assume that Φ^λ exists on $[0, l]$ and that

$$\limsup_{|\lambda|\to\infty} \|\Phi^\lambda\|_{L^\infty((0,l);H^1(\mathbb{R}^N))} < \infty$$

Then we have

$$\lim_{|\lambda|\to\infty} \left\| \begin{pmatrix} 1 \\ \nabla \\ |\cdot| \end{pmatrix} (\Phi^\lambda - U) \right\|_{L^q((0,l);L^r(\mathbb{R}^N))} = 0$$

for any admissible pairs (q, r). In particular $\Phi^\lambda \to U$ in $L^\infty((0, l); H^1(\mathbb{R}^N))$.

- Let us fix $0 < T < S_{max}$ and $M := \|U\|_{L^\infty((0,T);H^1(\mathbb{R}^N))}$
- Φ^λ exists in $[0,\delta]$ for all λ and furthermore $\sup_{\lambda\in\mathbb{R}} \|\Phi^\lambda\|_{L^\infty((0,\delta);H^1(\mathbb{R}^N))} \leq 2\|\varphi\|_\Sigma$.
- let $0 < l \leq T$ (we can always choose $l = \delta$) be such that
  - Φ^λ exists in $[0, l]$, and
  - that we have $\limsup_{|\lambda|\to\infty} \|\Phi^\lambda\|_{L^\infty((0,l);H^1(\mathbb{R}^N))} < \infty$
- with the Lemma we have convergence $\Phi^\lambda \to U$ in $L^q((0,l); L^r(\mathbb{R}^N))$ for all admissible pairs (q, r).
- In particular $\lim_{|\lambda|\to\infty} \|\Phi^\lambda(l) - U(l)\|_{H^1(\mathbb{R}^N)} = 0$.
  $\Rightarrow \sup_{|\lambda|\geq\Lambda} \|\Phi^\lambda(l)\|_{H^1(\mathbb{R}^N)} \leq M$ for $\Lambda > 0$ sufficiently large
- We can thus repeat the argument, starting at time $t = l$ ...
- Thus we repeat this argument to prove the result in the whole time interval $[0, T]$.

Properties of the Limiting System

- there are three conserved quantities: the mass of each component and the energy:

$$\begin{aligned} \|u_1(t)\|_2 &= \|u_1(0)\|_2, \\ \|u_2(t)\|_2 &= \|u_2(0)\|_2, \\ \tilde{E}(t) &= \tilde{E}(0); \end{aligned}$$

where

$$\begin{aligned} \tilde{E}(t) \;:=\; & \frac{1}{2}\int_{\mathbb{R}^N}\sum_{j=1}^{2}\Big[|\nabla u_j|^2 + \gamma^2|x|^2|u_j|^2 + \sigma_1|u_j|^4 \\ & + \sigma_2|u_1|^2|u_2|^2\Big](x,t)dx, \end{aligned}$$

Global Existence of the limiting system I

Let $U = (u_1, u_2)$ be the solution of the limiting system. Then there exists a global-in-time solution to in the following cases:

- $\sigma_1, \sigma_2 \geq 0$
- at least one $\sigma_i < 0$
  1. $\sigma_1 > 0$ and $|\sigma_2| < \sigma_1$
  2. $N = 1$
  3. $N = 2$ and
     - $M(0) < \frac{2}{C_2|\sigma_2|}$, if $\sigma_2 < 0$
     - $M(0) < \frac{1}{C_2|\sigma_1|}$, if $\sigma_1 < 0$
     - $M(0) < \frac{4}{C_2(2|\sigma_1|+|\sigma_2|)}$, if $\sigma_1 < 0$ and $\sigma_2 < 0$
  4. $N = 3$, $\|\nabla U(0)\|_2^2 \leq 2\tilde{E}(0)$, and

Global Existence of the limiting system II

- $M(0)\tilde{E}(0) < \frac{8}{27C_3^2\sigma_2^2}$, if $\sigma_2 < 0$
- $M(0)\tilde{E}(0) < \frac{2}{27C_3^2\sigma_1^2}$, if $\sigma_1 < 0$
- $M(0)\tilde{E}(0) < \frac{8}{27C_3^2(2|\sigma_1|+|\sigma_2|)^2}$, if $\sigma_1 < 0$ and $\sigma_2 < 0$

With this we have at least for large λ different parameter regimes, for which we expect global existence.

Example

Case: $\beta_{11} = -1$, $\beta_{22} = 1$, $\beta_{12} = 0$, thus we have:

$$\begin{aligned} i\partial_t \psi_1 &= -\frac{1}{2}\Delta\psi_1 + \frac{\gamma^2}{2}|x|^2\psi_1 - |\psi_1|^2\psi_1 + \lambda\psi_2 \\ i\partial_t \psi_2 &= -\frac{1}{2}\Delta\psi_2 + \frac{\gamma^2}{2}|x|^2\psi_2 + |\psi_2|^2\psi_2 + \lambda\psi_1 \\ & \psi_1(x,0) = \varphi_1(x), \quad \psi_2(x,0) = \varphi_2(x) \end{aligned}$$

Remember $\sigma_1 = \frac{\beta_{11}+2\beta_{12}+\beta_{22}}{4}$; $\quad \sigma_2 = \frac{\beta_{11}+\beta_{22}}{2}$
It follows for the limiting system when $|\lambda| \to \infty$:

$$\begin{aligned} i\partial_t u_1 &= -\tfrac{1}{2}\Delta u_1 + \tfrac{\gamma^2}{2}|x|^2 u_1 \\ i\partial_t u_2 &= -\tfrac{1}{2}\Delta u_2 + \tfrac{\gamma^2}{2}|x|^2 u_2 \end{aligned}$$

Alternative Transformation

We perform following transformation:

$$\begin{aligned}\phi_1 &= \cos(\lambda t)\psi_1 + i\sin(\lambda t)\psi_2\\ \phi_2 &= i\sin(\lambda t)\psi_1 + \cos(\lambda t)\psi_2\end{aligned}$$

$$\begin{aligned}i\partial_t\phi_1 &= -\frac{1}{2}\Delta\phi_1 + \frac{\gamma^2}{2}|x|^2\phi_1 + f_1(\lambda t)|\phi_1|^2\phi_1 + f_3(\lambda t)|\phi_2|^2\phi_1\\ &\quad + if_2(\lambda t)|\phi_1|^2\phi_2 + if_4(\lambda t)|\phi_2|^2\phi_2\\ &\quad -2f_2(\lambda t)\Im\left(\phi_1^*\phi_2\right)\phi_1 - if_5(\lambda t)\Im\left(\phi_1^*\phi_2\right)\phi_2\\ i\partial_t\phi_2 &= -\frac{1}{2}\Delta\phi_2 + \frac{\gamma^2}{2}|x|^2\phi_2 + f_6(\lambda t)|\phi_2|^2\phi_2 + f_3(\lambda t)|\phi_1|^2\phi_2\\ &\quad - if_2(\lambda t)|\phi_1|^2\phi_1 - if_4(\lambda t)|\phi_2|^2\phi_1\\ &\quad -2f_4(\lambda t)\Im\left(\phi_1^*\phi_2\right)\phi_2 + if_5(\lambda t)\Im\left(\phi_1^*\phi_2\right)\phi_1\end{aligned}$$

The coefficients depend on λ and t.

$$
\begin{aligned}
f_1(\lambda t) &= \beta_{11}\cos^4(\lambda t) + \beta_{22}\sin^4(\lambda t) + 2\beta_{12}\cos^2(\lambda t)\sin^2(\lambda t)\\
f_6(\lambda t) &= \beta_{11}\sin^4(\lambda t) + \beta_{22}\cos^4(\lambda t) + 2\beta_{12}\cos^2(\lambda t)\sin^2(\lambda t)\\
f_2(\lambda t) &= \sin(\lambda t)\cos(\lambda t)\left[-\beta_{11}\cos^2(\lambda t) + \beta_{22}\sin^2(\lambda t) + \beta_{12}\cos(2\lambda t)\right]\\
f_3(\lambda t) &= (\beta_{11} + \beta_{22})\cos^2(\lambda t)\sin^2(\lambda t) + \beta_{12}(\cos^4(\lambda t) + \sin^4(\lambda t))\\
f_4(\lambda t) &= \sin(\lambda t)\cos(\lambda t)\left[-\beta_{11}\sin^2(\lambda t) + \beta_{22}\cos^2(\lambda t) - \beta_{12}\cos(2\lambda t)\right]\\
f_5(\lambda t) &= 2\sin^2(\lambda t)\cos^2(\lambda t)\left[\beta_{11} + \beta_{22} - 2\beta_{12}\right]
\end{aligned}
$$

Formal Limit

$$
\begin{aligned}
i\partial_t u_1 &= -\frac{1}{2}\Delta u_1 + \frac{\gamma^2}{2}|x|^2 u_1 + \sigma_1 |u_1|^2 u_1 + \sigma_3 |u_2|^2 u_1 \\
&\quad -i\sigma_5 \Im\left(u_1^* u_2\right) u_2 \\
i\partial_t u_2 &= -\frac{1}{2}\Delta u_2 + \frac{\gamma^2}{2}|x|^2 u_2 + \sigma_1 |u_2|^2 u_2 + \sigma_3 |u_1|^2 u_2 \\
&\quad +i\sigma_5 \Im\left(u_1^* u_2\right) u_1
\end{aligned}
$$

with initial data $u_1(x,0) = \varphi_1(x)$ and $u_2(x,0) = \varphi_2(x)$ and

$$
\sigma_1 = \frac{3\beta_{11} + 3\beta_{22} + 2\beta_{12}}{8} \qquad \sigma_3 = \frac{\beta_{11} + \beta_{22} + 6\beta_{12}}{8}
$$

$$
\sigma_5 = \frac{\beta_{11} + \beta_{22} - 2\beta_{12}}{4}
$$

This system has three conserved quantities:

$$\begin{aligned}
\tilde{\tilde{E}}(t) &= \int_{\mathbb{R}^N} \Big[\sum_{j=1}^{2} \left(\frac{1}{2}|\nabla u_j|^2 + \frac{\gamma^2 |x|^2}{2}|u_j|^2 + \frac{\sigma_1}{2}|u_j|^4 \right) \\
&\quad + \sigma_3 |u_1|^2 |u_2|^2 + \sigma_5 \Im^2(u_1^* u_2) \Big] (x,t)dx \\
M(t) &= \int_{\mathbb{R}^N} \left(| \ u_1|^2 + |u_2|^2 \right)(x,t)dx \\
R(t) &= \Re \int_{\mathbb{R}^N} (u_1 u_2^*)(x,t)dx
\end{aligned}$$

- we can show the same convergence results as before
- global existence is in the same parameter regions as before

Conclusion

- we discussed the global existence and the blow-up alternative of the system
- semi-explicit formula describing the mass evolution, indicating the role of the Rabi frequency λ.
- we performed asymptotics for $|\lambda| \to \infty$
- proved the convergence locally in time in appropriate Strichartz' spaces.
- show existence of the system on a time interval strictly smaller than the existence interval of the limiting system. $\Rightarrow$ We expect the system to behave like the limiting system for $|\lambda|$ sufficiently large.

Thank you for your attention!