A NEW MULTIDIMENSIONAL MATRIX INVERSE WITH APPLICATIONS TO MULTIPLE q-SERIES

CHRISTIAN KRATTENTHALER† AND MICHAEL SCHLOSSER†

Institut für Mathematik der Universität Wien,
Strudlhofgasse 4, A-1090 Wien, Austria.
E-mail: kratt@pap.univie.ac.at, schlosse@pap.univie.ac.at
WWW: http://radon.mat.univie.ac.at/People/kratt
http://radon.mat.univie.ac.at/People/mschloss

Dedicated to H. W. Gould

ABSTRACT. We compute the inverse of a specific infinite r-dimensional matrix, extending a matrix inverse of Krattenthaler. Our inversion is different from the r-dimensional matrix inversion recently found by Schlosser but generalizes a multidimensional matrix inversion previously found by Chu. As applications of our matrix inversion we derive some multidimensional q-series identities. Among these are q-analogues of Carlitz' multidimensional Abel-type expansion formulas. Furthermore, we derive a q-analogue of MacMahon's Master Theorem.

1. INTRODUCTION

Matrix inversions are very important tools in combinatorics and special functions theory. In particular, it is a widely spread and often used method to derive and prove identities for (basic) hypergeometric series with the help of so-called "inverse relations" (see Section 4), which are immediate consequences of matrix inversions. (An inverse relation is in fact *equivalent* to its corresponding matrix inversion.) In order to be able to apply this method, *explicit* matrix inversions must be at hand.

At this point it seems appropriate to elaborate a little on the history of (explicit) matrix inversions and inverse relations, in particular, since H. W. Gould's name is inevitably tied with it. Over time, people came across an increasing number of such explicit matrix inversions. In the 1960s, in his book [53], Riordan provided lists of known matrix inversions and, in fact, dedicated two complete chapters of his book to inverse relations and their applications. (Riordan's inverse relations were classified and given a unified method of proof by Egorychev [16].) A prominent part of these inverse relations were due to Gould, who studied them in a series of papers [27], [28], [29], [30]. This study culminated in the important discovery, jointly with Hsu, of a very

† The authors were supported by the Austrian Science Foundation FWF, grant P12094-MAT
1991 *Mathematics Subject Classification*: Primary 33D70; Secondary 05A19 05A30 11B65 15A09 33C70 33D20.
Keywords: matrix inversion, inverse relations, basic hypergeometric series, q-Abel identities, q-Rothe identities, MacMahon's Master Theorem

general matrix inversion [32], which contained a lot of inverse relations of, what is now called, Gould-type and Abel-type as special cases. The problem, posed by Gould and Hsu, of finding a q-analogue of their formula was immediately solved thereafter by Carlitz [8]. He did not give any applications, however. The significance of Carlitz' matrix inversion showed up first when Andrews [1] discovered that the Bailey transform [3], [4], which is one of the corner stones in the development of the theory of (basic) hypergeometric series, is equivalent to a certain matrix inversion that is just a very special case of Carlitz'. Some time later, while further developing on Andrews' idea, Gessel and Stanton [22], [23] used another special case of Carlitz' matrix inversion (a bibasic extension of the inversion Andrews considered) to derive a number of basic hypergeometric summations and transformations, and identities of Rogers–Ramanujan type. Finite forms of identities of Rogers–Ramanujan type were considered by Bressoud [6]. The transform which he used to prove them is equivalent to a matrix inversion [7], which has some overlap with Carlitz' matrix inversion (namely in the one Andrews considered), but in general is not covered by Carlitz' result. A few years later, Gasper and Rahman proved a bibasic matrix inversion [19], [52] which unifies the matrix inversions of Gessel and Stanton, and Bressoud. It enabled them to derive numerous beautiful new quadratic, cubic, and quartic summation formulas for basic hypergeometric series. (They also extended this method to obtain bibasic, cubic, and quartic transformation formulas [20], [51], [21, Sec. 3.6].) The end of this line of development came with the attempt of the first author to combine all these recent matrix inversions into one formula. Indeed, in 1989, he discovered a matrix inversion, published in [42], which subsumes most of Riordan's inverse relations and all the other aforementioned matrix inversions, as it contains them all as special cases.

This matrix inversion is the following: The matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ ($\mathbb{Z}$ denotes the set of integers), are inverses of each other, where

$$f_{nk} = \frac{\prod_{j=k}^{n-1}(a_j + b_j c_k)}{\prod_{j=k+1}^{n}(c_j - c_k)}, \tag{1.1}$$

and

$$g_{kl} = \frac{(a_l + b_l c_l)}{(a_k + b_k c_k)} \frac{\prod_{j=l+1}^{k}(a_j + b_j c_k)}{\prod_{j=l}^{k-1}(c_j - c_k)}. \tag{1.2}$$

Starting in the late 1970s, Milne and co-authors, in a long series of papers (cf. [44], [48], [46], [47], [49], [50], and the references cited therein), developed a theory of multiple (basic) hypergeometric series associated to root systems. In order to have an equivalent of the (one-dimensional) Bailey transform at hand, to conveniently extend the development of the theory of (one-dimensional) basic hypergeometric series to an analogous theory for multiple series, matrix inversions in this multidimensional setting needed to be found. In [44] and [49], Lilly and Milne provided multidimensional extensions of the earlier mentioned matrix inversion that Andrews considered. Subsequently, Bhatnagar and Milne [5] found a matrix inversion that extended Gasper and Rahman's (one-dimensional) matrix inversion to the multidimensional setting. (According to our terminology "multidimensional" matrix inversions are matrix inversions that arise in the theory of multiple series, whereas "one-dimensional" matrix inversions are matrix inversions which arise in the theory of one-dimensional series; see Section 2 for a precise explanation.) At the end of this line of development stand the second author's matrix inversions [54]. Theorems 3.1 and 4.1 of [54] do indeed cover all previously mentioned matrix inversions in that area as special cases. In particular, these matrix inversions also contain the inversion (1.1)/(1.2) as one-dimensional special case.

The main result of this paper is another multidimensional extension of the matrix inverse (1.1)/(1.2) (see Theorem 3.1). This matrix inversion finds its applications in the theory of "ordinary" multiple series. It does not "belong", as far as we can tell, to the theory of multiple series associated to root systems. Also here, special cases of this matrix inversion appeared earlier in the literature. Aside from reducing to (1.1)/(1.2), the matrix inversion from [42], in the one-dimensional case, it also contains Chu's multidimensional matrix inversion [12] and a two-dimensional matrix inversion [40] by the first author. We demonstrate the usefulness of our new multidimensional matrix inverse by deriving several multidimensional q-series identities, among them q-analogues of Carlitz' multidimensional Abel-type expansion formulas, and a q-analogue of MacMahon's Master Theorem.

Our paper is organized as follows. In order to prove our matrix inversion, we need some preparations, which we provide in Section 2. There we review the first author's operator method [39]. We adapt a main theorem of [39] and add an appropriate multidimensional corollary (see Corollary 2.2). Then, in Section 3 we state and prove our multidimensional matrix inversion. We also add a companion inversion (Theorem 3.3) which we use later in the applications. In Section 4 the notion and use of inverse relations is explained, together with the standard basic hypergeometric notation. The following sections contain applications of our matrix inversion. In Section 5 we derive some new basic hypergeometric double summations. A multidimensional extension of a very-well-poised ${}_{10}\phi_9$-summation is the contents of Section 6. In Sections 7 and 10 we present q-analogues of Carlitz' multidimensional Abel-type expansion formulas [9], [10], [11]. These q-analogues are new even in the one-dimensional case. Related multiple q-Abel and q-Rothe summations are presented in Section 8. Finally, in Section 9, we find, for the first time, a (noncommutative) q-analogue of MacMahon's Master Theorem.

2. An operator method for proving matrix inversions

Let $F=(f_{\mathbf{n}\mathbf{k}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ (as before, $\mathbb{Z}$ denotes the set of integers) be an infinite lower-triangular r-dimensional matrix; i.e., $f_{\mathbf{n}\mathbf{k}}=0$ unless $\mathbf{n}\geq\mathbf{k}$, by which we mean $n_i\geq k_i$ for all $i=1,\dots,r$. The matrix $G=(g_{\mathbf{k}\mathbf{l}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ is said to be the *inverse matrix* of F if and only if

$$\sum_{\mathbf{n}\geq\mathbf{k}\geq\mathbf{l}} f_{\mathbf{n}\mathbf{k}}g_{\mathbf{k}\mathbf{l}}=\delta_{\mathbf{n}\mathbf{l}}$$

for all $\mathbf{n},\mathbf{l}\in\mathbb{Z}^r$, where $\delta_{\mathbf{n}\mathbf{l}}$ is the usual Kronecker delta.

In [39], the first author gave a method for solving Lagrange inversion problems, which are closely connected with the problem of inverting lower-triangular matrices. We will use his operator method for proving our new theorems.

First we need to introduce some notation and terminology. By a *formal Laurent series* we mean a series of the form $\sum_{\mathbf{n}\geq\mathbf{k}} a_{\mathbf{n}}\mathbf{z}^{\mathbf{n}}$, for some $\mathbf{k}\in\mathbb{Z}^r$, where $\mathbf{z}^{\mathbf{n}}=z_1^{n_1}z_2^{n_2}\cdots z_r^{n_r}$. Given the formal Laurent series $a(\mathbf{z})$ and $b(\mathbf{z})$ we introduce the bilinear form $\langle\, ,\, \rangle$ by

$$\langle a(\mathbf{z}), b(\mathbf{z})\rangle = \langle \mathbf{z}^{\mathbf{0}}\rangle (a(\mathbf{z})\cdot b(\mathbf{z})),$$

where $\langle \mathbf{z}^{\mathbf{0}}\rangle c(\mathbf{z})$ denotes the coefficient of $\mathbf{z}^{\mathbf{0}}$ in $c(\mathbf{z})$. Given any linear operator L acting on formal Laurent series, L^* denotes the adjoint of L with respect to $\langle\, ,\, \rangle$; i.e., $\langle La(\mathbf{z}), b(\mathbf{z})\rangle = \langle a(\mathbf{z}), L^* b(\mathbf{z})\rangle$ for all formal Laurent series $a(\mathbf{z})$ and $b(\mathbf{z})$. We need the following special case of [39, Theorem 1].

Lemma 2.1. *Let $F = (f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ be an infinite lower-triangular r-dimensional matrix with $f_{\mathbf{kk}} \neq 0$ for all $\mathbf{k} \in \mathbb{Z}^r$. For $\mathbf{k} \in \mathbb{Z}^r$, define the formal Laurent series $f_{\mathbf{k}}(\mathbf{z})$ and $g_{\mathbf{k}}(\mathbf{z})$ by $f_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{n}\geq\mathbf{k}} f_{\mathbf{nk}}\mathbf{z}^{\mathbf{n}}$ and $g_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\leq\mathbf{k}} g_{\mathbf{kl}}\mathbf{z}^{-\mathbf{l}}$, where $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ is the uniquely determined inverse matrix of F. Suppose that for $\mathbf{k} \in \mathbb{Z}^r$ a system of equations of the form*

$$U_j f_{\mathbf{k}}(\mathbf{z}) = c_j(\mathbf{k}) V f_{\mathbf{k}}(\mathbf{z}), \qquad j = 1,\dots,r,$$

holds, where U_j, V are linear operators acting on formal Laurent series, V being bijective, and where $(c_j(\mathbf{k}))_{\mathbf{k}\in\mathbb{Z}^r}$ are arbitrary sequences of constants. Moreover, we suppose that

$$\text{for all } \mathbf{m},\mathbf{n} \in \mathbb{Z}^r,\ \mathbf{m}\neq\mathbf{n}, \text{ there exists a } j \text{ with } 1\leq j\leq r \text{ and } c_j(\mathbf{m}) \neq c_j(\mathbf{n}). \tag{2.1}$$

Then, if $h_{\mathbf{k}}(\mathbf{z})$ is a solution of the dual system

$$U_j^* h_{\mathbf{k}}(\mathbf{z}) = c_j(\mathbf{k}) V^* h_{\mathbf{k}}(\mathbf{z}), \qquad j = 1,\dots,r,$$

with $h_{\mathbf{k}}(\mathbf{z}) \not\equiv 0$ for all $\mathbf{k} \in \mathbb{Z}^r$, the series $g_{\mathbf{k}}(\mathbf{z})$ are given by

$$g_{\mathbf{k}}(\mathbf{z}) = \frac{1}{\langle f_{\mathbf{k}}(\mathbf{z}), V^* h_{\mathbf{k}}(\mathbf{z})\rangle} V^* h_{\mathbf{k}}(\mathbf{z}).$$

We will use the following corollary of Lemma 2.1:

Corollary 2.2. *Let W_i, V_{ij} be linear operators acting on formal Laurent series, $c_j(\mathbf{k})$ arbitrary constants for $\mathbf{k} \in \mathbb{Z}^r$ and $i,j = 1,\dots,r$. Suppose the operators W_i, V_{ij}, $i,j = 1,\dots,r$, satisfy the commutation relations*

$$V_{i_1 j} W_{i_2} = W_{i_2} V_{i_1 j}, \qquad i_1 \neq i_2;\quad 1 \leq i_1, i_2, j \leq r, \tag{2.2}$$

$$V_{i_1 j_1} V_{i_2 j_2} = V_{i_2 j_2} V_{i_1 j_1}, \qquad i_1 \neq i_2;\quad 1 \leq i_1, i_2, j_1, j_2 \leq r. \tag{2.3}$$

Moreover, the $c_j(\mathbf{k})$ are assumed to satisfy (2.1), and $\det\limits_{1\leq i,j\leq r}(V_{ij})$ is assumed to be invertible. With the notation of Lemma 2.1, if

$$\sum_{j=1}^{r} c_j(\mathbf{k}) V_{ij} f_{\mathbf{k}}(\mathbf{z}) = W_i f_{\mathbf{k}}(\mathbf{z}), \qquad i = 1,\dots,r, \tag{2.4}$$

then

$$g_{\mathbf{k}}(\mathbf{z}) = \frac{1}{\langle f_{\mathbf{k}}(\mathbf{z}), \det(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z})\rangle} \det(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z}), \tag{2.5}$$

where $h_{\mathbf{k}}(\mathbf{z})$ is a solution of

$$\sum_{j=1}^{r} c_j(\mathbf{k}) V_{ij}^* h_{\mathbf{k}}(\mathbf{z}) = W_i^* h_{\mathbf{k}}(\mathbf{z}), \qquad i = 1,\dots,r, \tag{2.6}$$

with $h_{\mathbf{k}}(\mathbf{z}) \not\equiv 0$ *for all* $\mathbf{k} \in \mathbb{Z}^r$.

Proof. Due to (2.3), we can apply Cramer's rule to (2.4) to obtain

$$c_j(\mathbf{k}) \det_{1\le i,l\le r}(V_{il}) f_{\mathbf{k}}(\mathbf{z}) = \sum_{i=1}^{r} (-1)^{i+j} V^{(i,j)} W_i f_{\mathbf{k}}(\mathbf{z}),$$

for $j = 1, \dots, r$, $V^{(i,j)}$ being the minor of $(V_{st})_{1\le s,t\le r}$ with the i-th row and j-th column being omitted. The dual system (in the sense of Lemma 2.1) reads

$$\begin{aligned} c_j(\mathbf{k}) \det_{1\le i,l\le r}(V^*_{il}) h_{\mathbf{k}}(\mathbf{z}) &= \sum_{i=1}^{r} (-1)^{i+j} W^*_i V^{*(i,j)} h_{\mathbf{k}}(\mathbf{z}) \\ &= \sum_{i=1}^{r} (-1)^{i+j} V^{*(i,j)} W^*_i h_{\mathbf{k}}(\mathbf{z}), \end{aligned} \tag{2.7}$$

for $j = 1, \dots, r$, and is easily seen to be equivalent to (2.6). Notice that condition (2.3) justifies to write the dual of $\det(V_{il})$ as $\det(V^*_{il})$ (and similarly for $V^{(i,j)}$), and that, because of (2.2), we may commute W^*_i and $V^{*(i,j)}$ in (2.7). Now apply Lemma 2.1 with $V = \det(V_{ij})$ and $U_j = \sum_{i=1}^{r} (-1)^{i+j} V^{(i,j)} W_i$. $\square$

Remark 2.3. A slightly more general corollary is given in [54, Corollary 2.14] which was needed to prove another multidimensional matrix inversion which lead to the derivation of several interesting identities for multidimensional basic hypergeometric series associated to root systems.

3. A multidimensional matrix inversion

Theorem 3.1. *Let* $(a_i(t))_{t\in\mathbb{Z}}$, $(b_{ij}(t))_{t\in\mathbb{Z}}$, *and* $(c_i(t))_{t\in\mathbb{Z}}$, $i, j = 1, \dots, r$ *be arbitrary sequences such that* $c_i(s) \ne c_i(t)$ *for* $s \ne t$. *Then* $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ *and* $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ *are inverses of each other, where*

$$f_{\mathbf{nk}} = \prod_{i=1}^{r} \frac{\prod_{t_i=k_i}^{n_i-1} \left(a_i(t_i) + \sum_{j=1}^{r} b_{ij}(t_i) c_j(k_j)\right)}{\prod_{t_i=k_i+1}^{n_i} \left(c_i(t_i) - c_i(k_i)\right)} \tag{3.1}$$

and

$$\begin{aligned} g_{\mathbf{kl}} = {} & \frac{\det_{1\le i,j\le r}\Big(\big(a_i(l_i) + \sum_{s=1}^{r} b_{is}(l_i) c_s(k_s)\big)\delta_{ij} + b_{ij}(l_i)\big(c_i(l_i) - c_i(k_i)\big) \Big)}{\prod_{i=1}^{r} \left(a_i(k_i) + \sum_{j=1}^{r} b_{ij}(k_i) c_j(k_j)\right)} \\ & \times \prod_{i=1}^{r} \frac{\prod_{t_i=l_i+1}^{k_i} \left(a_i(t_i) + \sum_{j=1}^{r} b_{ij}(t_i) c_j(k_j)\right)}{\prod_{t_i=l_i}^{k_i-1} \left(c_i(t_i) - c_i(k_i)\right)}. \end{aligned} \tag{3.2}$$

Remark 3.2. For $r=1$, Theorem 3.1 reduces to the first author's matrix inverse (1.1)/(1.2). The special case $c_i(t)=t$, $i=1,\dots,r$, is equivalent to Chu's [12, Eqs. (2.3)/(2.4)] matrix inversion result. Setting $r=2$, $c_i(t)=q^t$, $a_i(t)=0$, $i=1,2$,

$$\big(b_{ij}(t_i)\big)_{1\le i,j\le 2}=\begin{pmatrix}-1 & q^{\mu+t_1}\\ q^{\lambda+t_2} & -1\end{pmatrix},$$

and simplifying a bit, we recover the first author's two-dimensional inversion [40, (4.15)/(4.16)], which was used there to derive many two-dimensional expansion formulas.

Proof of Theorem 3.1. We will use the operator method of Section 2. From (3.1) we deduce for $\mathbf{n}\ge\mathbf{k}$ the recursion

$$(c_i(n_i)-c_i(k_i))f_{\mathbf{nk}}=(a_i(n_i-1)+\textstyle\sum_{j=1}^r b_{ij}(n_i-1)c_j(k_j))f_{\mathbf{n}-\mathbf{e}_i,\mathbf{k}},\quad i=1,\dots,r,\tag{3.3}$$

where $\mathbf{e}_i$ denotes the vector of $\mathbb{Z}^r$ where all components are zero except the i-th, which is 1. We write

$$f_{\mathbf{k}}(\mathbf{z})=\sum_{\mathbf{n}\ge\mathbf{k}}f_{\mathbf{nk}}\mathbf{z}^{\mathbf{n}}=\sum_{\mathbf{n}\ge\mathbf{k}}\prod_{i=1}^r\frac{\prod\limits_{t_i=k_i}^{n_i-1}(a_i(t_i)+\sum_{j=1}^r b_{ij}(t_i)c_j(k_j))}{\prod\limits_{t_i=k_i+1}^{n_i}(c_i(t_i)-c_i(k_i))}\mathbf{z}^{\mathbf{n}}.$$

Moreover, we define linear operators $\mathcal{B}_{ij},\mathcal{A}_i,\mathcal{C}_i$ by $\mathcal{B}_{ij}\mathbf{z}^{\mathbf{n}}=b_{ij}(n_i)\mathbf{z}^{\mathbf{n}}$, $\mathcal{A}_i\mathbf{z}^{\mathbf{n}}=a_i(n_i)\mathbf{z}^{\mathbf{n}}$, and $\mathcal{C}_i\mathbf{z}^{\mathbf{n}}=c_i(n_i)\mathbf{z}^{\mathbf{n}}$, for all $i,j=1,\dots,r$. Then we may write (3.3) in the form

$$(\mathcal{C}_i-c_i(k_i))f_{\mathbf{k}}(\mathbf{z})=(z_i\mathcal{A}_i+z_i\textstyle\sum_{j=1}^r\mathcal{B}_{ij}c_j(k_j))f_{\mathbf{k}}(\mathbf{z}),\quad i=1,\dots,r,\tag{3.4}$$

valid for all $\mathbf{k}\in\mathbb{Z}^r$. We rewrite our system of equations in a way such that Corollary 2.2 is applicable:

$$(c_i(k_i)+z_i\textstyle\sum_{j=1}^r c_j(k_j)\mathcal{B}_{ij})f_{\mathbf{k}}(\mathbf{z})=(\mathcal{C}_i-z_i\mathcal{A}_i)f_{\mathbf{k}}(\mathbf{z}),\quad i=1,\dots,r.\tag{3.5}$$

Now (3.5) is a system of type (2.4) with $V_{ij}=\delta_{ij}+z_i\mathcal{B}_{ij}$, $W_i=\mathcal{C}_i-z_i\mathcal{A}_i$, and $c_j(\mathbf{k})=c_j(k_j)$. The conditions (2.1), (2.2), and (2.3) are satisfied. Hence we may apply Corollary 2.2. The dual system (2.6) for the auxiliary formal Laurent series $h_{\mathbf{k}}(\mathbf{z})$ in this case reads

$$(c_i(k_i)+\textstyle\sum_{j=1}^r c_j(k_j)\mathcal{B}_{ij}^*z_i)h_{\mathbf{k}}(\mathbf{z})=(\mathcal{C}_i^*-\mathcal{A}_i^*z_i)h_{\mathbf{k}}(\mathbf{z}),\quad i=1,\dots,r.$$

Equivalently, we have

$$(\mathcal{C}_i^*-c_i(k_i))h_{\mathbf{k}}(\mathbf{z})=(\mathcal{A}_i^*z_i+\textstyle\sum_{j=1}^r\mathcal{B}_{ij}^*c_j(k_j)z_i)h_{\mathbf{k}}(\mathbf{z}),\quad i=1,\dots,r,\tag{3.6}$$

for all $\mathbf{k}\in\mathbb{Z}^r$. As is easily seen, we have $\mathcal{B}_{ij}^*\mathbf{z}^{-\mathbf{l}}=b_{ij}(l_i)\mathbf{z}^{-\mathbf{l}}$, $\mathcal{A}_i^*\mathbf{z}^{-\mathbf{l}}=a_i(l_i)\mathbf{z}^{-\mathbf{l}}$, and $\mathcal{C}_i^*\mathbf{z}^{-\mathbf{l}}=c_i(l_i)\mathbf{z}^{-\mathbf{l}}$, for $i,j=1,\dots,r$. Thus, with $h_{\mathbf{k}}(\mathbf{z})=\sum_{\mathbf{l}\le\mathbf{k}}h_{\mathbf{kl}}\mathbf{z}^{-\mathbf{l}}$, by comparing coefficients of $\mathbf{z}^{-\mathbf{l}}$ in (3.6) we obtain

$$(c_i(l_i)-c_i(k_i))h_{\mathbf{kl}}=(a_i(l_i)+\textstyle\sum_{j=1}^r b_{ij}(l_i)c_j(k_j))h_{\mathbf{k},\mathbf{l}+\mathbf{e}_i},\quad i=1,\dots,r.$$

If we set $h_{\mathbf{kk}}=1$, we get

$$h_{\mathbf{kl}}=\prod_{i=1}^r\frac{\prod\limits_{t_i=l_i}^{k_i-1}(a_i(t_i)+\sum_{j=1}^r b_{ij}(t_i)c_j(k_j))}{\prod\limits_{t_i=l_i}^{k_i-1}(c_i(t_i)-c_i(k_i))}.$$

Taking into account (2.5), we have to compute the action of

$$\det_{1\le i,j\le r}(V_{ij}^*) = \det_{1\le i,j\le r}\left(\delta_{ij} + \mathcal{B}_{ij}^* z_i\right) \tag{3.7}$$

when applied to

$$h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} \prod_{i=1}^{r} \frac{\prod\limits_{t_i=l_i}^{k_i-1}(a_i(t_i) + \sum_{j=1}^r b_{ij}(t_i)c_j(k_j))}{\prod\limits_{t_i=l_i}^{k_i-1}(c_i(t_i) - c_i(k_i))} \mathbf{z}^{-\mathbf{l}}.$$

Since

$$z_i h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} \frac{(c_i(l_i) - c_i(k_i))}{(a_i(l_i) + \sum_{j=1}^r b_{ij}(l_i)c_j(k_j))} h_{\mathbf{kl}}\mathbf{z}^{-\mathbf{l}},$$

we conclude that

$$\det_{1\le i,j\le r}(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} \det_{1\le i,j\le r}\left(\delta_{ij} + \frac{b_{ij}(l_i)(c_i(l_i) - c_i(k_i))}{(a_i(l_i) + \sum_{s=1}^r b_{is}(l_i)c_s(k_s))}\right) h_{\mathbf{kl}}\mathbf{z}^{-\mathbf{l}}. \tag{3.8}$$

Note that since $f_{\mathbf{kk}} = 1$, the pairing $\langle f_{\mathbf{k}}(\mathbf{z}), \det(V_{ij}^*)h_{\mathbf{k}}(\mathbf{z})\rangle$ is simply the coefficient of $\mathbf{z}^{-\mathbf{k}}$ in (3.8) which is easily seen to be one. By taking the denominators out of the rows of the determinant, equation (2.5) is turned into

$$\begin{aligned} g_{\mathbf{k}}(\mathbf{z}) &= \det_{1\le i,j\le r}(V_{ij}^*) h_{\mathbf{k}}(\mathbf{z}) \\ &= \sum_{\mathbf{l}\le\mathbf{k}} \left(\frac{\det_{1\le i,j\le r}\left((a_i(l_i) + \sum_{s=1}^r b_{is}(l_i)c_s(k_s))\delta_{ij} + b_{ij}(l_i)(c_i(l_i) - c_i(k_i))\right)}{\prod\limits_{i=1}^{r}(a_i(k_i) + \sum_{j=1}^r b_{ij}(k_i)c_j(k_j))} \right. \\ &\qquad\qquad \left. \times \prod_{i=1}^{r} \frac{\prod\limits_{t_i=l_i+1}^{k_i}(a_i(t_i) + \sum_{j=1}^r b_{ij}(t_i)c_j(k_j))}{\prod\limits_{t_i=l_i}^{k_i-1}(c_i(t_i) - c_i(k_i))} \mathbf{z}^{-\mathbf{l}} \right), \end{aligned} \tag{3.9}$$

where $g_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{l}\le\mathbf{k}} g_{\mathbf{kl}}\mathbf{z}^{-\mathbf{l}}$. So, by extracting the coefficient of $\mathbf{z}^{-\mathbf{l}}$ in (3.9) we obtain exactly (3.2). $\square$

By a slightly modified application of the operator method of Section 2 one can show that the determinant in (3.2) can be "transferred" from $g_{\mathbf{kl}}$ to $f_{\mathbf{nk}}$. The corresponding Theorem reads as follows.

Theorem 3.3. *Assume the conditions of Theorem* 3.1. *Then* $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ *and* $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ *are inverses of each other, where*

$$f_{\mathbf{nk}} = \frac{\det\limits_{1\le i,j\le r}\Big(\big(a_i(n_i)+\sum_{s=1}^r b_{is}(n_i)c_s(k_s)\big)\delta_{ij}+b_{ij}(n_i)\big(c_i(n_i)-c_i(k_i)\big)\Big)}{\prod\limits_{i=1}^r\big(a_i(k_i)+\sum_{j=1}^r b_{ij}(k_i)c_j(k_j)\big)} \times\prod_{i=1}^r\frac{\prod\limits_{t_i=k_i}^{n_i-1}\big(a_i(t_i)+\sum_{j=1}^r b_{ij}(t_i)c_j(k_j)\big)}{\prod\limits_{t_i=k_i+1}^{n_i}\big(c_i(t_i)-c_i(k_i)\big)} \tag{3.10}$$

and

$$g_{\mathbf{kl}} = \prod_{i=1}^r\frac{\prod\limits_{t_i=l_i+1}^{k_i}\big(a_i(t_i)+\sum_{j=1}^r b_{ij}(t_i)c_j(k_j)\big)}{\prod\limits_{t_i=l_i}^{k_i-1}\big(c_i(t_i)-c_i(k_i)\big)}. \tag{3.11}$$

Remark 3.4. The special case $c_i(t)=t$, $i=1,\dots,r$, is equivalent to Chu's [12, Eq. (2.9)/(2.10)] companion matrix inversion result.

4. Preliminaries on inverse relations and basic hypergeometric notation

Here we introduce the basic concept of "inverse relations" and introduce some standard q-series notation.

There is a standard technique for deriving new summation formulas from known ones by using inverse matrices (cf. [1], [5], [13], [19], [20], [21, Sec. 3.8], [22], [23], [42], [44], [48], [49], [51], [52], [53], [54]). If $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ are lower triangular matrices being inverses of each other, then of course the following is true:

$$\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} f_{\mathbf{nk}}a_{\mathbf{k}} = b_{\mathbf{n}} \tag{4.1}$$

if and only if

$$\sum_{\mathbf{0}\le\mathbf{l}\le\mathbf{k}} g_{\mathbf{kl}}b_{\mathbf{l}} = a_{\mathbf{k}}. \tag{4.2}$$

If either (4.1) or (4.2) is known, then the other produces another summation formula. The less used dual version, the so-called "rotated inversion", can be used to derive nonterminating summations. It reads

$$\sum_{\mathbf{k}\le\mathbf{n}\le\boldsymbol{\infty}} f_{\mathbf{nk}}a_{\mathbf{n}} = b_{\mathbf{k}} \tag{4.3}$$

if and only if

$$\sum_{\mathbf{l}\le\mathbf{k}\le\boldsymbol{\infty}} g_{\mathbf{kl}}b_{\mathbf{k}} = a_{\mathbf{l}}, \tag{4.4}$$

subject to suitable convergence conditions. Again, if one of (4.3) or (4.4) is known, the other produces a possibly new identity.

In the subsequent sections we use special cases of our Theorems 3.1 and 3.3 to derive a couple of higher dimensional summations for q-series.

Before we start to develop the applications of our Theorems, we need to recall the standard basic hypergeometric notation (cf. [21]). Let q be a complex number such that $|q| < 1$. Define

$$(a;q)_\infty := \prod_{j\ge 0}(1-aq^j), \tag{4.5}$$

and,

$$(a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty} \tag{4.6}$$

$$= \prod_{j=0}^{k-1}(1-aq^j), \tag{4.7}$$

where the equality (4.7) holds when k is a non-negative integer. We also make use of the standard notation for basic hypergeometric series,

$${}_s\phi_t\left[\begin{matrix}a_1,\dots,a_s\\ b_1,\dots,b_t\end{matrix};q,z\right] := \sum_{k=0}^{\infty}\frac{(a_1;q)_k\,(a_2;q)_k\,\cdots\,(a_s;q)_k}{(q;q)_k\,(b_1;q)_k\,\cdots\,(b_t;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+t-s} z^k. \tag{4.8}$$

Finally, for multidimensional series, we also employ the notation $|\mathbf{k}|$ for $(k_1+\dots+k_r)$ where $\mathbf{k}=(k_1,\dots,k_r)$. Concerning the nonterminating multiple series given in this paper, we have stated their regions of convergence explicitly. The convergence of these series can be checked by application of the multiple power series ratio test [35], [38]. In cases where the summand of the multiple series contains a determinant we would first have to expand the determinant appearing in the summand according to its definition as a sum over the symmetric group, then interchange summations and apply the multiple power series ratio test to each of its resulting $r!$ multiple sums. In our proofs, however, we have not carried out such calculations explicitly. For explicit examples of how to use the multiple power series ratio test, see [48, Sec. 5].

5. Some identities for double series

In our first application we use the two-dimensional special case (i.e., the $r=2$ case) of our matrix inversion (3.1) to derive a few basic hypergeometric double summation theorems. These developments are very much in the spirit of [40], although the particular case of (3.1) that we consider here is a different one than in [40]. Namely, the particular choice of the parameters in (3.1) that we make is $a_1(t)=a_2(t)=0$, $c_1(t)=c_2(t)=q^t$, and

$$\Big(b_{ij}(t_i)\Big)_{1\le i,j\le 2} = \begin{pmatrix} Cq^{t_1} & -1\\ -1 & Dq^{t_2}\end{pmatrix}.$$

Thus, after little simplification, we obtain that *the matrices* $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^2}$ *and* $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^2}$ *are inverses of each other, where*

$$f_{\mathbf{nk}} = q^{(k_1-k_2)(n_2-k_2-n_1+k_1)}\frac{(Cq^{2k_1-k_2};q)_{n_1-k_1}\,(Dq^{2k_2-k_1};q)_{n_2-k_2}}{(q;q)_{n_1-k_1}\,(q;q)_{n_2-k_2}} \tag{5.1}$$

and

$$g_{\mathbf{kl}}=q^{(k_1-k_2)(k_2-l_2-k_1+l_1)}\frac{\big((Cq^{2l_1}-q^{k_2})(Dq^{2l_2}-q^{k_1})-(q^{l_1}-q^{k_1})(q^{l_2}-q^{k_2})\big)}{(Cq^{2k_1}-q^{k_2})(Dq^{2k_2}-q^{k_1})}\\ \times\frac{(Cq^{2k_1-k_2};q^{-1})_{k_1-l_1}\,(Dq^{2k_2-k_1};q^{-1})_{k_2-l_2}}{(q^{-1};q^{-1})_{k_1-l_1}\,(q^{-1};q^{-1})_{k_2-l_2}}. \tag{5.2}$$

Now, in (4.1) we take

$$a_{\mathbf{k}}=C^{k_2}D^{k_1}q^{-k_1^2+3k_1k_2-k_2^2}\frac{(C;q)_{2k_1-k_2}\,(D;q)_{2k_2-k_1}}{(q;q)_{k_1}\,(q;q)_{k_2}}. \tag{5.3}$$

By two-fold use of q-Chu–Vandermonde summation (cf. [21, Eq. (1.5.3); Appendix (II.6)]),

$${}_2\phi_1\left[\begin{matrix}a,q^{-n}\\ c\end{matrix};q,q\right]=\frac{a^n(c/a;q)_n}{(c;q)_n} \tag{5.4}$$

we have by (4.1) and a bit of manipulation

$$\begin{aligned}b_{\mathbf{n}}&=\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}q^{k_1k_2-k_1n_1+k_2n_1+k_1n_2-k_2n_2}C^{k_2}D^{k_1}\frac{(C;q)_{n_1+k_1-k_2}\,(D;q)_{n_2+k_2-k_1}}{(q;q)_{k_1}\,(q;q)_{k_2}\,(q;q)_{n_1-k_1}\,(q;q)_{n_2-k_2}}\\
&=\sum_{k_2=0}^{n_2}q^{k_2n_1-k_2n_2}C^{k_2}\frac{(C;q)_{n_1-k_2}\,(D;q)_{k_2+n_2}}{(q;q)_{k_2}\,(q;q)_{n_1}\,(q;q)_{n_2-k_2}}\,{}_2\phi_1\left[\begin{matrix}Cq^{n_1-k_2},q^{-n_1}\\ q^{1-k_2-n_2}/D\end{matrix};q,q\right]\\
&=\sum_{k_2=0}^{n_2}q^{{n_1}^2-k_2n_2}C^{k_2+n_1}\frac{(C;q)_{n_1-k_2}\,(D;q)_{k_2+n_2}\,(q^{1-n_1-n_2}/CD;q)_{n_1}}{(q;q)_{k_2}\,(q;q)_{n_1}\,(q;q)_{n_2-k_2}\,(q^{1-k_2-n_2}/D;q)_{n_1}}\\
&=q^{{n_1}^2}C^{n_1}\frac{(C;q)_{n_1}\,(D;q)_{n_2}\,(q^{1-n_1-n_2}/CD;q)_{n_1}}{(q;q)_{n_1}\,(q;q)_{n_2}\,(q^{1-n_2}/D;q)_{n_1}}\,{}_2\phi_1\left[\begin{matrix}Dq^{-n_1+n_2},q^{-n_2}\\ q^{1-n_1}/C\end{matrix};q,q\right]\\
&=q^{{n_1}^2-n_1n_2+{n_2}^2}C^{n_1}D^{n_2}\frac{(C;q)_{n_1}\,(D;q)_{n_2}\,(q^{1-n_1-n_2}/CD;q)_{n_1+n_2}}{(q;q)_{n_1}\,(q;q)_{n_2}\,(q^{1-n_1}/C;q)_{n_2}\,(q^{1-n_2}/D;q)_{n_1}}.\end{aligned} \tag{5.5}$$

Substituting this into the inverse relation (4.2) gives, after some simplification,

$$\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}q^{(k_1-k_2)(l_1-l_2)}\frac{\big((Cq^{2l_1}-q^{k_2})(Dq^{2l_2}-q^{k_1})-(q^{l_1}-q^{k_1})(q^{l_2}-q^{k_2})\big)}{(1-C)(1-D)}\\ \cdot\frac{(C;q)_{l_1-l_2}\,(D;q)_{l_2-l_1}\,(CD;q)_{l_1+l_2}\,(q^{-k_1};q)_{l_1}\,(q^{-k_2};q)_{l_2}}{(Cq^{1+k_1-k_2};q)_{l_1}\,(Dq^{1+k_2-k_1};q)_{l_2}\,(q;q)_{l_1}\,(q;q)_{l_2}}\\ =(-1)^{k_1+k_2}C^{k_2}D^{k_1}q^{k_1k_2-\binom{k_1}{2}-\binom{k_2}{2}}(Cq;q)_{k_1-k_2}\,(Dq;q)_{k_2-k_1}. \tag{5.6}$$

Setting $k_2=0$ in this identity, we obtain

$${}_5\phi_5\left[\begin{matrix}C,\sqrt{C}q,-\sqrt{C}q,CD,q^{-k_1}\\ \sqrt{C},-\sqrt{C},q/D,Cq^{1+k_1},0\end{matrix};q,\frac{q^{1+k_1}}{D}\right]=\frac{(qC;q)_{k_1}}{(q/D;q)_{k_1}}. \tag{5.7}$$

which is a terminating limiting case of the very-well-poised ${}_6\phi_5$-summation (cf. [21, Eq. (2.7.1); Appendix (II.20)]). Hence, our double sum identity (5.6) is a two-dimensional extension of the ${}_5\phi_5$-summation (5.7).

For our second application of the matrix inverse (5.1)/(5.2), in (4.1) we choose

$$a_{\mathbf{k}} = \delta_{k_1 k_2} A^{k_1} q^{k_1(k_1-1)} \frac{(C;q)_{k_1}\,(D;q)_{k_1}}{(q;q)_{k_1}\,(A;q)_{k_1}}, \tag{5.8}$$

where $\delta_{i,j}$ denotes the Kronecker delta, $\delta_{i,j}=1$ if $i=j$ and $\delta_{i,j}=0$ otherwise. Again, by using q-Chu–Vandermonde summation (5.4), we obtain from (4.1),

$$b_{\mathbf{n}} = \frac{(C;q)_{n_1}\,(D;q)_{n_2}\,(A;q)_{n_1+n_2}}{(q;q)_{n_1}\,(q;q)_{n_2}\,(A;q)_{n_1}\,(A;q)_{n_2}}. \tag{5.9}$$

With these values of $a_{\mathbf{k}}$ and $b_{\mathbf{n}}$, the inverse relation (4.2) then becomes

$$\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2} q^{(k_1-k_2)(l_1-l_2)} \frac{\big((Cq^{2l_1}-q^{k_2})(Dq^{2l_2}-q^{k_1})-(q^{l_1}-q^{k_1})(q^{l_2}-q^{k_2})\big)}{(1-C)(1-D)}$$
$$\cdot \frac{(A;q)_{l_1+l_2}\,(C;q)_{l_1}\,(D;q)_{l_2}\,(q^{-k_1};q)_{l_1}\,(q^{-k_2};q)_{l_2}}{(A;q)_{l_1}\,(A;q)_{l_2}\,(Cq^{1+k_1-k_2};q)_{l_1}\,(Dq^{1+k_2-k_1};q)_{l_2}\,(q;q)_{l_1}\,(q;q)_{l_2}}$$
$$= \delta_{k_1 k_2} A^{k_1} \frac{(q;q)_{k_1}}{(A)_{k_1}}. \tag{5.10}$$

This is a two-dimensional extension of the terminating very-well-poised ${}_4\phi_3$-summation (cf. [21, Eq. (2.3.4)]), which in Chapter 2 of [21] is used as one of the corner stones of building up the summation theory for very-well-poised basic hypergeometric series.

6. A "twisted" multidimensional extension of a very-well-poised ${}_{10}\phi_9$-summation

In this section we bring an application of the companion inversion (3.10)/(3.11). We start by making the replacements $c_i(t) \to 1/c_i(t) + A_i c_i(t)$, $a_i(t) \to (A_{i+1} + a_i(t)^2)/c_i(t+1)$, and

$$\big(b_{ij}(t_i)\big)_{1\le i,j\le r} = \begin{pmatrix} 0 & -\frac{a_1(t_1)}{c_1(t_1+1)} & 0 & \dots & \dots & 0 \\ & \ddots & -\frac{a_2(t_2)}{c_2(t_2+1)} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & 0 \\ 0 & & & & \ddots & -\frac{a_{r-1}(t_{r-1})}{c_{r-1}(t_{r-1}+1)} \\ -\frac{a_r(t_r)}{c_r(t_r+1)} & 0 & & \dots & & 0 \end{pmatrix} \tag{6.1}$$

in (3.3). Upon little simplification we obtain that *the matrices* $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ *and* $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$ *are inverses of each other, where*

$$f_{\mathbf{nk}} = \frac{\left(\prod_{i=1}^r c_i(n_i)A_{i+1}(1-a_i(n_i)c_{i+1}(k_{i+1}))(1-a_i(n_i)/A_{i+1}c_{i+1}(k_{i+1})) - \prod_{i=1}^r a_i(n_i)(1-A_ic_i(n_i)c_i(k_i))(1-c_i(n_i)/c_i(k_i))\right)}{\prod_{i=1}^r (1-a_i(k_i)c_{i+1}(k_{i+1}))(A_{i+1}c_{i+1}(k_{i+1})-a_i(k_i))}$$
$$\times \prod_{i=1}^r \frac{\prod_{t_i=k_i}^{n_i-1}(1-a_i(t_i)c_{i+1}(k_{i+1}))(1-a_i(t_i)/A_{i+1}c_{i+1}(k_{i+1}))}{\prod_{t_i=k_i+1}^{n_i}(1-A_ic_i(t_i)c_i(k_i))(1-c_i(t_i)/c_i(k_i))} \tag{6.2}$$

and

$$g_{\mathbf{kl}} = \prod_{i=1}^r \frac{\prod_{t_i=l_i+1}^{k_i}(1-a_i(t_i)c_{i+1}(k_{i+1}))(1-a_i(t_i)/A_{i+1}c_{i+1}(k_{i+1}))}{\prod_{t_i=l_i}^{k_i-1}(1-A_ic_i(t_i)c_i(k_i))(1-c_i(t_i)/c_i(k_i))}. \tag{6.3}$$

Here and in the following we make the convention that indices have to be taken modulo r, i.e., by k_{r+1} we mean k_1, etc. The above matrix inversion is a "twisted" extension to several dimensions of the "Bressoud-type" writing [42, (1.5)] of the one-dimensional matrix inversion (1.1)/(1.2), to which it reduces for $r=1$.

Now, in (6.2)/(6.3) we specialize $c_i(t)=q^t$ and $a_i(t)=a_iq^t$. Thus, we obtain the inverse pair of matrices $(f_{\mathbf{nk}})_{\mathbf{n},\mathbf{k}\in\mathbb{Z}^r}$ and $(g_{\mathbf{kl}})_{\mathbf{k},\mathbf{l}\in\mathbb{Z}^r}$, where

$$f_{\mathbf{nk}} = q^{|\mathbf{n}|}\frac{\left(\prod_{i=1}^r A_i(1-a_iq^{n_i+k_{i+1}})(1-a_iq^{n_i-k_{i+1}}/A_{i+1}) - \prod_{i=1}^r a_i(1-A_iq^{n_i+k_i})(1-q^{n_i-k_i})\right)}{\prod_{i=1}^r (1-a_iq^{k_i+k_{i+1}})(A_{i+1}q^{k_{i+1}}-a_iq^{k_i})}$$
$$\times \prod_{i=1}^r \frac{(a_iq^{k_i+k_{i+1}};q)_{n_i-k_i}\,(a_iq^{k_i-k_{i+1}}/A_{i+1};q)_{n_i-k_i}}{(A_iq^{2k_i+1};q)_{n_i-k_i}\,(q;q)_{n_i-k_i}} \tag{6.4}$$

and

$$g_{\mathbf{kl}} = \prod_{i=1}^r \frac{(a_iq^{k_i+k_{i+1}};q^{-1})_{k_i-l_i}\,(a_iq^{k_i-k_{i+1}}/A_{i+1};q^{-1})_{k_i-l_i}}{(A_iq^{2k_i-1};q^{-1})_{k_i-l_i}\,(q^{-1};q^{-1})_{k_i-l_i}}. \tag{6.5}$$

This matrix inversion is a "twisted" extension to several dimensions of Bressoud's matrix inversion [7], to which it reduces for $r=1$.

For our application of (6.4)/(6.5), in (4.2) we choose

$$b_{\mathbf{l}} = q^{|\mathbf{l}|}\prod_{i=1}^r \frac{(qa_i^2/A_iA_{i+1};q)_{l_i}}{(q;q)_{l_i}}. \tag{6.6}$$

Then the left-hand side of (4.2) can be written as

$$\prod_{i=1}^{r}\left(\frac{(q^{k_{i+1}+1}a_i;q)_{k_i}\,(a_iq^{1-k_{i+1}}/A_{i+1};q)_{k_i}}{(q^{-k_i};q)_{k_i}\,(A_iq^{k_i};q)_{k_i}}\,{}_3\phi_2\left[\begin{matrix}q^{k_i}A_i,\,qa_i^2/A_iA_{i+1},\,q^{-k_i}\\ q^{1-k_{i+1}}a_i/A_{i+1},\,q^{k_{i+1}+1}a_i\end{matrix};q,q\right]\right).$$

The ${}_3\phi_2$-series appearing in this expression can be evaluated by means of the q-Pfaff–Saalschütz summation (cf. [21, (1.7.2); Appendix (II.12)]),

$${}_3\phi_2\left[\begin{matrix}a,b,q^{-n}\\ c,abq^{1-n}/c\end{matrix};q,q\right]=\frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/ab;q)_n},$$

where n is a nonnegative integer. Thus we obtain

$$a_{\mathbf{k}}=\prod_{i=1}^{r}q^{k_i(k_i+k_{i+1}+1)}a_i^{k_i}\frac{(A_iq^{-k_{i+1}}/a_i;q)_{k_i}\,(q^{1-k_i-k_{i+1}}/A_iA_{i+1};q)_{k_i}}{(q^{k_i}A_i;q)_{k_i}\,(q;q)_{k_i}}. \tag{6.7}$$

Substitution of (6.6) and (6.7) in (4.1), and some simplification, leads to the following summation theorem.

Theorem 6.1. *Let a_i and A_i be indeterminates, $i=1,2,\dots,r$. Then*

$$\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\frac{\left(\prod_{i=1}^{r}A_i(1-a_iq^{n_i+k_{i+1}})(1-a_iq^{n_i-k_{i+1}}/A_{i+1})-\prod_{i=1}^{r}a_i(1-A_iq^{n_i+k_i})(1-q^{n_i-k_i})\right)}{\prod_{i=1}^{r}(1-a_iq^{k_i+k_{i+1}})(A_{i+1}q^{k_{i+1}}-a_iq^{k_i})}$$
$$\cdot\prod_{i=1}^{r}q^{2k_i}\left(\frac{a_i^2q^{n_i}}{A_iA_{i+1}}\right)^{k_i-k_{i+1}}\frac{(1-q^{2k_i}A_i)}{(1-A_i)}\frac{(A_i;q)_{k_i}\,(q^{-n_i};q)_{k_i}}{(q;q)_{k_i}\,(q^{1+n_i}A_i;q)_{k_i}}$$
$$\cdot\frac{(q^{n_i}a_i;q)_{k_{i+1}}\,(qa_i/A_i;q)_{k_{i+1}}\,(A_iA_{i+1}/a_i;q)_{k_i+k_{i+1}}\,(A_i/a_i;q)_{k_i-k_{i+1}}}{(q^{1-n_i}A_{i+1}/a_i;q)_{k_{i+1}}\,(A_iA_{i+1}/a_i;q)_{k_{i+1}}\,(a_i;q)_{k_i+k_{i+1}}\,(a_i/A_{i+1};q)_{k_i-k_{i+1}}}$$
$$=\prod_{i=1}^{r}\frac{(qa_i^2/A_iA_{i+1};q)_{n_i}\,(qA_i;q)_{n_i}}{(a_i;q)_{n_i}\,(a_i/A_{i+1};q)_{n_i}}, \tag{6.8}$$

where, by convention, $A_{r+1}=A_1$ and $k_{r+1}=k_1$.

Remark 6.2. The special case $r=1$ of (6.8) can be rewritten as (when writing a for a_1, A for A_1, and n for n_1)

$${}_{10}\phi_9\left[\begin{matrix}A,\sqrt{A}q,-\sqrt{A}q,A/\sqrt{a},-A/\sqrt{a},A\sqrt{q}/\sqrt{a},-A\sqrt{q}/\sqrt{a},aq/A,aq^n,q^{-n}\\ \sqrt{A},-\sqrt{A},\sqrt{a}q,-\sqrt{a}q,\sqrt{aq},-\sqrt{aq},A^2/a,Aq^{1-n}/a,Aq^{1+n}\end{matrix};q,q\right]$$
$$=\frac{(1-a)}{(1-aq^{2n})}\frac{(qa^2/A^2;q)_n\,(qA;q)_n}{(a;q)_n\,(a/A;q)_n}. \tag{6.9}$$

This ${}_{10}\phi_9$-summation can be obtained by specializing Bailey's transformation formula [2] (see [21, (2.8.5); Appendix (III.27)]; the specializations that have to be performed there are $d=1$, $b\to a^2q/cA$, $a\to A^2/aq$, in that order).

7. q-Analogues of Carlitz' Abel-type expansion formulas

In [10], Carlitz gave multidimensional extensions of Euler's formulas

$$e^{AZ}=\sum_{k=0}^{\infty}\frac{A(A+Bk)^{k-1}}{k!}Z^ke^{-BZk}, \tag{7.1}$$

where $\left|BZe^{1-BZ}\right|<1$ [17, p. 354] (cf. [53, Sec. 4.5]), and

$$(1+Z)^A=\sum_{k=0}^{\infty}\frac{A}{A+Bk}\binom{A+Bk}{k}Z^k(1+Z)^{-Bk}, \tag{7.2}$$

where $\left|\frac{(B-1)Z}{(1+Z)^B}\right|<1$ [17, p. 350] (cf. [53, Sec. 4.5]). The purpose of this section is to present q-analogues of (7.1) and (7.2) and to derive q-analogues of Carlitz' multidimensional extensions thereof.

First we derive a simple multidimensional extension of the expansion formula

$$1=\sum_{k=0}^{\infty}\frac{(a+b)(a+bq^k)^{k-1}}{(q;q)_k}(z(a+bq^k);q)_\infty\, z^k, \tag{7.3}$$

valid for $|az|<1$, which is a q-analogue of Euler's formula (7.1).

To see that (7.3) is a q-analogue of (7.1), do the replacements $a\to 1-q^A+\frac{1-q^B}{1-q}$, $b\to -\frac{1-q^B}{1-q}$, $z\to Z$ and then let $q\to 1$. In this case, $\lim_{q\to1}\frac{a+bq^k}{1-q}=A+Bk$. Also, recall that $\lim_{q\to1}((1-q)Z;q)_\infty=e^{-Z}$.

The second formula that we extend to several dimensions is

$$(z;q)_\infty=\sum_{k=0}^{\infty}\frac{1-(a+b)}{1-(aq^{-k}+b)}\frac{(aq^{-k}+b;q)_k}{(q;q)_k}(-1)^kq^{\binom k2}(z(a+bq^k);q)_\infty\, z^k, \tag{7.4}$$

valid for $|az|<1$, which in turn is a q-analogue of Euler's formula (7.2).

To see that (7.4) is a q-analogue of (7.2), do the replacements $a\to q^A-\frac{1-q^B}{1-q}$, $b\to\frac{1-q^B}{1-q}$, $z\to -Z$ and then let $q\to1$. In this case, $\lim_{q\to1}\frac{1-(aq^{-k}+b)q^j}{1-q}=A+Bk+j-k$. Furthermore, we use $\lim_{q\to1}\frac{(z(a+bq^k);q)_\infty}{(z;q)_\infty}=(1+Z)^{-A-Bk}$. Similar limiting processes apply for others of the identities given in the following sections, especially for our multidimensional formulas.

Now we state our multiple extension of (7.3):

Theorem 7.1. *Let a_i,b_{ij},z_i, $i,j=1,\dots,r$, be indeterminate. Then there holds*

$$1=\sum_{k_1,\dots,k_r=0}^{\infty}\Bigg(\det_{1\le i,j\le r}\left(\left(a_i+\textstyle\sum_{s=1}^r b_{is}q_s^{k_s}\right)\delta_{ij}+b_{ij}(1-q_i^{k_i})\right)\\ \times\prod_{i=1}^r\frac{\left(a_i+\sum_{j=1}^r b_{ij}q_j^{k_j}\right)^{k_i-1}}{(q_i;q_i)_{k_i}}\left(z_i(a_i+\textstyle\sum_{j=1}^r b_{ij}q_j^{k_j});q_i\right)_\infty z_i^{k_i}\Bigg), \tag{7.5}$$

provided $|a_iz_i|<1$ for $i=1,\dots,r$.

Remark 7.2. The expansion (7.5) is a q-analogue of Carlitz' formula [10, Eq. (3.5)] which he derived via MacMahon's Master Theorem. To obtain his result we would have to do the replacements $a_i \to 1 - q_i^{A_i} + \sum_{j=1}^r \frac{1-q_i^{B_{ij}}}{1-q_i}$, $b_{ij} \to -\frac{1-q_i^{B_{ij}}}{1-q_i}$, and then let $q_i \to 1$ for $i = 1, \dots, r$ (compare with our observation concerning equation (7.3)).

Proof of Theorem 7.1. Setting $c_i(t_i) \to q_i^{t_i}$, $a_i(t_i) \to a_i$, $b_{ij}(t_i) \to b_{ij}$, $i, j = 1, \dots, r$ in Theorem 3.1 we see that the following pair of matrices are inverses of each other:

$$f_{\mathbf{nk}} = (-1)^{|\mathbf{n}|-|\mathbf{k}|} \prod_{i=1}^r \frac{\left(a_i + \sum_{j=1}^r b_{ij} q_j^{k_j}\right)^{n_i - k_i}}{(q_i; q_i)_{n_i - k_i}} q_i^{\binom{n_i - k_i}{2}}$$

and

$$g_{\mathbf{kl}} = \frac{\det_{1 \le i,j \le r} \left(\left(a_i + \sum_{s=1}^r b_{is} q_s^{k_s}\right) \delta_{ij} + b_{ij}(q_i^{l_i} - q_i^{k_i}) \right)}{\prod_{i=1}^r \left(a_i + \sum_{j=1}^r b_{ij} q_j^{k_j}\right)} \prod_{i=1}^r \frac{\left(a_i + \sum_{j=1}^r b_{ij} q_j^{k_j}\right)^{k_i - l_i}}{(q_i; q_i)_{k_i - l_i}}.$$

Now (4.3) holds for

$$a_{\mathbf{n}} = \prod_{i=1}^r z_i^{n_i} \qquad \text{and} \qquad b_{\mathbf{k}} = \prod_{i=1}^r z_i^{k_i} \left(z_i(a_i + \textstyle\sum_{j=1}^r b_{ij} q_j^{k_j}); q_i\right)_\infty,$$

by r-fold application of the ${}_0\phi_0$-summation (which is a q-analogue of the exponential function [21, Eq. (1.3.16); Appendix (II.2)]). This implies the inverse relation (4.4), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After shifting the indices $k_i \to k_i + l_i$, $i = 1, \dots, r$, and substituting the variables $b_{ij} \to b_{ij} q_j^{-l_j}$, $i, j = 1, \dots, r$, we get rid of the l_i and eventually obtain (7.5). □

Next we give a multidimensional extension of (7.4):

Theorem 7.3. *Let a_i, b_{ij}, z_i, $i, j = 1, \dots, r$, be indeterminate. Then there holds*

$$\prod_{i=1}^r (z_i; q_i)_\infty = \sum_{k_1, \dots, k_r = 0}^\infty \Bigg(\frac{\det_{1 \le i,j \le r} \left(\left(a_i - 1 + \sum_{s=1}^r b_{is} q_s^{k_s}\right) \delta_{ij} + b_{ij}(1 - q_i^{k_i}) \right)}{\prod_{i=1}^r \left(a_i - q_i^{k_i} + \sum_{j=1}^r b_{ij} q_j^{k_j}\right)}$$
$$\times \prod_{i=1}^r \frac{\left((a_i + \sum_{j=1}^r b_{ij} q_j^{k_j}) q_i^{-k_i}; q_i\right)_{k_i}}{(q_i; q_i)_{k_i}} q_i^{\binom{k_i+1}{2}} \cdot (-1)^{|\mathbf{k}|}$$
$$\times \prod_{i=1}^r \left(z_i(a_i + \textstyle\sum_{j=1}^r b_{ij} q_j^{k_j}); q_i\right)_\infty z_i^{k_i} \Bigg), \tag{7.6}$$

provided $|a_i z_i| < 1$ for $i = 1, \dots, r$.

Remark 7.4. The expansion (7.6) is a q-analogue of Carlitz' formula [10, Eq. (6.5)] which he also derived via MacMahon's Master Theorem. To obtain his result we would have to do the replacements $a_i \to q_i^{A_i} - \sum_{j=1}^r \frac{1-q_i^{B_{ij}}}{1-q_i}$, $b_{ij} \to \frac{1-q_i^{B_{ij}}}{1-q_i}$, $z_i \to -Z_i$, and then let $q_i \to 1$ for $i = 1, \dots, r$ (compare with our observation concerning equation (7.4)).

Proof of Theorem 7.3. Setting $c_i(t_i) \to q_i^{t_i}$, $a_i(t_i) \to a_i - q_i^{t_i}$, $b_{ij}(t_i) \to b_{ij}$, $i,j = 1,\dots,r$ in Theorem 3.1 we see that the following pair of matrices are inverses of each other:

$$f_{\mathbf{nk}} = (-1)^{|\mathbf{n}|-|\mathbf{k}|} \prod_{i=1}^{r} \frac{\left(q_i^{k_i}/(a_i+\sum_{j=1}^r b_{ij}q_j^{k_j});q_i\right)_{n_i-k_i}}{(q_i;q_i)_{n_i-k_i}} \left(a_i+\textstyle\sum_{j=1}^r b_{ij}q_j^{k_j}\right)^{n_i-k_i} q_i^{\binom{n_i-k_i}{2}}$$

and

$$g_{\mathbf{kl}} = \frac{\det\limits_{1\le i,j\le r}\left(\left(a_i - q_i^{l_i} + \sum_{s=1}^r b_{is}q_s^{k_s}\right)\delta_{ij} + b_{ij}(q_i^{l_i}-q_i^{k_i})\right)}{\prod\limits_{i=1}^{r}\left(a_i - q_i^{k_i}+\sum_{j=1}^r b_{ij}q_j^{k_j}\right)}$$
$$\times (-1)^{|\mathbf{k}|-|\mathbf{l}|} \prod_{i=1}^{r} \frac{\left((a_i+\sum_{j=1}^r b_{ij}q_j^{k_j})q_i^{-k_i};q_i\right)_{k_i-l_i}}{(q_i;q_i)_{k_i-l_i}} q_i^{\binom{k_i+1}{2}-\binom{l_i+1}{2}}.$$

Now (4.3) holds for

$$a_{\mathbf{n}} = \prod_{i=1}^{r} \frac{z_i^{n_i}}{(z_i;q_i)_{n_i}} \qquad \text{and} \qquad b_{\mathbf{k}} = \prod_{i=1}^{r} \frac{\left(z_i(a_i+\sum_{j=1}^r b_{ij}q_j^{k_j});q_i\right)_\infty}{(z_i;q_i)_\infty} z_i^{k_i},$$

by r-fold application of the ${}_1\phi_1$-summation (see [21, Appendix (II.5)]). This implies the inverse relation (4.4), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After shifting the indices $k_i \to k_i + l_i$, $i = 1,\dots,r$, and substituting the variables $z_i \to z_i q_i^{-l_i}$, $a_i \to a_i q_i^{l_i}$, $b_{ij} \to b_{ij} q_i^{l_i} q_j^{-l_j}$, $i,j = 1,\dots,r$, we get rid of the l_i and eventually obtain (7.6). □

We consider Theorems 7.1 and 7.3 as being of “simple type”, as we have started with products of classical (one-dimensional) summations, even with independent bases $(q_1,\dots,q_r)$, and then applied inversion. Our next theorem, a multiple extension of (7.4), is not as “simple” in this respect, as we start with a genuine r-dimensional summation which needs to have only one base q. For the derivation of our multiple extension of (7.4), in Theorem 7.6 we apply rotated inversion to the following multidimensional ${}_1\phi_1$-sum:

Lemma 7.5 (An A_r ${}_1\phi_1$-sum). *Let $x_1,\dots,x_r$, a and c be indeterminate. There holds the following summation:*

$$\sum_{k_1,\dots,k_r=0}^{\infty}\left(\prod_{1\le i<j\le r}\left(\frac{q^{-k_i}/x_i - q^{-k_j}/x_j}{1/x_i-1/x_j}\right)\prod_{i=1}^{r}\frac{(ax_i;q)_{k_i}}{(q;q)_{k_i}(cx_i;q)_{k_i}}\cdot(-1)^{|\mathbf{k}|}q^{\sum_{i=1}^r\binom{k_i}{2}}\left(\frac{c}{a}\right)^{|\mathbf{k}|}\right)$$
$$= \prod_{i=1}^{r}\frac{(cq^{1-i}/a;q)_\infty}{(cx_i;q)_\infty}. \tag{7.7}$$

Lemma 7.5 (which extends the classical ${}_1\phi_1$-summation formula [21, Appendix (II.5)]) is a special case of a more general multidimensional q-Gauß summation formula which originally came up in [41, (4.3.12)]. Such type of identities occurred there in combinatorial studies of generating functions for specific plane partitions. In the sequel the special type of series has been studied by Gustafson and the first author [33], [34], and more recently by the second author [55].

Here is our other extension of (7.4):

Theorem 7.6. *Let b_i, x_i, $i = 1, \dots, r$, a and z be indeterminate. Then there holds*

$$\prod_{i=1}^{r}(zx_i;q)_\infty = \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg(\prod_{1\le i<j\le r} \left(\frac{q^{-k_i}/x_i - q^{-k_j}/x_j}{1/x_i - 1/x_j}\right) (-1)^{|\mathbf{k}|} q^{\sum_{i=1}^r \binom{k_i}{2}} \\ \times \left(1 - \sum_{i=1}^{r} \frac{b_i(1-q^{k_i})}{(x_i - a - \sum_{j=1}^r b_j q^{k_j})}\right) \prod_{i=1}^{r} \frac{\left((a+\sum_{j=1}^r b_j q^{k_j})q^{1-k_i}/x_i;q\right)_{k_i}}{(q;q)_{k_i}} \\ \times \prod_{i=1}^{r} \left(zq^{1-i}(a+\textstyle\sum_{j=1}^r b_j q^{k_j});q\right)_\infty x_i^{k_i} \cdot z^{|\mathbf{k}|} \Bigg), \tag{7.8}$$

provided $|azq^{1-r}| < 1$.

Remark 7.7. To the authors' knowledge, the expansion (7.8) is not a q-analogue of any of the identities which have appeared in literature yet. In particular, it is of different type than Carlitz' formulas in [9], [10].

Proof of Theorem 7.6. Setting $c_i(t_i) \to q^{t_i}$, $a_i(t_i) \to a - x_i q^{t_i}$, $b_{ij}(t_i) \to b_j$, $i, j = 1, \dots, r$ in Theorem 3.1 we see that the following pair of matrices are inverses of each other:

$$f_{\mathbf{nk}} = (-1)^{|\mathbf{n}|-|\mathbf{k}|} \prod_{i=1}^{r} \frac{\left(x_i q^{k_i}/(a+\sum_{j=1}^r b_j q^{k_j});q\right)_{n_i-k_i}}{(q;q)_{n_i-k_i}} \left(a+\textstyle\sum_{j=1}^r b_j q^{k_j}\right)^{n_i-k_i} q^{\binom{n_i-k_i}{2}}$$

and

$$g_{\mathbf{kl}} = \frac{\det_{1\le i,j\le r}\left(\left(a - x_i q^{l_i} + \sum_{s=1}^r b_s q^{k_s}\right)\delta_{ij} + b_j(q^{l_i} - q^{k_i})\right)}{\prod_{i=1}^{r}\left(a - x_i q^{k_i} + \sum_{j=1}^r b_j q^{k_j}\right)} \\ \times (-1)^{|\mathbf{k}|-|\mathbf{l}|} \prod_{i=1}^{r} \frac{\left((a+\sum_{j=1}^r b_j q^{k_j})q^{-k_i}/x_i;q\right)_{k_i-l_i}}{(q;q)_{k_i-l_i}} x_i^{k_i-l_i} q^{\binom{k_i+1}{2}-\binom{l_i+1}{2}}.$$

Now (4.3) holds for

$$a_{\mathbf{n}} = \prod_{1\le i<j\le r}\left(q^{-n_i}/x_i - q^{-n_j}/x_j\right) \frac{z^{|\mathbf{n}|}}{\prod_{i=1}^r (zx_i;q)_{n_i}}$$

and

$$b_{\mathbf{k}} = \prod_{1\le i<j\le r}\left(q^{-k_i}/x_i - q^{-k_j}/x_j\right) \frac{z^{|\mathbf{k}|}}{\prod_{i=1}^r (zx_i;q)_\infty} \prod_{i=1}^{r}\left(zq^{1-i}(a+\textstyle\sum_{j=1}^r b_j q^{k_j});q\right)_\infty,$$

by Lemma 7.5. This implies the inverse relation (4.4), with the above values of $a_{\mathbf{n}}$ and $b_{\mathbf{k}}$. After shifting the indices $k_i \to k_i + l_i$, $i = 1, \dots, r$, and substituting the variables $x_i \to x_i q^{-l_i}$, $b_i \to b_i q^{-l_i}$, $i = 1, \dots, r$, we get rid of the l_i. In addition, we can simplify our determinant due to the rule

$$\det_{1\le i,j\le r}\left(A_i\delta_{ij} + B_i C_j\right) = \left(1 + \textstyle\sum_{i=1}^r \frac{B_i C_i}{A_i}\right) \prod_{i=1}^r A_i, \tag{7.9}$$

and eventually obtain (7.8). □

Of course, by specializing equation (7.8) we may also obtain an interesting formula for ordinary series. If we do the replacements $a \to q^A - \sum_{j=1}^r \frac{1-q^{B_j}}{1-q}$, $b_i \to \frac{1-q^{B_i}}{1-q}$, $x_i \to q^{A-A_i}$, $i = 1, \dots, r$, $z \to -Z$, then let $q \to 1$ and rewrite (compare with Remark 7.4), we obtain the following multidimensional generalization of (7.2):

Theorem 7.8. *Let A_i, B_i, $i = 1, \dots, r$, and Z be indeterminate. Then there holds*

$$(1+Z)^{\binom{r}{2}+\sum_{i=1}^r A_i} = \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg(\prod_{1\le i<j\le r} \left(\frac{A_i - k_i - A_j + k_j}{A_i - A_j} \right) \cdot \left(1 - \sum_{i=1}^r \frac{B_i k_i}{(A_i + \sum_{j=1}^r B_j k_j)} \right) \\ \times \prod_{i=1}^r \binom{A_i + \sum_{j=1}^r B_j k_j}{k_i} Z^{k_i} (1+Z)^{-B_i k_i} \Bigg), \quad (7.10)$$

provided $\left| \frac{(B_i-1)Z}{(1+Z)^{B_i}} \right| < 1$ for $i = 1, \dots, r$.

8. Multiple q-Abel and q-Rothe summations

We can use the expansions (7.3) and (7.4) to obtain terminating q-Abel and q-Rothe summations, respectively (for q-summations of this type, also see [5], [36], [37]). Later, we will apply the same method to derive multidimensional generalizations of these formulas.

First, we apply the q-binomial theorem in the form

$$(z(a+bq^k);q)_\infty = (z;q)_\infty \sum_{j=0}^{\infty} \frac{(a+bq^k;q)_j}{(q;q)_j} z^j$$

to the right-hand side of (7.3) and move $(z;q)_\infty$ to the left-hand side of (7.3). Next, by equating coefficients of $z^n/(q;q)_n$ in the resulting identity (again making use of the q-binomial theorem), and employing the q-binomial coefficient notation

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

for nonnegative integers $k \le n$ (cf. [21, Appendix (I.39)]), we arrive at the following terminating summation:

$$1 = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (a+b)(a+bq^k)^{k-1}(a+bq^k;q)_{n-k} \quad (8.1)$$

(see [36]). This is a q-analogue of Abel's theorem

$$(A+C)^n = \sum_{k=0}^{n} \binom{n}{k} A(A+Bk)^{k-1}(C-Bk)^{n-k}. \quad (8.2)$$

For, (8.2) can easily be obtained from (8.1) via the substitutions $a \to \frac{1-q^{A/(A+C)}}{1-q} + \frac{1-q^{B/(A+C)}}{(1-q)^2}$, $b \to -\frac{1-q^{B/(A+C)}}{(1-q)^2}$, and then letting $q \to 1$.

On the other hand, if we iterate (7.4) $r-1$ times we get

$$(z;q)_\infty = \sum_{k_1,\dots,k_r=0}^{\infty} \left(\prod_{i=1}^{r} \frac{1-(a_i+b_i)}{1-(a_iq^{-k_i}+b_i)} \frac{(a_iq^{-k_i}+b_i;q)_{k_i}}{(q;q)_{k_i}} (-1)^{k_i} q^{\binom{k_i}{2}} \right. \\ \left. \times \left(z\prod_{i=1}^r(a_i+b_iq^{k_i});q \right)_\infty z^{|\mathbf{k}|} \prod_{i=1}^{r} \left(a_i+b_iq^{k_i}\right)^{\sum_{j=i+1}^r k_j} \right). \tag{8.3}$$

Now, after the following application of the q-binomial theorem

$$\left(z\prod_{i=1}^r(a_i+b_iq^{k_i});q \right)_\infty = (z/c;q)_\infty \sum_{j=0}^{\infty} \frac{\left(c\prod_{i=1}^r(a_i+b_iq^{k_i});q \right)_j}{(q;q)_j} \left(\frac{z}{c}\right)^j$$

to the right-hand side of (8.3) we may put $(z/c;q)_\infty$ to the left-hand side of (8.3). Next, by equating coefficients of $(z/c)^N$ (again making use of the q-binomial theorem), we arrive at the following terminating summation:

$$\frac{(c;q)_N}{(q;q)_N} = \sum_{\substack{k_1,\dots,k_r\ge 0\\ 0\le|\mathbf{k}|\le N}} \left(\prod_{i=1}^{r} \frac{1-(a_i+b_i)}{1-(a_iq^{-k_i}+b_i)} \frac{(a_iq^{-k_i}+b_i;q)_{k_i}}{(q;q)_{k_i}} (-1)^{k_i} q^{\binom{k_i}{2}} \right. \\ \left. \times \frac{\left(c\prod_{i=1}^r(a_i+b_iq^{k_i});q \right)_{N-|\mathbf{k}|}}{(q;q)_{N-|\mathbf{k}|}} \prod_{i=1}^{r} \left(a_i+b_iq^{k_i}\right)^{\sum_{j=i+1}^r k_j} c^{|\mathbf{k}|} \right). \tag{8.4}$$

Identity (8.4) may be viewed as a Gould-type generalization of the q-multinomial theorem. The case $r=1$,

$$\frac{(c;q)_n}{(q;q)_n} = \sum_{k=0}^{n} \frac{1-(a+b)}{1-(aq^{-k}+b)} \frac{(aq^{-k}+b;q)_k}{(q;q)_k} \frac{(c\,(a+bq^k);q)_{n-k}}{(q;q)_{n-k}} (-1)^k q^{\binom{k}{2}} c^k \tag{8.5}$$

(see [37]), is a q-analogue of the (Hagen-)Rothe summation formula [26]

$$\binom{A+C}{n} = \sum_{k=0}^{n} \frac{A}{A+Bk} \binom{A+Bk}{k} \binom{C-Bk}{n-k}, \tag{8.6}$$

for (8.6) can be obtained from (8.5) via the substitutions $a\to q^A-\frac{1-q^B}{1-q}$, $b\to\frac{1-q^B}{1-q}$, $c\to q^{-A-C}$, and then letting $q\to 1$.

Many more similar convolution formulas, in the $q=1$ case, are listed in [31] (also see [58]), whereas more q-Abel and q-Rothe summations can be found in [37], where these are derived by means of umbral calculus.

We start our multidimensional exposition with a multiple q-Abel summation:

Theorem 8.1. *Let a_i, b_{ij}, $i,j=1,\dots,r$, be indeterminate, and let $n_1,\dots,n_r$ be nonnegative integers. Then there holds*

$$1=\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\Bigg(\det_{1\le i,j\le r}\Big(\Big(a_i+\textstyle\sum_{s=1}^r b_{is}q_s^{k_s}\Big)\delta_{ij}+b_{ij}(1-q_i^{k_i})\Big)\\ \times\prod_{i=1}^r\begin{bmatrix}n_i\\k_i\end{bmatrix}_{q_i}\Big(a_i+\textstyle\sum_{j=1}^r b_{ij}q_j^{k_j}\Big)^{k_i-1}\Big(a_i+\textstyle\sum_{j=1}^r b_{ij}q_j^{k_j};q_i\Big)_{n_i-k_i}\Bigg).\qquad(8.7)$$

Starting with the identity (7.5), Theorem 8.1 is proved exactly as in the one variable case (8.1), by an r-fold application of the q-binomial theorem and a comparison of coefficients. Our multidimensional q-Abel summation theorem is a q-analogue of Carlitz' formula [10, Eq. (3.8)] which basically can be obtained from (8.7) via the substitutions $a_i\to\frac{1-q^{A_i/(A_i+C_i)}}{1-q_i}+\sum_{j=1}^r\frac{1-q^{B_{ij}/(A_i+C_i)}}{(1-q_i)^2}$, $b_{ij}\to-\frac{1-q^{B_{ij}/(A_i+C_i)}}{(1-q_i)^2}$, and then letting $q_i\to1$, for $i=1,\dots,r$.

A multidimensional q-Abel summation of a different type is given in Bhatnagar and Milne [5].

Concerning the next two theorems, we could also have given multidimensional generalizations of the Gould-type q-multinomial convolution (8.4), but have decided to restrict ourselves to stating the special cases which are multiple q-Rothe summations:

Theorem 8.2. *Let a_i, b_{ij}, c_i, $i,j=1,\dots,r$, be indeterminate, and let $n_1,\dots,n_r$ be nonnegative integers. Then there holds*

$$\prod_{i=1}^r\frac{(c_i;q_i)_{n_i}}{(q_i;q_i)_{n_i}}=\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\Bigg(\frac{\det_{1\le i,j\le r}\Big(\Big(a_i-1+\sum_{s=1}^r b_{is}q_s^{k_s}\Big)\delta_{ij}+b_{ij}(1-q_i^{k_i})\Big)}{\prod_{i=1}^r\Big(a_i-q_i^{k_i}+\sum_{j=1}^r b_{ij}q_j^{k_j}\Big)}\\ \times\prod_{i=1}^r\frac{\Big((a_i+\sum_{j=1}^r b_{ij}q_j^{k_j})q_i^{-k_i};q_i\Big)_{k_i}}{(q_i;q_i)_{k_i}}q_i^{\binom{k_i+1}{2}}\cdot(-1)^{|\mathbf{k}|}\\ \times\prod_{i=1}^r\frac{\Big(c_i(a_i+\sum_{j=1}^r b_{ij}q_j^{k_j});q_i\Big)_{n_i-k_i}}{(q_i;q_i)_{n_i-k_i}}c_i^{k_i}\Bigg).\qquad(8.8)$$

Starting with the identity (7.6), Theorem 8.2 is proved exactly as in the one variable case (8.5), by an r-fold application of the q-binomial theorem and a comparison of coefficients. Our multidimensional q-Rothe summation theorem is a q-analogue of Carlitz' formula [10, Eq. (6.10)] which basically can be obtained from (8.8) via the substitutions $a_i\to q^{A_i}-\sum_{j=1}^r\frac{1-q^{B_{ij}}}{1-q_i}$, $b_{ij}\to\frac{1-q^{B_{ij}}}{1-q_i}$, $c_i\to q_i^{-A_i-C_i}$, and then letting $q_i\to1$, for $i=1,\dots,r$.

Theorem 8.3. *Let b_i, c_i, x_i, $i=1,\dots,r$, and a be indeterminate, and let $n_1,\dots,n_r$ be nonneg-*

ative integers. Then there holds

$$\prod_{i=1}^{r} \frac{(c_i;q)_{n_i}}{(q;q)_{n_i}} = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \Bigg(\prod_{1\le i<j\le r} \left(\frac{q^{-k_i}/x_i - q^{-k_j}/x_j}{1/x_i - 1/x_j} \right) (-1)^{|\mathbf{k}|} q^{\sum_{i=1}^r \binom{k_i}{2}}$$
$$\times \left(1 - \sum_{i=1}^{r} \frac{b_i(1-q^{k_i})}{(x_i - a - \sum_{j=1}^r b_j q^{k_j})} \right) \prod_{i=1}^{r} \frac{\left((a+\sum_{j=1}^r b_j q^{k_j}) q^{1-k_i}/x_i;q\right)_{k_i}}{(q;q)_{k_i}}$$
$$\times \prod_{i=1}^{r} \frac{\left((a+\sum_{j=1}^r b_j q^{k_j}) c_i q^{1-i}/x_i;q\right)_{n_i-k_i}}{(q;q)_{n_i-k_i}} c_i^{k_i} \Bigg). \quad (8.9)$$

Starting with the identity (7.8), Theorem 8.3 is proved exactly as in the one variable case (8.5), by an r-fold application of the q-binomial theorem and a comparison of coefficients. In identity (8.9), if we make the substitutions $a \to q^A - \sum_{j=1}^r \frac{1-q^{B_j}}{1-q}$, $b_i \to \frac{1-q^{B_i}}{1-q}$, $c_i \to q^{-A_i-C_i}$, $x_i \to q^{A-A_i}$, $i=1,\dots,r$, and then let $q \to 1$, we obtain the following nice multidimensional Rothe summation:

Theorem 8.4. *Let A_i, B_i, C_i, $i=1,\dots,r$, be indeterminate, and let $n_1,\dots,n_r$ be nonnegative integers. Then there holds*

$$\prod_{i=1}^{r} \binom{A_i+C_i}{n_i} = \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \Bigg(\prod_{1\le i<j\le r} \left(\frac{A_i - k_i - A_j + k_j}{A_i - A_j} \right) \cdot \left(1 - \sum_{i=1}^{r} \frac{B_i k_i}{(A_i + \sum_{j=1}^r B_j k_j)} \right)$$
$$\times \prod_{i=1}^{r} \binom{A_i + \sum_{j=1}^r B_j k_j}{k_i} \binom{C_i + i - 1 - \sum_{j=1}^r B_j k_j}{n_i - k_i} \Bigg). \quad (8.10)$$

Remark 8.5. In this section, we derived (multiple) q-Abel and q-Rothe summations by manipulating the series expansions we had obtained by rotated inversion in Section 7 and then extracting coefficients from them. However, we also could have derived these terminating summations directly by applying the inverse relations (4.1)/(4.2) combined with terminating q-binomial and q-Chu–Vandermonde summations. In this case we would have utilized the companion matrix inversion in Theorem 3.3.

9. A q-analogue of MacMahon's Master Theorem

Here we derive a q-extension of MacMahon's Master Theorem [45]. Chu [12, Sec. 5] observed that inverse relations imply MacMahon's Master Theorem. Basically, he recovered Carlitz' multidimensional extension of (7.1) [10, Eq. (3.5)] (or rather the related formula in [9, Eq. (4.3)]) by inverse relations, which by some further manipulations he showed to be equivalent to MacMahon's celebrated theorem.

Letting $\langle \mathbf{z}^\mathbf{n} \rangle f(\mathbf{z})$ denote the coefficient of $\mathbf{z}^\mathbf{n}$ in $f(\mathbf{z})$, the classical version of MacMahon's Master Theorem can be stated as follows:

Theorem 9.1. *Let z_i, b_{ij}, $i,j=1,\dots,r$, be indeterminate, and let $n_1,\dots,n_r$ be arbitrary nonnegative integers. Then there holds*

$$\langle \mathbf{z}^\mathbf{n} \rangle \prod_{i=1}^{r} \left(\sum\nolimits_{j=1}^r b_{ij} z_j \right)^{n_i} = \langle \mathbf{z}^\mathbf{n} \rangle \left(\det_{1\le i,j\le r} (\delta_{ij} - z_i b_{ij}) \right)^{-1}.$$

In deriving our q-analogue, Theorem 9.2, we basically "q-extend" Chu's analysis, but if we would perform the whole matter with Theorem 7.1, a q-extension of Carlitz' identity mentioned above, we would just end up with the classical version of the Master Theorem. Instead, in our derivation we utilize Theorem 7.3, a multiple extension of the q-expansion (7.4) (i.e., a q-analogue of [10, Eq. (6.5)]), and are able to extend the whole analysis with additional bases $q_1, q_2, \dots, q_r$. In our case certain q-operators come into the game.

Defining the shift operators $\mathcal{E}_{(b)}^{q_i}$ by $\mathcal{E}_{(b)}^{q_i} b = q_i b$, our derivation is based on rewriting the identity (7.6) of Theorem 7.3 in the form

$$\prod_{i=1}^r (z_i;q_i)_\infty = \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg(\det_{1\le i,j\le r} \Big(\delta_{ij} + z_i b_{ij} \mathcal{E}_{(z_i)}^{q_i} \mathcal{E}_{(a_i)}^{1/q_i} \textstyle\prod_{s\ne i} \mathcal{E}_{(b_{is})}^{q_s} \mathcal{E}_{(b_{is})}^{1/q_i}\Big) \\ \times \prod_{i=1}^r \frac{\big((a_i + \sum_{j=1}^r b_{ij} q_j^{k_j}) q_i^{1-k_i}; q_i\big)_{k_i}}{(q_i;q_i)_{k_i}} q_i^{\binom{k_i}{2}} \cdot (-1)^{|\mathbf{k}|} \\ \times \prod_{i=1}^r \Big(z_i(a_i + \textstyle\sum_{j=1}^r b_{ij} q_j^{k_j}); q_i\Big)_\infty z_i^{k_i} \Bigg).$$

This is achieved by moving all the terms of the summand in (7.6) inside the determinant using linearity in the rows, by termwise rewriting of the expressions in the determinant, thereby introducing the shift operators, and then moving terms again outside of the determinant by linearity in the rows. For our purpose it is particularly pleasant that now the determinant does not depend on the summation indices. If we transfer $\prod_{i=1}^r (z_i;q_i)_\infty$ to the right-hand side, we obtain

$$1 = \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg(\det_{1\le i,j\le r} \Big((1 - z_i/q_i)\delta_{ij} + z_i b_{ij} \mathcal{E}_{(z_i)}^{q_i} \mathcal{E}_{(a_i)}^{1/q_i} \textstyle\prod_{s\ne i} \mathcal{E}_{(b_{is})}^{q_s} \mathcal{E}_{(b_{is})}^{1/q_i}\Big) \\ \times \prod_{i=1}^r \frac{\big((a_i + \sum_{j=1}^r b_{ij} q_j^{k_j}) q_i^{1-k_i}; q_i\big)_{k_i}}{(q_i;q_i)_{k_i}} q_i^{\binom{k_i}{2}} \cdot (-1)^{|\mathbf{k}|} \\ \times \prod_{i=1}^r \frac{\big(z_i(a_i + \sum_{j=1}^r b_{ij} q_j^{k_j}); q_i\big)_\infty}{(z_i/q_i;q_i)_\infty} z_i^{k_i} \Bigg). \tag{9.1}$$

Since the determinant does not depend on the summation indices we can multiply both sides of (9.1) with the operator inverse of the determinant. Then we obtain, after having replaced z_i by $z_i q_i$, a_i by a_i/q_i, and b_{ij} by b_{ij}/q_i, for $i,j = 1,\dots,r$, respectively,

$$\Big(\det_{1\le i,j\le r} \Big((1 - z_i)\delta_{ij} + z_i b_{ij} \mathcal{E}_{(z_i)}^{q_i}\Big)\Big)^{-1} \mathbf{1} = \sum_{k_1,\dots,k_r=0}^{\infty} \Bigg(\prod_{i=1}^r \frac{\big((a_i + \sum_{j=1}^r b_{ij} q_j^{k_j}) q_i^{-k_i}; q_i\big)_{k_i}}{(q_i;q_i)_{k_i}} q_i^{\binom{k_i+1}{2}} \\ \times (-1)^{|\mathbf{k}|} \prod_{i=1}^r \frac{\big(z_i(a_i + \sum_{j=1}^r b_{ij} q_j^{k_j}); q_i\big)_\infty}{(z_i;q_i)_\infty} z_i^{k_i} \Bigg), \tag{9.2}$$

where we have readily cancelled the operators in the determinant which produce no powers of q in the left-hand side expansion, since the operator is applied to the constant polynomial 1 which we have denoted by $\mathbf{1}$. For our q-Master Theorem we set $a_i \to 0$, $i = 1, \dots, r$, in (9.2). Then, we expand the right-hand side further by means of the q-binomial theorem in order to extract the coefficient of $\mathbf{z}^{\mathbf{n}}$:

$$\begin{aligned}
&\left(\det_{1\le i,j\le r}\left((1-z_i)\delta_{ij} + z_i b_{ij}\mathcal{E}^{q_i}_{(z_i)}\right)\right)^{-1}\mathbf{1}\\
&\quad = \sum_{k_1,\dots,k_r=0}^{\infty}\prod_{i=1}^{r}\left(\frac{(\sum_{j=1}^r b_{ij}q_j^{k_j}q_i^{-k_i};q_i)_{k_i}}{(q_i;q_i)_{k_i}}(-1)^{k_i}q_i^{\binom{k_i+1}{2}}z_i^{k_i}\sum_{l_i=0}^{\infty}\frac{(\sum_{j=1}^r b_{ij}q_j^{k_j};q_i)_{l_i}}{(q_i;q_i)_{l_i}}z_i^{l_i}\right)\\
&\quad = \sum_{n_1,\dots,n_r=0}^{\infty}\prod_{i=1}^{r}\left(\frac{z_i^{n_i}}{(q_i;q_i)_{n_i}}\sum_{k_i=0}^{n_i}\begin{bmatrix}n_i\\k_i\end{bmatrix}_{q_i}(-1)^{k_i}q_i^{\binom{k_i+1}{2}}(\textstyle\sum_{j=1}^r b_{ij}q_j^{k_j}q_i^{-k_i};q_i)_{n_i}\right). \qquad (9.3)
\end{aligned}$$

Observe that

$$\prod_{i=1}^{r}\frac{1}{(q_i;q_i)_{n_i}}\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}}\prod_{i=1}^{r}\left(\begin{bmatrix}n_i\\k_i\end{bmatrix}_{q_i}(-1)^{k_i}q_i^{\binom{k_i+1}{2}-n_ik_i}\right)p(z_1q_1^{k_1},\dots,z_rq_r^{k_r})\bigg|_{z_1=\dots z_r=1} = \langle\mathbf{z}^{\mathbf{n}}\rangle\, p(z_1,\dots,z_r), \qquad (9.4)$$

for polynomials $p(z_1, \dots, z_r)$ of degree $\le |\mathbf{n}|$, by iterated application of the q-binomial theorem and linearity. Besides, note that by defining the partial difference operators $\mathcal{D}_i$ by $\mathcal{D}_i = ((q_i - 1)z_i)^{-1}(\mathcal{E}^{q_i}_{(z_i)} - I)$, where I denotes the identity operator (cf. [14]), and using Schützenberger's [56] observation that if $yx = qxy$, then

$$(x+y)^n = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q x^k y^{n-k}, \qquad (9.5)$$

equation (9.4) may also be expressed more compactly as

$$\prod_{i=1}^{r}\frac{(1-q_i)^{n_i}}{(q_i;q_i)_{n_i}}\mathcal{D}_i^{n_i}\,p(\mathbf{z})\bigg|_{z_1=\dots z_r=1} = \langle\mathbf{z}^{\mathbf{n}}\rangle\, p(\mathbf{z}).$$

Back to (9.3), we recognize that we can apply (9.4) to the inner sum on the right-hand side of (9.3), with the instance

$$p(z_1,\dots,z_r) = \textstyle\prod_{i=1}^r z_i^{n_i}(\sum_{j=1}^r b_{ij}z_j/z_i;q_i)_{n_i}.$$

After this observation we have arrived at:

Theorem 9.2 (A q-analogue of MacMahon's Master Theorem). *Let z_i, b_{ij}, for $i, j = 1, \dots, r$, be indeterminate, and let $n_1, \dots, n_r$ be arbitrary nonnegative integers. Then there holds*

$$\langle\mathbf{z}^{\mathbf{0}}\rangle\prod_{i=1}^{r}\left(\textstyle\sum_{j=1}^r b_{ij}z_j/z_i;q_i\right)_{n_i} = \langle\mathbf{z}^{\mathbf{n}}\rangle\left(\det_{1\le i,j\le r}\left((1-z_i)\delta_{ij}+z_ib_{ij}\mathcal{E}_i\right)\right)^{-1}\mathbf{1}, \qquad (9.6)$$

where $\mathcal{E}_i$ denotes the q-shift operator defined by $\mathcal{E}_i z_i = q_i z_i$, and $\mathbf{1}$ denotes the constant polynomial 1.

Theorem 9.2 indeed includes MacMahon's Master Theorem as a special case. Namely, if we write (9.6) in the way

$$\langle \mathbf{z}^{\mathbf{n}} \rangle \prod_{i=1}^{r} \prod_{s=1}^{n_i} \left(z_i - \sum\nolimits_{j=1}^{r} b_{ij} z_j q_i^{s-1} \right) \;=\; \langle \mathbf{z}^{\mathbf{n}} \rangle \left(\det_{1\le i,j\le r} \Big((1-z_i)\delta_{ij} + z_i b_{ij} \mathcal{E}_i \Big) \right)^{-1} \mathbf{1},$$

we conveniently can see that the substitutions $q_i \to 1$, $z_i \to a z_i$, $b_{ij} \to -b_{ij}/a$, $i,j = 1,\dots,r$, then $a \to 0$, specialize to the classical case, Theorem 9.1.

For illustration, we quickly verify the statement of Theorem 9.2 for one dimension ($r=1$). Writing $b_{11} = b$, $z_1 = z$, and $\mathcal{E}_1 = \mathcal{E}$, we want to check

$$(b;q)_n \;=\; \langle z^n \rangle \,(1 - z + zb\mathcal{E})^{-1}\, \mathbf{1}.$$

We have

$$(1-z+zb\mathcal{E})^{-1}\mathbf{1} = \sum_{n=0}^{\infty} (z - zb\mathcal{E})^n \mathbf{1} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q z^k (-zb\mathcal{E})^{n-k} \mathbf{1} = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} b^k,$$

the second equality due to (9.5). The last inner sum evaluates to the desired quantity by the q-binomial theorem.

We plan to give a more detailed discussion of our q-analogue of MacMahon's Master Theorem including several applications in a forthcoming paper [43].

10. Additional expansion formulas

We want to mention some formulas which are closely related to those we used and derived in Section 7, and which may also be used to derive additional identities.

The expansion formulas (see [53, Sec. 4.5])

$$\frac{e^{AZ}}{1-BZ} = \sum_{k=0}^{\infty} \frac{(A+Bk)^k}{k!} Z^k e^{-BZk}, \tag{10.1}$$

where $\left| BZe^{1-BZ} \right| < 1$, and

$$\frac{(1+Z)^A}{1 - \frac{BZ}{1+Z}} = \sum_{k=0}^{\infty} \binom{A+Bk}{k} Z^k (1+Z)^{-Bk}, \tag{10.2}$$

where $\left| \frac{(B-1)Z}{(1+Z)^B} \right| < 1$, are companion identities of (7.1) and (7.2), respectively. q-Analogues of these identities are

$$\sum_{k=0}^{\infty} (-1)^k q^{\binom{k}{2}} b^k z^k = \sum_{k=0}^{\infty} \frac{(a+bq^k)^k}{(q;q)_k} (z(a+bq^k);q)_\infty \, z^k, \tag{10.3}$$

and

$$(z;q)_\infty \sum_{k=0}^{\infty} (b;q)_k z^k = \sum_{k=0}^{\infty} \frac{(aq^{-k}+b;q)_k}{(q;q)_k} (-1)^k q^{\binom{k+1}{2}} (z(a+bq^k);q)_\infty \, z^k, \tag{10.4}$$

respectively, both being valid for $|az| < 1$. To see that these formulas are q-analogues of the above we can make similar substitutions that were needed ealier in the respective cases where we showed that (7.3) and (7.4) are q-analoques of (7.1) and (7.2).

We have already given a multidimensional version of (10.4), see equation (9.2), involving q-shift operators, which appeared in our derivation of the q-Master Theorem.

The multidimensional version of (10.3),

$$\Big(\det_{1\le i,j\le r}\Big(\delta_{ij}+z_ib_{ij}\mathcal{E}^{q_i}_{(z_i)}\Big)\Big)^{-1}\mathbf{1}$$
$$=\sum_{k_1,\dots,k_r=0}^{\infty}\left(\prod_{i=1}^{r}\frac{\big(a_i+\sum_{j=1}^r b_{ij}q_j^{k_j};q_i\big)^{k_i}}{(q_i;q_i)_{k_i}}\Big(z_i(a_i+\textstyle\sum_{j=1}^r b_{ij}q_j^{k_j});q_i\Big)_\infty z_i^{k_i}\right), \qquad (10.5)$$

can easily be deduced from (7.5) in the same manner as (9.2) was deduced from (7.6).

Identities (10.5) and (9.2) themselves can be used to derive additional higher-dimensional (terminating) convolutions with the method we demonstrated in Section 8. These would include q-extensions of Carlitz' other multidimensional convolution formulas [10, Eqs. (3.9) and (6.9)].

It is also interesting to look for nontrivial cases where the determinant in the multiple identities simplify. These cases have higher chances to occur naturally in combinatorial enumeration problems. The specific type of multiple series we consider occur, e.g., in [57]. We already had a case of nearly total factorization of the determinant in Theorem 7.6, where we could simplify the determinant due to (7.9). Another case concerns the determinant of a matrix having entries $\neq 0$ only in the principal diagonal and the diagonal above (indices modulo r). It is easy to see that in this case the determinant can be reduced to the difference of two products (see for example Section 6).

The following theorem provides such an example, where we use the q-binomial coefficient notation

$$\begin{bmatrix}\alpha\\k\end{bmatrix}_q:=\frac{(q^{1+\alpha-k};q)_k}{(q;q)_k}=\frac{(q^{-\alpha};q)_k}{(q;q)_k}(-q^\alpha)^k q^{-\binom{k}{2}},$$

for nonnegative integer k and arbitrary α (cf. [21, Appendix (I.42),(I.43)]).

Theorem 10.1. *Let α_i, z_i, $i=1,\dots,r$, be indeterminate. Then there holds*

$$\sum_{m=0}^{\infty}q^{r\binom{m}{2}+m\sum_{i=1}^r\alpha_i}\prod_{i=1}^{r}\frac{z_i^m}{(-z_i;q)_m}=\sum_{k_1,\dots,k_r=0}^{\infty}\prod_{i=1}^{r}q^{\binom{k_i}{2}}\begin{bmatrix}\alpha_i+k_{i+1}\\k_i\end{bmatrix}_q\frac{z_i^{k_i}}{(-z_i;q)_{\alpha_i+k_{i+1}}}, \qquad (10.6)$$

where the indices are written modulo r.

Proof. We consider the special case of formula (9.2), where we have $q_1,\dots,q_r=q$, $z_i\to -z_i/q$, $a_i=0$, $b_{i,i+1}=q^{1+\alpha_i}$, for $i=1,\dots,r$ (mod r), and where $b_{ij}=0$ if $j\neq i+1$ (mod r). We also write $\mathcal{E}_i$ instead of $\mathcal{E}^q_{(z_i)}$, for short. In this special case, the left-hand side of (9.2) is

$$\Big(\prod_{i=1}^r(1+z_i/q)-\prod_{i=1}^r q^{\alpha_i}z_i\mathcal{E}_i\Big)^{-1}\mathbf{1}=\Big(1-\prod_{i=1}^r(1+z_i/q)^{-1}q^{\alpha_i}z_i\mathcal{E}_i\Big)^{-1}\prod_{i=1}^r(1+z_i/q)^{-1}\,\mathbf{1}$$
$$=\sum_{m=0}^{\infty}\prod_{i=1}^{r}\Big((1+z_i/q)^{-1}q^{\alpha_i}z_i\mathcal{E}_i\Big)^m(1+z_i/q)^{-1}\,\mathbf{1}=\sum_{m=0}^{\infty}q^{r\binom{m}{2}+m\sum_{i=1}^r\alpha_i}\prod_{i=1}^{r}\frac{z_i^m}{(-z_i/q;q)_{m+1}}.$$

It is even more straightforward to compute the right-hand side of (9.2) for our particular choice of parameters. Finally, we multiply both sides of the resulting identity by $\prod_{i=1}^r(1+z_i/q)$ to obtain (10.6). $\square$

Remark 10.2. Identity (10.6) is a q-analogue of a special case of Carlitz' formula [11, $\lambda = 1$ in Eq. (2.6)],

$$\frac{\prod_{i=1}^{r}(1+z_i)^{\alpha_i+1}}{\prod_{i=1}^{r}(1+z_i)-\prod_{i=1}^{r}z_i}=\sum_{k_1,\dots,k_r=0}^{\infty}\prod_{i=1}^{r}\binom{\alpha_i+k_{i+1}}{k_i}\left(\frac{z_i}{1+z_{i-1}}\right)^{k_i} \tag{10.7}$$

(again, indices are written modulo r), which simply follows from (10.6) by the limit $q \to 1$. Also Carlitz derived his formula by specializing a more general expansion. It is worth noting that (10.7) (or rather an identity equivalent to (10.7) via substitutions) was given combinatorial proofs [15], [24], [57]. It would also be interesting to find a combinatorial proof of the multiple q-series identity (10.6).

References

1. G. E. Andrews, "Connection coefficient problems and partitions", D. Ray-Chaudhuri, ed., *Proc. Symp. Pure Math.*, vol. 34, Amer. Math. Soc., Providence, R. I., 1979, 1–24.
2. W. N. Bailey, "A transformation of nearly-poised basic hypergeometric series", *J. London Math. Soc.* **22** (1947), 237–240.
3. W. N. Bailey, "Some identities in combinatory analysis", *Proc. London Math. Soc.* (2) **49** (1947), 421–435.
4. W. N. Bailey, "Identities of the Roger–Ramanujan type", *Proc. London Math. Soc.* (2) **50** (1949), 1–10.
5. G. Bhatnagar and S. C. Milne, "Generalized bibasic hypergeometric series and their $U(n)$ extensions", *Adv. in Math.* **131** (1997), 188–252.
6. D. M. Bressoud, "Some identities for terminating q-series", *Math. Proc. Cambridge Philos. Soc.* **89** (1981), 211–223.
7. D. M. Bressoud, "A matrix inverse", *Proc. Amer. Math. Soc.* **88** (1983), 446–448.
8. L. Carlitz, "Some inverse relations", *Duke Math. J.* **40** (1973), 893–901.
9. L. Carlitz, "An application of MacMahon's Master Theorem", *SIAM J. Appl. Math.* **26** (1974), 431–436.
10. L. Carlitz, "Some expansions and convolution formulas related to MacMahon's Master Theorem", *SIAM J. Appl. Math.* **8** (1977), 320–336.
11. L. Carlitz, "Multiple binomial and power sums", *Houston J. Math.* **6** (1980), 331–354.
12. W. Chu, "Some multifold reciprocal transformations with applications to series expansions", *Europ. J. Comb.* **12** (1991), 1–8.
13. W. Chu, "Inversion techniques and combinatorial identities – Jackson's q-analogue of the Dougall-Dixon theorem and the dual formulae", *Compositio Math.* **95** (1995), 43–68.
14. J. Cigler, "Operatormethoden für q-Identitäten", *Monatsh. Math.* **88** (1979), 87–105.
15. I. Constantineau, "A generalization of the Pfaff–Saalschütz theorem", *Stud. Appl. Math.* **85** (1991), 243–248.
16. G. P. Egorychev, *Integral representation and the computation of combinatorial sums*, "Nauka" Sibirsk. Otdel., Novosibirsk, 1977; translated from the Russian: Translations of Mathematical Monographs, 59, Amer. Math. Soc., Providence, R. I., 1984.
17. L. Euler, *De serie Lambertiana plurimisque eius insignibus proprietatibus*, Acta Acad. Sci. Petro, 1779; II, 1783, pp. 29–51; reprinted in *Opera Omnia* Ser. I, vol. 6, Teubner, Leibzig, 1921, pp. 350–369.
18. G. Gasper, "Summation formulas for basic hypergeometric series", *SIAM J. Math. Anal.* **12** (1981), 196–200.
19. G. Gasper, "Summation, transformation and expansion formulas for bibasic series", *Trans. Amer. Soc.* **312** (1989), 257-278.
20. G. Gasper and M. Rahman, "An indefinite bibasic summation formula and some quadratic, cubic and quartic summation and transformation formulae", *Canad. J. Math.* **42** (1990), 1–27.
21. G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics And Its Applications 35, Cambridge University Press, Cambridge, 1990.
22. I. M. Gessel and D. Stanton, "Application of q-Lagrange inversion to basic hypergeometric series", *Trans. Amer. Math. Soc.* **277** (1983), 173–203.

23. I. M. Gessel and D. Stanton, "Another family of q-Lagrange inversion formulas", *Rocky Moutain J. Math.* **16** (1986), 373–384.
24. I. M. Gessel and D. Sturtevant, "A combinatorial proof of Saalschütz's theorem", unpublished.
25. H. W. Gould, "Some generalizations of Vandermonde's convolution", *Amer. Math. Monthly* **63** (1956), 84–91.
26. H. W. Gould, "Final analysis of Vandermonde's convolution", *Amer. Math. Monthly* **64** (1957), 409–415.
27. H. W. Gould, "A series transformation for finding convolution identities", *Duke Math. J.* **28** (1961), 193–202.
28. H. W. Gould, "A new convolution formula and some new orthogonal relations for inversion of series", *Duke Math. J.* **29** (1962), 393–404.
29. H. W. Gould, "A new series transform with application to Bessel, Legrende, and Tchebychev polynomials", *Duke Math. J.* **31** (1964), 325–334.
30. H. W. Gould, "Inverse series relations and other expansions involving Humbert polynomials", *Duke Math. J.* **32** (1965), 691–711.
31. H. W. Gould, *Combinatorial Identities*, Morgantown, 1972.
32. H. W. Gould and L. C. Hsu, "Some new inverse series relations", *Duke Math. J.* **40** (1973), 885–891.
33. R. A. Gustafson and C. Krattenthaler, "Heine transformations for a new kind of basic hypergeometric series in $U(n)$", *J. Comput. Appl. Math.* **68** (1996), 151–158.
34. R. A. Gustafson and C. Krattenthaler, "Determinant evaluations and $U(n)$ extensions of Heine's ${}_2\phi_1$-transformations", in: Special Functions, q-Series and Related Topics (M. E. H. Ismail, D. R. Masson and M. Rahman, eds.), Fields Institute Communications, Amer. Math. Soc., Providence, R. I., **14** (1997), 83–89.
35. J. Horn, "Ueber die Convergenz der hypergeometrischen Reihen zweier und dreier Veränderlichen", *Math. Ann.* **34** (1889), 544–600.
36. F. H. Jackson, "A q-generalization of Abel's series", *Rend. Circ. Math. Palermo* **29** (1910), 340–346.
37. W. P. Johnson, "q-Extensions of identities of Abel–Rothe type", *Discrete Math.* **159** (1996), 161–177.
38. P. W. Karlsson and H. M. Srivastava, *Multiple Gaussian hypergeometric series*, Ellis Horwood Series: Mathematics and Its Applications, Halsted Press, J. Wiley & Sons, New York, 1985.
39. C. Krattenthaler, "Operator methods and Lagrange inversion, a unified approach to Lagrange formulas", *Trans. Amer. Math. Soc.* **305** (1988), 325–334.
40. C. Krattenthaler, "Q-analogue of a two variable inverse pair of series with applications to basic double hypergeometric series", *Canad. J. Math.* **4** (1989), 743–768.
41. C. Krattenthaler, "The major counting of nonintersecting lattice paths and generating functions for tableaux", *Mem. Amer. Math. Soc.* **155** No. 552 (1995).
42. C. Krattenthaler, "A new matrix inverse", *Proc. Amer. Math. Soc.* **124** (1996), 47–59.
43. C. Krattenthaler and M. Schlosser, "A q-analogue of MacMahon's Master Theorem and applications", in preparation.
44. G. M. Lilly and S. C. Milne, "The C_l Bailey Transform and Bailey Lemma", *Constr. Approx.* **9** (1993), 473–500.
45. P. A. MacMahon, *Combinatory analysis*, 2 vols., Cambridge Univ. Press, London, 1915–1916; reprint, Chelsea, New York 1960.
46. S. C. Milne, "Basic hypergeometric series very well-poised in $U(n)$", *J. Math. Anal. Appl.* **122** (1987), 223–256.
47. S. C. Milne, "q-analog of a Whipple's transformation for hypergeometric series in $U(n)$", *Adv. in Math.* **108** (1994), 1–76.
48. S. C. Milne, "Balanced ${}_3\phi_2$ summation theorems for $U(n)$ basic hypergeometric series", *Adv. in Math.* **131** (1997), 93–187.
49. S. C. Milne and G. M. Lilly, "Consequences of the A_l and C_l Bailey Transform and Bailey Lemma", *Discrete Math.* **139** (1995), 319–346.
50. S. C. Milne and J. W. Newcomb, "$U(n)$ very-well-poised ${}_{10}\phi_9$ transformations", *J. Comput. Appl. Math.* **68** (1996), 239–285.
51. M. Rahman, "Some cubic summation formulas for basic hypergeometric series", *Utilitas Math.* **36** (1989), 161–172.
52. M. Rahman, "Some quadratic and cubic summation formulas for basic hypergeometric series", *Can. J.*

Math. **45** (1993), 394–411.
53. J. Riordan, *Combinatorial identities*, J. Wiley, New York, 1968.
54. M. Schlosser, "Multidimensional matrix inversions and A_r and D_r basic hypergeometric series", *The Ramanujan J.* **1** (1997), 243–274.
55. M. Schlosser, "Summation theorems for multidimensional basic hypergeometric series by determinant evaluations", submitted.
56. M.-P. Schützenberger, "Une interprétation de certain solutions de l'équation fonctionnelle: $F(x+y) = F(x)F(y)$", *C. R. Acad. Sci. Paris* **236** (1953), 352–353.
57. V. Strehl, *Zykel-Enumeration bei lokal-strukturierten Funktionen*, Habilitationsschrift, Universität Erlangen, 1990.
58. V. Strehl, "Identities of Rothe–Abel–Schläfli–Hurwitz-type", *Discrete Math.* **99** (1992), 321–340.

Institut für Mathematik der Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria.
E-mail address: KRATT@Pap.Univie.Ac.At, SCHLOSSE@Pap.Univie.Ac.At,
WWW: `http://radon.mat.univie.ac.at/People/kratt`,
`http://radon.mat.univie.ac.at/People/mschloss`