A FAMILY OF q-HYPERGEOMETRIC CONGRUENCES MODULO THE FOURTH POWER OF A CYCLOTOMIC POLYNOMIAL

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Abstract. We prove a two-parameter family of q-hypergeometric congruences modulo the fourth power of a cyclotomic polynomial. Crucial ingredients in our proof are George Andrews' multiseries extension of the Watson transformation, and a Karlsson–Minton type summation for very-well-poised basic hypergeometric series due to George Gasper. The new family of q-congruences is then used to prove two conjectures posed earlier by the authors.

1. Introduction

In 1914, Ramanujan [26] presented a number of fast approximations of $1/\pi$, including

$$\sum_{k=0}^{\infty}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}=\frac{4}{\pi}, \tag{1.1}$$

where $(a)_n=a(a+1)\cdots(a+n-1)$ denotes the rising factorial. In 1997, Van Hamme [31] proposed 13 interesting p-adic analogues of Ramanujan-type formulas, such as

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}\equiv p(-1)^{(p-1)/2} \pmod{p^4}, \tag{1.2}$$

where $p>3$ is a prime. Van Hamme's supercongruence (1.2) was first proved by Long [21]. It should be pointed out that all of the 13 supercongruences have been proved by different techniques (see [25,29]). For some background on Ramanujan-type supercongruences, the reader is referred to Zudilin's paper [33].

In 2016, Long and Ramakrishna [22, Thm. 2] proved the following supercongruence:

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac{1}{3})_k^6}{k!^6}\equiv\begin{cases}-p\Gamma_p\left(\dfrac{1}{3}\right)^9 \pmod{p^6}, & \text{if } p\equiv 1 \pmod{6},\\[2ex] -\frac{10p^4}{27}\Gamma_p\left(\dfrac{1}{3}\right)^9 \pmod{p^6}, & \text{if } p\equiv 5 \pmod{6},\end{cases} \tag{1.3}$$

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where $\Gamma_p(x)$ is the p-adic Gamma function. This result for $p \equiv 1 \pmod 6$ confirms the (D.2) supercongruence of Van Hamme, which asserts a congruence modulo p^4.

During the past few years, many congruences and supercongruences were generalized to the q-setting by various authors (see, for instance, [4–18, 23, 24, 28, 30]). In particular, the authors [15, Thm. 2.3] proposed the following partial q-analogue of Long and Ramakrishna's supercongruence (1.3):

$$\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^3)_k^6}{(q^3;q^3)_k^6}q^{3k} \equiv \begin{cases} 0 \pmod{[n]}, & \text{if } n\equiv 1 \pmod 3, \\ 0 \pmod{[n]\Phi_n(q)}, & \text{if } n\equiv 2 \pmod 3. \end{cases} \tag{1.4}$$

Here and throughout the paper, we adopt the standard q-notation (cf. [3]): For an indeterminate q, let

$$(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$$

be the *q-shifted factorial*. For convenience, we compactly write

$$(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n.$$

Moreover, $[n]=[n]_q=1+q+\cdots+q^{n-1}$ denotes the *q-integer* and $\Phi_n(q)$ the n-th *cyclotomic polynomial* in q, which may be defined as

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity.

The authors [15, Conjectures 12.10 and 12.11] also proposed the following conjectures, the first one generalizing the q-congruence (1.4) for $n\equiv 2 \pmod 3$.

Conjecture 1. *Let $d\geqslant 3$ and n be positive integers with $n\equiv -1 \pmod d$. Then*

$$\sum_{k=0}^{M}[2dk+1]\frac{(q;q^d)_k^{2d}}{(q^d;q^d)_k^{2d}}q^{d(d-2)k}\equiv 0 \pmod{[n]\Phi_n(q)^3},$$

where $M=((d-1)n-1)/d$ or $n-1$.

Conjecture 2. *Let $d\geqslant 3$ and $n>1$ be integers with $n\equiv 1 \pmod d$. Then*

$$\sum_{k=0}^{M}[2dk-1]\frac{(q^{-1};q^d)_k^{2d}}{(q^d;q^d)_k^{2d}}q^{d^2k}\equiv 0 \pmod{[n]\Phi_n(q)^3},$$

where $M=((d-1)n+1)/d$ or $n-1$.

Note that Conjecture 1 does not hold for $d=2$ while Conjecture 2 is still true for $d=2$. In fact, the first author and Wang [17] proved that

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}\equiv q^{(1-n)/2}[n]+\frac{(n^2-1)(1-q)^2}{24}q^{(1-n)/2}[n]^3 \pmod{[n]\Phi_n(q)^3}$$

for odd n, and the authors [14] showed that

$$\sum_{k=0}^{(n+1)/2}[4k-1]\frac{(q^{-1};q^2)_k^4}{(q^2;q^2)_k^4}q^{4k}\equiv -(1+3q+q^2)[n]^4 \pmod{[n]^4\Phi_n(q)}$$

for odd $n>1$.

The last two q-congruences are quite special, as they are rare examples of q-hypergeometric congruences that were rigorously shown to hold modulo a high (fourth and even fifth) power of a cyclotomic polynomial. The main purpose of this paper is to add a complete two-parameter family of q-hypergeometric congruences to the list of such q-congruences (see Theorem 1).

We shall also prove that Conjectures 1 and 2 are true. Our proof relies on the following result:

Theorem 1. *Let d,r,n be integers satisfying $d\geqslant 3$, $r\leqslant d-2$ (in particular, r may be negative), and $n\geqslant d-r$, such that d and r are coprime, and $n\equiv -r \pmod d$. Then*

$$\sum_{k=0}^{n-1}[2dk+r]\frac{(q^r;q^d)_k^{2d}}{(q^d;q^d)_k^{2d}}q^{d(d-1-r)k}\equiv 0 \pmod{\Phi_n(q)^4}. \tag{1.5}$$

This result is similar in nature to the two-parameter result in [8, Thm. 1.1] which, however, only concerned a q-congruence modulo $\Phi_n(q)^2$.

Note that the q-congruence (1.5) is still true when the sum is over k from 0 to $((d-1)n-r)/d$, since $(q^r;q^d)_k/(q^d;q^d)_k\equiv 0 \pmod{\Phi_n(q)}$ for $((d-1)n-r)/d<k\leqslant n-1$. (Also, we must have $((d-1)n-r)/d\le n-1$ since $n\geqslant d-r$.) Thus, Theorem 1 implies that Conjectures 1 and 2 hold modulo $\Phi_n(q)^4$.

To prove that Conjectures 1 and 2 also hold modulo $[n]$ (which in conjunction with Theorem 1 would fully establish the validity of the conjectures), we need to prove the following result.

Theorem 2. *Let $d\geqslant 3$ and n be positive integers with $\gcd(d,n)=1$. Then*

$$\sum_{k=0}^{n-1}[2dk+1]\frac{(q;q^d)_k^{2d}}{(q^d;q^d)_k^{2d}}q^{d(d-2)k}\equiv 0 \pmod{\Phi_n(q)}, \tag{1.6}$$

and

$$\sum_{k=0}^{n-1}[2dk-1]\frac{(q^{-1};q^d)_k^{2d}}{(q^d;q^d)_k^{2d}}q^{d^2k}\equiv 0 \pmod{\Phi_n(q)}. \tag{1.7}$$

We shall prove Theorem 1 in Section 2 by making a careful use of Andrews' multiseries generalization (2.2) of the Watson transformation [1, Theorem 4], combined with a special case of Gasper's very-well-poised Karlsson–Minton type summation [2, Eq. (5.13)]. We point out that Andrews' transformation plays an important role in combinatorics and number theory. For example, this transformation was utilized by Zudilin [32] to solve a problem of Asmus Schmidt. It was used by Krattenthaler and Rivoal [20] to provide an alternative proof of a result by Zudilin that relates a very-well-poised hypergeometric series with a Vasilenko–Vasilev-type multiple integral, the latter serving as a tool in the study of the arithmetic behaviour of values of the Riemann zeta function at integers. Andrews' transformation was also used by the first author, Jouhet and Zeng [11] to prove some q-congruences involving q-binomial coefficients. The couple Hessami Pilehrood [19] used this transformation to give a short proof of a theorem of Zagier. Recently, the present authors [13, 16] applied Andrews' transformation to establish some q-congruences for truncated basic hypergeometric series. We shall prove Theorem 2 in Section 3. The proof of Conjectures 1 and 2 will be given in Section 4. We conclude this short paper by Section 5, where we state an open problem involving a q-hypergeometric congruence modulo the fifth power of a cyclotomic polynomial.

2. Proof of Theorem 1

We first give a simple q-congruence modulo $\Phi_n(q)^2$, which was already used in [16].

Lemma 1. *Let α, r be integers and n a positive integer. Then*

$$(q^{r-\alpha n}, q^{r+\alpha n}; q^d)_k \equiv (q^r; q^d)_k^2 \pmod{\Phi_n(q)^2}. \tag{2.1}$$

Proof. For any integer j, it is easy to check that

$$(1-q^{\alpha n-dj+d-r})(1-q^{\alpha n+dj-d+r}) + (1-q^{dj-d+r})^2 q^{\alpha n-dj+d-r} = (1-q^{\alpha n})^2$$

and $1-q^{\alpha n} \equiv 0 \pmod{\Phi_n(q)}$, and so

$$(1-q^{\alpha n-dj+d-r})(1-q^{\alpha n+dj-d+r}) \equiv -(1-q^{dj-d+r})^2 q^{\alpha n-dj+d-r} \pmod{\Phi_n(q)^2}.$$

The proof then follows easily from the above q-congruence. □

We will make use of a powerful transformation formula due to Andrews [1, Theorem 4], which can be stated as follows:

$$\begin{aligned}
&\sum_{k\geqslant 0} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, \dots, b_m, c_m, q^{-N}; q)_k}{(q, \sqrt{a}, -\sqrt{a}, aq/b_1, aq/c_1, \dots, aq/b_m, aq/c_m, aq^{N+1}; q)_k} \left(\frac{a^m q^{m+N}}{b_1 c_1 \cdots b_m c_m}\right)^k \\
&\quad = \frac{(aq, aq/b_m c_m; q)_N}{(aq/b_m, aq/c_m; q)_N} \sum_{j_1,\dots,j_{m-1}\geqslant 0} \frac{(aq/b_1 c_1; q)_{j_1} \cdots (aq/b_{m-1} c_{m-1}; q)_{j_{m-1}}}{(q;q)_{j_1} \cdots (q;q)_{j_{m-1}}} \\
&\qquad \times \frac{(b_2, c_2; q)_{j_1} \dots (b_m, c_m; q)_{j_1+\cdots+j_{m-1}}}{(aq/b_1, aq/c_1; q)_{j_1} \dots (aq/b_{m-1}, aq/c_{m-1}; q)_{j_1+\cdots+j_{m-1}}} \\
&\qquad \times \frac{(q^{-N}; q)_{j_1+\cdots+j_{m-1}}}{(b_m c_m q^{-N}/a; q)_{j_1+\cdots+j_{m-1}}} \frac{(aq)^{j_{m-2}+\cdots+(m-2)j_1} q^{j_1+\cdots+j_{m-1}}}{(b_2 c_2)^{j_1} \cdots (b_{m-1} c_{m-1})^{j_1+\cdots+j_{m-2}}}.
\end{aligned} \tag{2.2}$$

This transformation actually constitutes a multiseries generalization of Watson's ${}_8\phi_7$ transformation formula (see [3, Appendix (III.18)]) which we state here in standard notation for basic hypergeometric series [3, Section 1]:

$$
{}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix} ; q,\ \frac{a^2q^{n+2}}{bcde}\right]
$$

$$
= \frac{(aq, aq/de;q)_n}{(aq/d, aq/e;q)_n}\, {}_4\phi_3\left[\begin{matrix} aq/bc,\ d,\ e,\ q^{-n} \\ aq/b,\ aq/c,\ deq^{-n}/a \end{matrix} ; q,\ q\right]. \tag{2.3}
$$

Next, we recall the following very-well-poised Karlsson–Minton type summation by Gasper [2, Eq. (5.13)] (see also [3, Ex. 2.33 (i)]).

$$
\sum_{k=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, a/b, d, e_1, aq^{n_1+1}/e_1, \dots, e_m, aq^{n_m+1}/e_m;q)_k}{(q, \sqrt{a}, -\sqrt{a}, aq/b, bq, aq/d, aq/e_1, e_1q^{-n_1}, \dots, aq/e_m, e_mq^{-n_m};q)_k} \left(\frac{q^{1-\nu}}{d}\right)^k
$$

$$
= \frac{(q, aq, aq/bd, bq/d;q)_\infty}{(bq, aq/b, aq/d, q/d;q)_\infty} \prod_{j=1}^{m} \frac{(aq/be_j, bq/e_j;q)_{n_j}}{(aq/e_j, q/e_j;q)_{n_j}}, \tag{2.4}
$$

where $n_1, \dots, n_m$ are nonnegative integers, $\nu = n_1 + \dots + n_m$, and $|q^{1-\nu}/d| < 1$ when the series does not terminate. For an elliptic extension of the terminating $d = q^{-\nu}$ case of (2.4), see [27, Eq. (1.7)].

In particular, we note that for $d = bq$ the right-hand side of (2.4) vanishes. Putting in addition $b = q^{-N}$ we obtain the following terminating summation:

$$
\sum_{k=0}^{N} \frac{(a, q\sqrt{a}, -q\sqrt{a}, e_1, aq^{n_1+1}/e_1, \dots, e_m, aq^{n_m+1}/e_m, q^{-N};q)_k}{(q, \sqrt{a}, -\sqrt{a}, aq/e_1, e_1q^{-n_1}, \dots, aq/e_m, e_mq^{-n_m}, aq^{N+1};q)_k} q^{(N-\nu)k} = 0, \tag{2.5}
$$

valid for $N > \nu = n_1 + \dots + n_m$.

By suitably combining (2.2) with (2.5), we obtain the following multiseries summation formula:

Lemma 2. *Let $m \geqslant 2$. Let q, a and $e_1, \dots, e_{m+1}$ be arbitrary parameters with $e_{m+1} = e_1$, and let $n_1, \dots, n_m$ and N be nonnegative integers such that $N > n_1 + \dots + n_m$. Then*

$$
0 = \sum_{j_1,\dots,j_{m-1}\geqslant 0} \frac{(e_1q^{-n_1}/e_2;q)_{j_1} \cdots (e_{m-1}q^{-n_{m-1}}/e_m;q)_{j_{m-1}}}{(q;q)_{j_1} \cdots (q;q)_{j_{m-1}}}
$$

$$
\times \frac{(aq^{n_2+1}/e_2, e_3;q)_{j_1} \dots (aq^{n_m+1}/e_m, e_{m+1};q)_{j_1+\dots+j_{m-1}}}{(e_1q^{-n_1}, aq/e_2;q)_{j_1} \dots (e_{m-1}q^{-n_{m-1}}, aq/e_m;q)_{j_1+\dots+j_{m-1}}}
$$

$$
\times \frac{(q^{-N};q)_{j_1+\dots+j_{m-1}}}{(e_1q^{n_m-N+1}/e_m;q)_{j_1+\dots+j_{m-1}}} \frac{(aq)^{j_{m-2}+\dots+(m-2)j_1} q^{j_1+\dots+j_{m-1}}}{(aq^{n_2+1}e_3/e_2)^{j_1} \cdots (aq^{n_{m-1}+1}e_m/e_{m-1})^{j_1+\dots+j_{m-2}}}. \tag{2.6}
$$

Proof. By specializing the parameters in the multisum transformation (2.2) by $b_i \mapsto aq^{n_i+1}/e_i$, $c_i \mapsto e_{i+1}$, for $1 \le i \le m$ (where $e_{m+1} = e_1$), and dividing both sides of the identity by the prefactor of the multisum, we obtain that the series on the right-hand side of (2.6) equals

$$\frac{(e_mq^{-n_m}, aq/e_1;q)_N}{(aq, e_mq^{-n_m}/e_1;q)_N} \times\sum_{k=0}^{N}\frac{(a, q\sqrt{a}, -q\sqrt{a}, e_1, aq^{n_1+1}/e_1,\dots,e_m, aq^{n_m+1}/e_m, q^{-N};q)_k}{(q,\sqrt{a},-\sqrt{a}, aq/e_1, e_1q^{-n_1},\dots,aq/e_m, e_mq^{-n_m}, aq^{N+1};q)_k}q^{(N-\nu)k},$$

with $\nu = n_1+\cdots+n_m$. Now the last sum vanishes by the special case of Gasper's summation stated in (2.5). □

We collected enough ingredients and are ready to prove Theorem 1.

Proof of Theorem 1. The left-hand side of (1.5) can be written as the following multiple of a terminating ${}_{2d+4}\phi_{2d+3}$ series:

$$\frac{1-q^r}{1-q}\sum_{k=0}^{((d-1)n-r)/d}\frac{(q^r, q^{d+\frac{r}{2}}, -q^{d+\frac{r}{2}}, q^r,\dots,q^r, q^{d+(d-1)n}, q^{r-(d-1)n};q^d)_k}{(q^d, q^{\frac{r}{2}}, -q^{\frac{r}{2}}, q^d,\dots,q^d, q^{r-(d-1)n}, q^{d+(d-1)n};q^d)_k}q^{d(d-1-r)k}.$$

Here, the $q^r,\dots,q^r$ in the numerator refers to $2d-1$ instances of q^r, and similarly, the $q^d,\dots,q^d$ in the denominator to $2d-1$ instances of q^d. Now, by the $m=d$ case of Andrews' transformation (2.2), we can write the above expression as

$$\frac{(1-q^r)(q^{d+r}, q^{-(d-1)n};q^d)_{((d-1)n-r)/d}}{(1-q)(q^d, q^{r-(d-1)n};q^d)_{((d-1)n-r)/d}}\sum_{j_1,\dots,j_{d-1}\geqslant 0}\frac{(q^{d-r};q^d)_{j_1}\cdots(q^{d-r};q^d)_{j_{d-1}}}{(q^d;q^d)_{j_1}\cdots(q^d;q^d)_{j_{d-1}}}$$
$$\times\frac{(q^r,q^r;q^d)_{j_1}\dots(q^r,q^r;q^d)_{j_1+\cdots+j_{d-2}}(q^r, q^{d+(d-1)n};q^d)_{j_1+\cdots+j_{d-1}}}{(q^d,q^d;q^d)_{j_1}\dots(q^d,q^d;q^d)_{j_1+\cdots+j_{d-1}}}$$
$$\times\frac{(q^{r-(d-1)n};q^d)_{j_1+\cdots+j_{d-1}}}{(q^{d+r};q^d)_{j_1+\cdots+j_{d-1}}}\frac{q^{(d+r)(j_{d-2}+\cdots+(d-2)j_1)}q^{d(j_1+\cdots+j_{d-1})}}{q^{2rj_1}\dots q^{2r(j_1+\cdots+j_{d-2})}}. \tag{2.7}$$

It is easy to see that the q-shifted factorial $(q^{d+r};q^d)_{((d-1)n-r)/d}$ contains the factor $1-q^{(d-1)n}$ which is a multiple of $1-q^n$. Similarly, the q-shifted factorial $(q^{-(d-1)n};q^d)_{((d-1)n-r)/d}$ contains the factor $1-q^{-(d-1)n}$ (again being a multiple of $1-q^n$) since $((d-1)n-r)/d\geqslant 1$ holds due to the conditions $d\geqslant 3$, $r\leqslant d-2$, and $n\geqslant d-r$. This means that the q-factorial $(q^{d+r}, q^{-(d-1)n};q^d)_{((d-1)n-r)/d}$ in the numerator of the fraction before the multisummation is divisible by $\Phi_n(q)^2$. Moreover, it is easily seen that the q-factorial $(q^d, q^{r-(d-1)n};q^d)_{((d-1)n-r)/d}$ in the denominator is coprime with $\Phi_n(q)$.

Note that the non-zero terms in the multisummation in (2.7) are those indexed by $(j_1,\dots,j_{d-1})$ that satisfy $j_1+\cdots+j_{d-1}\leqslant((d-1)n-r)/d$ because of the appearance of the factor $(q^{r-(d-1)n};q^d)_{j_1+\cdots+j_{d-1}}$ in the numerator. None of the factors appearing in the denominator of the multisummation of (2.7) contain a factor of the form $1-q^{\alpha n}$ (and are therefore coprime with $\Phi_n(q)$), except for $(q^{d+r};q^d)_{j_1+\cdots+j_{d-1}}$ when $j_1+\cdots+j_{d-1}=$

$((d-1)n-r)/d$. (In this case, the factor $1-q^{(d-1)n}$ appears in the numerator.) Writing $n=ad-r$ (with $a\geqslant 1$), we have $j_1+\cdots+j_{d-1}=a(d-1)-r$. Since $r\le d-2$, there must be an i with $j_i\geqslant a$. Then $(q^{d-r};q^d)_{j_i}$ contains the factor $1-q^{d-r+d(a-1)}=1-q^n$ which is a multiple of $\Phi_n(q)$. So the denominator of the reduced form of the multisum in (2.7) is coprime with $\Phi_n(q)$. What remains is to show that the multisum in (2.7), without the prefactor, is divisible by $\Phi_n(q)^2$, i.e. vanishes modulo $\Phi_n(q)^2$.

By repeated applications of Lemma 1, the mulitsum in (2.7), without the prefactor, is modulo $\Phi_n(q)^2$ congruent to

$$\sum_{j_1,\dots,j_{d-1}\geqslant 0}\frac{(q^{d-r};q^d)_{j_1}\cdots(q^{d-r};q^d)_{j_{d-1}}}{(q^d;q^d)_{j_1}\cdots(q^d;q^d)_{j_{d-1}}}$$
$$\times\frac{(q^{r-(d-2)n},q^{r+(d-2)n};q^d)_{j_1}\dots(q^{r-n},q^{r+n};q^d)_{j_1+\cdots+j_{d-2}}(q^r,q^{d+(d-1)n};q^d)_{j_1+\cdots+j_{d-1}}}{(q^{d+(d-1)n},q^{d-(d-1)n};q^d)_{j_1}\dots(q^{d+2n},q^{d-2n};q^d)_{j_1+\cdots+j_{d-2}}(q^{d+n},q^{d-n};q^d)_{j_1+\cdots+j_{d-1}}}$$
$$\times\frac{(q^{r-(d-1)n};q^d)_{j_1+\cdots+j_{d-1}}}{(q^{d+r};q^d)_{j_1+\cdots+j_{d-1}}}\frac{q^{(d+r)(j_{d-2}+\cdots+(d-2)j_1)}q^{d(j_1+\cdots+j_{d-1})}}{q^{2rj_1}\cdots q^{2r(j_1+\cdots+j_{d-2})}}.$$

However, this sum vanishes due to the $m=d$, $q\mapsto q^d$, $a\mapsto q^r$, $e_1\mapsto q^{d+(d-1)n}$, $e_i\mapsto q^{r+(d-i+1)n}$, $n_1=0$, $n_i\mapsto(n+r-d)/d$, $2\leqslant i\leqslant d$, $N=((d-1)n-r)/d$, case of Lemma 2. ∎

3. Proof of Theorem 2

We first give the following result, which is a generalization of [15, Lemma 3.1].

Lemma 3. *Let d, m and n be positive integers with $m\leqslant n-1$ and $dm\equiv -1 \pmod n$. Then, for $0\leqslant k\leqslant m$, we have*

$$\frac{(aq;q^d)_{m-k}}{(q^d/a;q^d)_{m-k}}\equiv(-a)^{m-2k}\frac{(aq;q^d)_k}{(q^d/a;q^d)_k}q^{m(dm-d+2)/2+(d-1)k}\pmod{\Phi_n(q)}.$$

Proof. In view of $q^n\equiv 1\pmod{\Phi_n(q)}$, we have

$$\begin{aligned}\frac{(aq;q^d)_m}{(q^d/a;q^d)_m}&=\frac{(1-aq)(1-aq^{d+1})\cdots(1-aq^{dm-d+1})}{(1-q^d/a)(1-q^{2d}/a)\cdots(1-q^{dm}/a)}\\&\equiv\frac{(1-aq)(1-aq^{d+1})\cdots(1-aq^{dm-d+1})}{(1-q^{d-dm-1}/a)(1-q^{2d-dm-1}/a)\cdots(1-q^{-1}/a)}\\&=(-a)^mq^{m(dm-d+2)/2}\pmod{\Phi_n(q)}.\end{aligned}\tag{3.1}$$

Furthermore, modulo $\Phi_n(q)$, we get

$$\begin{aligned}\frac{(aq;q^d)_{m-k}}{(q^d/a;q^d)_{m-k}} &= \frac{(aq;q^d)_m}{(q^d/a;q^d)_m}\frac{(1-q^{dm-dk+d}/a)(1-q^{dm-dk+2d}/a)\cdots(1-q^{dm}/a)}{(1-aq^{dm-dk+1})(1-aq^{dm-dk+d+1})\cdots(1-aq^{dm-d+1})}\\ &\equiv \frac{(aq;q^d)_m}{(q^d/a;q^d)_m}\frac{(1-q^{d-dk-1}/a)(1-q^{2d-dk-1}/a)\cdots(1-q^{-1}/a)}{(1-aq^{-dk})(1-aq^{d-dk})\cdots(1-aq^{-d})}\\ &= \frac{(aq;q^d)_m}{(q^d/a;q^d)_m}\frac{(aq;q^d)_k}{(q^d/a;q^d)_k}a^{-2k}q^{(d-1)k},\end{aligned}$$

which together with (3.1) establishes the assertion. □

Similarly, we have the following q-congruence.

Lemma 4. *Let d, m and n be positive integers with $m \leqslant n-1$ and $dm \equiv 1 \pmod{n}$. Then, for $0 \leqslant k \leqslant m$, we have*

$$\frac{(aq^{-1};q^d)_{m-k}}{(q^d/a;q^d)_{m-k}} \equiv (-a)^{m-2k}\frac{(aq^{-1};q^d)_k}{(q^d/a;q^d)_k}q^{m(dm-d-2)/2+(d+1)k} \pmod{\Phi_n(q)}.$$

The proof of Lemma 4 is completely analogous to that of Lemma 3 and thus omitted.

Proof of Theorem 2. Since $\gcd(d,n)=1$, there exists a positive integer $m \leqslant n-1$ such that $dm \equiv -1 \pmod{n}$. By the $a=1$ case of Lemma 3 one sees that, for $0 \leqslant k \leqslant m$, the k-th and $(m-k)$-th terms on the left-hand side of (1.6) cancel each other modulo $\Phi_n(q)$, i.e.,

$$[2d(m-k)+1]\frac{(q;q^d)_{m-k}^{2d}}{(q^d;q^d)_{m-k}^{2d}}q^{d(d-2)(m-k)} \equiv -[2dk+1]\frac{(q;q^d)_k^{2d}}{(q^d;q^d)_k^{2d}}q^{d(d-2)k} \pmod{\Phi_n(q)}.$$

This proves that

$$\sum_{k=0}^{m}[2dk+1]\frac{(q;q^d)_k^{2d}}{(q^d;q^d)_k^{2d}}q^{d(d-2)k} \equiv 0 \pmod{\Phi_n(q)}. \tag{3.2}$$

Moreover, since $dm \equiv -1 \pmod{n}$, the expression $(q;q^d)_k$ contains a factor of the form $1-q^{\alpha n}$ for $m<k\leqslant n-1$, and is therefore congruent to 0 modulo $\Phi_n(q)$. At the same time the expression $(q^d;q^d)_k$ is relatively prime to $\Phi_n(q)$ for $m<k\leqslant n-1$. Therefore, each summand in (1.6) with k in the range $m<k\leqslant n-1$ is congruent to 0 modulo $\Phi_n(q)$. This together with (3.2) establishes the q-congruence (1.6).

Similarly, we can use Lemma 4 to prove (1.7). The proof of the theorem is complete. □

4. Proof of Conjectures 1 and 2

As mentioned in the introduction, we only need to show that Conjectures 1 and 2 are also true modulo $[n]$. We first give a detailed proof of the q-congruences modulo $[n]$ in Conjecture 1.

Proof of Conjecture 1. We need to show that

$$\sum_{k=0}^{((d-1)n-1)/d}[2dk+1]\frac{(q;q^d)_k^{2d}}{(q^d;q^d)_k^{2d}}q^{d(d-2)k}\equiv 0 \pmod{[n]}, \tag{4.1}$$

and

$$\sum_{k=0}^{n-1}[2dk+1]\frac{(q;q^d)_k^{2d}}{(q^d;q^d)_k^{2d}}q^{d(d-2)k}\equiv 0 \pmod{[n]}. \tag{4.2}$$

Let $\zeta\neq 1$ be an n-th root of unity, not necessarily primitive. Clearly, ζ is a primitive root of unity of degree s with $s\mid n$ and $s>1$. Let $c_q(k)$ denote the k-th term on the left-hand side of (4.1) or (4.2), i.e.,

$$c_q(k)=[2dk+1]\frac{(q;q^d)_k^{2d}}{(q^d;q^d)_k^{2d}}q^{d(d-2)k}.$$

The q-congruences (3.2) and (1.6) with $n\mapsto s$ imply that

$$\sum_{k=0}^{m}c_\zeta(k)=\sum_{k=0}^{s-1}c_\zeta(k)=0,$$

where $dm\equiv -1 \pmod{s}$ and $1\leqslant m\leqslant s-1$. Observing that

$$\frac{c_\zeta(\ell s+k)}{c_\zeta(\ell s)}=\lim_{q\to\zeta}\frac{c_q(\ell s+k)}{c_q(\ell s)}=c_\zeta(k), \tag{4.3}$$

we have

$$\sum_{k=0}^{n-1}c_\zeta(k)=\sum_{\ell=0}^{n/s-1}\sum_{k=0}^{s-1}c_\zeta(\ell s+k)=\sum_{\ell=0}^{n/s-1}c_\zeta(\ell s)\sum_{k=0}^{s-1}c_\zeta(k)=0, \tag{4.4}$$

and

$$\sum_{k=0}^{((d-1)n-1)/d}c_\zeta(k)=\sum_{\ell=0}^{N-1}c_\zeta(\ell s)\sum_{k=0}^{s-1}c_\zeta(k)+c_\zeta(Ns)\sum_{k=0}^{m}c_\zeta(k)=0,$$

where

$$N=\frac{(d-1)n-dm-1}{ds}.$$

(It is easy to check that N is a positive integer.) This means that the sums $\sum_{k=0}^{n-1}c_q(k)$ and $\sum_{k=0}^{((d-1)n-1)/d}c_q(k)$ are both divisible by the cyclotomic polynomial $\Phi_s(q)$. Since this is true for any divisor $s>1$ of n, we deduce that they are divisible by

$$\prod_{s\mid n,\, s>1}\Phi_s(q)=[n],$$

thus establishing the q-congruences (4.1) and (4.2). □

Similarly, we can prove Conjecture 2.

Proof of Conjecture 2. This time we need to show that

$$\sum_{k=0}^{((d-1)n+1)/d} [2dk-1]\frac{(q^{-1};q^d)_k^{2d}}{(q^d;q^d)_k^{2d}} q^{d^2k} \equiv 0 \pmod{[n]},$$

and

$$\sum_{k=0}^{n-1} [2dk-1]\frac{(q^{-1};q^d)_k^{2d}}{(q^d;q^d)_k^{2d}} q^{d^2k} \equiv 0 \pmod{[n]}.$$

Again, let ζ be a primitive root of unity of degree s with $s \mid n$ and $s>1$, and let

$$c_q(k)=[2dk-1]\frac{(q^{-1};q^d)_k^{2d}}{(q^d;q^d)_k^{2d}} q^{d^2k}.$$

Just like before, we have

$$\sum_{k=0}^{m} c_\zeta(k)=\sum_{k=0}^{s-1} c_\zeta(k)=0,$$

where $dm\equiv 1 \pmod{s}$ and $1\leqslant m\leqslant s-1$. Furthermore, we also have (4.3), (4.4), and

$$\sum_{k=0}^{((d-1)n+1)/d} c_\zeta(k)=\sum_{\ell=0}^{N-1} c_\zeta(\ell s)\sum_{k=0}^{s-1} c_\zeta(k)+c_\zeta(Ns)\sum_{k=0}^{m} c_\zeta(k)=0,$$

where $N=\frac{(d-1)n-dm+1}{ds}$ this time. The rest is exactly the same as in the proof of Conjecture 1 and is omitted here. □

5. An open problem

Recently, the first author [9, Theorem 5.4] proved that

$$\sum_{k=0}^{M}[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^4}{(q^4;q^4)_k^4}q^{4k}\equiv 0 \pmod{[n]_{q^2}\Phi_n(q^2)^2},$$

where n is odd and $M=(n+1)/2$ or $n-1$. We take this opportunity to propose a unified generalization of [9, Conjectures 6.3 and 6.4], involving a remarkable q-hypergeometric congruence modulo the fifth power of a cyclotomic polynomial:

Conjecture 3. *Let $n>1$ be an odd integer. Then*

$$\sum_{k=0}^{M}[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^4}{(q^4;q^4)_k^4}q^{4k}\equiv (2q+2q^{-1}-1)[n]_{q^2}^4 \pmod{[n]_{q^2}^4\Phi_n(q^2)},$$

where $M=(n+1)/2$ or $n-1$.

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