CURIOUS EXTENSIONS OF RAMANUJAN'S ${}_1\psi_1$ SUMMATION FORMULA

VICTOR J. W. GUO* AND MICHAEL J. SCHLOSSER**

Abstract. We deduce new q-series identities by applying inverse relations to certain identities for basic hypergeometric series. The identities obtained themselves do not belong to the hierarchy of basic hypergeometric series. We extend two of our identities, by analytic continuation, to bilateral summation formulae which contain Ramanujan's ${}_1\psi_1$ summation and a very-well-poised ${}_4\psi_6$ summation as special cases.

1. Introduction

Ramanujan's ${}_1\psi_1$ summation (cf. [3, Eq. (5.2.1)])

$$\sum_{k=-\infty}^{\infty}\frac{(a;q)_k}{(b;q)_k}z^k=\frac{(q;q)_\infty(b/a;q)_\infty(az;q)_\infty(q/az;q)_\infty}{(b;q)_\infty(q/a;q)_\infty(z;q)_\infty(b/az;q)_\infty}, \tag{1.1}$$

where $|q|<1$ and $|b/a|<|z|<1$, and where $(x;q)_\infty=\prod_{j\geq 0}(1-xq^j)$ and $(x;q)_k=(x;q)_\infty/(xq^k;q)_\infty$ for integer k, is one of the most important and beautiful identities in the theory of basic hypergeometric series, see [3].

Concerning hypergeometric and basic hypergeometric identities, there is a dual hierarchy of certain identities closely related to these but which themselves do not belong to the hierarchy of hypergeometric or basic hypergeometric series. These identities can be obtained by applying *inverse relations* to the respective (basic) hypergeometric identities. For instance, "dual" to the *binomial theorem*,

$$(a+c)^n=\sum_{k=0}^{n}\binom{n}{k}a^k\,c^{n-k},$$

Date: August 1, 2006.

2000 *Mathematics Subject Classification.* Primary 33D15; Secondary 15A09, 33D99.

Key words and phrases. q-series, basic hypergeometric series, bilateral series, Ramanujan's ${}_1\psi_1$ summation, matrix inversion.

*The first author was supported by a Junior Research Fellowship at ESI (Erwin Schrödinger Institute), enabling his visit to Vienna where this research was carried out.

**The second author was partly supported by FWF Austrian Science Fund grants P17563-N13, and S9607 (the second is part of the Austrian National Research Network "Analytic Combinatorics and Probabilistic Number Theory").

there is Abel's summation formula,

$$(a+c)^n=\sum_{k=0}^{n}\binom{n}{k}a(a+bk)^{k-1}(c-bk)^{n-k} \tag{1.2}$$

(cf. [8, Sec. 1.5]), which, containing an extra parameter b, is even more general than the binomial theorem. Similarly, dual to the Chu–Vandermonde summation,

$$\binom{a+c}{n}=\sum_{k=0}^{n}\binom{a}{k}\binom{c}{n-k},$$

we have the Hagen–Rothe summation,

$$\binom{a+c}{n}=\sum_{k=0}^{n}\frac{a}{a+bk}\binom{a+bk}{k}\binom{c-bk}{n-k} \tag{1.3}$$

(cf. [5]), which, containing an extra parameter b, is even more general than the Chu–Vandermonde summation. Here we would like to point out that Abel's summation can be deduced from the Hagen–Rothe summation. Indeed, replacing a, b and c by ma, mb and mc, respectively, in (1.3), and dividing both sides by m^n, we obtain

$$\binom{ma+mc}{n}m^{-n}=\sum_{k=0}^{n}\frac{ma}{ma+mbk}\binom{ma+mbk}{k}\binom{mc-mbk}{n-k}m^{-n}. \tag{1.4}$$

Letting $m\to\infty$ in (1.4), we immediately get (1.2).

Furthermore, dual to the Pfaff–Saalschütz summation,

$$\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}=\sum_{k=0}^{n}\frac{(a)_k(b)_k(-n)_k}{(1)_k(c)_k(a+b-c+1-n)_k} \tag{1.5}$$

(cf. [1, Thm. 2.2.6]), where $(a)_0=1$ and $(a)_k=a(a+1)\dots(a+k-1)$ for positive integer k, we have the following identity which was derived in [9, Thm. 7.8],

$$\begin{aligned}\frac{(2c+1)_n}{(c+1)_n}=\sum_{k=0}^{n}&\left(\frac{b+(a-c)a}{b+(a-c)(a+k)}\right)\left(\frac{b+(a+k)^2}{b+a(a+k)}\right)\\&\times\frac{(-n)_k\,(c)_k\,\left(a-c+\frac{b}{a+k}\right)_k}{(1)_k\,(-c-n)_k\,\left(a+c+\frac{b}{a+k}+1\right)_k}\,\frac{\left(a+c+\frac{b}{a+k}+1\right)_n}{\left(a+\frac{b}{a+k}+1\right)_n}.\end{aligned} \tag{1.6}$$

Corresponding to the summations in (1.2), (1.3) and (1.6), there exist contiguous identities (with slightly modified summand, usually involving some additional linear factors, or "shifts" on some of the parameters), nonterminating summations (expansions), and basic (q-)versions, see [9]. For instance, by inverting the q-Pfaff–Saalschütz summation,

$$\frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/ab;q)_n}=\sum_{k=0}^{n}\frac{(a;q)_k(b;q)_k(q^{-n};q)_k}{(q;q)_k(c;q)_k(abq^{1-n}/c;q)_k}q^k \tag{1.7}$$

(cf. [3, Eq. (1.7.2)]), the following identity was derived in [9, Thm. 7.34],

$$\frac{(c^2q;q)_n}{(cq;q)_n}=\sum_{k=0}^{n}\frac{(b+(a-c)(a-1))}{(b+(a-c)(a-q^{-k}))}\frac{(b+(a-q^{-k})^2)}{(b+(a-1)(a-q^{-k}))}$$
$$\times\frac{(q^{-n};q)_k\,(c;q)_k\,\left(\frac{b+a(a-q^{-k})}{c(a-q^{-k})};q\right)_k}{(q;q)_k\,(q^{-n}/c;q)_k\,\left(cq\frac{b+a(a-q^{-k})}{(a-q^{-k})};q\right)_k}\frac{\left(cq\frac{b+a(a-q^{-k})}{(a-q^{-k})};q\right)_n}{\left(q\frac{b+a(a-q^{-k})}{(a-q^{-k})};q\right)_n}q^k \tag{1.8}$$

(by which we correct some misprints which appeared in the printed version of [9]). This identity can be compared to Equation (3.1) in Theorem 3.1 of this paper, which is different but somewhat similar to (1.8). To give another example, by inverting the q-Gauß summation,

$$\frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}=\sum_{k=0}^{\infty}\frac{(a;q)_k(b;q)_k}{(q;q)_k(c;q)_k}\left(\frac{c}{ab}\right)^k, \tag{1.9}$$

where $|c/ab|<1$ (cf. [3, Eq. (1.5.1)]), the following nonterminating identity was derived in [9, Thm. 7.16],

$$\frac{(b^2q;q)_\infty}{(bq;q)_\infty}=\sum_{k=0}^{\infty}\frac{(c-(a+1)(a+b))}{(c-(a+1)(a+bq^k))}\frac{(c-(a+bq^k)^2)}{(c-(a+b)(a+bq^k))}$$
$$\times\frac{(b;q)_k\left(\frac{(a+bq^k)}{c-a(a+bq^k)};q\right)_k\left(\frac{(a+bq^k)b^2q^{k+1}}{c-a(a+bq^k)};q\right)_\infty}{(q;q)_k\left(\frac{(a+bq^k)bq}{c-a(a+bq^k)};q\right)_\infty}(bq)^k, \tag{1.10}$$

where $|bq|<1$. This identity can be compared to Equations (3.2) and (3.2), in Theorems 3.2 and 3.3, respectively, of this paper, which are different but somewhat similar to (1.10).

In [9] also *multidimensional* identities associated with root systems of Abel-, Rothe- and the above "curious" type are derived. Related beta type integrals are deduced in [4] and [10].

Concerning *bilateral* summations, a "curious" generalization of Jacobi's triple product identity was given in [11]. However, so far no "curious" extensions of the more general ${}_1\psi_1$ summation formula (1.1) have been given. Also, so far none of the existing *very-well-poised* summations have been inverted to obtain non-hypergeometric identities of the above "curious" type. In this paper, we provide for the first time such extensions. Our formulae, see Theorems 3.4 and 3.5, not only generalize Ramanujan's bilateral summation (1.1), but also contain a very-well-poised ${}_4\psi_6$ summation formula as special case.

Our paper is organized as follows. In Section 2, we recall some standard facts about basic hypergeometric series and list some of the identities we will be dealing with. In the same section, we also explain the concept of inverse relations and display some specific matrix inverses (which are in fact special cases of Krattenthaler's [7] matrix inverse) we need. These matrix inverses are utilized in Section 3, where via inverse relations we deduce from known summations a couple of new "curious" summations which do not belong to the hierarchy of basic hypergeometric series. In particular, by inverting the terminating very-well-poised ${}_6\phi_5$ summation we obtain a new terminating "curious" summation. Similarly, by inverting the nonterminating very-well-poised ${}_5\phi_5$ summation we obtain a new nonterminating "curious" summation. As a limiting case of the new terminating curious summation, we deduce yet another nonterminating curious summation. We extend by analytic continuation suitable special cases of both of these nonterminating unilateral summations to bilateral summation formulae, which on one hand contain Ramanujan's ${}_1\psi_1$ summation and on the other hand also a very-well-poised ${}_4\psi_6$ summation as special cases.

As a matter of fact, we were not able to find a likewise "curious" nonhypergeometric generalization of the ${}_6\psi_6$ summation (2.4), for reasons of convergence. Such a generalization may still exist but its proof (assuming it involves inverse relations) would require a matrix inverse different from Corollaries 2.2 or 2.3. However, already the "curious" extensions of Ramanujan's ${}_1\psi_1$ summation in Theorems 3.4 and 3.5 came to us as a big surprise.

Concluding this introduction, we would like to add that the identities derived in this paper have been checked numerically by Mathematica.

2. Preliminaries

2.1. **Basic hypergeometric series.** Let q (the "base") be a complex number such that $0<|q|<1$. Define the *q-shifted factorial* by

$$(a;q)_\infty := \prod_{j\geq 0}(1-aq^j) \qquad \text{and} \qquad (a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty}$$

for integer k. The *basic hypergeometric ${}_r\phi_s$ series* with numerator parameters $a_1,\dots,a_r$, denominator parameters $b_1,\dots,b_s$, base q, and argument z is defined by

$${}_r\phi_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,z\right] := \sum_{k=0}^{\infty}\frac{(a_1;q)_k\dots(a_r;q)_k}{(q;q)_k(b_1;q)_k\dots(b_s;q)_k}\left((-1)^kq^{\binom{k}{2}}\right)^{1+s-r}z^k.$$

The ${}_r\phi_s$ series terminates if one of the numerator parameters is of the form q^{-n} for a nonnegative integer n. If the series does not terminate, it converges for $r=s+1$ when $|z|<1$. For $r\leq s$, it converges everywhere. The *bilateral basic hypergeometric ${}_r\psi_s$ series* with numerator parameters $a_1,\dots,a_r$, denominator parameters $b_1, \dots, b_s$, base q, and argument z is defined by

$$ {}_r\psi_s\left[\begin{matrix}a_1, \dots, a_r\\ b_1, \dots, b_s\end{matrix}; q, z\right] := \sum_{k=-\infty}^{\infty} \frac{(a_1;q)_k \dots (a_r;q)_k}{(b_1;q)_k \dots (b_s;q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{s-r} z^k. $$

The ${}_r\psi_s$ series reduces to a unilateral ${}_r\phi_s$ series if one of its lower parameters is q. If the series does not terminate, it converges for $r = s$ when $|z| < 1$ and $|z| > |b_1 \dots b_r/a_1 \dots a_r|$. For $r < s$, it converges everywhere when $|z| > |b_1 \dots b_s/a_1 \dots a_r|$.

For a thorough exposition on basic hypergeometric series (or, synonymously, *q-hypergeometric series*), including a list of several selected summation and transformation formulas, we refer the reader to [3].

We list some specific identities which we will utilize in this paper.

We start with the terminating very-well-poised ${}_6\phi_5$ summation (cf. [3, Eq. (2.4.2)]).

$$ {}_6\phi_5\left[\begin{matrix}a,\, q\sqrt{a}, -q\sqrt{a}, b, c, q^{-n}\\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{1+n}\end{matrix}; q, \frac{aq^{1+n}}{bc}\right] = \frac{(aq;q)_k (aq/bc;q)_k}{(aq/b;q)_k (aq/c;q)_k}. \tag{2.1} $$

This can be extended to the following nonterminating very-well-poised ${}_6\phi_5$ summation (cf. [3, Eq. (2.7.1)]):

$$ {}_6\phi_5\left[\begin{matrix}a,\, q\sqrt{a}, -q\sqrt{a}, b, c, d\\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d\end{matrix}; q, \frac{aq}{bcd}\right] = \frac{(aq;q)_\infty (aq/bc;q)_\infty (aq/bd;q)_\infty (aq/cd;q)_\infty}{(aq/b;q)_\infty (aq/c;q)_\infty (aq/d;q)_\infty (aq/bcd;q)_\infty}, \tag{2.2} $$

valid for $|aq/bcd| < 1$. Clearly, (2.2) reduces to (2.1) for $d = q^{-k}$.

Rather than (2.2), we will need the following nonterminating very-well-poised ${}_5\phi_5$ summation, resulting from (2.2) as the special case where $d \to \infty$ (cf. [3, Ex. 2.22, 2nd Eq.]):

$$ {}_5\phi_5\left[\begin{matrix}a,\, q\sqrt{a}, -q\sqrt{a}, b, c\\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, 0\end{matrix}; q, \frac{aq}{bc}\right] = \frac{(aq;q)_\infty (aq/bc;q)_\infty}{(aq/b;q)_\infty (aq/c;q)_\infty}. \tag{2.3} $$

The nonterminating ${}_6\phi_5$ summation in (2.2) can yet be further extended to Bailey's very-well-poised ${}_6\psi_6$ summation (cf. [3, Eq. (5.3.1)]):

$$ \begin{aligned} {}_6\psi_6&\left[\begin{matrix}q\sqrt{a}, -q\sqrt{a}, b, c, d, e\\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e\end{matrix}; q, \frac{a^2q}{bcde}\right] \\ &= \frac{(q;q)_\infty (aq;q)_\infty (q/a;q)_\infty (aq/bc;q)_\infty}{(q/b;q)_\infty (q/c;q)_\infty (q/d;q)_\infty (q/e;q)_\infty} \\ &\times \frac{(aq/bd;q)_\infty (aq/be;q)_\infty (aq/cd;q)_\infty (aq/ce;q)_\infty (aq/de;q)_\infty}{(aq/b;q)_\infty (aq/c;q)_\infty (aq/d;q)_\infty (aq/e;q)_\infty (a^2q/bcde;q)_\infty}, \end{aligned} \tag{2.4} $$

valid for $|aq^2/bcde| < 1$. Clearly, (2.4) reduces to (2.2) for $e = a$.

We will in particular refer to the following very-well-poised ${}_4\psi_6$ summation formula, obtained as the $d, e \to \infty$ special case of (2.4):

$$
{}_4\psi_6\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, 0, 0 \end{matrix}; q, \frac{a^2q}{bc}\right]
= \frac{(q;q)_\infty (aq;q)_\infty (q/a;q)_\infty (aq/bc;q)_\infty}{(q/b;q)_\infty (q/c;q)_\infty (aq/b;q)_\infty (aq/c;q)_\infty}. \tag{2.5}
$$

2.2. **Inverse relations.** Let $\mathbb{Z}$ denote the set of integers and $F = (f_{nk})_{n,k\in\mathbb{Z}}$ be an infinite lower-triangular matrix; i.e. $f_{nk} = 0$ unless $n \geq k$. The matrix $G = (g_{kl})_{k,l\in\mathbb{Z}}$ is said to be the *inverse matrix* of F if and only if

$$
\sum_{l\leq k\leq n} f_{nk} g_{kl} = \delta_{nl} \tag{2.6}
$$

for all $n, l \in \mathbb{Z}$, where δ_{nl} is the usual Kronecker delta. Since F anf G are both lower-triangular, the dual orthogonality relation,

$$
\sum_{l\leq k\leq n} g_{nk} f_{kl} = \delta_{nl}, \tag{2.7}
$$

automatically must hold at the same time.

The method of applying *inverse relations* [8] is a well-known technique for proving identities, or for producing new ones from given ones. It is an immediate consequence of the orthogonality relation (2.6), that if $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are lower-triangular matrices that are inverses of each other, then

$$
\sum_{k=0}^{n} f_{nk} a_k = b_n \tag{2.8a}
$$

if and only if

$$
\sum_{l=0}^{k} g_{kl} b_l = a_k. \tag{2.8b}
$$

Another variant of inverse relations which we will also utilize in this paper involves infinite sums and reads as follows:

$$
\sum_{n\geq k} f_{nk} a_n = b_k \tag{2.9a}
$$

if and only if

$$
\sum_{k\geq l} g_{kl} b_k = a_l, \tag{2.9b}
$$

subject to suitable convergence conditions. For some applications of (2.9) see e.g. [7, 8, 9].

It is clear that in order to apply (2.8) (or (2.9)) effectively, one should have some explicit matrix inversion at hand.

Lemma 2.1 (Krattenthaler [7]). *Let $(a_j)_{j\in\mathbb{Z}}$, $(c_j)_{j\in\mathbb{Z}}$ be arbitrary sequences and d an arbitrary indeterminate. Then the infinite matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are inverses of each other, where*

$$f_{nk} = \frac{\prod_{j=k}^{n-1}(a_j - d/c_k)(a_j - c_k)}{\prod_{j=k+1}^{n}(c_j - d/c_k)(c_j - c_k)},$$

$$g_{kl} = \frac{(a_l c_l - d)(a_l - c_l)}{(a_k c_k - d)(a_k - c_k)} \frac{\prod_{j=l+1}^{k}(a_j - d/c_k)(a_j - c_k)}{\prod_{j=l}^{k-1}(c_j - d/c_k)(c_j - c_k)}.$$

Krattenthaler's matrix inverse is very general as it contains a vast number of other known explicit infinite matrix inversions. Several of its useful special cases are of (basic) hypergeometric type. The following special case of Lemma 2.1 has not been considered explicitly before. It is exceptional in the sense that although it involves powers of q, it is *not* to be considered a q-hypergeometric inversion. (More precisely, the following special case serves as a bridge between q-hypergeometric and certain non-q-hypergeometric identities. For some other such matrix inverses, see [9].)

In particular, we set

$$a_j = \frac{1-bc}{1-acq^j}, \qquad c_j = 1 - cq^{-j}, \qquad d = 1 - bc, \tag{2.11}$$

for all integers j.

To give a flavor of the elementary computations involved, we show explicitly how to compute $\prod_{j=l+1}^{k}(a_j - c_k)$:

$$\prod_{j=l+1}^{k}(a_j - c_k) = \prod_{j=l+1}^{k}\left(\frac{1-bc}{1-acq^j} - \frac{q^k - c}{q^k}\right)$$
$$= \prod_{j=l+1}^{k}\frac{q^k - bcq^k - q^k + c - ac^2q^j + acq^{k+j}}{(1-acq^j)q^k} = \prod_{j=l+1}^{k}\frac{c\big(1 - bq^k - (c-q^k)aq^j\big)}{(1-acq^j)q^k}$$
$$= \left(\frac{c(1-bq^k)}{q^k}\right)^{k-l}\frac{\big(\frac{c-q^k}{1-bq^k}aq^{1+l};q\big)_{k-l}}{(acq^{1+l};q)_{k-l}}.$$

Similarly, we compute the other products appearing in (2.10). After transferring some factors which depend only on one index from one matrix to the other (which corresponds to simultaneously multiplying one of the matrices by a suitable diagonal matrix and multiplying the other matrix by the inverse of that diagonal matrix) we obtain the following result:

Corollary 2.2. *Let*

$$f_{nk} = \frac{(1-bq^n)}{(1-bq^k)}\left(\frac{1-bq^k}{c-q^k}\right)^n \frac{\Big(1 - \frac{c-q^n}{1-bq^n}aq^n\Big)}{\Big(1 - \frac{c-q^k}{1-bq^k}a\Big)} \frac{\Big(1 - \frac{1-bq^k}{c-q^k}q^k\Big)}{\Big(1 - \frac{1-bq^k}{c-q^k}\Big)} q^k$$

$$\times \frac{(q^{-n};q)_k\,(aq^n;q)_k}{(q;q)_k\,(aq;q)_k}\frac{\left(\frac{c-q^k}{1-bq^k}a;q\right)_n}{\left(\frac{1-bq^k}{c-q^k}q;q\right)_n}, \tag{2.12a}$$

$$g_{kl}=\left(\frac{c-q^k}{1-bq^k}\right)^l q^{kl}\,\frac{(1-aq^{2l})}{(1-a)}\frac{(a;q)_l\,(q^{-k};q)_l}{(q;q)_l\,(aq^{1+k};q)_l}\frac{\left(\frac{1-bq^k}{c-q^k};q\right)_l}{\left(\frac{c-q^k}{1-bq^k}aq;q\right)_l}. \tag{2.12b}$$

Then the infinite matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are inverses of each other.

For convenience, we also display another version of Corollary 2.2, easily obtained from the above matrix inverse by transferring some factors from one matrix to the other.

Corollary 2.3. *Let*

$$f_{nk}=\left(\frac{1-bq^k}{c-q^k}q^{-k}\right)^{n-k}\frac{(1-aq^{2n})}{(1-aq^{2k})}\frac{(aq^{2k};q)_{n-k}}{(q;q)_{n-k}}\frac{\left(\frac{c-q^k}{1-bq^k}aq^k;q\right)_{n-k}}{\left(\frac{1-bq^k}{c-q^k}q^{1+k};q\right)_{n-k}}, \tag{2.13a}$$

$$\begin{aligned}g_{kl}=(-1)^{k-l}q^{\binom{l}{2}-\binom{k}{2}}&\frac{(aq;q)_{2k}}{(q;q)_{k-l}\,(aq;q)_{k+l}}\frac{(1-bq^l)}{(1-bq^k)}\left(\frac{1-bq^k}{c-q^k}\right)^{k-l}\\&\times\frac{\left(1-\frac{c-q^l}{1-bq^l}aq^l\right)}{\left(1-\frac{c-q^k}{1-bq^k}aq^k\right)}\frac{\left(\frac{c-q^k}{1-bq^k}aq;q\right)_k}{\left(\frac{1-bq^k}{c-q^k};q\right)_k}\frac{\left(\frac{1-bq^k}{c-q^k};q\right)_l}{\left(\frac{c-q^k}{1-bq^k}aq;q\right)_l}.\end{aligned} \tag{2.13b}$$

Then the infinite matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are inverses of each other.

3. Some curious q-series identities

We start with a terminating summation, obtained by inverting the terminating very-well-poised ${}_6\phi_5$ summation using Corollary 2.2.

Theorem 3.1. *Let a, b, c, and d be indeterminates, and let n be a nonnegative integer. Then*

$$\begin{aligned}\frac{(q/d;q)_n}{(ad;q)_n}(ad)^n=\sum_{k=0}^n&\frac{1-bq^n}{1-bq^k}\left(\frac{1-bq^k}{c-q^k}\right)^n\frac{(q^{-n};q)_k\,(aq^n;q)_k}{(q;q)_k\,(ad;q)_k}\\&\times\frac{\left(1-\frac{c-q^n}{1-bq^n}aq^n\right)}{\left(1-\frac{c-q^k}{1-bq^k}aq^k\right)}\frac{\left(1-\frac{1-bq^k}{c-q^k}q^k\right)}{\left(1-\frac{1-bq^k}{c-q^k}\right)}\frac{\left(\frac{c-q^k}{1-bq^k}ad;q\right)_k}{\left(\frac{c-q^k}{1-bq^k}a;q\right)_k}\frac{\left(\frac{c-q^k}{1-bq^k}a;q\right)_n}{\left(\frac{1-bq^k}{c-q^k}q;q\right)_n}q^k.\end{aligned} \tag{3.1}$$

For $c=1/b$, (3.1) reduces to the q-Pfaff-Saalschütz summation (cf. [3, Eq. (1.7.2)]). On the other hand, performing the substitution $c\mapsto -b/c$ and then letting $b\to\infty$ gives the terminating very-well-poised ${}_6\phi_5$ summation (2.1).

Proof of Theorem 3.1. Let the inverse matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ be defined as in Equations (2.12). Then (2.8b) holds for

$$a_k = \frac{(aq;q)_k}{(ad;q)_k}\frac{\left(\frac{c-q^k}{1-bq^k}ad;q\right)_k}{\left(\frac{c-q^k}{1-bq^k}aq;q\right)_k} \qquad \text{and} \qquad b_l = \frac{(q/d;q)_l}{(ad;q)_l}(ad)^l$$

by the terminating ${}_6\phi_5$ summation in (2.1). This implies the inverse relation (2.8a) with the above values of a_k and b_l. After some minor simplifications we readily arrive at (3.1). □

Next, we deduce a nonterminating summation, obtained by inverting the nonterminating very-well-poised ${}_5\phi_5$ summation using Corollary 2.3. (We point out that the likewise inversion of the nonterminating very-well-poised ${}_6\phi_5$ summation fails dues to reasons of convergence.)

Theorem 3.2. *Let a, b, c, and d be indeterminates. Then*

$$\frac{1}{(1-b+a(1-c))}\frac{(ad;q)_\infty}{(aq;q)_\infty} = \sum_{k=0}^{\infty}\frac{1}{1-bq^k}\left(\frac{1-bq^k}{c-q^k}\right)^k\left(1-\frac{1-bq^k}{c-q^k}q^k\right)$$
$$\times\frac{(q/d;q)_k}{(q;q)_k\,(aq;q)_k}\frac{\left(\frac{c-q^k}{1-bq^k}aq;q\right)_{k-1}\left(\frac{1-bq^k}{c-q^k}d;q\right)_\infty}{\left(\frac{1-bq^k}{c-q^k};q\right)_\infty}d^k, \quad (3.2)$$

where $|d/c|<1$.

For $c=1/b$, (3.2) reduces to the q-Gauß summation (cf. [3, Eq. (1.5.1)]). On the other hand, multiplying both sides by b, performing the substitution $c\mapsto -b/c$ and then letting $b\to\infty$ gives the very-well-poised ${}_5\phi_5$ summation (2.3).

Proof of Theorem 3.2. Let the inverse matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ be defined as in Equations (2.13). Then (2.9a) holds for

$$a_n = (-1)^n q^{\binom{n}{2}}\frac{(q/d;q)_n}{(ad;q)_n}d^n$$

and

$$b_k = (-1)^k q^{\binom{k}{2}} d^k\,(q/d;q)_k\frac{(aq^{1+2k};q)_\infty}{(ad;q)_\infty}\frac{\left(\frac{1-bq^k}{c-q^k}d;q\right)_\infty}{\left(\frac{1-bq^k}{c-q^k}q^{1+k};q\right)_\infty}$$

by the nonterminating ${}_5\phi_5$ summation in (2.3). This implies the inverse relation (2.9b) with the above values of a_k and b_l. After the substitutions $a\mapsto aq^{-2l}$, $b\mapsto bq^{-l}$, $c\mapsto cq^l$, and $d\mapsto dq^l$, we can get rid of l, and after some simplifications we readily arrive at (3.2). □

We now derive a summation "contiguous" to Theorem 3.2. For this, we simultaneously perform the substitutions $a \mapsto q^{-n}/d$ and $d \mapsto adq^{1+n}$ in (3.2), multiply both sides by $(-1)^n q^{\binom{n}{2}-n} d^n/(c-d)$ and let $n \to \infty$ (while appealing to Tannery's theorem [2] for justification of taking term-wise limits). We have the following:

Theorem 3.3. *Let a, b, c, and d be indeterminates. Then*

$$\frac{1}{(c-d)}\frac{(adq;q)_\infty}{(aq;q)_\infty} = \sum_{k=0}^{\infty} \frac{1}{c-q^k}\left(\frac{1-bq^k}{c-q^k}\right)^k \left(1-\frac{1-bq^k}{c-q^k}q^k\right) \\ \times \frac{(1/d;q)_k}{(q;q)_k\,(aq;q)_k} \frac{\left(\frac{c-q^k}{1-bq^k}aq;q\right)_k \left(\frac{1-bq^k}{c-q^k}dq;q\right)_\infty}{\left(\frac{1-bq^k}{c-q^k};q\right)_\infty} d^k, \quad (3.3)$$

where $|d/c|<1$.

Again, for $c = 1/b$, (3.3) reduces to the q-Gauß summation. Also, multiplying both sides by c, performing the substitution $c \mapsto -b/c$ and then letting $b \to \infty$ gives the very-well-poised ${}_5\phi_5$ summation (2.3).

Now we extend the $a = 0$ case of the unilateral summation in Theorem 3.2 to a bilateral identity by analytic continuation. (The likewise extension of the full summation in Theorem 3.2 to a bilateral identity by analytic continuation fails due to reasons of convergence and analyticity.)

Theorem 3.4. *Let b, c, d, and e be indeterminates. Then*

$$\frac{1}{(e-b)}\frac{(q;q)_\infty(de;q)_\infty}{(d;q)_\infty(eq;q)_\infty} = \sum_{k=-\infty}^{\infty} \frac{1}{(1-bq^k)}\left(1-\frac{1-bq^k}{c-q^k}q^k\right) \\ \times \frac{\left(\frac{1-bq^k}{c-q^k}d;q\right)_\infty \left(\frac{c-q^k}{1-bq^k}eq;q\right)_\infty}{\left(\frac{1-bq^k}{c-q^k};q\right)_\infty \left(\frac{c-q^k}{1-bq^k}q;q\right)_\infty} \frac{(q/d;q)_k}{(eq;q)_k}\left(\frac{1-bq^k}{c-q^k}d\right)^k, \quad (3.4)$$

where $|d/c|<1$ and $|e/b|<1$.

Clearly, Theorem 3.4 reduces to the $a = 0$ case of Theorem 3.2 for $e = 1$. The $c = 1/b$ case of (3.4) gives, after some rewriting, Ramanujan's ${}_1\psi_1$ summation (1.1). On the other hand, performing the substitution $c \mapsto -b/c$ and then letting $b \to \infty$ gives the very-well-poised ${}_4\psi_6$ summation in (2.5). We find this unification of (1.1) and (2.5) quite surprising due to the fact that the ${}_1\psi_1$ summation is *not* a special case of the very-well-poised ${}_4\psi_6$ summation, in fact, not even of Bailey's very-well-poised ${}_6\psi_6$ summation formula.

Proof of Theorem 3.4. We apply Ismail's argument [6] to the parameter e using the $a = 0$ case of the nonterminating identity in Theorem 3.2. Both sides of (3.4) are analytic in the parameter e in a domain around the origin.

This follows from expanding the products depending on k by (special cases of) the q-binomial theorem. Now, the identity is true for $e=q^l$, by the $a=0$ case of Theorem 3.2 (see the next paragraph for the details). This holds for all $l\geq 0$. Since $\lim_{l\to\infty} q^l=0$ is an interior point in the domain of analyticity of e, by the identity theorem, we obtain an identity for general e.

The details are displayed as follows. Setting $e=q^l$, the right-hand side of (3.4) becomes

$$\sum_{k=-l}^{\infty}\frac{1}{(1-bq^k)}\left(1-\frac{1-bq^k}{c-q^k}q^k\right)\\ \times\frac{\left(\frac{1-bq^k}{c-q^k}d;q\right)_\infty}{\left(\frac{1-bq^k}{c-q^k};q\right)_\infty\left(\frac{c-q^k}{1-bq^k}q;q\right)_l}\frac{(q/d;q)_k}{(q^{1+l};q)_k}\left(\frac{1-bq^k}{c-q^k}d\right)^k. \tag{3.5}$$

We shift the summation index in (3.5) by $k\mapsto k-l$, and obtain

$$\frac{(q/d;q)_{-l}}{(q^{1+l};q)_{-l}}\sum_{k=0}^{\infty}\frac{1}{(1-bq^{k-l})}\left(1-\frac{1-bq^{k-l}}{c-q^{k-l}}q^{k-l}\right)\\ \times\frac{\left(\frac{1-bq^{k-l}}{c-q^{k-l}}d;q\right)_\infty}{\left(\frac{1-bq^{k-l}}{c-q^{k-l}};q\right)_\infty\left(\frac{c-q^{k-l}}{1-bq^{k-l}}q;q\right)_l}\frac{(q^{1-l}/d;q)_k}{(q;q)_k}\left(\frac{1-bq^{k-l}}{c-q^{k-l}}d\right)^{k-l}\\ =\frac{(q;q)_l}{(d;q)_l}q^{-l}\sum_{k=0}^{\infty}\frac{1}{(1-bq^{k-l})}\left(1-\frac{1-bq^{k-l}}{c-q^{k-l}}q^{k-l}\right)\\ \times\frac{\left(\frac{1-bq^{k-l}}{c-q^{k-l}}d;q\right)_\infty}{\left(\frac{1-bq^{k-l}}{c-q^{k-l}}q^{-l};q\right)_\infty}\frac{(q^{1-l}/d;q)_k}{(q;q)_k}\left(\frac{1-bq^{k-l}}{c-q^{k-l}}d\right)^{k}.$$

Next, we apply the $a\mapsto 0$, $b\mapsto bq^{-l}$, $c\mapsto cq^l$, $d\mapsto dq^l$, case of Theorem 3.2, simplify, and obtain for the last expression

$$\frac{(q;q)_l}{(d;q)_l}\frac{q^{-l}}{(1-bq^{-l})}=\frac{1}{(q^l-b)}\frac{(q;q)_\infty(dq^l;q)_\infty}{(d;q)_\infty(q^{1+l};q)_\infty},$$

which is exactly the $e=q^l$, case of the left-hand side of (3.4). $\square$

Similarly, we can extend the $a=0$ case of the unilateral summation in Theorem 3.3 to a bilateral identity by analytic continuation. (Again, the likewise extension of the full summation in Theorem 3.3 to a bilateral identity by analytic continuation fails due to reasons of convergence and analyticity.)

Theorem 3.5. *Let b, c, d, and e be indeterminates. Then*

$$\frac{1}{(c-d)}\frac{(q;q)_\infty(deq;q)_\infty}{(dq;q)_\infty(eq;q)_\infty}=\sum_{k=-\infty}^{\infty}\frac{1}{(c-q^k)}\left(1-\frac{1-bq^k}{c-q^k}q^k\right)$$

$$\times \frac{\left(\frac{1-bq^k}{c-q^k}dq;q\right)_\infty \left(\frac{c-q^k}{1-bq^k}eq;q\right)_\infty}{\left(\frac{1-bq^k}{c-q^k};q\right)_\infty \left(\frac{c-q^k}{1-bq^k}q;q\right)_\infty} \frac{(1/d;q)_k}{(eq;q)_k} \left(\frac{1-bq^k}{c-q^k}d\right)^k, \quad (3.6)$$

where $|d/c| < 1$ *and* $|e/b| < 1$.

The proof is similar to the proof of Theorem 3.4. We therefore omit the details. Theorem 3.5 reduces to the $a = 0$ case of Theorem 3.3 for $e = 1$. Similar to (3.4), the $c = 1/b$ case of (3.6) gives, after some rewriting, Ramanujan's ${}_1\psi_1$ summation (1.1). Also, after multiplying both sides by c, performing the substitution $c \mapsto -b/c$ and then letting $b \to \infty$, one obtains the very-well-poised ${}_4\psi_6$ summation in (2.5).

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E-mail address: jwguo1977@yahoo.com.cn
URL: http://math.univ-lyon1.fr/~guo

Institut für Mathematik der Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria
E-mail address: michael.schlosser@univie.ac.at
URL: http://www.mat.univie.ac.at/~schlosse