# Scalar Waves on a Naked Singularity Background


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[1] Joint work with A. Shadi Tahvildar-Zadeh *Class. Quant. Grav.* 2004

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- There are no troublesome indices.
- People (mostly Wald[2] and students) suggest well-posedness of wave equations as a substitute for geodesic completeness.

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- Dispersive Estimates ($L^\infty$ decay):

$$\|u(t)\|_{L^\infty(\mathbf{R}^3)} \leq Ct^{-1}\left(\|\nabla u(0)\|_{L^1(\mathbf{R}^3)} + \|\partial_t u(0)\|_{L^1(\mathbf{R}^3)}\right)$$

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One nice thing about this Strichartz is that it is Lorentz invariant. Strichartz estimates, like dispersive estimates, can be used to prove stability for non-linear wave equations, but Strichartz estimates are often true in contexts where dispersive estimates fail.

# Scalar Wave Equation in Spherical Symmetry

Metric in isothermal coordinates:

$$g_{\mu\nu} dx^\mu dx^\nu = \alpha(r)^2 \left(-dt^2 + dr^2\right) + \rho(r)^2 \left(d\varphi^2 + \sin^2\varphi \, d\theta^2\right)$$

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(Massless, Chargeless) Scalar Wave Equation:

$$g^{\mu\nu}\psi_{;\mu\nu} = 0$$

or

$$\partial_t^2\psi - \frac{1}{\rho^2}\partial_r\left(\rho^2\partial_r\psi\right) - \frac{\alpha^2}{\rho^2}\Delta_{\mathrm{Sph}}\psi = 0.$$

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$$u = \rho\psi/r$$

satisfies

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Compared to the scalar wave equation in Minkowski space, there is an additional scalar potential

$$V(r) = \rho''(r)/\rho(r).$$

# Reissner-Nordström

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- If $|e| = m$, the extremal case, then the quadratic has a double root at $\rho = m$, which is again a horizon and the metric is valid outside the horizon.
- If $|e| > m$, the super-extremal case, then there are no horizons and the metric above is valid for all $r > 0$, but is highly singular at $r = 0$.

# Super-extremal Case

$$V = \frac{2m}{\rho^3} - \frac{2e^2 + 4m^2}{\rho^4} + \frac{6me^2}{\rho^5} - \frac{2e^4}{\rho^6}$$

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$$\lim_{r \to 0^+} r^2 V(r) = -\frac{2}{9}$$

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We are forced by the problem to choose

$$Au = -\frac{1}{r^2}\partial_r(r^2\partial_r u) - \frac{\alpha^2}{\rho^2}\Delta_{\mathrm{Sph}} u + V(r)u,$$

but with what domain?

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For smooth functions $u$, $v$ supported on compact subsets of $\mathbf{R}^3 - 0$, we have

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Almost all symmetric differential operators appearing in Mathematical Physics are essentially self-adjoint.

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An alternate characterisation, due to Krein, is this: If $A_F$ is the Friedrichs extension and $A_E$ is any other positive self-adjoint extension of $A$ then

$$u \in \mathrm{Dom}(A_F) \implies u \in \mathrm{Dom}(A_E) \, \mathrm{and} \, \langle u, A_F u \rangle \leq \langle u, A_E u \rangle \, .$$

[3] See Seggev, 2003.

# BPST-Z

Now that we have a well defined problem, we can use a theorem we proved earlier with Burq and Planchon:[4]
Let $V \in C^1(\mathbf{R}^+)$ satisfy

- $\sup_{r\in\mathbf{R}^+} r^2 V(r) < \infty$
- $\inf_{r\in\mathbf{R}^+} r^2 V(r) > -1/4$
- $\sup_{r\in\mathbf{R}^+} r^2 \frac{d}{dr}(rV(r)) < 1/4$,

let $P = -\Delta + V$, and let $P_F$ be the Friedrichs extension of $P$. Then there exists a $C$ such that if

$$\partial_t^2 u + P_F u = 0$$

then

$$\|u\|_{L^4(\mathbf{R}^{1+3})} \leq C\mathbf{E}_{1/2}[u].$$

[4] Indiana J of Math, 2004, but use Arxiv instead!

# What needs to be done?

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  and $\frac{\alpha r}{\rho}$ is bounded.
- We need to check the three hypotheses on our $V$.

# Checking $V$

Recall:

$$V = \frac{2m}{\rho^3} - \frac{2e^2 + 4m^2}{\rho^4} + \frac{6me^2}{\rho^5} - \frac{2e^4}{\rho^6}$$

$$r = \rho - r_0 + m \log\left(\frac{\rho^2 - 2m\rho + e^2}{e^2 - m^2}\right) + \frac{2m^2 - e^2}{\sqrt{e^2 - m^2}} \arctan \frac{\rho - m}{\sqrt{e^2 - m^2}},$$

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We still have nasty transcendental functions of the two variables $r/e$ and $m/e$ whose zeroes we need to find.

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$$r = \rho - r_0 + m \log\left(\frac{\rho^2 - 2m\rho + e^2}{e^2 - m^2}\right) + \frac{2m^2 - e^2}{\sqrt{e^2 - m^2}} \arctan \frac{\rho - m}{\sqrt{e^2 - m^2}},$$

$$r_0 = m \log\left(\frac{e^2}{e^2 - m^2}\right) - \frac{2m^2 - e^2}{\sqrt{e^2 - m^2}} \arctan \frac{m}{\sqrt{e^2 - m^2}}$$

We can eliminate one variable by scaling.
We still have nasty transcendental functions of the two variables $r/e$ and $m/e$ whose zeroes we need to find.
Idea: Forget the transcendental relation between the two and treat $r/e$ and $\rho/e$ as independent. That gives us an algebraic problem in the three variables $\rho/e$, $r/e$ and $m/e$.

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Doesn't quite work as written, but something very similar does, at least for $|e| \geq 2m$.

# Algebra

Eventually, after much algebra, the problem reduces to one of real algebraic geometry:
Is the curve

$$
\begin{aligned}
&72x^{14} - (576 + 432y^2)x^{13} + (1947 + 3552y^2 + 1152y^4)x^{12} \\
&- (3504 + 11988y^2 + 10464y^4 + 1440y^6)x^{11} \\
&+ (3452 + 20360y^2 + 38762y^4 + 15384y^6 + 720y^8)x^{10} \\
&- (1536 + 16456y^2 + 71800y^4 + 66316y^6 + 10536y^8)x^9 \\
&+ (2040y^2 + 62966y^4 + 143492y^6 + 57803y^8 + 2160y^{10})x^8 \\
&- (-4608y^2 + 8608y^4 + 153832y^6 + 154672y^8 + 21648y^{10})x^7 \\
&+ (-20100y^4 + 48272y^6 + 208760y^8 + 83120y^{10} + 2760y^{12})x^6 \\
&- (-36120y^6 + 104440y^8 + 151552y^{10} + 20824y^{12})x^5 \\
&+ (-33769y^8 + 109100y^{10} + 58958y^{12} + 1908y^{14})x^4 \\
&- (-17900y^{10} + 62848y^{12} + 11680y^{14})x^3 \\
&+ (-5530y^{12} + 20912y^{14} + 944y^{16})x^2 \\
&- (-972y^{14} + 3888y^{16})x + (-81y^{16} + 324y^{18}) = 0
\end{aligned}
$$

compact?

# Algebra (Continued)

Oddly, that question seems never to have been considered.

[5] Forum Mathematicum, 2007

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Definition: The *Newton polygon* of a polynomial $p(x, y)$ in two variables is the convex hull of the set of $\mathbf{Z}^2$ lattice points $(k, l)$ where the coefficient of $x^k y^l$ in $p$ is non-zero.

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Theorem:[5] The curve $C = \{(x, y) \in \mathbf{R}^2 \colon p(x, y) = 0\}$ is compact if $p$ is not divisible by $x$ or $y$ and none of its edge polynomials have real zeroes.

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Note that there is a classical algorithm for checking the existence of real zeroes of polynomials in one variable, the *Sturm test*.

[5] Forum Mathematicum, 2007

# So now what?

For the scalar wave equation in super-extremal Reissner-Nordström,

- energy estimates are trivial, once you figure out how to get a well-defined problem,
- Strichartz estimates, somewhat miraculously, are true, at least for spherically symmetric data,
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There are some interesting new non-linear stability results for MBI, by Speck.

Questions?