Some curious extensions of the classical beta integral evaluation

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ABSTRACT: *We deduce curious q-series identities by applying an inverse relation to a certain identity for basic hypergeometric series. After rewriting some of these identities in terms of q-integrals, we obtain, in the limit $q \to 1$, curious integral identities which generalize the classical beta integral evaluation.*

1 Introduction

Euler's beta integral evaluation (cf. [1, Eq. (1.1.13)])

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \Re(\alpha), \Re(\beta) > 0, \tag{1}$$

is one of the most important and prominent identities in special functions. In Andrews, Askey and Roy's modern treatise [1], the beta integral (and its various extensions) runs like a thread through their whole exposition.

An unusual extension of (1) was recently found by George Gasper and the present author in [4, Th. 5.1] and reads as follows.

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = (c-(a+1)^2) \int_0^1 \frac{(c-a(a+t))^{\beta}\,(c-(a+1)(a+t))^{\beta-1}}{(c-(a+t)^2)^{2\beta}} \times {}_2F_1\left[\begin{matrix}\alpha-\beta-1, -\beta \\ \alpha\end{matrix}; \frac{(a+t)t}{c-a(a+t)}\right] t^{\alpha-1}\,(1-t)^{\beta-1}\,dt, \tag{2}$$

provided $\Re(\alpha), \Re(\beta) > 0$. It is clear that (2) reduces to (1) when either $c \to \infty$ or $a \to \infty$. Two special cases of (2) where the ${}_2F_1$ in the integrand can be simplified are $\alpha = \beta + 1$ and $\alpha = \beta$. Specifically, we have

$$\frac{\Gamma(\beta)\Gamma(\beta)}{2\,\Gamma(2\beta)} = (c-(a+1)^2) \int_0^1 \frac{(c-a(a+t))^{\beta}\,(c-(a+1)(a+t))^{\beta-1}}{(c-(a+t)^2)^{2\beta}} \times t^{\beta}\,(1-t)^{\beta-1}\,dt \tag{3}$$

and

$$\frac{\Gamma(\beta)\Gamma(\beta)}{\Gamma(2\beta)} = (c-(a+1)^2) \int_0^1 \frac{(c-a(a+t))^{\beta-1}\,(c-(a+1)(a+t))^{\beta-1}}{(c-(a+t)^2)^{2\beta}} \times (c-(a-t)(a+t))\,t^{\beta-1}\,(1-t)^{\beta-1}\,dt, \tag{4}$$

where in each case $\Re(\beta) > 0$.

In an early version of [4] we claimed that the integral evaluations (3) and (4), proved by the same procedure as the integral identities in this paper, "seem to be difficult to prove by standard methods". However, after seeing our preprint [4], Mizan Rahman [7] communicated to us a remarkable proof of (3) which involves a sequence of manipulations of hypergeometric series [2].

Another beta-type integral evaluation which has some similarity to (2), is [4, Th. 5.2]. It reads as follows. Let m be a nonnegative integer. Then

$$\frac{\Gamma(\beta)\Gamma(\beta)}{2\,\Gamma(2\beta)} = (c-(a+1)^2)\int_0^1 \frac{(c-a(a+t))^\beta\,(c-(a+1)(a+t))^{\beta-1}}{(c-(a+t)^2)^{2\beta}} \times {}_2F_1\left[\begin{matrix}-\beta,-m\\-2\beta\end{matrix};\frac{c-(a+t)^2}{(c-a(a+t))(1-et)}\right]\left(\frac{1-et}{1-e}\right)^m t^{\beta-m}\,(1-t)^{\beta-1}\,dt, \quad (5)$$

provided $\Re(\beta) > \max(0, m-1)$. Some special cases are considered in [4, Sec. 5].

In this paper, we generalize both identities (2) and (5), see Corollary 5.3 and Theorem 5.1, respectively. While (5) does not extend the classical beta integral evaluation (1), its extension in Theorem 5.1 now does. In order to deduce our results, we apply essentially the same machinery which was utilized in [4] with the difference that our derivation now makes use of a more general basic hypergeometric identity (namely, (6)).

We start with some preliminaries on hypergeometric and basic hypergeometric series, see Section 2. In the same section we also exhibit an explicit matrix inverse which will be crucial in our further analysis. This matrix inverse is applied in Section 3 to derive a new q-series identity which we list together with some corollaries. In Section 4 we rewrite two of the obtained identities in terms of q-integrals. From these we deduce in Section 5, by letting $q \to 1$, new beta-type integral identities by which we generalize the results from [4].

2 Preliminaries

2.1 Hypergeometric and basic hypergeometric series

For a complex number a, define the *shifted factorial*

$$(a)_0 := 1, \qquad (a)_k := a(a+1)\ldots(a+k-1),$$

where k is a positive integer. Let r be a positive integer. The *hypergeometric* ${}_rF_{r-1}$ *series* with numerator parameters $a_1,\ldots,a_r$, denominator parameters $b_1,\ldots,b_{r-1}$, and argument z is defined by

$${}_rF_{r-1}\left[\begin{matrix}a_1,\ldots,a_r\\b_1,\ldots,b_{r-1}\end{matrix};z\right] := \sum_{k\ge 0}\frac{(a_1)_k\ldots(a_r)_k}{k!\,(b_1)_k\ldots(b_{r-1})_k}\,z^k.$$

The ${}_rF_{r-1}$ series terminates if one of the numerator parameters is of the form $-n$ for a nonnegative integer n. If the series does not terminate, it converges when $|z|<1$, and also when $|z|=1$ and $\Re[b_1+b_2+\cdots+b_{r-1}-(a_1+a_2+\cdots+a_r)]>0$. See [2, 10] for a classic texts on (ordinary) hypergeometric series.

Let q (the "base") be a complex number such that $0<|q|<1$. Define the *q-shifted factorial* by

$$(a;q)_\infty := \prod_{j\ge 0}(1-aq^j) \qquad \text{and} \qquad (a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty}$$

for integer k. The *basic hypergeometric* ${}_r\phi_{r-1}$ *series* with numerator parameters $a_1,\dots,a_r$, denominator parameters $b_1,\dots,b_{r-1}$, base q, and argument z is defined by

$$
{}_r\phi_{r-1}\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_{r-1}\end{matrix};q,z\right]:=\sum_{k\ge 0}\frac{(a_1;q)_k\dots(a_r;q)_k}{(q;q)_k(b_1;q)_k\dots(b_{r-1};q)_k}\,z^k.
$$

The ${}_r\phi_{r-1}$ series terminates if one of the numerator parameters is of the form q^{-n} for a nonnegative integer n. If the series does not terminate, it converges when $|z|<1$. For a thorough exposition on basic hypergeometric series (or, synonymously, *q-hypergeometric series*), including a list of several selected summation and transformation formulas, we refer the reader to [3].

We list two specific identities which we utilize in this paper.

First, we have the following three-term transformation (cf. [3, Eq. (III.34)]),

$$
\begin{aligned}
{}_3\phi_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix};q,\frac{de}{abc}\right]&=\frac{(e/b;q)_\infty(e/c;q)_\infty}{(e;q)_\infty(e/bc;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}d/a,b,c\\ d,bcq/e\end{matrix};q,q\right]\\
&\quad+\frac{(d/a;q)_\infty(b;q)_\infty(c;q)_\infty(de/bc;q)_\infty}{(d;q)_\infty(e;q)_\infty(bc/e;q)_\infty(de/abc;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}e/b,e/c,de/abc\\ de/bc,eq/bc\end{matrix};q,q\right],
\end{aligned}
\tag{6}
$$

where $|de/abc|<1$. Further, we need (cf. [3, Eq. (III.9)])

$$
{}_3\phi_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix};q,\frac{de}{abc}\right]=\frac{(e/a;q)_\infty(de/bc;q)_\infty}{(e;q)_\infty(de/abc;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}a,d/b,d/c\\ d,de/bc\end{matrix};q,\frac{e}{a}\right],
\tag{7}
$$

where $|de/abc|,|e/a|<1$.

2.2 Inverse relations

Let $\mathbb{Z}$ denote the set of integers and $F=(f_{nk})_{n,k\in\mathbb{Z}}$ be an infinite lower-triangular matrix; i.e. $f_{nk}=0$ unless $n\ge k$. The matrix $G=(g_{kl})_{k,l\in\mathbb{Z}}$ is said to be the *inverse matrix* of F if and only if

$$
\sum_{l\le k\le n} f_{nk}g_{kl}=\delta_{nl}
$$

for all $n,l\in\mathbb{Z}$, where δ_{nl} is the usual Kronecker delta.

The method of applying *inverse relations* [8] is a well-known technique for proving identities, or for producing new ones from given ones.

If $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are lower-triangular matrices that are inverses of each other, then

$$
\sum_{n\ge k} f_{nk}a_n=b_k
\tag{8a}
$$

if and only if

$$
\sum_{k\ge l} g_{kl}b_k=a_l,
\tag{8b}
$$

subject to suitable convergence conditions. For some applications of (8) see e.g. [6, 8, 9].

Note that in the literature it is actually more common to consider the following inverse relations involving finite sums,

$$\sum_{k=0}^{n} f_{nk} a_k = b_n \qquad \textit{if and only if} \qquad \sum_{l=0}^{k} g_{kl} b_l = a_k. \tag{9}$$

It is clear that in order to apply (8) (or (9)) effectively, one should have some explicit matrix inversion at hand. The following result, which is a special case of Krattenthaler's matrix inverse [6], will be crucial in our derivation of new identities. It can be regarded as a bridge between q-hypergeometric and certain non-q-hypergeometric identities. (For some other such matrix inverses, see [9].)

Lemma 2.1 (MS [9, Eqs. (7.18)/(7.19)]) *Let*

$$f_{nk} = \frac{(1/b;q)_{n-k} \left(\frac{(a+bq^k)q^k}{c-a(a+bq^k)};q\right)_{n-k}}{(q;q)_{n-k} \left(\frac{(a+bq^k)bq^{k+1}}{c-a(a+bq^k)};q\right)_{n-k}},$$

$$g_{kl} = (-1)^{k-l} q^{\binom{k-l}{2}} \frac{(c-(a+bq^l)(a+q^l))}{(c-(a+bq^k)(a+q^k))} \frac{(q^{l-k+1}/b;q)_{k-l} \left(\frac{(a+bq^k)q^{l+1}}{c-a(a+bq^k)};q\right)_{k-l}}{(q;q)_{k-l} \left(\frac{(a+bq^k)bq^l}{c-a(a+bq^k)};q\right)_{k-l}}.$$

Then the infinite matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ are inverses of each other.

3 Some curious q-series expansions

Proposition 3.1 *Let a, b, c, d and e be indeterminate. Then*

$$\begin{aligned}
1 = \frac{(bq;q)_\infty}{(b^2q;q)_\infty} \sum_{k=0}^{\infty} & \frac{(c-(a+1)(a+b))}{(c-(a+1)(a+bq^k))} \frac{(c-(a+bq^k)^2)}{(c-(a+b)(a+bq^k))} \\
& \times {}_3\phi_2 \left[\begin{matrix} e/d, 1/b, \frac{(a+bq^k)q^k}{c-a(a+bq^k)} \\ eq^k, 1/b^2 \end{matrix} ; q, q \right] \\
\times & \frac{(b;q)_k\,(d;q)_k \left(\frac{(a+bq^k)}{c-a(a+bq^k)};q\right)_k \left(\frac{(a+bq^k)b^2q^{k+1}}{c-a(a+bq^k)};q\right)_\infty}{(q;q)_k\,(e;q)_k \left(\frac{(a+bq^k)bq}{c-a(a+bq^k)};q\right)_\infty} \left(\frac{beq}{d}\right)^k \\
+ & \frac{(e/d;q)_\infty\,(1/b;q)_\infty\,(b^2eq;q)_\infty}{(1/b^2q;q)_\infty\,(b^2eq/d;q)_\infty\,(e;q)_\infty} \sum_{k=0}^{\infty} \frac{(c-(a+1)(a+b))}{(c-(a+1)(a+bq^k))} \\
\times & \frac{(c-(a+bq^k)^2)}{(c-(a+b)(a+bq^k))} {}_3\phi_2 \left[\begin{matrix} bq, b^2eq/d, \frac{(a+bq^k)b^2q^{k+1}}{c-a(a+bq^k)} \\ b^2q^2, b^2eq^{k+1} \end{matrix} ; q, q \right] \\
& \times \frac{(b;q)_k\,(d;q)_k \left(\frac{(a+bq^k)}{c-a(a+bq^k)};q\right)_\infty}{(q;q)_k\,(b^2eq;q)_k \left(\frac{(a+bq^k)bq}{c-a(a+bq^k)};q\right)_\infty} \left(\frac{beq}{d}\right)^k,
\end{aligned} \tag{11}$$

provided $|beq/d| < 1$.

Proof of Proposition 3.1. Let the inverse matrices $(f_{nk})_{n,k\in\mathbb{Z}}$ and $(g_{kl})_{k,l\in\mathbb{Z}}$ be defined as in Corollary 2.1. Then (8a) holds for

$$a_n = \frac{(d;q)_n}{(e;q)_n}\left(\frac{b^2eq}{d}\right)^n$$

and

$$b_k = \frac{(d;q)_k}{(e;q)_k}\left(\frac{b^2eq}{d}\right)^k \frac{(bq;q)_\infty \left(\frac{(a+bq^k)b^2q^{k+1}}{c-a(a+bq^k)};q\right)_\infty}{(b^2q;q)_\infty \left(\frac{(a+bq^k)bq^{k+1}}{c-a(a+bq^k)};q\right)_\infty} {}_3\phi_2\left[\begin{matrix} e/d, 1/b, \frac{(a+bq^k)q^k}{c-a(a+bq^k)} \\ eq^k, 1/b^2 \end{matrix}; q, q\right]$$

$$+ \frac{(e/d;q)_\infty\,(1/b;q)_\infty\,(b^2eq^{k+1};q)_\infty \left(\frac{(a+bq^k)q^k}{c-a(a+bq^k)};q\right)_\infty}{(1/b^2q;q)_\infty\,(b^2eq/d;q)_\infty\,(eq^k;q)_\infty \left(\frac{(a+bq^k)bq^{k+1}}{c-a(a+bq^k)};q\right)_\infty}$$

$$\times \frac{(d;q)_k}{(e;q)_k}\left(\frac{b^2eq}{d}\right)^k {}_3\phi_2\left[\begin{matrix} bq, b^2eq/d, \frac{(a+bq^k)q^k}{c-a(a+bq^k)} \\ b^2q^2, b^2eq^{k+1} \end{matrix}; q, q\right]$$

by (6). This implies the inverse relation (8b), with the above values of a_n and b_k. After performing the shift $k \mapsto k+l$, and the substitutions $a \mapsto aq^l$, $c \mapsto cq^{2l}$, $e \mapsto eq^{-l}$, we get rid of l and eventually obtain (11). $\square$

Corollary 3.2 *Let a, b, c, d and e be indeterminate. Then*

$$1 = \sum_{k=0}^{\infty} \frac{(c-(a+1)(a+b))}{(c-(a+1)(a+bq^k))} \frac{(c-(a+bq^k)^2)}{(c-(a+b)(a+bq^k))} \frac{(b;q)_k\,(d;q)_k}{(q;q)_k\,(e;q)_k}$$

$$\times {}_3\phi_2\left[\begin{matrix} 1/b, dq^k, \frac{(a+bq^k)q^k}{c-a(a+bq^k)} \\ eq^k, \frac{(a+bq^k)bq^{k+1}}{c-a(a+bq^k)} \end{matrix}; q, \frac{b^2eq}{d}\right] \frac{\left(\frac{(a+bq^k)}{c-a(a+bq^k)};q\right)_\infty}{\left(\frac{(a+bq^k)bq}{c-a(a+bq^k)};q\right)_\infty} \left(\frac{beq}{d}\right)^k, \qquad (12)$$

provided $|beq/d| < 1$ and $|b^2eq/d| < 1$.

Proof. Apply (6) to the right-hand side of (11), with respect to the simultaneous substitutions $a \mapsto dq^k$, $b \mapsto 1/b$, $c \mapsto (a+bq^k)q^k/(c-a(a+bq^k))$, $d \mapsto eq^k$, $e \mapsto (a+bq^k)bq^{k+1}/(c-a(a+bq^k))$. $\square$

Corollary 3.3 *Let a, b, c, d and e be indeterminate. Then*

$$\frac{(e;q)_\infty\,(b^2eq/d;q)_\infty}{(be;q)_\infty\,(beq/d;q)_\infty} = \sum_{k=0}^{\infty} \frac{(c-(a+1)(a+b))}{(c-(a+1)(a+bq^k))} \frac{(c-(a+bq^k)^2)}{(c-(a+b)(a+bq^k))}$$

$$\times \frac{(b;q)_k\,(d;q)_k}{(q;q)_k\,(be;q)_k} {}_3\phi_2\left[\begin{matrix} 1/b, bq, \frac{(a+bq^k)bq}{c-a(a+bq^k)} \\ beq/d, \frac{(a+bq^k)bq^{k+1}}{c-a(a+bq^k)} \end{matrix}; q, beq^k\right] \frac{\left(\frac{(a+bq^k)}{c-a(a+bq^k)};q\right)_\infty}{\left(\frac{(a+bq^k)bq}{c-a(a+bq^k)};q\right)_\infty} \left(\frac{beq}{d}\right)^k, \qquad (13)$$

provided $|beq/d| < 1$ and $|be| < 1$.

Proof. Apply (7) to the ${}_3\phi_2$ on the right-hand side of (12), with respect to the simultaneous substitutions $a \mapsto 1/b$, $b \mapsto dq^k$, $c \mapsto (a+bq^k)q^k/(c-a(a+bq^k))$, $d \mapsto (a+bq^k)bq^{k+1}/(c-a(a+bq^k))$, $e \mapsto eq^k$, and divide both sides of the resulting identity by $(be;q)_\infty(beq/d;q)_\infty/(e;q)_\infty(b^2eq/d;q)_\infty$. $\square$

We will make use of Proposition 3.1 and of Corollary 3.3 in our derivation of new beta integral identities.

4 q-Integrals

In the following we restrict ourselves to *real* q with $0 < q < 1$.

Thomae [11] introduced the q-integral defined by

$$\int_0^1 f(t)d_qt = (1-q)\sum_{k=0}^{\infty} f(q^k)q^k. \tag{14}$$

Later Jackson [5] gave a more general q-integral which however we do not need here.

By considering the Riemann sum for a continuous function f over the closed interval $[0,1]$, partitioned by the points q^k, $k \geq 0$, one easily sees that

$$\lim_{q\to 1^-}\int_0^1 f(t)d_qt = \int_0^1 f(t)dt.$$

It is well known that many identities for q-series can be written in terms of q-integrals, which then may be specialized (as $q \to 1$) to ordinary integrals. For instance, the q-binomial theorem (cf. [3, Eq. (II.3)])

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \qquad |z| < 1, \tag{15}$$

can be written, when $a \mapsto q^\beta$ and $z \mapsto q^\alpha$, as

$$\int_0^1 \frac{(qt;q)_\infty}{(q^\beta t;q)_\infty}t^{\alpha-1}d_qt = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}, \tag{16}$$

where

$$\Gamma_q(x) := (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty} \tag{17}$$

is the q-gamma function, introduced by Thomae [11], see also [1, § 10.3] and [3, § 1.11]. In fact, (16) is a q-extension of the beta integral evaluation (1).

We will rewrite the identities in Propostion 3.1 and in Corollary 3.3 in terms of q-integrals. These will then be utilized in Section 5 to obtain new extensions of the beta integral evaluation.

Starting with (11), if we replace b by q^β, d by $eq^{\beta+1-\alpha}$, and multiply both sides of the identity by

$$\frac{(e;q)_\infty}{(eq^{\beta+1-\alpha};q)_\infty},$$

we obtain the following q-beta-type integral identity:

$$\begin{aligned}
\frac{(e;q)_\infty}{(eq^{\beta+1-\alpha};q)_\infty} &= \frac{\Gamma_q(2\beta+1)}{\Gamma_q(\beta+1)\Gamma_q(\beta)} \int_0^1 \frac{(c-(a+1)(a+q^\beta))}{(c-(a+1)(a+q^\beta t))} \\
&\quad \times \frac{(c-(a+q^\beta t)^2)}{(c-(a+q^\beta)(a+q^\beta t))}\, {}_3\phi_2\left[\begin{matrix} q^{\alpha-\beta-1}, q^{-\beta}, \frac{(a+q^\beta t)t}{c-a(a+q^\beta t)} \\ et, q^{-2\beta} \end{matrix}; q, q\right] \\
&\quad \times \frac{(qt;q)_\infty\,(et;q)_\infty\,\left(\frac{(a+q^\beta t)}{c-a(a+q^\beta t)};q\right)_\infty \left(\frac{(a+q^\beta t)q^{2\beta+1}t}{c-a(a+q^\beta t)};q\right)_\infty}{(q^\beta t;q)_\infty\,(eq^{\beta+1-\alpha}t;q)_\infty\,\left(\frac{(a+q^\beta t)q^{\beta+1}}{c-a(a+q^\beta t)};q\right)_\infty \left(\frac{(a+q^\beta t)t}{c-a(a+q^\beta t)};q\right)_\infty}\, t^{\alpha-1} d_q t \\
&\quad + \frac{\Gamma_q(-2\beta-1)\,\Gamma_q(\alpha+\beta)}{\Gamma_q(\beta)\,\Gamma_q(\alpha-\beta-1)\,\Gamma_q(-\beta)} \int_0^1 \frac{(c-(a+1)(a+q^\beta))}{(c-(a+1)(a+q^\beta t))} \\
&\quad \times \frac{(c-(a+q^\beta t)^2)}{(c-(a+q^\beta)(a+q^\beta t))}\, {}_3\phi_2\left[\begin{matrix} q^{\beta+1}, q^{\alpha+\beta}, \frac{(a+q^\beta t)q^{2\beta+1}t}{c-a(a+q^\beta t)} \\ q^{2\beta+2}, eq^{2\beta+1}t \end{matrix}; q, q\right] \\
&\quad \times \frac{(qt;q)_\infty\,(eq^{2\beta+1}t;q)_\infty\,\left(\frac{(a+q^\beta t)}{c-a(a+q^\beta t)};q\right)_\infty}{(q^\beta t;q)_\infty\,(eq^{\beta+1-\alpha}t;q)_\infty\,\left(\frac{(a+q^\beta t)q^{\beta+1}}{c-a(a+q^\beta t)};q\right)_\infty}\, t^{\alpha-1} d_q t. \qquad (18)
\end{aligned}$$

Similarly, starting with (13), if we replace b by q^β, d by $eq^{\beta+1-\alpha}$, and multiply both sides of the identity by

$$(1-q)\,\frac{(q;q)_\infty\,(eq^\beta;q)_\infty}{(q^\beta;q)_\infty\,(eq^{\beta+1-\alpha};q)_\infty},$$

we obtain the following q-beta-type integral evaluation:

$$\begin{aligned}
\frac{\Gamma_q(\alpha)\,\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}\,\frac{(e;q)_\infty}{(eq^{\beta+1-\alpha};q)_\infty} &= \int_0^1 \frac{(c-(a+1)(a+q^\beta))}{(c-(a+1)(a+q^\beta t))} \\
&\quad \times \frac{(c-(a+q^\beta t)^2)}{(c-(a+q^\beta)(a+q^\beta t))}\, {}_3\phi_2\left[\begin{matrix} q^{-\beta}, q^{\beta+1}, \frac{(a+q^\beta t)q^\alpha}{e(c-a(a+q^\beta t))} \\ q^\alpha, \frac{(a+q^\beta t)q^\beta t}{c-a(a+q^\beta t)} \end{matrix}; q, eq^\beta t\right] \\
&\quad \times \frac{(qt;q)_\infty\,(eq^\beta t;q)_\infty\,\left(\frac{(a+q^\beta t)}{c-a(a+q^\beta t)};q\right)_\infty \left(\frac{(a+q^\beta t)q^{\beta+1}t}{c-a(a+q^\beta t)};q\right)_\infty}{(q^\beta t;q)_\infty\,(eq^{\beta+1-\alpha}t;q)_\infty\,\left(\frac{(a+q^\beta t)q^{\beta+1}}{c-a(a+q^\beta t)};q\right)_\infty \left(\frac{(a+q^\beta t)t}{c-a(a+q^\beta t)};q\right)_\infty}\, t^{\alpha-1} d_q t. \qquad (19)
\end{aligned}$$

5 Curious beta-type integrals

Observe that $\lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x)$ (see [3, (1.10.3)]) and

$$\lim_{q\to 1^-} \frac{(q^\alpha u;q)_\infty}{(u;q)_\infty} = (1-u)^{-\alpha}$$

for constant u (with $|u|<1$), due to (15) and its $q\to 1$ limit, the ordinary binomial theorem.

We thus immediately deduce, as consequences of our q-integral identities from Section 4, new beta integral identies. We implicitly assume that the integrals are well defined, in particular that the parameters are chosen such that no poles occur on the path of integration $t \in [0,1]$ and the integrals converge.

We first consider the beta-type integral identity obtained from multiplying both sides of (18) by

$$\frac{\Gamma(\beta)\,\Gamma(\beta+1)}{\Gamma(2\beta+1)},$$

and letting $q \to 1^-$.

Theorem 5.1 *Let $\Re(\alpha), \Re(\beta) > 0$. Then*

$$\begin{aligned}
&\frac{\Gamma(\beta)\,\Gamma(\beta)}{2\,\Gamma(2\beta)}\,(1-e)^{\beta+1-\alpha}\\
&\qquad = (c-(a+1)^2)\int_0^1 \frac{(c-a(a+t))^{\beta}\,(c-(a+1)(a+t))^{\beta-1}}{(c-(a+t)^2)^{2\beta}}\\
&\times {}_2F_1\left[\begin{matrix}\alpha-\beta-1,-\beta\\ -2\beta\end{matrix};\frac{c-(a+t)^2}{(c-a(a+t))(1-et)}\right](1-et)^{\beta+1-\alpha}\,t^{\alpha-1}\,(1-t)^{\beta-1}\,dt\\
&\qquad + \frac{\Gamma(\beta)\,\Gamma(-2\beta-1)\,\Gamma(\alpha+\beta)}{2\,\Gamma(2\beta)\,\Gamma(-\beta)\,\Gamma(\alpha-\beta-1)}\,(c-(a+1)^2)\int_0^1 (c-(a+t)^2)\\
&\quad\times \frac{(c-(a+1)(a+t))^{\beta-1}}{(c-a(a+t))^{\beta+1}}\,{}_2F_1\left[\begin{matrix}\beta+1,\alpha+\beta\\ 2\beta+2\end{matrix};\frac{c-(a+t)^2}{(c-a(a+t))(1-et)}\right]\\
&\qquad\qquad\qquad \times (1-et)^{-\alpha-\beta}\,t^{\alpha-1}\,(1-t)^{\beta-1}\,dt. \qquad (20)
\end{aligned}$$

Note that (20) can be further rewritten using Legendre's duplication formula

$$\Gamma(2\beta) = \frac{1}{\sqrt{\pi}}\,2^{2\beta-1}\Gamma(\beta)\Gamma(\beta+\tfrac{1}{2}),$$

after which the left hand side becomes

$$\frac{\sqrt{\pi}}{4^{\beta}}\,\frac{\Gamma(\beta)}{\Gamma(\beta+\frac{1}{2})}\,(1-e)^{\beta+1-\alpha}.$$

Clearly, (20) reduces to (5) if $\alpha-\beta-1=m$, a nonnegative integer.

Observe that (20) reduces to the classical beta integral evaluation (1) for $e=0$ and $c\to\infty$ due to the *Gauß summation*

$${}_2F_1\left[\begin{matrix}A,B\\ C\end{matrix};1\right] = \frac{\Gamma(C)\,\Gamma(C-A-B)}{\Gamma(C-A)\,\Gamma(C-B)},$$

where $\Re(C-A-B)>0$, the *reflection formula*

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin\pi z},$$

where z is not an integer, and some elementary identities for trigonometric functions, such as

$$\sin(x+y)+\sin(x-y) = \sin x\,\frac{\sin 2y}{\sin y}.$$

Next, we have the beta-type integral identity obtained from (19) by letting $q \to 1^-$.

Theorem 5.2 *Let $\Re(\alpha), \Re(\beta) > 0$. Then*

$$\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,(1-e)^{\beta+1-\alpha} = (c-(a+1)^2)\int_0^1 \frac{(c-(a+1)(a+t))^{\beta-1}}{(c-(a+t)^2)^{\beta}}$$
$$\times\, {}_2F_1\left[\begin{matrix}-\beta, \beta+1\\ \alpha\end{matrix}; \frac{(ce-(1+ae)(a+t))t}{c-(a+t)^2}\right](1-et)^{1-\alpha}\,t^{\alpha-1}\,(1-t)^{\beta-1}\,dt. \quad (21)$$

Clearly, (21) reduces to (1) when $e = 0$ and $c \to \infty$.

Corollary 5.3 *Let $\Re(\alpha), \Re(\beta) > 0$. Then*

$$\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} = (c-(a+1)^2)\int_0^1 \frac{(c-a(a+t))^{\beta}\,(c-(a+1)(a+t))^{\beta-1}}{(c-(a+t)^2)^{2\beta}}$$
$$\times\, {}_2F_1\left[\begin{matrix}-\beta, \alpha-\beta-1\\ \alpha\end{matrix}; \frac{((1+ae)(a+t)-ce)t}{(c-a(a+t))(1-et)}\right]$$
$$\times\left(\frac{1-et}{1-e}\right)^{\beta+1-\alpha} t^{\alpha-1}\,(1-t)^{\beta-1}\,dt. \quad (22)$$

Proof. We apply the transformation [2, p. 10, Eq. 2.4(1)]

$$(1-z)^{-A}\,{}_2F_1\left[\begin{matrix}A, B\\ C\end{matrix}; \frac{-z}{1-z}\right] = {}_2F_1\left[\begin{matrix}A, C-B\\ C\end{matrix}; z\right], \quad (23)$$

valid for $|z| < 1$ and $\Re(z) < \frac{1}{2}$ (conditions which we implicitly assume), to the ${}_2F_1$ on the right-hand side of (21) and divide both sides by $(1-e)^{\beta+1-\alpha}$. □

Clearly, (22) reduces to (2) for $e = 0$.

As in [4], we observe that by performing various substitutions one may change the form and path of integration of the considered integrals. In particular, using $t \mapsto s/(s+1)$ these integrals then run over the half line $s \in [0, \infty)$.

Acknowledgements

We thank George Gasper for comments.

The author was fully supported by an APART fellowship of the Austrian Academy of Sciences.

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