C_n AND D_n VERY-WELL-POISED ${}_{10}\phi_9$ TRANSFORMATIONS

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Abstract. In this paper we derive multivariable generalizations of Bailey's classical terminating balanced very-well-poised ${}_{10}\phi_9$ transformation. We work in the setting of multiple basic hypergeometric series very-well-poised on the root systems A_n, C_n, and D_n. Following the distillation of Bailey's ideas by Gasper and Rahman [11], we use a suitable interchange of multisums. We obtain C_n and D_n ${}_{10}\phi_9$ transformations combined with A_n, C_n, and D_n extensions of Jackson's ${}_8\phi_7$ summation. Milne and Newcomb have previously obtained an analogous formula for A_n series. Special cases of our ${}_{10}\phi_9$ transformations include several new multivariable generalizations of Watson's transformation of an ${}_8\phi_7$ into a multiple of a ${}_4\phi_3$ series. We also deduce multidimensional extensions of Sears' ${}_4\phi_3$ transformation formula, the second iterate of Heine's transformation, the q-Gauss summation theorem, and of the q-binomial theorem.

1. Introduction

The Rogers–Ramanujan identities are perhaps the most celebrated identities involving basic hypergeometric series. G. N. Watson [35] proved these identities by first finding a q-analogue of Whipple's [36] transformation formula, taking suitable limiting cases, and then invoking Jacobi's triple product identity [1, 18]. Watson's proof of his transformation formula was analogous to Whipple's [37] second proof of his identity. He used a polynomial argument and Jackson's [16] q-Pfaff–Saalschütz theorem, a summation theorem for a terminating and balanced ${}_3\phi_2$ series. Bailey [4] observed that Whipple's ideas could be

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1991 *Mathematics Subject Classification.* Primary: 33D70; Secondary: 05A19, 33D20.

Key words and phrases. multiple basic hypergeometric series associated to root systems A_n, C_n, and D_n, Jackson's ${}_8\phi_7$ summations, terminating ${}_{10}\phi_9$ transformations, Watson's transformations, Sears' ${}_4\phi_3$ transformations, Heine's ${}_2\phi_1$ transformation, q-Gauss summation, q-binomial theorem.

[†] M. Schlosser was supported by the Austrian Science Foundation FWF, grant P10191-MAT.

extended further, by using Jackson's [17] q-Dougall summation theorem, which is a summation theorem for a terminating and balanced ${}_8\phi_7$ sum, and which contains the q-Pfaff–Saalschütz summation as a special case. The result was Bailey's ${}_{10}\phi_9$ transformation formula, which converts a terminating, balanced, and very-well-poised ${}_{10}\phi_9$ series into a multiple of a series of the same kind.

In this paper, we generalize Bailey's ${}_{10}\phi_9$ transformation formula to multiple series. The type of series appearing in our extensions of Bailey's formula are referred to as multiple basic hypergeometric series, very-well-poised over the root systems A_n, C_n, or D_n. As special cases of our results, we find several new multivariable extensions of Watson's transformation formula.

In an important paper, Andrews [2] indicated that it may be desirable to know all the multivariable generalizations of Watson's formula. In particular, he asked for the q-analogue of the multivariable generalization of Whipple's transformation found by Gustafson [12] for the type of series considered by Holman, Biedenharn and Louck [15]. This was done by Milne [21, 24], who adopted the terminology in [15] and called his series $U(n+1)$ series. That there was some relation of these series with root systems of type A_n was also shown by Milne [20], and later Gustafson [13] defined multiple basic hypergeometric series over the other root systems. However, it turns out that the difference between, for instance, A_n and C_n series, is largely one of name. Indeed, Milne's elementary approach to A_n series appears to work as well on C_n and D_n series: these techniques led to C_n and D_n generalizations of Watson's formula found by Milne and Lilly [25], and Bhatnagar [7], respectively.

Previously, Milne and Newcomb [26] have generalized Bailey's formula to $U(n+1)$ multiple series, or equivalently to A_n series. As special cases, they also recover many A_n generalizations of Watson's transformation found earlier by Milne [22, 24]. We use similar techniques and find C_n and D_n generalizations of the ${}_{10}\phi_9$ transformation formula. Several different generalizations of Watson's transformation follow from any one multivariable extension of Bailey's formula. Thus we are able to unify all the previously known A_n, C_n, and D_n generalizations of Watson's transformations [7, 21, 22, 24, 25]. In addition, we find 14 new A_n and D_n generalizations of Watson's transformation.

The proofs of our results are generalizations of Bailey's [3, 5] second proof of his transformation formula. Following the distillation of Bailey's ideas by Gasper and Rahman [11], we use a suitable multivariable extension of the elementary interchange of summation argument:

$$\sum_{m=0}^{n} \sum_{k=0}^{m} A(m)\, C(k,m) = \sum_{k=0}^{n} \sum_{m=0}^{n-k} A(m+k)\, C(k,m+k). \tag{1.1}$$

When A and C are chosen so that the inner sums are both summable using Jackson's q-Dougall sum, we obtain the ${}_{10}\phi_9$ transformation formula. Essentially, Milne and Newcomb [26] use the same model in their derivation, except that they follow the organization of Bailey's ideas in Slater's [33] book, rather than the simpler exposition by Gasper and Rahman [11].

Milne and Newcomb [26] used Milne's [21] A_n generalization of Jackson's ${}_8\phi_7$ sum to sum up the inner sum on both sides of a suitable multivariable generalization of (1.1). However, this is not the only multivariable generalization of Jackson's sum. A C_n extension was found independently by Denis and Gustafson [10] and by Milne and Lilly [25], and more recently, Bhatnagar [7] and Schlosser [29] have found D_n extensions of Jackson's sum. To derive our C_n and D_n ${}_{10}\phi_9$ transformations, we have used all possible combinations of these A_n, C_n and D_n summations. (The C_n extension of Jackson's sum given by Schlosser [30, 31] is for a different type of series, and cannot be used to derive a multiple ${}_{10}\phi_9$ transformation.) As one strange consequence of combining C_n and D_n theorems, we find a generalization of Bailey's transformation, which converts a C_n ${}_{10}\phi_9$ series into a multiple of an A_n ${}_{10}\phi_9$ series. In this paper, while we follow the convention set by previous authors [7, 13, 25] of labeling our formulas as A_n, C_n, or D_n theorems, we do not hesitate to mix the different types of series if necessary.

The rest of this paper is organized as follows. In §2, we find a multivariable ${}_{10}\phi_9$ transformation formula, which transforms a C_n ${}_{10}\phi_9$ into a multiple of an A_n ${}_{10}\phi_9$. In §3, we derive several D_n ${}_{10}\phi_9$ transformation formulas. In §4, we present several Watson's transformations which follow from our results in §2 and §3.

In §5, we present some A_n and D_n extensions of Sears' [32] ${}_4\phi_3$ transformation formula which follow immediately from our ${}_{10}\phi_9$ transformations. In addition, we present a set of transformation and summation theorems which seem related to Milne's earlier work, but have not been noticed previously. These include an A_n generalization of the second iterate of Heine's transformation [14], an A_n extension of the q-Gauss summation [14], and an A_n extension of Cauchy's [8] q-binomial theorem.

Finally, in Appendix A, we collect a useful lemma from Milne [23], and A_n, C_n, and D_n generalizations of Jackson's sum from [7, 21, 25, 29].

Many of the calculations we use in our work are summarized in Appendix I of Gasper and Rahman's [11] text on basic hypergeometric series. We will always refer to standard summation and transformation theorems from classical (one-variable) basic hypergeometric series from [11]. In addition, important multivariable techniques have been given by Milne [23], and summarized in [6, Chapter II].

This paper is part of the second author's Ph.D. thesis [30], written under the direction of Professor C. Krattenthaler. We thank him for his helpful comments. We also thank Professor S. C. Milne for showing us his notes for [24].

In the rest of this section, we introduce some notation, and outline the main ideas we use from the theory of (classical) basic hypergeometric series [11]. We also indicate the conventions used in naming series appearing in this paper as A_n, C_n, or D_n series [7].

We recall the standard definition of the q-rising factorial. Let q be a complex number such that $|q| < 1$. Define

$$(a;q)_\infty := \prod_{j\geq 0}(1-aq^j),$$

and,

$$\begin{aligned}(a;q)_k &:= \frac{(a;q)_\infty}{(aq^k;q)_\infty} \qquad (1.2)\\ &= \prod_{j=0}^{k-1}(1-aq^j), \qquad (1.3)\end{aligned}$$

where the equality (1.3) holds when k is a non-negative integer. Classical basic hypergeometric series (q-hypergeometric series) with r numerator parameters $a_1,\dots,a_r$ and s denominator parameters $b_1,\dots,b_s$ are defined as

Definition 1.4 (${}_r\phi_s$ basic hypergeometric series).

$${}_r\phi_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,z\right] := \sum_{k=0}^{\infty}\frac{(a_1;q)_k\,(a_2;q)_k\,\cdots\,(a_r;q)_k}{(q;q)_k\,(b_1;q)_k\,\cdots\,(b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k, \qquad (1.5)$$

with $\binom{k}{2} = k(k-1)/2$, where $q \neq 0$ when $r > s+1$. The parameters $b_1,\dots,b_s$ are such that the denominator factors in the terms of the series (1.5) are never zero.

Since $(q^{-n};q)_k = 0$, if $k = n+1, n+2, \ldots$, an $_r\phi_s$ series terminates if one of its numerator parameters is of the form q^{-n} with $n = 0, 1, 2, \ldots$, and $q \neq 0$. See [11] for the convergence criteria of these series when they do not terminate.

The theory of the (classical) basic hypergeometric series consists of several summation and transformation formulas involving $_r\phi_s$ series. Most of the fundamental summation and transformation formulas involve series where $r = s+1$. The classical summation theorems for terminating $_3\phi_2$, $_6\phi_5$, and $_8\phi_7$ series require that the parameters satisfy the additional condition of being either balanced and/or very-well-poised. An $_{r+1}\phi_r$ basic hypergeometric series is called *balanced* if $b_1 \cdots b_r = a_1 \cdots a_{r+1} q$ and $z = q$. An $_{r+1}\phi_r$ series is *well-poised* if $a_1 q = a_2 b_1 = \cdots = a_{r+1} b_r$. It is called *very-well-poised* if it is well-poised and if $a_2 = q\sqrt{a_1}$ and $a_3 = -q\sqrt{a_1}$. Note that the factor

$$\frac{1-a_1 q^{2k}}{1-a_1} \tag{1.6}$$

appears in a very-well-poised series. The parameter a_1 is usually referred to as the *special parameter* of such a series.

With this notation, we now state Bailey's [4] $_{10}\phi_9$ transformation formula, which transforms a terminating $_{10}\phi_9$ series, which is both balanced and very-well-poised, into a multiple of a series of the same type [11, equation (2.9.1)].

Theorem 1.7 (Bailey's classical $\boldsymbol{_{10}\phi_9}$ transformation)**.** *Let a, b, c, d, e, and f be indeterminate, let n be a nonnegative integer, and suppose that none of the denominators in* (1.8) *vanish. Then*

$$\begin{aligned}
&{}_{10}\phi_9\left[\begin{matrix} a,\, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f, \lambda a q^{n+1}/ef, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f, efq^{-n}/\lambda, aq^{n+1} \end{matrix}; q, q\right] \\
&\qquad = \frac{(aq;q)_n\,(aq/ef;q)_n\,(\lambda q/e;q)_n\,(\lambda q/f;q)_n}{(\lambda q/ef;q)_n\,(\lambda q;q)_n\,(aq/f;q)_n\,(aq/e;q)_n} \\
&\times {}_{10}\phi_9\left[\begin{matrix} \lambda,\, q\sqrt{\lambda}, -q\sqrt{\lambda}, \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda a q^{n+1}/ef, q^{-n} \\ \sqrt{\lambda}, -\sqrt{\lambda}, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f, efq^{-n}/a, \lambda q^{n+1} \end{matrix}; q, q\right],
\end{aligned} \tag{1.8}$$

where $\lambda = qa^2/bcd$.

Proof. Following Gasper and Rahman [11], we indicate the main ideas in the proof of Theorem 1.7.

To derive (1.8), we start by writing the sum on the left hand side of (1.1) in the form

$$\sum_{m=0}^{n} A(m)\, B(m), \tag{1.9}$$

where

$$B(m) = \sum_{k=0}^{m} C(k,m).$$

We delay the choice of $A(m)$ for now. Choose $B(m)$ as

$$B(m) = \frac{(b;q)_m\,(c;q)_m\,(d;q)_m\,(\lambda q;q)_m}{(aq/b;q)_m\,(aq/c;q)_m\,(aq/d;q)_m\,(a/\lambda;q)_m}.$$

The choice of $B(m)$ is motivated by the product side of Jackson's [17] terminating balanced very-well-poised ${}_8\phi_7$ summation:

Theorem 1.10 (Jackson's classical ${}_8\phi_7$ summation)**.** *Let a, b, c, and d be indeterminate, let n be a nonnegative integer and suppose that none of the denominators in* (1.11) *vanish. Then*

$$\begin{aligned}{}_8\phi_7&\left[\begin{matrix}a,\, q\sqrt{a}, -q\sqrt{a}, b, c, d, a^2q^{n+1}/bcd, q^{-n}\\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, bcdq^{-n}/a, aq^{n+1}\end{matrix}; q, q\right]\\ &= \frac{(aq;q)_n\,(aq/bc;q)_n\,(aq/bd;q)_n\,(aq/cd;q)_n}{(aq/bcd;q)_n\,(aq/d;q)_n\,(aq/c;q)_n\,(aq/b;q)_n}.\end{aligned} \tag{1.11}$$

Theorem 1.10 is equation (2.6.2) of [11], where we have chosen to replace e by a^2q^{n+1}/bcd explicitly.

We continue our derivation of Theorem 1.7. $B(m)$ is the $a \mapsto \lambda$, $b \mapsto \lambda b/a$, $c \mapsto \lambda c/a$, $d \mapsto \lambda d/a$, and $n \mapsto m$ case of the product side of (1.11), provided $\lambda = qa^2/bcd$. Replace $B(m)$ in (1.9), by this case of the ${}_8\phi_7$ sum in (1.11), and obtain a double sum in the same form as the left hand side of (1.1). Here, $C(k,m)$ represents the summand in the above specialization of the ${}_8\phi_7$.

Next, we interchange the summation as in (1.1). Now it is not difficult to choose $A(m)$ so that the inner sum is also summable by using Jackson's sum. The choice of $A(m)$ which works is given by

$$A(m) = \frac{(1-aq^{2m})\,(a;q)_m\,(e;q)_m\,(f;q)_m\,(a\lambda q^{1+n}/ef;q)_m\,(a/\lambda;q)_m\,(q^{-n};q)_m}{(1-a)\,(q;q)_m\,(aq/e;q)_m\,(aq/f;q)_m\,(efq^{-n}/\lambda;q)_m\,(\lambda q;q)_m\,(aq^{1+n};q)_m}\, q^m.$$

With the above choice of $A(m)$ and $B(m)$, (1.9) becomes the ${}_{10}\phi_9$ on the left hand side of Theorem 1.7. Finally, we use the $a \mapsto aq^k$, $b \mapsto eq^k$, $c \mapsto fq^k$, $d \mapsto a/\lambda$, and $n \mapsto n-k$ case of (1.11) to sum the inner sum on the right hand side of (1.1). Some elementary simplification using the identities in [11, Appendix I] completes the proof of Theorem 1.7. For more details, see [11]. □

Next, we note the conventions for naming our series as A_n, C_n, or D_n basic hypergeometric series. We consider multiple series of the form

$$\sum_{\substack{k_i\geq 0\\ i=1,\dots,n}} S(\mathbf{k}), \tag{1.12}$$

where $\mathbf{k}=(k_1,\dots,k_n)$, which reduce to classical basic hypergeometric series when $n=1$. We call such a multiple basic hypergeometric series *balanced* if it reduces to a balanced series when $n=1$. Well-poised and very-well-poised series are defined similarly.

Further, such a multiple series is called a C_n basic hypergeometric series if the summand $S(\mathbf{k})$ contains the factor

$$\prod_{1\leq i<j\leq n}\left[\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right]\prod_{1\leq i<j\leq n}\left[\frac{1-q^{k_i+k_j}x_ix_j}{1-x_ix_j}\right]\prod_{i=1}^{n}\left[\frac{1-x_i^2q^{2k_i}}{1-x_i^2}\right]. \tag{1.13}$$

Note that when $n=1$, the first two factors disappear, and (1.13) reduces to

$$\frac{1-x_1^2q^{2k_1}}{1-x_1^2}. \tag{1.14}$$

The ratio (1.14) is reminiscent of (1.6). Indeed, in C_n series, x_1^2 acts like the special parameter of a very-well poised series. In our statements of C_n theorems, we set $x_i\mapsto\sqrt{a}x_i$ for $i=1,\dots,n$, and make similar changes to other parameters in $S(\mathbf{k})$. This is done in order to follow the classical notation in [11] as closely as possible. See our remarks after Theorem 2.1 and Theorem 3.1. A typical example of a C_n basic hypergeometric series is the left hand side of (A.7).

D_n multiple basic series are closely related to C_n series. Instead of (1.13), $S(\mathbf{k})$ only has the following factors:

$$\prod_{1\leq i<j\leq n}\left[\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right]\prod_{1\leq i<j\leq n}\left[\frac{1-q^{k_i+k_j}x_ix_j}{1-x_ix_j}\right]. \tag{1.15}$$

A typical example is the left hand side of (A.13).

Finally, A_n basic hypergeometric series only have

$$\prod_{1\leq i<j\leq n}\left[\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right] \tag{1.16}$$

as a factor of $S(\mathbf{k})$. A typical example is the left hand side of (A.4). The other differences between A_n, C_n, and D_n series are best understood by comparing the different types of series appearing in this paper. A reason for naming these series as A_n, C_n, or D_n series is that (1.16), (1.13), and (1.15) are closely associated with the product side of the Weyl denominator formulas for the respective root systems, see [7, 34].

We terminate our multiple series in two ways: If the summand contains the factor

$$\prod_{i,j=1}^{n} (q^{-N_j} x_i/x_j; q)_{k_i}$$

in the numerator, then it vanishes if $k_i > N_i$ for $i = 1, 2, \ldots, n$. In this case we have the natural bounds $0 \leq k_i \leq N_i$, for $i = 1, 2, \ldots, n$, and we say that the series is summed over an n-rectangle. On the other hand, if the summand contains the factor

$$(q^{-N}; q)_{k_1+\cdots+k_n}$$

in the numerator, then it vanishes if $k_1 + \cdots + k_n > N$. In this case we have the natural bound $0 \leq k_1 + \cdots + k_n \leq N$ for all $k_i \geq 0$, and we say that the series is summed over an n-tetrahedron. Many identities involving series summed over an n-rectangle are equivalent to a corresponding identity, where the the series involved are summed over an n-tetrahedron, see e.g. Theorem 3.1 and Theorem 3.7, or Theorem 3.9 and Theorem 3.11.

Finally, we mention some notation used to simplify our displays. We employ the notation $|\mathbf{k}|$ for $(k_1 + \cdots + k_n)$, where $\mathbf{k} = (k_1, \ldots, k_n)$. This notation is also applied to the vectors $\mathbf{N}$ and $\mathbf{m}$. We also find it useful to denote by $\mathbf{m} + \mathbf{k}$ the vector $(m_1 + k_1, \ldots, m_n + k_n)$ formed by component-wise addition of the two vectors.

2. A C_n ${}_{10}\phi_9$ TRANSFORMATION

In this section, we present a C_n generalization of Theorem 1.7. We use a straightforward multiple series extension of (1.1), and two multiple series generalizations of Theorem 1.10. These are: A D_n extension of Jackson's sum found by Schlosser [29, 30]; and a C_n summation theorem found independently by Denis and Gustafson [10] and Milne and Lilly [25]. These results are presented in the appendix, see Theorem A.12 and Theorem A.6.

Consider the substitutions $a \mapsto qa^2/bcd$, $b \mapsto aq/cd$, $c \mapsto aq/bd$, $d \mapsto aq/bc$ in Theorem 1.7. With these substitutions, λ becomes a, and the ${}_{10}\phi_9$ on the right hand side of (1.8) gets transformed into the one on the left, and vice-versa. Once again, we obtain Bailey's formula (1.8). This is quite a striking symmetry of Bailey's transformation. This symmetry holds for all the multiple series extensions of (1.8) in

[26] and in §3. Remarkably enough, it does not hold in the following generalization of Theorem 1.7:

Theorem 2.1 (A $\boldsymbol{C_n}$ $\boldsymbol{{}_{10}\phi_9}$ transformation)**.** *Let a, b, c, d, e, f, and $x_1,\dots,x_n$ be indeterminate, let N_i be nonnegative integers for $i=1,2,\dots,n$ with $n\geq 1$, and suppose that none of the denominators in (2.2) vanish. Then*

$$\begin{aligned}
&\sum_{\substack{0\leq k_i\leq N_i\\ i=1,2,\dots,n}} \Bigg(\prod_{i=1}^{n} \left(\frac{1-ax_i^2q^{2k_i}}{1-ax_i^2}\right) \prod_{1\leq i<j\leq n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\,\frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j}\right)\\
&\qquad\times \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j;q)_{k_i}\,(ax_ix_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq^{1+N_j};q)_{k_i}}\\
&\qquad\times \prod_{i=1}^{n} \frac{(bx_i;q)_{k_i}\,(cx_i;q)_{k_i}\,(dx_i;q)_{k_i}}{(ax_iq/b;q)_{k_i}\,(ax_iq/c;q)_{k_i}\,(ax_iq/d;q)_{k_i}}\\
&\qquad\times \prod_{i=1}^{n} \frac{(ex_i;q)_{k_i}\,(fx_i;q)_{k_i}\,(a\lambda x_iq^{1+|\mathbf{N}|}/ef;q)_{k_i}}{(ax_iq/e;q)_{k_i}\,(ax_iq/f;q)_{k_i}\,(efx_iq^{-|\mathbf{N}|}/\lambda;q)_{k_i}}\, q^{\sum_{i=1}^n ik_i}\Bigg)\\
&= \prod_{1\leq i<j\leq n} (ax_ix_jq;q)^{-1}_{N_i+N_j} \prod_{i,j=1}^{n} (ax_ix_jq;q)_{N_i}\\
&\times \frac{(\lambda q/e;q)_{|\mathbf{N}|}\,(\lambda q/f;q)_{|\mathbf{N}|}\,(aq/ef;q)_{|\mathbf{N}|}}{\prod_{i=1}^{n}\left[(\lambda x_iq;q)_{N_i}\,(ax_iq/e;q)_{N_i}\,(ax_iq/f;q)_{N_i}\,(\lambda q^{1+|\mathbf{N}|-\mathbf{N_i}}/efx_i;q)_{N_i}\right]}\\
&\times \sum_{\substack{0\leq k_i\leq N_i\\ i=1,2,\dots,n}} \Bigg(\prod_{i=1}^{n} \left(\frac{1-\lambda x_iq^{k_i+|\mathbf{k}|}}{1-\lambda x_i}\right) \prod_{1\leq i<j\leq n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\\
&\qquad\times \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{n} \frac{(\lambda x_i;q)_{|\mathbf{k}|}}{(\lambda x_iq^{1+N_i};q)_{|\mathbf{k}|}}\\
&\qquad\times \prod_{i=1}^{n} \frac{(ex_i;q)_{k_i}\,(fx_i;q)_{k_i}\,(a\lambda x_iq^{1+|\mathbf{N}|}/ef;q)_{k_i}}{(ax_iq/b;q)_{k_i}\,(ax_iq/c;q)_{k_i}\,(ax_iq/d;q)_{k_i}}\\
&\qquad\times \frac{(\lambda b/a;q)_{|\mathbf{k}|}\,(\lambda c/a;q)_{|\mathbf{k}|}\,(\lambda d/a;q)_{|\mathbf{k}|}}{(\lambda q/e;q)_{|\mathbf{k}|}\,(\lambda q/f;q)_{|\mathbf{k}|}\,(efq^{-|\mathbf{N}|}/a;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^n ik_i}\Bigg), \qquad (2.2)
\end{aligned}$$

where $\lambda = qa^2/bcd$.

Remark 2.3. Theorem 2.1 is equivalent to its special case where $a=1$. However, we have stated it with one extra parameter in order to closely follow Gasper and Rahman [11]. When $a=1$ in (2.2), it is easy to find

(1.13) on the left hand side of the resulting transformation. Thus we call it a C_n sum. The sum on the right hand side of (2.2) is an A_n sum.

Remark 2.4. We call Theorem 2.1 a C_n theorem, since a limiting case yields Milne and Lilly's [25] C_n generalization of Watson's transformation. However, note that (2.2) transforms a terminating, balanced, and very-well-poised C_n ${}_{10}\phi_9$ sum into a multiple of a terminating A_n series, which is also balanced and very-well-poised. In view of our earlier remarks, it may also be written so that it transforms an A_n series into a C_n series.

Proof. The left side of (2.2) can be written in the form

$$\sum_{\substack{0\le m_i\le N_i\\ i=1,2,\dots,n}} A(\mathbf{m})\, B(\mathbf{m}), \tag{2.5}$$

where

$$\begin{aligned} A(\mathbf{m}) = \prod_{i=1}^{n}\left(\frac{1-ax_i^2q^{2m_i}}{1-ax_i^2}\right) &\prod_{1\le i<j\le n}\left(\frac{1-q^{m_i-m_j}x_i/x_j}{1-x_i/x_j}\,\frac{1-ax_ix_jq^{m_i+m_j}}{1-ax_ix_j}\right)\\ \times\prod_{i=1}^{n}&\frac{(ex_i;q)_{m_i}\,(fx_i;q)_{m_i}\,(a\lambda x_iq^{1+|\mathbf{N}|}/ef;q)_{m_i}\,(ax_i/\lambda;q)_{m_i}}{(ax_iq/e;q)_{m_i}\,(ax_iq/f;q)_{m_i}\,(efx_iq^{-|\mathbf{N}|}/\lambda;q)_{m_i}\,(\lambda x_iq;q)_{m_i}}\\ &\times\prod_{i,j=1}^{n}\frac{(q^{-N_j}x_i/x_j;q)_{m_i}\,(ax_ix_j;q)_{m_i}}{(qx_i/x_j;q)_{m_i}\,(ax_ix_jq^{1+N_j};q)_{m_i}}\;q^{\sum_{i=1}^n i\,m_i} \end{aligned}$$

and

$$B(\mathbf{m}) = \prod_{i=1}^{n}\frac{(bx_i;q)_{m_i}\,(cx_i;q)_{m_i}\,(dx_i;q)_{m_i}\,(\lambda x_iq;q)_{m_i}}{(ax_iq/b;q)_{m_i}\,(ax_iq/c;q)_{m_i}\,(ax_iq/d;q)_{m_i}\,(ax_i/\lambda;q)_{m_i}}.$$

By Theorem A.12 we may substitute

$$\begin{aligned} \sum_{\substack{0\le k_i\le m_i\\ i=1,2,\dots,n}}\Bigg(&\prod_{i=1}^{n}\left(\frac{1-\lambda x_iq^{k_i+|\mathbf{k}|}}{1-\lambda x_i}\right)\prod_{1\le i<j\le n}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\\ &\times\prod_{1\le i<j\le n}(ax_ix_j;q)^{-1}_{k_i+k_j}\prod_{i,j=1}^{n}(ax_ix_jq^{m_j};q)_{k_i}\\ &\times\prod_{i,j=1}^{n}\frac{(q^{-m_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\prod_{i=1}^{n}\frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(\lambda q/ax_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}{(\lambda x_iq^{1+m_i};q)_{|\mathbf{k}|}\,(\lambda q^{1-m_i}/ax_i;q)_{|\mathbf{k}|}}\\ &\times\frac{(\lambda b/a;q)_{|\mathbf{k}|}\,(\lambda c/a;q)_{|\mathbf{k}|}\,(\lambda d/a;q)_{|\mathbf{k}|}}{\prod_{i=1}^{n}\left[(ax_iq/b;q)_{k_i}\,(ax_iq/c;q)_{k_i}\,(ax_iq/d;q)_{k_i}\right]}\;q^{\sum_{i=1}^n i\,k_i}\Bigg) \end{aligned} \tag{2.6}$$

for $B(\mathbf{m})$ in (2.5). Write $C(\mathbf{k},\mathbf{m})$ for the summand in (2.6). Thus, the left side of (2.2) equals

$$\sum_{\substack{0\le m_i\le N_i\\ i=1,\dots,n}} \sum_{\substack{0\le k_i\le m_i\\ i=1,\dots,n}} A(\mathbf{m})\, C(\mathbf{k},\mathbf{m}).$$

Now we interchange the summations

$$\sum_{\substack{0\le k_i\le N_i\\ i=1,\dots,n}} \sum_{\substack{k_i\le m_i\le N_i\\ i=1,\dots,n}} A(\mathbf{m})\, C(\mathbf{k},\mathbf{m}),$$

and after shifting each m_i to m_i+k_i in the inner sum while changing the range of summation we obtain

$$\sum_{\substack{0\le k_i\le N_i\\ i=1,\dots,n}} \sum_{\substack{0\le m_i\le N_i-k_i\\ i=1,\dots,n}} A(\mathbf{m}+\mathbf{k})\, C(\mathbf{k},\mathbf{m}+\mathbf{k}). \tag{2.7}$$

Explicitly, after rearranging terms, (2.7) is

$$\begin{aligned}
&\sum_{\substack{0\le k_i\le N_i\\ i=1,2,\dots,n}} \left(\prod_{i=1}^{n} \left(\frac{1-\lambda x_i q^{k_i+|\mathbf{k}|}}{1-\lambda x_i} \right) \prod_{1\le i<j\le n} \left(\frac{1-q^{k_i-k_j} x_i/x_j}{1-x_i/x_j} \right) \right.\\
&\qquad \times \prod_{1\le i<j\le n} (a x_i x_j;q)^{-1}_{k_i+k_j} \prod_{i,j=1}^{n} (a x_i x_j q^{k_j};q)_{k_i}\\
&\quad \times \prod_{i,j=1}^{n} \frac{(q^{-k_j} x_i/x_j;q)_{k_i}}{(q x_i/x_j;q)_{k_i}} \prod_{i=1}^{n} \frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(\lambda q/a x_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}{(\lambda x_i q^{1+k_i};q)_{|\mathbf{k}|}\,(\lambda q^{1-k_i}/a x_i;q)_{|\mathbf{k}|}}\\
&\quad \times \frac{(\lambda b/a;q)_{|\mathbf{k}|}\,(\lambda c/a;q)_{|\mathbf{k}|}\,(\lambda d/a;q)_{|\mathbf{k}|}}{\prod_{i=1}^{n} [(a x_i q/b;q)_{k_i}\,(a x_i q/c;q)_{k_i}\,(a x_i q/d;q)_{k_i}]}\, q^{\sum_{i=1}^{n} i k_i}\\
&\quad \times \prod_{i=1}^{n} \left(\frac{1-a x_i^2 q^{2k_i}}{1-a x_i^2} \right) \prod_{1\le i<j\le n} \left(\frac{1-q^{k_i-k_j} x_i/x_j}{1-x_i/x_j}\, \frac{1-a x_i x_j q^{k_i+k_j}}{1-a x_i x_j} \right)\\
&\quad \times \prod_{i=1}^{n} \frac{(e x_i;q)_{k_i}\,(f x_i;q)_{k_i}\,(a\lambda x_i q^{1+|\mathbf{N}|}/ef;q)_{k_i}\,(a x_i/\lambda;q)_{k_i}}{(a x_i q/e;q)_{k_i}\,(a x_i q/f;q)_{k_i}\,(ef x_i q^{-|\mathbf{N}|}/\lambda;q)_{k_i}\,(\lambda x_i q;q)_{k_i}}\\
&\qquad \times \prod_{i,j=1}^{n} \frac{(q^{-N_j} x_i/x_j;q)_{k_i}\,(a x_i x_j;q)_{k_i}}{(q x_i/x_j;q)_{k_i}\,(a x_i x_j q^{1+N_j};q)_{k_i}}\, q^{\sum_{i=1}^{n} i k_i}\\
&\times \sum_{\substack{0\le m_i\le N_i-k_i\\ i=1,2,\dots,n}} \prod_{1\le i<j\le n} \left(\frac{1-q^{m_i-m_j+k_i-k_j} x_i/x_j}{1-q^{k_i-k_j} x_i/x_j}\, \frac{1-a x_i x_j q^{k_i+k_j+m_i+m_j}}{1-a x_i x_j q^{k_i+k_j}} \right)
\end{aligned}$$

$$\times \prod_{i=1}^{n}\left(\frac{1-ax_i^2q^{2k_i+2m_i}}{1-ax_i^2q^{2k_i}}\right)\prod_{i=1}^{n}\frac{(ex_iq^{k_i};q)_{m_i}\,(fx_iq^{k_i};q)_{m_i}}{(ax_iq^{1+k_i}/e;q)_{m_i}\,(ax_iq^{1+k_i}/f;q)_{m_i}}$$
$$\times \prod_{i=1}^{n}\frac{(a\lambda x_iq^{1+k_i+|\mathbf{N}|}/ef;q)_{m_i}\,(ax_iq^{k_i-|\mathbf{k}|}/\lambda;q)_{m_i}}{(efx_iq^{k_i-|\mathbf{N}|}/\lambda;q)_{m_i}\,(\lambda x_iq^{1+k_i+|\mathbf{k}|};q)_{m_i}}$$
$$\times \prod_{i,j=1}^{n}\frac{(q^{k_i-N_j}x_i/x_j;q)_{m_i}\,(ax_ix_jq^{k_i+k_j};q)_{m_i}}{(q^{1+k_i-k_j}x_i/x_j;q)_{m_i}\,(ax_ix_jq^{1+k_i+N_j};q)_{m_i}}\,q^{\sum_{i=1}^{n} im_i}\Bigg).$$

Summing the inner sum by means of the

$$\begin{aligned}&b\mapsto e, \qquad c\mapsto f, \qquad d\mapsto aq^{-|\mathbf{k}|}/\lambda,\\&x_i\mapsto x_iq^{k_i}, \quad N_i\mapsto N_i-k_i, \quad \text{for } i=1,2,\dots,n\end{aligned}$$

case of Theorem A.6 and simplifying terms by Lemma A.1 and some elementary manipulations (cf. [11, Appendix I]), we obtain the right side of (2.2). □

Remark 2.8. An alternative way to derive Theorem 2.1 is to use Theorem A.6 in the inner sum, and then employ Bhatnagar's D_n ${}_8\phi_7$ summation, Theorem A.9, after exchanging sums. Then we would arrive at identity (2.2) too, but with the right hand side on the left and vice-versa.

3. Some D_n ${}_{10}\phi_9$ transformations

In this section, we follow the method of §2 and derive some D_n ${}_{10}\phi_9$ transformations. We use various A_n and D_n extensions of Jackson's sum which are collected together in Appendix A. For instance, by combining the D_n summation theorems of Bhatnagar [7] and Schlosser [29], we obtain Theorem 3.1. Instead, by combining Bhatnagar's [7] D_n Jackson's sum with Milne's [21] A_n sum, we get another D_n transformation, Theorem 3.9. In addition, by reversing series, and relabeling parameters, we obtain Theorem 3.13.

The transformation formulas obtained in this manner involve series which are summed over an n-rectangle. A standard polynomial argument, see for example [23], leads to an equivalent formulation of each of these transformations, where the series involved are summed over an n-tetrahedron.

We begin by combining both D_n ${}_8\phi_7$ summations, Theorem A.9 and Theorem A.12, to obtain the following D_n transformation.

Theorem 3.1 (A $\boldsymbol{D_n}$ ${}_{\boldsymbol{10}}\boldsymbol{\phi}_{\boldsymbol{9}}$ transformation)**.** *Let a, b, c, d, e, f and $x_1,\dots,x_n$ be indeterminate, let N_i be nonnegative integers for*

$i = 1, 2, \ldots, n$ with $n \geq 1$, and suppose that none of the denominators in (3.2) *vanish. Then*

$$
\begin{aligned}
&\sum_{\substack{0 \leq k_i \leq N_i \\ i=1,2,\ldots,n}} \Bigg(\prod_{i=1}^n \left(\frac{1 - a x_i q^{k_i + |\mathbf{k}|}}{1 - a x_i} \right) \prod_{1 \leq i < j \leq n} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right) \\
&\qquad \times \prod_{1 \leq i < j \leq n} \frac{(a x_i x_j q / d; q)_{k_i + k_j}}{(\lambda a x_i x_j q / ef; q)_{k_i + k_j}} \prod_{i=1}^n \frac{(ef / \lambda x_i; q)_{|\mathbf{k}| - \mathbf{k_i}}}{(d / x_i; q)_{|\mathbf{k}| - \mathbf{k_i}}} \\
&\qquad \times \prod_{i,j=1}^n \frac{(q^{-N_j} x_i / x_j; q)_{k_i} \, (\lambda a x_i x_j q^{1 + N_j} / ef; q)_{k_i}}{(q x_i / x_j; q)_{k_i} \, (a x_i x_j q / d; q)_{k_i}} \\
&\qquad \times \prod_{i=1}^n \frac{(a x_i; q)_{|\mathbf{k}|} \, (d / x_i; q)_{|\mathbf{k}|}}{(a x_i q^{1 + N_i}; q)_{|\mathbf{k}|} \, (ef q^{-N_i} / \lambda x_i; q)_{|\mathbf{k}|}} \\
&\qquad \times \prod_{i=1}^n \frac{(b x_i; q)_{k_i} \, (c x_i; q)_{k_i}}{(a x_i q / e; q)_{k_i} \, (a x_i q / f; q)_{k_i}} \\
&\qquad \times \frac{(e; q)_{|\mathbf{k}|} \, (f; q)_{|\mathbf{k}|}}{(aq/b; q)_{|\mathbf{k}|} \, (aq/c; q)_{|\mathbf{k}|}} \, q^{\sum_{i=1}^n i k_i} \Bigg) \\
&= \prod_{i=1}^n \frac{(a x_i q; q)_{N_i} \, (a x_i q / ef; q)_{N_i} \, (\lambda x_i q / e; q)_{N_i} \, (\lambda x_i q / f; q)_{N_i}}{(\lambda x_i q / ef; q)_{N_i} \, (\lambda x_i q; q)_{N_i} \, (a x_i q / f; q)_{N_i} \, (a x_i q / e; q)_{N_i}} \\
&\times \sum_{\substack{0 \leq k_i \leq N_i \\ i=1,2,\ldots,n}} \Bigg(\prod_{i=1}^n \left(\frac{1 - \lambda x_i q^{k_i + |\mathbf{k}|}}{1 - \lambda x_i} \right) \prod_{1 \leq i < j \leq n} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right) \\
&\qquad \times \prod_{1 \leq i < j \leq n} \frac{(a x_i x_j q / d; q)_{k_i + k_j}}{(\lambda a x_i x_j q / ef; q)_{k_i + k_j}} \prod_{i=1}^n \frac{(ef / a x_i; q)_{|\mathbf{k}| - \mathbf{k_i}}}{(\lambda d / a x_i; q)_{|\mathbf{k}| - \mathbf{k_i}}} \\
&\qquad \times \prod_{i,j=1}^n \frac{(q^{-N_j} x_i / x_j; q)_{k_i} \, (\lambda a x_i x_j q^{1 + N_j} / ef; q)_{k_i}}{(q x_i / x_j; q)_{k_i} \, (a x_i x_j q / d; q)_{k_i}} \\
&\qquad \times \prod_{i=1}^n \frac{(\lambda x_i; q)_{|\mathbf{k}|} \, (\lambda d / a x_i; q)_{|\mathbf{k}|}}{(\lambda x_i q^{1 + N_i}; q)_{|\mathbf{k}|} \, (ef q^{-N_i} / a x_i; q)_{|\mathbf{k}|}} \\
&\qquad \times \prod_{i=1}^n \frac{(\lambda b x_i / a; q)_{k_i} \, (\lambda c x_i / a; q)_{k_i}}{(\lambda x_i q / e; q)_{k_i} \, (\lambda x_i q / f; q)_{k_i}} \\
&\qquad \times \frac{(e; q)_{|\mathbf{k}|} \, (f; q)_{|\mathbf{k}|}}{(aq/b; q)_{|\mathbf{k}|} \, (aq/c; q)_{|\mathbf{k}|}} \, q^{\sum_{i=1}^n i k_i} \Bigg), \qquad (3.2)
\end{aligned}
$$

where $\lambda = qa^2/bcd$.

Remark 3.3. Theorem 3.1 is equivalent to its special case obtained by setting $d \mapsto aq/d$, and then $d = 1$. However, we have stated it with one extra parameter in order to closely follow Gasper and Rahman [11]. In the above special case of (3.2), it is easy to find (1.15) in both sums of the resulting transformation. Thus we call it a transformation formula for D_n series. A similar remark applies to all the multivariable identities in this paper.

Proof. The left side of (3.2) can be written in the form (2.5) where

$$A(\mathbf{m}) = \prod_{i=1}^{n} \left(\frac{1 - ax_i q^{m_i + |\mathbf{m}|}}{1 - ax_i} \right) \prod_{1 \le i < j \le n} \left(\frac{1 - q^{m_i - m_j} x_i/x_j}{1 - x_i/x_j} \right)$$
$$\times \prod_{1 \le i < j \le n} (\lambda a x_i x_j q/ef; q)^{-1}_{m_i + m_j} \prod_{i,j=1}^{n} (\lambda a x_i x_j q^{1+N_j}/ef; q)_{m_i}$$
$$\times \prod_{i,j=1}^{n} \frac{(q^{-N_j} x_i/x_j; q)_{m_i}}{(qx_i/x_j; q)_{m_i}} \prod_{i=1}^{n} \frac{(ax_i; q)_{|\mathbf{m}|} \, (ef/\lambda x_i; q)_{|\mathbf{m}| - \mathbf{m_i}}}{(ax_i q^{1+N_i}; q)_{|\mathbf{m}|} \, (efq^{-N_i}/\lambda x_i; q)_{|\mathbf{m}|}}$$
$$\times \frac{(e; q)_{|\mathbf{m}|} \, (f; q)_{|\mathbf{m}|} \, (a/\lambda; q)_{|\mathbf{m}|}}{\prod_{i=1}^{n} \left[(ax_i q/e; q)_{m_i} \, (ax_i q/f; q)_{m_i} \, (\lambda x_i q; q)_{m_i} \right]} \, q^{\sum_{i=1}^{n} i m_i}$$

and

$$B(\mathbf{m}) = \prod_{1 \le i < j \le n} (ax_i x_j q/d; q)_{m_i + m_j} \prod_{i,j=1}^{n} (ax_i x_j q/d; q)^{-1}_{m_i}$$
$$\times \frac{\prod_{i=1}^{n} \left[(bx_i; q)_{m_i} \, (cx_i; q)_{m_i} \, (dq^{|\mathbf{m}| - \mathbf{m_i}}/x_i; q)_{m_i} \, (\lambda x_i q; q)_{m_i} \right]}{(aq/b; q)_{|\mathbf{m}|} \, (aq/c; q)_{|\mathbf{m}|} \, (a/\lambda; q)_{|\mathbf{m}|}}.$$

By Theorem A.9 we may substitute

$$\sum_{\substack{0 \le k_i \le m_i \\ i = 1, 2, \ldots, n}} \left(\prod_{i=1}^{n} \left(\frac{1 - \lambda x_i q^{k_i + |\mathbf{k}|}}{1 - \lambda x_i} \right) \prod_{1 \le i < j \le n} \left(\frac{1 - q^{k_i - k_j} x_i/x_j}{1 - x_i/x_j} \right) \right.$$
$$\times \prod_{1 \le i < j \le n} (ax_i x_j q/d; q)_{k_i + k_j} \prod_{i,j=1}^{n} (ax_i x_j q/d; q)^{-1}_{k_i}$$
$$\times \prod_{i,j=1}^{n} \frac{(q^{-m_j} x_i/x_j; q)_{k_i}}{(qx_i/x_j; q)_{k_i}} \prod_{i=1}^{n} \frac{(\lambda x_i; q)_{|\mathbf{k}|} \, (\lambda d/ax_i; q)_{|\mathbf{k}|}}{(\lambda x_i q^{1+m_i}; q)_{|\mathbf{k}|} \, (\lambda d/ax_i; q)_{|\mathbf{k}| - \mathbf{k_i}}}$$

$$\times \frac{\prod_{i=1}^{n} \left[(\lambda b x_i/a;q)_{k_i}\,(\lambda c x_i/a;q)_{k_i}\,(a x_i q^{|\mathbf{m}|};q)_{k_i}\right]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}\,(\lambda q^{1-|\mathbf{m}|}/a;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^n i k_i}\Bigg) \tag{3.4}$$

for $B(\mathbf{m})$ in (2.5). Write $C(\mathbf{k},\mathbf{m})$ for the summand in (3.4). As in the proof of Theorem 2.1, the left side of (3.2) equals

$$\sum_{\substack{0\le k_i\le N_i\\ i=1,\ldots,n}} \sum_{\substack{0\le m_i\le N_i-k_i\\ i=1,\ldots,n}} A(\mathbf{m}+\mathbf{k})\, C(\mathbf{k},\mathbf{m}+\mathbf{k}). \tag{3.5}$$

Explicitly, after rearranging terms, (3.5) is

$$\sum_{\substack{0\le k_i\le N_i\\ i=1,2,\ldots,n}} \Bigg(\prod_{i=1}^{n} \left(\frac{1-\lambda x_i q^{k_i+|\mathbf{k}|}}{1-\lambda x_i}\right) \prod_{1\le i<j\le n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times \prod_{1\le i<j\le n} (a x_i x_j q/d;q)_{k_i+k_j} \prod_{i,j=1}^{n} (a x_i x_j q/d;q)_{k_i}^{-1}$$
$$\times \prod_{i,j=1}^{n} \frac{(q^{-k_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{n} \frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(\lambda d/a x_i;q)_{|\mathbf{k}|}}{(\lambda x_i q^{1+k_i};q)_{|\mathbf{k}|}\,(\lambda d/a x_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}$$
$$\times \frac{\prod_{i=1}^{n} \left[(\lambda b x_i/a;q)_{k_i}\,(\lambda c x_i/a;q)_{k_i}\,(a x_i q^{|\mathbf{k}|};q)_{k_i}\right]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}\,(\lambda q^{1-|\mathbf{k}|}/a;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^n i k_i}$$
$$\times \prod_{i=1}^{n} \left(\frac{1-a x_i q^{k_i+|\mathbf{k}|}}{1-a x_i}\right) \prod_{1\le i<j\le n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times \prod_{1\le i<j\le n} (\lambda a x_i x_j q/ef;q)_{k_i+k_j}^{-1} \prod_{i,j=1}^{n} (\lambda a x_i x_j q^{1+N_j}/ef;q)_{k_i}$$
$$\times \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{n} \frac{(a x_i;q)_{|\mathbf{k}|}\,(ef/\lambda x_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}{(a x_i q^{1+N_i};q)_{|\mathbf{k}|}\,(efq^{-N_i}/\lambda x_i;q)_{|\mathbf{k}|}}$$
$$\times \frac{(e;q)_{|\mathbf{k}|}\,(f;q)_{|\mathbf{k}|}\,(a/\lambda;q)_{|\mathbf{k}|}}{\prod_{i=1}^{n} \left[(a x_i q/e;q)_{k_i}\,(a x_i q/f;q)_{k_i}\,(\lambda x_i q;q)_{k_i}\right]}\, q^{\sum_{i=1}^n i k_i}$$
$$\times \sum_{\substack{0\le m_i\le N_i-k_i\\ i=1,2,\ldots,n}} \prod_{i=1}^{n} \left(\frac{1-a x_i q^{k_i+|\mathbf{k}|+m_i+|\mathbf{m}|}}{1-a x_i q^{k_i+|\mathbf{k}|}}\right) \prod_{1\le i<j\le n} \left(\frac{1-q^{m_i-m_j+k_i-k_j}x_i/x_j}{1-q^{k_i-k_j}x_i/x_j}\right)$$

$$\times \prod_{1\le i<j\le n} (\lambda a x_i x_j q^{1+k_i+k_j}/ef;q)^{-1}_{m_i+m_j} \prod_{i,j=1}^{n} (\lambda a x_i x_j q^{1+k_i+N_j}/ef;q)_{m_i}$$
$$\times \prod_{i,j=1}^{n} \frac{(q^{k_i-N_j}x_i/x_j;q)_{m_i}}{(q^{1+k_i-k_j}x_i/x_j;q)_{m_i}} \prod_{i=1}^{n} \frac{(ax_iq^{k_i+|\mathbf{k}|};q)_{|\mathbf{m}|}\,(efq^{|\mathbf{k}|-k_i}/\lambda x_i;q)_{|\mathbf{m}|-\mathbf{m_i}}}{(ax_iq^{1+|\mathbf{k}|+N_i};q)_{|\mathbf{m}|}\,(efq^{|\mathbf{k}|-N_i}/\lambda x_i;q)_{|\mathbf{m}|}}$$
$$\times \frac{(eq^{|\mathbf{k}|};q)_{|\mathbf{m}|}\,(fq^{|\mathbf{k}|};q)_{|\mathbf{m}|}\,(a/\lambda;q)_{|\mathbf{m}|}}{\prod_{i=1}^{n}\left[(ax_iq^{1+k_i}/e;q)_{m_i}\,(ax_iq^{1+k_i}/f;q)_{m_i}\,(\lambda x_iq^{1+k_i+|\mathbf{k}|};q)_{m_i}\right]}\, q^{\sum_{i=1}^{n} i m_i}\Bigg).$$

Summing the inner sum by means of the

$$\begin{aligned} &a\mapsto aq^{|\mathbf{k}|}, \qquad b\mapsto eq^{|\mathbf{k}|}, \qquad c\mapsto fq^{|\mathbf{k}|}, \quad d\mapsto a/\lambda, \\ &x_i\mapsto x_iq^{k_i}, \quad N_i\mapsto N_i-k_i, \quad \text{for } i=1,2,\dots,n \end{aligned}$$

case of Theorem A.12 and simplifying terms by Lemma A.1 and some elementary manipulations, we obtain the right side of (3.2). □

Remark 3.6. Some special limiting cases of Theorem 3.1, and the other transformation formulas in this section, yield many generalizations of Watson's transformation, see §4.

By using a polynomial argument we easily obtain

Theorem 3.7 (A D_n $_{10}\phi_9$ transformation)**.** *Let a, b, c, d, e, f_1, …, f_n and x_1, …, x_n be indeterminate, let N be a nonnegative integer, let $n\ge 1$, and suppose that none of the denominators in* (3.8) *vanish. Then*

$$\sum_{\substack{k_1,k_2,\dots,k_n\ge 0\\ 0\le|\mathbf{k}|\le \mathbf{N}}} \Bigg(\prod_{i=1}^{n}\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\le i<j\le n}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times \prod_{1\le i<j\le n} \frac{(ax_ix_jq/d;q)_{k_i+k_j}}{(\lambda ax_ix_jq^{1+N}/e;q)_{k_i+k_j}} \prod_{i=1}^{n} \frac{(eq^{-N}/\lambda x_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}{(d/x_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}$$
$$\times \prod_{i,j=1}^{n} \frac{(f_jx_i/x_j;q)_{k_i}\,(\lambda ax_ix_jq^{1+N}/ef_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq/d;q)_{k_i}}$$
$$\times \prod_{i=1}^{n} \frac{(ax_i;q)_{|\mathbf{k}|}\,(d/x_i;q)_{|\mathbf{k}|}}{(ax_iq/f_i;q)_{|\mathbf{k}|}\,(ef_iq^{-N}/\lambda x_i;q)_{|\mathbf{k}|}}$$
$$\times \prod_{i=1}^{n} \frac{(bx_i;q)_{k_i}\,(cx_i;q)_{k_i}}{(ax_iq/e;q)_{k_i}\,(ax_iq^{1+N};q)_{k_i}}$$

$$\times \frac{(e;q)_{|\mathbf{k}|}\,(q^{-N};q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^n i k_i}\Bigg)$$

$$= \prod_{i=1}^n \frac{(ax_iq;q)_N\,(ax_iq/ef_i;q)_N\,(\lambda x_iq/e;q)_N\,(\lambda x_iq/f_i;q)_N}{(\lambda x_iq/ef_i;q)_N\,(\lambda x_iq;q)_N\,(ax_iq/f_i;q)_N\,(ax_iq/e;q)_N}$$

$$\times \sum_{\substack{k_1,k_2,\ldots,k_n\geq 0\\ 0\leq|\mathbf{k}|\leq\mathbf{N}}} \Bigg(\prod_{i=1}^n\left(\frac{1-\lambda x_iq^{k_i+|\mathbf{k}|}}{1-\lambda x_i}\right)\prod_{1\leq i<j\leq n}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$

$$\times \prod_{1\leq i<j\leq n}\frac{(ax_ix_jq/d;q)_{k_i+k_j}}{(\lambda ax_ix_jq^{1+N}/e;q)_{k_i+k_j}}\prod_{i=1}^n\frac{(eq^{-N}/ax_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}{(\lambda d/ax_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}$$

$$\times \prod_{i,j=1}^n \frac{(f_jx_i/x_j;q)_{k_i}\,(\lambda ax_ix_jq^{1+N}/ef_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq/d;q)_{k_i}}$$

$$\times \prod_{i=1}^n \frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(\lambda d/ax_i;q)_{|\mathbf{k}|}}{(\lambda x_iq/f_i;q)_{|\mathbf{k}|}\,(ef_iq^{-N}/ax_i;q)_{|\mathbf{k}|}}$$

$$\times \prod_{i=1}^n \frac{(\lambda bx_i/a;q)_{k_i}\,(\lambda cx_i/a;q)_{k_i}}{(\lambda x_iq/e;q)_{k_i}\,(\lambda x_iq^{1+N};q)_{k_i}}$$

$$\times \frac{(e;q)_{|\mathbf{k}|}\,(q^{-N};q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^n i k_i}\Bigg), \tag{3.8}$$

where $\lambda = qa^2/bcd$.

Proof. First we write the product in front of the sum of the right side of (3.8) as quotient of infinite products using (1.2). Then by the $f = q^{-N}$ case of Theorem 3.1 it follows that the identity (3.8) holds for $f_j = q^{-N_j}$, $j = 1,\ldots,n$. By clearing out denominators in (3.8), we get a polynomial equation in f_1, which is true for q^{-N_1}, $N_1 = 0,1,\ldots$. Thus we obtain an identity in f_1. By carrying out this process for $f_2, f_3, \ldots, f_n$ also, we obtain Theorem 3.7. □

Next we use Theorem A.3 and Theorem A.9 to obtain

Theorem 3.9 (A $\boldsymbol{D_n}$ ${}_{\mathbf{10}}\boldsymbol{\phi}_{\mathbf{9}}$ transformation)**.** *Let a, b, c, d, e, f and $x_1,\ldots,x_n$ be indeterminate, let N_i be nonnegative integers for $i = 1,2,\ldots,n$ with $n \geq 1$, and suppose that none of the denominators in* (3.10) *vanish. Then*

$$\sum_{\substack{0\leq k_i\leq N_i\\ i=1,2,\ldots,n}} \Bigg(\prod_{i=1}^n\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right)\prod_{1\leq i<j\leq n}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$

$$
\times \prod_{1\le i<j\le n} (ax_ix_jq/d;q)_{k_i+k_j} \prod_{i,j=1}^{n} (ax_ix_jq/d;q)_{k_i}^{-1}
$$

$$
\times \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{n} \frac{(ax_i;q)_{|\mathbf{k}|}\,(d/x_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+N_i};q)_{|\mathbf{k}|}\,(d/x_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}
$$

$$
\times \frac{\prod_{i=1}^{n}\left[(bx_i;q)_{k_i}\,(cx_i;q)_{k_i}\,(ex_i;q)_{k_i}\,(\lambda ax_iq^{1+|\mathbf{N}|}/ef;q)_{k_i}\right]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}\,(aq/e;q)_{|\mathbf{k}|}\,(efq^{-|\mathbf{N}|}/\lambda;q)_{|\mathbf{k}|}}
$$

$$
\left.\times \frac{(f;q)_{|\mathbf{k}|}}{\prod_{i=1}^{n}(ax_iq/f;q)_{k_i}}\, q^{\sum_{i=1}^{n} i\,k_i}\right)
$$

$$
= \frac{(aq/ef;q)_{|\mathbf{N}|}\,(\lambda q/e;q)_{|\mathbf{N}|}}{(\lambda q/ef;q)_{|\mathbf{N}|}\,(aq/e;q)_{|\mathbf{N}|}} \prod_{i=1}^{n} \frac{(ax_iq;q)_{N_i}\,(\lambda x_iq/f;q)_{N_i}}{(\lambda x_iq;q)_{N_i}\,(ax_iq/f;q)_{N_i}}
$$

$$
\times \sum_{\substack{0\le k_i\le N_i\\ i=1,2,\dots,n}} \left(\prod_{i=1}^{n} \left(\frac{1-\lambda x_iq^{k_i+|\mathbf{k}|}}{1-\lambda x_i}\right) \prod_{1\le i<j\le n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\right.
$$

$$
\times \prod_{1\le i<j\le n} (ax_ix_jq/d;q)_{k_i+k_j} \prod_{i,j=1}^{n} (ax_ix_jq/d;q)_{k_i}^{-1}
$$

$$
\times \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{n} \frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(\lambda d/ax_i;q)_{|\mathbf{k}|}}{(\lambda x_iq^{1+N_i};q)_{|\mathbf{k}|}\,(\lambda d/ax_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}
$$

$$
\times \frac{\prod_{i=1}^{n}\left[(\lambda bx_i/a;q)_{k_i}\,(\lambda cx_i/a;q)_{k_i}\,(ex_i;q)_{k_i}\,(\lambda ax_iq^{1+|\mathbf{N}|}/ef;q)_{k_i}\right]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}\,(\lambda q/e;q)_{|\mathbf{k}|}\,(efq^{-|\mathbf{N}|}/a;q)_{|\mathbf{k}|}}
$$

$$
\left.\times \frac{(f;q)_{|\mathbf{k}|}}{\prod_{i=1}^{n}(\lambda x_iq/f;q)_{k_i}}\, q^{\sum_{i=1}^{n} i\,k_i}\right), \quad (3.10)
$$

where $\lambda = qa^2/bcd$*.*

Proof. The left side of (3.10) can be written in the form (2.5) where

$$
A(\mathbf{m}) = \prod_{i=1}^{n} \left(\frac{1-ax_iq^{m_i+|\mathbf{m}|}}{1-ax_i}\right) \prod_{1\le i<j\le n} \left(\frac{1-q^{m_i-m_j}x_i/x_j}{1-x_i/x_j}\right)
$$

$$
\times \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j;q)_{m_i}}{(qx_i/x_j;q)_{m_i}} \prod_{i=1}^{n} \frac{(ax_i;q)_{|\mathbf{m}|}}{(ax_iq^{1+N_i};q)_{|\mathbf{m}|}}
$$

$$\times \prod_{i=1}^{n} \frac{(ex_i;q)_{m_i}\,(\lambda a x_i q^{1+|\mathbf{N}|}/ef;q)_{m_i}}{(ax_iq/f;q)_{m_i}\,(\lambda x_i q;q)_{m_i}}$$

$$\times \frac{(f;q)_{|\mathbf{m}|}\,(a/\lambda;q)_{|\mathbf{m}|}}{(aq/e;q)_{|\mathbf{m}|}\,(efq^{-|\mathbf{N}|}/\lambda;q)_{|\mathbf{m}|}}\; q^{\sum_{i=1}^n i\,m_i}$$

and

$$B(\mathbf{m}) = \prod_{1\le i<j\le n} (ax_ix_jq/d;q)_{m_i+m_j} \prod_{i,j=1}^{n} (ax_ix_jq/d;q)_{m_i}^{-1}$$
$$\times \frac{\prod_{i=1}^{n}\left[(bx_i;q)_{m_i}\,(cx_i;q)_{m_i}\,(dq^{|\mathbf{m}|-\mathbf{m_i}}/x_i;q)_{m_i}\,(\lambda x_i q;q)_{m_i}\right]}{(aq/b;q)_{|\mathbf{m}|}\,(aq/c;q)_{|\mathbf{m}|}\,(a/\lambda;q)_{|\mathbf{m}|}}.$$

By Theorem A.9 we may substitute (3.4) for $B(\mathbf{m})$ in (2.5). Write $C(\mathbf{k},\mathbf{m})$ for the summand in (3.4). As in the proofs of Theorems 2.1 and 3.1, the left side of (3.10) equals

$$\sum_{\substack{0\le k_i\le N_i\\ i=1,\dots,n}} \sum_{\substack{0\le m_i\le N_i-k_i\\ i=1,\dots,n}} A(\mathbf{m}+\mathbf{k})\,C(\mathbf{k},\mathbf{m}+\mathbf{k}).$$

Explicitly, after rearranging terms, this is

$$\sum_{\substack{0\le k_i\le N_i\\ i=1,2,\dots,n}} \Bigg(\prod_{i=1}^{n}\left(\frac{1-\lambda x_i q^{k_i+|\mathbf{k}|}}{1-\lambda x_i}\right) \prod_{1\le i<j\le n}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$

$$\times \prod_{1\le i<j\le n} (ax_ix_jq/d;q)_{k_i+k_j} \prod_{i,j=1}^{n} (ax_ix_jq/d;q)_{k_i}^{-1}$$

$$\times \prod_{i,j=1}^{n} \frac{(q^{-k_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{n} \frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(\lambda d/ax_i;q)_{|\mathbf{k}|}}{(\lambda x_i q^{1+k_i};q)_{|\mathbf{k}|}\,(\lambda d/ax_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}$$

$$\times \frac{\prod_{i=1}^{n}\left[(\lambda b x_i/a;q)_{k_i}\,(\lambda c x_i/a;q)_{k_i}\,(ax_iq^{|\mathbf{k}|};q)_{k_i}\right]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}\,(\lambda q^{1-|\mathbf{k}|}/a;q)_{|\mathbf{k}|}}\; q^{\sum_{i=1}^n i\,k_i}$$

$$\times \prod_{i=1}^{n}\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\le i<j\le n}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$

$$\times \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{n} \frac{(ax_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+N_i};q)_{|\mathbf{k}|}}$$

$$\times \prod_{i=1}^{n} \frac{(ex_i;q)_{k_i}\,(\lambda a x_i q^{1+|\mathbf{N}|}/ef;q)_{k_i}}{(ax_iq/f;q)_{k_i}\,(\lambda x_i q;q)_{k_i}}$$

$$\times \frac{(f;q)_{|\mathbf{k}|}\,(a/\lambda;q)_{|\mathbf{k}|}}{(aq/e;q)_{|\mathbf{k}|}\,(efq^{-|\mathbf{N}|}/\lambda;q)_{|\mathbf{k}|}}\; q^{\sum_{i=1}^n i\,k_i}$$

$$\times \sum_{\substack{0\le m_i\le N_i-k_i\\ i=1,2,\dots,n}} \prod_{i=1}^n \left(\frac{1-ax_iq^{k_i+|\mathbf{k}|+\mathbf{m_i}+|\mathbf{m}|}}{1-ax_iq^{k_i+|\mathbf{k}|}}\right) \prod_{1\le i<j\le n} \left(\frac{1-q^{m_i-m_j+k_i-k_j}x_i/x_j}{1-q^{k_i-k_j}x_i/x_j}\right)$$

$$\times \prod_{i,j=1}^n \frac{(q^{k_i-N_j}x_i/x_j;q)_{m_i}}{(q^{1+k_i-k_j}x_i/x_j;q)_{m_i}} \prod_{i=1}^n \frac{(ax_iq^{k_i+|\mathbf{k}|};q)_{|\mathbf{m}|}}{(ax_iq^{1+|\mathbf{k}|+N_i};q)_{|\mathbf{m}|}}$$

$$\times \prod_{i=1}^n \frac{(ex_iq^{k_i};q)_{m_i}\,(\lambda ax_iq^{1+k_i+|\mathbf{N}|}/ef;q)_{m_i}}{(ax_iq^{1+k_i}/f;q)_{m_i}\,(\lambda x_iq^{1+k_i+|\mathbf{k}|};q)_{m_i}}$$

$$\times \frac{(fq^{|\mathbf{k}|};q)_{|\mathbf{m}|}\,(a/\lambda;q)_{|\mathbf{m}|}}{(aq^{1+|\mathbf{k}|}/e;q)_{|\mathbf{m}|}\,(efq^{|\mathbf{k}|-|\mathbf{N}|}/\lambda;q)_{|\mathbf{m}|}}\; q^{\sum_{i=1}^n i\,m_i}\Bigg).$$

Summing the inner sum by means of the

$$a\mapsto aq^{|\mathbf{k}|},\qquad b\mapsto fq^{|\mathbf{k}|},\qquad c\mapsto a/\lambda,\quad d\mapsto e,$$
$$x_i\mapsto x_iq^{k_i},\quad N_i\mapsto N_i-k_i,\quad \text{for } i=1,2,\dots,n$$

case of Theorem A.3 and simplifying terms by Lemma A.1 and some elementary manipulations, we obtain the right side of (3.10). ☐

By using a polynomial argument we obtain

Theorem 3.11 (A $\boldsymbol{D_n}$ $_{\mathbf{10}}\boldsymbol{\phi}_{\mathbf{9}}$ transformation)**.** *Let a, b, c, d, e, f_1, …, f_n and x_1, …, x_n be indeterminate, let N be a nonnegative integer, let $n\ge 1$, and suppose that none of the denominators in* (3.12) *vanish. Then*

$$\sum_{\substack{k_1,k_2,\dots,k_n\ge 0\\ 0\le|\mathbf{k}|\le\mathbf{N}}} \Bigg(\prod_{i=1}^n \left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\le i<j\le n}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$

$$\times \prod_{1\le i<j\le n} (ax_ix_jq/d;q)_{k_i+k_j} \prod_{i,j=1}^n (ax_ix_jq/d;q)_{k_i}^{-1}$$

$$\times \prod_{i,j=1}^n \frac{(f_jx_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^n \frac{(ax_i;q)_{|\mathbf{k}|}\,(d/x_i;q)_{|\mathbf{k}|}}{(ax_iq/f_i;q)_{|\mathbf{k}|}\,(d/x_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}$$

$$\times \frac{\prod_{i=1}^n \left[(bx_i;q)_{k_i}\,(cx_i;q)_{k_i}\,(ex_i;q)_{k_i}\,(\lambda ax_iq^{1+N}/ef_1\cdots f_n;q)_{k_i}\right]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}\,(aq/e;q)_{|\mathbf{k}|}\,(ef_1\cdots f_nq^{-N}/\lambda;q)_{|\mathbf{k}|}}$$

$$\times \frac{(q^{-N};q)_{|\mathbf{k}|}}{\prod_{i=1}^{n}(ax_iq^{1+N};q)_{k_i}}\, q^{\sum_{i=1}^n i k_i}\Bigg)$$

$$= \frac{(aq/ef_1\cdots f_n;q)_N\,(\lambda q/e;q)_N}{(\lambda q/ef_1\cdots f_n;q)_N\,(aq/e;q)_N}\prod_{i=1}^{n}\frac{(ax_iq;q)_N\,(\lambda x_iq/f_i;q)_N}{(\lambda x_iq;q)_N\,(ax_iq/f_i;q)_N}$$

$$\times \sum_{\substack{k_1,k_2,\ldots,k_n\ge 0\\ 0\le|\mathbf{k}|\le\mathbf{N}}}\Bigg(\prod_{i=1}^{n}\left(\frac{1-\lambda x_iq^{k_i+|\mathbf{k}|}}{1-\lambda x_i}\right)\prod_{1\le i<j\le n}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$

$$\times \prod_{1\le i<j\le n}(ax_ix_jq/d;q)_{k_i+k_j}\prod_{i,j=1}^{n}(ax_ix_jq/d;q)_{k_i}^{-1}$$

$$\times \prod_{i,j=1}^{n}\frac{(f_jx_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\prod_{i=1}^{n}\frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(\lambda d/ax_i;q)_{|\mathbf{k}|}}{(\lambda x_iq/f_i;q)_{|\mathbf{k}|}\,(\lambda d/ax_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}$$

$$\times \frac{\prod_{i=1}^{n}\left[(\lambda bx_i/a;q)_{k_i}\,(\lambda cx_i/a;q)_{k_i}\,(ex_i;q)_{k_i}\,(\lambda ax_iq^{1+N}/ef_1\cdots f_n;q)_{k_i}\right]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}\,(\lambda q/e;q)_{|\mathbf{k}|}\,(ef_1\cdots f_nq^{-N}/a;q)_{|\mathbf{k}|}}$$

$$\times \frac{(q^{-N};q)_{|\mathbf{k}|}}{\prod_{i=1}^{n}(\lambda x_iq^{1+N};q)_{k_i}}\, q^{\sum_{i=1}^n i k_i}\Bigg), \tag{3.12}$$

where $\lambda = qa^2/bcd$.

Proof. First we write the product in front of the sum of the right side of (3.12) as quotient of infinite products using (1.2). Then by the $f = q^{-N}$ case of Theorem 3.9 it follows that the identity (3.12) holds for $f_j = q^{-N_j}$, $j = 1,\ldots,n$. By clearing out denominators in (3.12), we get a polynomial equation in f_1, which is true for q^{-N_1}, $N_1 = 0,1,\ldots$. Thus we obtain an identity in f_1. By carrying out this process for $f_2, f_3, \ldots, f_n$ also, we obtain Theorem 3.11. □

By reversing sums in Theorem 3.9 we get

Theorem 3.13 (A $\boldsymbol{D_n}$ ${}_{\mathbf{10}}\boldsymbol{\phi}_{\mathbf{9}}$ transformation)**.** *Let a, b, c, d, e, f and $x_1,\ldots,x_n$ be indeterminate, let N_i be nonnegative integers for $i = 1,2,\ldots,n$ with $n \ge 1$, and suppose that none of the denominators in* (3.14) *vanish. Then*

$$\sum_{\substack{0\le k_i\le N_i\\ i=1,2,\ldots,n}}\Bigg(\prod_{i=1}^{n}\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right)\prod_{1\le i<j\le n}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$

$$
\begin{aligned}
&\times \prod_{1\le i<j\le n} (\lambda a x_i x_j q/ef;q)^{-1}_{k_i+k_j} \prod_{i,j=1}^n (\lambda a x_i x_j q^{1+N_j}/ef;q)_{k_i}\\
&\times \prod_{i,j=1}^n \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^n \frac{(ax_i;q)_{|\mathbf{k}|}\,(ef/\lambda x_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}{(ax_iq^{1+N_i};q)_{|\mathbf{k}|}\,(efq^{-N_i}/\lambda x_i;q)_{|\mathbf{k}|}}\\
&\times \frac{(c;q)_{|\mathbf{k}|}\,(d;q)_{|\mathbf{k}|}\,(e;q)_{|\mathbf{k}|}\,(f;q)_{|\mathbf{k}|}}{\prod_{i=1}^n\left[(ax_iq/c;q)_{k_i}\,(ax_iq/d;q)_{k_i}\,(ax_iq/e;q)_{k_i}\,(ax_iq/f;q)_{k_i}\right]}\\
&\qquad\qquad\qquad \times \frac{\prod_{i=1}^n (bx_i;q)_{k_i}}{(aq/b;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^n i\,k_i}\Bigg)\\
&= \prod_{i=1}^n \frac{(ax_iq;q)_{N_i}\,(ax_iq/ef;q)_{N_i}\,(\lambda x_iq/e;q)_{N_i}\,(\lambda x_iq/f;q)_{N_i}}{(\lambda x_iq/ef;q)_{N_i}\,(\lambda x_iq;q)_{N_i}\,(ax_iq/f;q)_{N_i}\,(ax_iq/e;q)_{N_i}}\\
&\times \sum_{\substack{0\le k_i\le N_i\\ i=1,2,\dots,n}} \Bigg(\prod_{i=1}^n \left(\frac{1-\lambda x_i q^{k_i+|\mathbf{k}|}}{1-\lambda x_i}\right) \prod_{1\le i<j\le n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\\
&\times \prod_{1\le i<j\le n} (\lambda a x_i x_j q/ef;q)^{-1}_{k_i+k_j} \prod_{i,j=1}^n (\lambda a x_i x_j q^{1+N_j}/ef;q)_{k_i}\\
&\times \prod_{i,j=1}^n \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^n \frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(ef/ax_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}{(\lambda x_iq^{1+N_i};q)_{|\mathbf{k}|}\,(efq^{-N_i}/ax_i;q)_{|\mathbf{k}|}}\\
&\times \frac{(\lambda c/a;q)_{|\mathbf{k}|}\,(\lambda d/a;q)_{|\mathbf{k}|}\,(e;q)_{|\mathbf{k}|}\,(f;q)_{|\mathbf{k}|}}{\prod_{i=1}^n\left[(ax_iq/c;q)_{k_i}\,(ax_iq/d;q)_{k_i}\,(\lambda x_iq/e;q)_{k_i}\,(\lambda x_iq/f;q)_{k_i}\right]}\\
&\qquad\qquad\qquad\qquad \times \frac{\prod_{i=1}^n (\lambda bx_i/a;q)_{k_i}}{(aq/b;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^n i\,k_i}\Bigg), \qquad (3.14)
\end{aligned}
$$

where $\lambda = qa^2/bcd$.

Proof. First, replace k_i by $N_i - k_i$, for $i = 1, 2, \dots, n$, on both sides of Theorem 3.9 and simplify terms by Lemma A.1 and some elementary manipulations (cf. [11, Appendix I]). Finally, relabel

$$
\begin{aligned}
&a \mapsto q^{-|\mathbf{N}|}/a, \quad b \mapsto c/a, \quad c \mapsto d/a, \quad d \mapsto \lambda q^{1-|\mathbf{N}|}/ef,\\
&e \mapsto e/a, \quad f \mapsto bq^{-|\mathbf{N}|}/a, \quad x_i \mapsto q^{-N_i}/x_i, \quad \text{for } i = 1, 2, \dots, n
\end{aligned}
$$

in the resulting identity to obtain (3.14). $\square$

Remark 3.15. An alternative way to derive this is to combine Milne's A_n ${}_8\phi_7$ summation, Theorem A.3, and Schlosser's ${}_8\phi_7$ summation, Theorem A.12, with the interchange of summation argument.

By using a polynomial argument we obtain

Theorem 3.16 (A $\boldsymbol{D_n}$ ${}_{\mathbf{10}}\boldsymbol{\phi}_{\mathbf{9}}$ transformation)**.** *Let a, b, c, d, e, $f_1, \ldots,$ f_n and $x_1, \ldots, x_n$ be indeterminate, let N be a nonnegative integer, let $n \geq 1$, and suppose that none of the denominators in* (3.17) *vanish. Then*

$$
\begin{aligned}
&\sum_{\substack{k_1,k_2,\ldots,k_n\geq 0\\ 0\leq|\mathbf{k}|\leq \mathbf{N}}} \Bigg(\prod_{i=1}^n \left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\leq i<j\leq n}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\\
&\quad\times \prod_{1\leq i<j\leq n} (\lambda a x_i x_j q^{1+N}/e;q)^{-1}_{k_i+k_j} \prod_{i,j=1}^n (\lambda a x_i x_j q^{1+N}/ef_j;q)_{k_i}\\
&\quad\times \prod_{i,j=1}^n \frac{(f_jx_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^n \frac{(ax_i;q)_{|\mathbf{k}|}\,(eq^{-N}/\lambda x_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}{(ax_iq/f_i;q)_{|\mathbf{k}|}\,(ef_iq^{-N}/\lambda x_i;q)_{|\mathbf{k}|}}\\
&\quad\times \frac{(c;q)_{|\mathbf{k}|}\,(d;q)_{|\mathbf{k}|}\,(e;q)_{|\mathbf{k}|}\,(q^{-N};q)_{|\mathbf{k}|}}{\prod_{i=1}^n\left[(ax_iq/c;q)_{k_i}\,(ax_iq/d;q)_{k_i}\,(ax_iq/e;q)_{k_i}\,(ax_iq^{1+N};q)_{k_i}\right]}\\
&\qquad\qquad\times \frac{\prod_{i=1}^n (bx_i;q)_{k_i}}{(aq/b;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^n i k_i}\Bigg)\\
&= \prod_{i=1}^n \frac{(ax_iq;q)_N\,(ax_iq/ef_i;q)_N\,(\lambda x_iq/e;q)_N\,(\lambda x_iq/f_i;q)_N}{(\lambda x_iq/ef_i;q)_N\,(\lambda x_iq;q)_N\,(ax_iq/f_i;q)_N\,(ax_iq/e;q)_N}\\
&\times \sum_{\substack{k_1,k_2,\ldots,k_n\geq 0\\ 0\leq|\mathbf{k}|\leq \mathbf{N}}} \Bigg(\prod_{i=1}^n \left(\frac{1-\lambda x_iq^{k_i+|\mathbf{k}|}}{1-\lambda x_i}\right) \prod_{1\leq i<j\leq n}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\\
&\quad\times \prod_{1\leq i<j\leq n} (\lambda a x_i x_j q^{1+N}/e;q)^{-1}_{k_i+k_j} \prod_{i,j=1}^n (\lambda a x_i x_j q^{1+N}/ef_j;q)_{k_i}\\
&\quad\times \prod_{i,j=1}^n \frac{(f_jx_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^n \frac{(\lambda x_i;q)_{|\mathbf{k}|}\,(eq^{-N}/ax_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}{(\lambda x_iq/f_i;q)_{|\mathbf{k}|}\,(ef_iq^{-N}/ax_i;q)_{|\mathbf{k}|}}\\
&\quad\times \frac{(\lambda c/a;q)_{|\mathbf{k}|}\,(\lambda d/a;q)_{|\mathbf{k}|}\,(e;q)_{|\mathbf{k}|}\,(q^{-N};q)_{|\mathbf{k}|}}{\prod_{i=1}^n\left[(ax_iq/c;q)_{k_i}\,(ax_iq/d;q)_{k_i}\,(\lambda x_iq/e;q)_{k_i}\,(\lambda x_iq^{1+N};q)_{k_i}\right]}
\end{aligned}
$$

$$\times \frac{\prod_{i=1}^{n} (\lambda b x_i/a;q)_{k_i}}{(aq/b;q)_{|\mathbf{k}|}} \, q^{\sum_{i=1}^{n} i k_i} \Bigg), \quad (3.17)$$

where $\lambda = qa^2/bcd$.

Proof. First we write the product in front of the sum of the right side of (3.17) as quotient of infinite products using (1.2). Then by the $f = q^{-N}$ case of Theorem 3.13 it follows that the identity (3.17) holds for $f_j = q^{-N_j}$, $j = 1, \ldots, n$. By clearing out denominators in (3.17), we get a polynomial equation in f_1, which is true for q^{-N_1}, $N_1 = 0, 1, \ldots$. Thus we obtain an identity in f_1. By carrying out this process for $f_2, f_3, \ldots, f_n$ also, we obtain Theorem 3.16. $\square$

4. Some Watson's transformations

In this section, we find the multivariable generalizations of Watson's transformation which follow from our results in §2 and §3. As special cases of our generalizations of Bailey's ${}_{10}\phi_9$ transformation, we obtain A_n, C_n, and D_n generalizations of Watson's transformation found by Milne [21, 22, 24], Milne and Lilly [25], and Bhatnagar [7], respectively. See also [19, 26, 28, 30] for some of Milne's A_n generalizations of Watson's theorem. In addition to recovering all such transformations known previously, we also obtain two new A_n, and several D_n generalizations of Watson's theorem.

Watson's formula follows from Bailey's ${}_{10}\phi_9$ transformation in one of many ways. For instance, if we take the limit as $d \to \infty$ in Theorem 1.7, and replace f by d in the resulting identity, we obtain [11, equation (2.5.1)]:

Theorem 4.1 (Watson's classical q-Whipple transformation)**.** *Let a, b, c, d, and e be indeterminate, let n be a nonnegative integer, and suppose that none of the denominators in* (4.2) *vanish. Then*

$$ {}_8\phi_7 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{n+1} \end{matrix} ; q, \frac{a^2q^{2+n}}{bcde} \right] $$
$$ = \frac{(aq;q)_n\,(aq/de;q)_n}{(aq/d;q)_n\,(aq/e;q)_n}\, {}_4\phi_3 \left[\begin{matrix} aq/bc, d, e, q^{-n} \\ aq/b, aq/c, deq^{-n}/a \end{matrix} ; q, q \right]. \quad (4.2)$$

This is not the only the only way to obtain Theorem 4.1 from (1.8). We may also take the limit $b \to \infty$, or $c \to \infty$. Further, we may first restate Theorem 1.7 by replacing e or f by $\lambda aq^{n+1}/ef$ in (1.8), and then take the limits as b, c, or $d \to \infty$, to obtain Watson's transformation.

All the above limiting cases of Bailey's formula lead to the same identity, upto a trivial relabeling of parameters. However, in our multivariable generalizations of Bailey's formula, some of its symmetry is broken, and we obtain many different generalizations of Watson's transformation. For example, while the b, c, or $d \to \infty$ case of Theorem 2.1 yields the same identity, letting $b \to \infty$ and $d \to \infty$ in Theorem 3.1 yield different generalizations of Theorem 4.1.

Finally, as noted in §2, we may set $a \mapsto qa^2/bcd$, $b \mapsto aq/cd$, $c \mapsto aq/bd$, $d \mapsto aq/bc$ in Theorem 1.7. With these substitutions, λ becomes a, and the ${}_{10}\phi_9$ on the right hand side of (1.8) gets transformed into the one on the left, and vice-versa. Once again, we obtain Bailey's ${}_{10}\phi_9$ formula. Now, we may take limits as above and obtain Watson's transformation. This observation does not lead to any further generalizations of (4.2) from our results in §3, but it does apply to Theorem 2.1.

If we perform the above substitutions and let $d \to \infty$ in Theorem 2.1, we obtain one of Milne's [24] A_n Watson's transformations, see [26, Theorem 5.1]. Instead, if we simply let $d \to \infty$ in Theorem 2.1, we obtain Milne and Lilly's [25, Theorem 6.6] C_n generalization of Theorem 4.1.

Next we consider the multivariable generalizations of Watson's transformations following from Theorem 3.1. First we have an A_n extension of Theorem 4.1.

Theorem 4.3 (An $\boldsymbol{A_n}$ Watson's transformation)**.** *Let a, b, c, d, e and $x_1, \ldots, x_n$ be indeterminate, let N_i be nonnegative integers for $i = 1, 2, \ldots, n$ with $n \geq 1$, and suppose that none of the denominators in (4.4) vanish. Then*

$$\sum_{\substack{0 \leq k_i \leq N_i \\ i=1,2,\ldots,n}} \Bigg(\prod_{i=1}^{n} \left(\frac{1 - a x_i q^{k_i + |\mathbf{k}|}}{1 - a x_i} \right) \prod_{1 \leq i < j \leq n} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right)$$
$$\times \prod_{i,j=1}^{n} \frac{(q^{-N_j} x_i / x_j; q)_{k_i}}{(q x_i / x_j; q)_{k_i}} \prod_{i=1}^{n} \frac{(a x_i; q)_{|\mathbf{k}|}}{(a x_i q^{1+N_i}; q)_{|\mathbf{k}|}}$$
$$\times \prod_{i=1}^{n} \frac{(b x_i; q)_{k_i} \, (c x_i; q)_{k_i}}{(a x_i q/d; q)_{k_i} \, (a x_i q/e; q)_{k_i}}$$
$$\times \frac{(d; q)_{|\mathbf{k}|} \, (e; q)_{|\mathbf{k}|}}{(aq/b; q)_{|\mathbf{k}|} \, (aq/c; q)_{|\mathbf{k}|}} \left(\frac{a^2 q^{1+|\mathbf{N}|}}{bcde} \right)^{|\mathbf{k}|} q^{\sum_{i=1}^{n} i k_i} \Bigg)$$

$$
\begin{aligned}
&= \prod_{i=1}^{n} \left[\frac{(ax_i q;q)_{N_i}\,(ax_i q/de;q)_{N_i}}{(ax_i q/d;q)_{N_i}\,(ax_i q/e;q)_{N_i}} \right] \\
&\times \sum_{\substack{0\le k_i\le N_i\\ i=1,2,\dots,n}} \Bigg(\prod_{1\le i<j\le n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j} \right) \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \\
&\qquad \times \prod_{i=1}^{n} \left[\frac{(de/ax_i;q)_{|\mathbf{k}|-k_i}\,(aq/bcx_i;q)_{|\mathbf{k}|}}{(aq/bcx_i;q)_{|\mathbf{k}|-k_i}\,(deq^{-N_i}/ax_i;q)_{|\mathbf{k}|}} \right] \\
&\qquad\qquad \times \frac{(d;q)_{|\mathbf{k}|}\,(e;q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^{n} i\,k_i} \Bigg). \qquad (4.4)
\end{aligned}
$$

Proof. We first replace λ by qa^2/bcd in Theorem 3.1, and let $d\to\infty$. Finally, relabel $f\mapsto d$ in the resulting identity to obtain (4.4). $\square$

Remark 4.5. Theorem 4.3 concerns series summed over an n-rectangle. By using a polynomial argument similar to the proof of Theorem 3.7, we may find another A_n extension of Theorem 4.1, where the series involved are summed over an n-tetrahedron. This may also be accomplished by taking the limit as $d\to\infty$ in (3.8).

Remark 4.6. Theorem 4.3 is different from any of the generalizations of Theorem 4.1 which appears in [24]. The series on the left hand side of (4.4) is the same as two such transformations appearing in [26, Theorems A10 and 5.1]. By comparing the other two sides we obtain generalizations of the Sears ${}_4\phi_3$ transformations [32], [11, equations (III.15) and (III.16)], see [26] for an example of similar calculations.

The rest of our extensions of Theorem 4.1 concern D_n series. After taking limits, we sometimes use

$$\binom{|\mathbf{k}|}{2} - e_2(\mathbf{k}) = \binom{k_1}{2} + \binom{k_2}{2} + \cdots + \binom{k_n}{2}$$

to simplify the powers of q.

Theorem 4.7 (A $\boldsymbol{D_n}$ Watson's transformation)**.** *Let a, b, c, d, e and $x_1,\dots,x_n$ be indeterminate, let N_i be nonnegative integers for $i=1,2,\dots,n$ with $n\ge 1$, and suppose that none of the denominators in* (4.8) *vanish. Then*

$$\sum_{\substack{0\le k_i\le N_i\\ i=1,2,\dots,n}} \Bigg(\prod_{i=1}^{n} \left(\frac{1-ax_i q^{k_i+|\mathbf{k}|}}{1-ax_i} \right) \prod_{1\le i<j\le n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j} \right)$$

$$\times \prod_{1\le i<j\le n} (ax_ix_jq/c;q)_{k_i+k_j} \prod_{i=1}^n \frac{1}{(c/x_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}$$
$$\times \prod_{i,j=1}^n \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq/c;q)_{k_i}}$$
$$\times \prod_{i=1}^n \frac{(ax_i;q)_{|\mathbf{k}|}\,(c/x_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+N_i};q)_{|\mathbf{k}|}}$$
$$\times \prod_{i=1}^n \left[\frac{(bx_i;q)_{k_i}}{(ax_iq/d;q)_{k_i}\,(ax_iq/e;q)_{k_i}}\right] \frac{(d;q)_{|\mathbf{k}|}\,(e;q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}}$$
$$\times \left(\frac{a^2q^{1+|\mathbf{N}|}}{bcde}\right)^{|\mathbf{k}|} q^{\sum_{i=1}^n i\,k_i} q^{-2e_2(\mathbf{k})} \prod_{i=1}^n x_i^{2k_i}\Bigg)$$
$$= \prod_{i=1}^n \left[\frac{(ax_iq;q)_{N_i}\,(ax_iq/de;q)_{N_i}}{(ax_iq/d;q)_{N_i}\,(ax_iq/e;q)_{N_i}}\right]$$
$$\times \sum_{\substack{0\le k_i\le N_i\\ i=1,2,\dots,n}} \Bigg(\prod_{1\le i<j\le n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\le i<j\le n} (ax_ix_jq/c;q)_{k_i+k_j}$$
$$\times \prod_{i,j=1}^n \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq/c;q)_{k_i}}$$
$$\times \prod_{i=1}^n \left[\frac{(de/ax_i;q)_{|\mathbf{k}|-k_i}\,(ax_iq/bc;q)_{k_i}}{(deq^{-N_i}/ax_i;q)_{|\mathbf{k}|}}\right]$$
$$\times \frac{(d;q)_{|\mathbf{k}|}\,(e;q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}} q^{\sum_{i=1}^n i\,k_i}\Bigg), \qquad (4.8)$$

where $e_2(\mathbf{k})$ *is the second elementary symmetric function of* $\mathbf{k}$.

Proof. We first replace λ by qa^2/bcd in Theorem 3.1, and let $c\to\infty$. Finally, relabel $d\mapsto c$, and $f\mapsto d$ in the resulting identity to obtain (4.8). □

Remark 4.9. Theorem 4.7 concerns series summed over an n-rectangle. By using a polynomial argument similar to the proof of Theorem 3.7, we may find another transformation formula between series summed over an n-tetrahedron. This may also be accomplished by taking the limit as $c\to\infty$ in (3.8).

Next, we consider the special cases of the equivalent formulation of Theorem 3.1, obtained by replacing e by $\lambda aq/ef$ in (3.2).

Theorem 4.10 (A $\boldsymbol{D_n}$ Watson's transformation)**.** *Let a, b, c, d, e and $x_1, \ldots, x_n$ be indeterminate, let N_i be nonnegative integers for $i = 1, 2, \ldots, n$ with $n \geq 1$, and suppose that none of the denominators in* (4.11) *vanish. Then*

$$
\begin{aligned}
\sum_{\substack{0 \leq k_i \leq N_i \\ i=1,2,\ldots,n}} \Bigg(\prod_{i=1}^{n} \left(\frac{1 - a x_i q^{k_i + |\mathbf{k}|}}{1 - a x_i} \right) \prod_{1 \leq i < j \leq n} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right) \\
\times \prod_{1 \leq i < j \leq n} \frac{(a x_i x_j q / c; q)_{k_i + k_j}}{(e x_i x_j; q)_{k_i + k_j}} \prod_{i=1}^{n} \frac{(aq/ex_i; q)_{|\mathbf{k}| - \mathbf{k_i}}}{(c/x_i; q)_{|\mathbf{k}| - \mathbf{k_i}}} \\
\times \prod_{i,j=1}^{n} \frac{(q^{-N_j} x_i / x_j; q)_{k_i} \, (e x_i x_j q^{N_j}; q)_{k_i}}{(q x_i / x_j; q)_{k_i} \, (a x_i x_j q / c; q)_{k_i}} \\
\times \prod_{i=1}^{n} \frac{(a x_i; q)_{|\mathbf{k}|} \, (c/x_i; q)_{|\mathbf{k}|}}{(a x_i q^{1+N_i}; q)_{|\mathbf{k}|} \, (a q^{1-N_i} / e x_i; q)_{|\mathbf{k}|}} \\
\times \prod_{i=1}^{n} \left[\frac{(b x_i; q)_{k_i}}{(a x_i q / d; q)_{k_i}} \right] \frac{(d; q)_{|\mathbf{k}|}}{(aq/b; q)_{|\mathbf{k}|}} \left(\frac{q a^2}{bcde} \right)^{|\mathbf{k}|} q^{\sum_{i=1}^{n} i k_i} \Bigg) \\
= \prod_{i=1}^{n} \left[\frac{(a x_i q; q)_{N_i} \, (d e x_i / a; q)_{N_i}}{(e x_i / a; q)_{N_i} \, (a x_i q / d; q)_{N_i}} \right] \left(\frac{1}{d} \right)^{|\mathbf{N}|} \\
\times \sum_{\substack{0 \leq k_i \leq N_i \\ i=1,2,\ldots,n}} \Bigg(\prod_{1 \leq i < j \leq n} \left(\frac{1 - q^{k_i - k_j} x_i / x_j}{1 - x_i / x_j} \right) \prod_{1 \leq i < j \leq n} \frac{(a x_i x_j q / c; q)_{k_i + k_j}}{(e x_i x_j; q)_{k_i + k_j}} \\
\times \prod_{i,j=1}^{n} \frac{(q^{-N_j} x_i / x_j; q)_{k_i} \, (e x_i x_j q^{N_j}; q)_{k_i}}{(q x_i / x_j; q)_{k_i} \, (a x_i x_j q / c; q)_{k_i}} \\
\times \prod_{i=1}^{n} \left[\frac{(a x_i q / bc; q)_{k_i}}{(d e x_i / a; q)_{k_i}} \right] \frac{(d; q)_{|\mathbf{k}|}}{(aq/b; q)_{|\mathbf{k}|}} q^{\sum_{i=1}^{n} i k_i} \Bigg). \qquad (4.11)
\end{aligned}
$$

Proof. We first replace e by $\lambda aq/ef$ in (3.2) to obtain an equivalent formulation of Theorem 3.1. Next, we replace λ by qa^2/bcd, and let $c \to \infty$. Finally, relabel $f \mapsto d$ and $d \mapsto c$ in the resulting identity to obtain (4.11). □

Remark 4.12. Theorem 4.10 concerns series summed over an n-rectangle. By using a polynomial argument similar to the proof of Theorem 3.7, we may find another transformation formula between series summed over an n-tetrahedron. This may also be accomplished by first replacing e by $\lambda aq/ef_1\cdots f_n$ in (3.8), and then taking the limit as $c\to\infty$ in (3.8).

Theorem 4.13 (A $\boldsymbol{D_n}$ Watson's transformation)**.** *Let a, b, c, d, e and $x_1,\ldots,x_n$ be indeterminate, let N_i be nonnegative integers for $i=1,2,\ldots,n$ with $n\geq 1$, and suppose that none of the denominators in* (4.14) *vanish. Then*

$$\sum_{\substack{0\leq k_i\leq N_i\\ i=1,2,\ldots,n}}\Bigg(\prod_{i=1}^n\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right)\prod_{1\leq i<j\leq n}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times\prod_{1\leq i<j\leq n}\frac{1}{(ex_ix_j;q)_{k_i+k_j}}\prod_{i=1}^n(aq/ex_i;q)_{|\mathbf{k}|-\mathbf{k_i}}$$
$$\times\prod_{i,j=1}^n\frac{(q^{-N_j}x_i/x_j;q)_{k_i}\,(ex_ix_jq^{N_j};q)_{k_i}}{(qx_i/x_j;q)_{k_i}}$$
$$\times\prod_{i=1}^n\frac{(ax_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+N_i};q)_{|\mathbf{k}|}\,(aq^{1-N_i}/ex_i;q)_{|\mathbf{k}|}}$$
$$\times\prod_{i=1}^n\left[\frac{(bx_i;q)_{k_i}\,(cx_i;q)_{k_i}}{(ax_iq/d;q)_{k_i}}\right]\frac{(d;q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}}$$
$$\times\left(\frac{qa^2}{bcde}\right)^{|\mathbf{k}|}q^{\sum_{i=1}^n i\,k_i}q^{2e_2(\mathbf{k})}\prod_{i=1}^n x_i^{-2k_i}\Bigg)$$
$$=\prod_{i=1}^n\left[\frac{(ax_iq;q)_{N_i}\,(dex_i/a;q)_{N_i}}{(ex_i/a;q)_{N_i}\,(ax_iq/d;q)_{N_i}}\right]\left(\frac{1}{d}\right)^{|\mathbf{N}|}$$
$$\times\sum_{\substack{0\leq k_i\leq N_i\\ i=1,2,\ldots,n}}\Bigg(\prod_{1\leq i<j\leq n}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{1\leq i<j\leq n}\frac{1}{(ex_ix_j;q)_{k_i+k_j}}$$
$$\times\prod_{i,j=1}^n\frac{(q^{-N_j}x_i/x_j;q)_{k_i}\,(ex_ix_jq^{N_j};q)_{k_i}}{(qx_i/x_j;q)_{k_i}}$$
$$\times\prod_{i=1}^n\left[\frac{(aq/bcx_i;q)_{|\mathbf{k}|}}{(aq/bcx_i;q)_{|\mathbf{k}|-k_i}\,(dex_i/a;q)_{k_i}}\right]$$
$$\times\frac{(d;q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}}\,q^{\sum_{i=1}^n i\,k_i}\Bigg),\quad(4.14)$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\mathbf{k}$.

Proof. We first replace e by $\lambda aq/ef$ in (3.2) to obtain an equivalent formulation of Theorem 3.1. Next, we replace λ by qa^2/bcd, and let $d \to \infty$. Finally, relabel $f \mapsto d$ in the resulting identity to obtain (4.14). $\square$

Remark 4.15. Theorem 4.13 concerns series summed over an n-rectangle. By using a polynomial argument similar to the proof of Theorem 3.7, we may find another transformation formula between series summed over an n-tetrahedron. This may also be accomplished by first replacing e by $\lambda aq/ef_1 \cdots f_n$ in (3.8), and then taking the limit as $d \to \infty$ in the resulting identity.

Next, we obtain the Watson's transformations which follow from Theorem 3.9. If we take the limit as $d \to \infty$ in Theorem 3.9, we obtain one of Milne's transformations, see [28, Theorem 4.5]. Another limiting case yields:

Theorem 4.16 (A $\boldsymbol{D_n}$ Watson's transformation)**.** *Let a, b, c, d, e and $x_1, \ldots, x_n$ be indeterminate, let N_i be nonnegative integers for $i = 1, 2, \ldots, n$ with $n \geq 1$, and suppose that none of the denominators in* (4.17) *vanish. Then*

$$\sum_{\substack{0 \leq k_i \leq N_i \\ i=1,2,\ldots,n}} \Bigg(\prod_{i=1}^n \left(\frac{1 - ax_i q^{k_i + |\mathbf{k}|}}{1 - ax_i} \right) \prod_{1 \leq i < j \leq n} \left(\frac{1 - q^{k_i - k_j} x_i/x_j}{1 - x_i/x_j} \right)$$

$$\times \prod_{1 \leq i < j \leq n} (ax_i x_j q/c; q)_{k_i + k_j} \prod_{i=1}^n \frac{1}{(c/x_i; q)_{|\mathbf{k}| - \mathbf{k_i}}}$$

$$\times \prod_{i,j=1}^n \frac{(q^{-N_j} x_i/x_j; q)_{k_i}}{(qx_i/x_j; q)_{k_i} \, (ax_i x_j q/c; q)_{k_i}}$$

$$\times \prod_{i=1}^n \frac{(ax_i; q)_{|\mathbf{k}|} \, (c/x_i; q)_{|\mathbf{k}|}}{(ax_i q^{1+N_i}; q)_{|\mathbf{k}|}}$$

$$\times \frac{\prod_{i=1}^n \left[(bx_i; q)_{k_i} \, (ex_i; q)_{k_i} \right]}{(aq/b; q)_{|\mathbf{k}|} \, (aq/e; q)_{|\mathbf{k}|}} \frac{(d; q)_{|\mathbf{k}|}}{\prod_{i=1}^n (ax_i q/d)_{k_i}}$$

$$\times \left(\frac{a^2 q^{1+|\mathbf{N}|}}{bcde} \right)^{|\mathbf{k}|} q^{\sum_{i=1}^n i k_i} q^{-e_2(\mathbf{k})} \prod_{i=1}^n x_i^{k_i} \Bigg)$$

$$= \prod_{i=1}^n \left[\frac{(ax_i q; q)_{N_i}}{(ax_i q/d; q)_{N_i}} \right] \frac{(aq/de; q)_{|\mathbf{N}|}}{(aq/e; q)_{|\mathbf{N}|}}$$

$$\times \sum_{\substack{0\le k_i\le N_i\\ i=1,2,\ldots,n}} \Bigg(\prod_{1\le i<j\le n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\le i<j\le n} (ax_ix_jq/c;q)_{k_i+k_j}$$

$$\times \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq/c;q)_{k_i}}$$

$$\times \prod_{i=1}^{n} [(ax_iq/bc;q)_{k_i}\,(ex_i;q)_{k_i}]$$

$$\times \frac{(d;q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}\,(deq^{-|\mathbf{N}|}/a;q)_{|\mathbf{k}|}} q^{\sum_{i=1}^{n} i\,k_i} \Bigg), \quad (4.17)$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\mathbf{k}$.

Proof. We first replace λ by qa^2/bcd in Theorem 3.9, and let $c \to \infty$. Finally, relabel $d \mapsto c$, and $f \mapsto d$ in the resulting identity to obtain (4.17). □

Remark 4.18. Theorem 4.16 concerns series summed over an n-rectangle. By using a polynomial argument similar to the proof of Theorem 3.11, we may find another transformation formula between series summed over an n-tetrahedron. This may also be accomplished by taking the limit as $c \to \infty$ in (3.12).

Next, we consider the special cases of the equivalent formulation of Theorem 3.9, obtained by replacing f by $\lambda aq^{1+|\mathbf{N}|}/ef$ in (3.10). If we take the limit as $c \to \infty$ in the resulting identity, we obtain a D_n generalization of Theorem 4.1 found by Bhatnagar [7]. Instead, if we take the limit as $d \to \infty$, we obtain an A_n transformation theorem found by Milne [24].

Further, if we take the limit as $d \to \infty$ in Theorem 3.13, we obtain an A_n generalization of Theorem 4.1 found by Milne [24]. Leininger and Milne [19] have found some elegant applications of this case of Theorem 3.13. The $b \to \infty$ case of Theorem 3.13 was also found by Milne [24].

Finally, we consider the equivalent formulation of Theorem 3.13, obtained by replacing e by $\lambda aq/ef$ in (3.14).

Theorem 4.19 (A $\boldsymbol{D_n}$ Watson's transformation)**.** *Let a, b, c, d, e and $x_1,\ldots,x_n$ be indeterminate, let N_i be nonnegative integers for $i = 1,2,\ldots,n$ with $n \ge 1$, and suppose that none of the denominators in*

(4.20) *vanish. Then*

$$\sum_{\substack{0\le k_i\le N_i\\ i=1,2,\dots,n}} \Bigg(\prod_{i=1}^n \left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\le i<j\le n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times \prod_{1\le i<j\le n} \frac{1}{(ex_ix_j;q)_{k_i+k_j}} \prod_{i=1}^n (aq/ex_i;q)_{|\mathbf{k}|-\mathbf{k_i}}$$
$$\times \prod_{i,j=1}^n \frac{(q^{-N_j}x_i/x_j;q)_{k_i}\,(ex_ix_jq^{N_j};q)_{k_i}}{(qx_i/x_j;q)_{k_i}}$$
$$\times \prod_{i=1}^n \frac{(ax_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+N_i};q)_{|\mathbf{k}|}\,(aq^{1-N_i}/ex_i;q)_{|\mathbf{k}|}}$$
$$\times \frac{(b;q)_{|\mathbf{k}|}\,(c;q)_{|\mathbf{k}|}\,(d;q)_{|\mathbf{k}|}}{\prod_{i=1}^n [(ax_iq/b;q)_{k_i}\,(ax_iq/c;q)_{k_i}(ax_iq/d;q)_{k_i}]}$$
$$\times \left(\frac{qa^2}{bcde}\right)^{|\mathbf{k}|} q^{\sum_{i=1}^n i\,k_i} \Bigg)$$
$$= \prod_{i=1}^n \left[\frac{(ax_iq;q)_{N_i}\,(dex_i/a;q)_{N_i}}{(ex_i/a;q)_{N_i}\,(ax_iq/d;q)_{N_i}}\right] \left(\frac{1}{d}\right)^{|\mathbf{N}|}$$
$$\times \sum_{\substack{0\le k_i\le N_i\\ i=1,2,\dots,n}} \Bigg(\prod_{1\le i<j\le n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\le i<j\le n} \frac{1}{(ex_ix_j;q)_{k_i+k_j}}$$
$$\times \prod_{i,j=1}^n \frac{(q^{-N_j}x_i/x_j;q)_{k_i}\,(ex_ix_jq^{N_j};q)_{k_i}}{(qx_i/x_j;q)_{k_i}}$$
$$\times \prod_{i=1}^n \left[\frac{(ax_iq/bc;q)_{k_i}}{(ax_iq/b;q)_{k_i}\,(ax_iq/c;q)_{k_i}\,(dex_i/a;q)_{k_i}}\right]$$
$$\times (d;q)_{|\mathbf{k}|}\, q^{\sum_{i=1}^n i\,k_i} \Bigg). \quad (4.20)$$

Proof. We first replace e by $\lambda aq/ef$ in (3.14) to obtain an equivalent formulation of Theorem 3.13. Next, we replace λ by qa^2/bcd, and let $b\to\infty$. Finally, relabel $d\mapsto b$, $f\mapsto d$ in the resulting identity to obtain (4.20). □

Remark 4.21. Theorem 4.19 concerns series summed over an n-rectangle. By using a polynomial argument similar to the proof of Theorem 3.16, we may find another transformation formula between series summed over an n-tetrahedron. This may also be accomplished by first replacing e by $\lambda aq/ef_1\cdots f_n$ in (3.17), and then taking the limit as $b\to\infty$ in the resulting identity.

Theorem 4.22 (A $\boldsymbol{D_n}$ Watson's transformation)**.** *Let a, b, c, d, e and $x_1,\ldots,x_n$ be indeterminate, let N_i be nonnegative integers for $i=1,2,\ldots,n$ with $n\geq 1$, and suppose that none of the denominators in* (4.23) *vanish. Then*

$$\begin{aligned}
&\sum_{\substack{0\leq k_i\leq N_i\\ i=1,2,\ldots,n}}\Bigg(\prod_{i=1}^n\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right)\prod_{1\leq i<j\leq n}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\\
&\qquad\times\prod_{1\leq i<j\leq n}\frac{1}{(ex_ix_j;q)_{k_i+k_j}}\prod_{i=1}^n(aq/ex_i;q)_{|\mathbf{k}|-\mathbf{k_i}}\\
&\qquad\times\prod_{i,j=1}^n\frac{(q^{-N_j}x_i/x_j;q)_{k_i}\,(ex_ix_jq^{N_j};q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\\
&\qquad\times\prod_{i=1}^n\frac{(ax_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+N_i};q)_{|\mathbf{k}|}\,(aq^{1-N_i}/ex_i;q)_{|\mathbf{k}|}}\\
&\qquad\times\prod_{i=1}^n\left[\frac{(bx_i;q)_{k_i}}{(ax_iq/c;q)_{k_i}\,(ax_iq/d;q)_{k_i}}\right]\frac{(c;q)_{|\mathbf{k}|}\,(d;q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}}\\
&\qquad\times\left(\frac{qa^2}{bcde}\right)^{|\mathbf{k}|}q^{\sum_{i=1}^n ik_i}q^{e_2(\mathbf{k})}\prod_{i=1}^n x_i^{-k_i}\Bigg)\\
&=\prod_{i=1}^n\left[\frac{(ax_iq;q)_{N_i}\,(dex_i/a;q)_{N_i}}{(ex_i/a;q)_{N_i}\,(ax_iq/d;q)_{N_i}}\right]\left(\frac{1}{d}\right)^{|\mathbf{N}|}\\
&\times\sum_{\substack{0\leq k_i\leq N_i\\ i=1,2,\ldots,n}}\Bigg(\prod_{1\leq i<j\leq n}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{1\leq i<j\leq n}\frac{1}{(ex_ix_j;q)_{k_i+k_j}}\\
&\qquad\times\prod_{i,j=1}^n\frac{(q^{-N_j}x_i/x_j;q)_{k_i}\,(ex_ix_jq^{N_j};q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\\
&\qquad\times\prod_{i=1}^n\left[\frac{1}{(ax_iq/c;q)_{k_i}\,(dex_i/a;q)_{k_i}}\right]\\
&\qquad\times\frac{(aq/bc;q)_{|\mathbf{k}|}\,(d;q)_{|\mathbf{k}|}}{(aq/b;q)_{|\mathbf{k}|}}\,q^{\sum_{i=1}^n ik_i}\Bigg),
\end{aligned}\tag{4.23}$$

where $e_2(\mathbf{k})$ *is the second elementary symmetric function of* $\mathbf{k}$.

Proof. We first replace e by $\lambda aq/ef$ in (3.14) to obtain an equivalent formulation of Theorem 3.13. Next, we replace λ by qa^2/bcd, and let $d \to \infty$. Finally, relabel $f \mapsto d$ in the resulting identity to obtain (4.23). □

Remark 4.24. Theorem 4.22 concerns series summed over an n-rectangle. By using a polynomial argument similar to the proof of Theorem 3.16, we may find another transformation formula between series summed over an n-tetrahedron. This may also be accomplished by first replacing e by $\lambda aq/ef_1 \cdots f_n$ in (3.17), and then taking the limit as $d \to \infty$ in the resulting identity.

Before closing this section, we indicate the effect of reversing the sum, on the various generalizations of Theorem 4.1 presented above. Theorem 4.3 and Theorem 4.10 are invariant if we reverse the series on both sides of the transformation, and relabel parameters. If we do the same for Theorem 4.7, we obtain Theorem 4.13, and vice-versa. Similarly, Theorem 4.16 and Theorem 4.22 are transformed into each other by reversing series. Finally, Theorem 4.19 and Bhatnagar's [7] D_n Watson's transformation are similarly seen to be equivalent.

5. Some Sears' transformations

Multivariable generalizations of Sears' [32] ${}_4\phi_3$ transformations follow directly from Bailey's ${}_{10}\phi_9$ transformation formula. In this section, we present a few A_n and D_n Sears transformations which follow directly from two transformations from §3. We do not make an exhaustive list of such transformation formulas, but simply indicate some of the possibilities. In addition, we specialize one of our transformation formulas and obtain a transformation formula for non-terminating A_n ${}_3\phi_2$ series ([11, equation (3.2.7)]), and a generalization of a transformation formula of Heine [11, equation (1.4.3)]. These transformation formulas specialize further to give a generalization of Heine's [11, equation (1.5.1)] q-Gauss sum, and the q-binomial theorem [11, equation (1.3.2)].

To obtain Sears' ${}_4\phi_3$ transformation from Bailey's ${}_{10}\phi_9$ transformation, replace b by aq/b and e by aq/e in (1.8), and then take the limit as $a \to 0$. After relabeling of parameters, $b \mapsto d$, $d \mapsto b$, and $f \mapsto a$, we obtain [11, equation (3.2.1)]:

Theorem 5.1 (Sears' classical $\boldsymbol{{}_4\phi_3}$ transformation)**.** *Let* a*,* b*,* c*,* d*, and* e *be indeterminate, let* n *be a nonnegative integer, and suppose* *that none of the denominators in* (5.2) *vanish. Then*

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, a, b, c \\ d, e, abcq^{1-n}/de \end{matrix}; q, q\right] = \frac{(e/a;q)_n\,(de/bc;q)_n}{(e;q)_n\,(de/abc;q)_n}\,{}_4\phi_3\left[\begin{matrix} q^{-n}, a, d/b, d/c \\ d, aq^{1-n}/e, de/bc \end{matrix}; q, q\right]. \tag{5.2}
$$

In the multivariable case, by varying the above calculations slightly, we obtain distinct generalizations of (5.2). Further, Theorem 5.1 may also be obtained from Theorem 4.1. Previously, A_n extensions of Sears transformation have been obtained from multivariable extensions of Watson's transformation. See Milne and Newcomb [26], Milne and Lilly [25] and Bhatnagar [7] for examples of such calculations.

We begin with two immediate consequences of Theorem 3.1.

Theorem 5.3 (An $\boldsymbol{A_n}$ Sears' ${}_{\boldsymbol 4}\boldsymbol{\phi}_{\boldsymbol 3}$ transformation)**.** *Let a, b, c, d, e and $x_1, \ldots, x_n$ be indeterminate, let $N_1, \ldots, N_n$ be nonnegative integers with $n \geq 1$, and suppose that none of the denominators in* (5.4) *vanish. Then*

$$
\begin{aligned}
&\sum_{\substack{0\le k_i\le N_i\\ i=1,2,\ldots,n}} \Bigg(\prod_{1\le i<j\le n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{i=1}^n \frac{(abcq/dex_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}{(b/x_i;q)_{|\mathbf{k}|-\mathbf{k_i}}} \\
&\qquad \times \prod_{i,j=1}^n \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^n \frac{(b/x_i;q)_{|\mathbf{k}|}}{(abcq^{1-N_i}/dex_i;q)_{|\mathbf{k}|}} \\
&\qquad\qquad \times \prod_{i=1}^n \frac{(cx_i;q)_{k_i}}{(ex_i;q)_{k_i}} \cdot \frac{(a;q)_{|\mathbf{k}|}}{(d;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^n ik_i} \Bigg) \\
&\qquad\qquad = \prod_{i=1}^n \frac{(ex_i/a;q)_{N_i}\,(dex_i/bc;q)_{N_i}}{(ex_i;q)_{N_i}\,(dex_iq/abc;q)_{N_i}} \\
&\times \sum_{\substack{0\le k_i\le N_i\\ i=1,2,\ldots,n}} \Bigg(\prod_{1\le i<j\le n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{i=1}^n \frac{(aq/ex_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}{(d/cx_i;q)_{|\mathbf{k}|-\mathbf{k_i}}} \\
&\qquad \times \prod_{i,j=1}^n \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^n \frac{(d/cx_i;q)_{|\mathbf{k}|}}{(aq^{1-N_i}/ex_i;q)_{|\mathbf{k}|}} \\
&\qquad\qquad \times \prod_{i=1}^n \frac{(dx_i/b;q)_{k_i}}{(dex_i/bc;q)_{k_i}} \cdot \frac{(a;q)_{|\mathbf{k}|}}{(d;q)_{|\mathbf{k}|}}\, q^{\sum_{i=1}^n ik_i} \Bigg).
\end{aligned} \tag{5.4}
$$

Proof. We first replace b by aq/b and e by aq/e in (3.2), and then take the limit $a \to 0$. After relabeling of parameters, $b \mapsto d$, $d \mapsto b$, and $f \mapsto a$, we obtain (5.4). □

Remark 5.5. Theorem 5.3 concerns series summed over an n-rectangle. By using a polynomial argument similar to the proof of Theorem 3.16, we may find another transformation formula between series summed over an n-tetrahedron. This may also be accomplished by first replacing b by aq/b and e by aq/e in (3.8), and then taking the limit as $a \to 0$ in the resulting identity.

The above generalization of Sears' identity cannot, at present, be obtained from a multivariable Watson's transformation. It would be interesting to obtain a generalization of Theorem 4.1 which implies Theorem 5.3. The following is a transformation of D_n series and also does not follow from any known generalization of Watson's transformation.

Theorem 5.6 (A $\boldsymbol{D_n}$ Sears' $\boldsymbol{{}_4\phi_3}$ transformation)**.** *Let a, b, c, d, e and $x_1, \dots, x_n$ be indeterminate, let $N_1, \dots, N_n$ be nonnegative integers with $n \geq 1$, and suppose that none of the denominators in* (5.7) *vanish. Then*

$$
\begin{aligned}
&\sum_{\substack{0\leq k_i\leq N_i\\ i=1,2,\dots,n}} \Bigg(\prod_{1\leq i<j\leq n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\leq i<j\leq n} (dx_ix_j;q)_{k_i+k_j} \\
&\qquad \times \prod_{i=1}^n \frac{(abcq/dex_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}{(abcq^{1-N_i}/dex_i;q)_{|\mathbf{k}|}} \prod_{i,j=1}^n \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(dx_ix_j;q)_{k_i}} \\
&\qquad\qquad \times \prod_{i=1}^n \frac{(bx_i;q)_{k_i}\,(cx_i;q)_{k_i}}{(ex_i;q)_{k_i}} \cdot (a;q)_{|\mathbf{k}|}\, q^{\sum_{i=1}^n ik_i} \Bigg) \\
&\qquad\qquad\quad = \prod_{i=1}^n \frac{(ex_i/a;q)_{N_i}\,(dex_i/bc;q)_{N_i}}{(ex_i;q)_{N_i}\,(dex_iq/abc;q)_{N_i}} \\
&\times \sum_{\substack{0\leq k_i\leq N_i\\ i=1,2,\dots,n}} \Bigg(\prod_{1\leq i<j\leq n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{1\leq i<j\leq n} (dx_ix_j;q)_{k_i+k_j} \\
&\qquad \times \prod_{i=1}^n \frac{(aq/ex_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}{(aq^{1-N_i}/ex_i;q)_{|\mathbf{k}|}} \prod_{i,j=1}^n \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(dx_ix_j;q)_{k_i}} \\
&\qquad\qquad \times \prod_{i=1}^n \frac{(dx_i/b;q)_{k_i}\,(dx_i/c;q)_{k_i}}{(dex_i/bc;q)_{k_i}} \cdot (a;q)_{|\mathbf{k}|}\, q^{\sum_{i=1}^n ik_i} \Bigg). \qquad (5.7)
\end{aligned}
$$

Proof. We first replace d by aq/d and e by aq/e in (3.2), and then take the limit $a \to 0$. After relabeling, $f \mapsto a$, we obtain (5.7). ☐

Remark 5.8. Theorem 5.6 concerns series summed over an n-rectangle. By using a polynomial argument similar to the proof of Theorem 3.16, we may find another transformation formula between series summed over an n-tetrahedron. This may also be accomplished by first replacing d by aq/d and e by aq/e in (3.8), and then taking the limit as $a \to 0$ in the resulting identity.

Next, we list several related transformation and summation formulas which follow from Theorem 3.16.

Theorem 5.9 (An $\boldsymbol{A_n}$ Sears ${}_{\boldsymbol{4}}\boldsymbol{\phi}_{\boldsymbol{3}}$ transformation)**.** *Let $a_1, \ldots, a_n$, b, c, d, e, and $x_1, \ldots, x_n$ be indeterminate, let N be a nonnegative integer, let $n \geq 1$, and suppose that none of the denominators in (5.10) vanish. Then*

$$\begin{aligned}\sum_{\substack{k_1,k_2,\ldots,k_n\geq 0\\ 0\leq|\mathbf{k}|\leq N}} \Bigg(&\prod_{1\leq i<j\leq n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \prod_{i,j=1}^{n} \frac{(a_jx_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \\ &\times \prod_{i=1}^{n} \frac{(bcq^{1-N}/dex_i;q)_{|\mathbf{k}|-k_i}}{(a_ibcq^{1-N}/dex_i;q)_{|\mathbf{k}|}} (c;q)_{|\mathbf{k}|}\, (q^{-N};q)_{|\mathbf{k}|} \\ &\times \prod_{i=1}^{n} \left[\frac{(bx_i;q)_{k_i}}{(dx_i;q)_{k_i}\,(ex_i;q)_{k_i}}\right] q^{\sum_{i=1}^{n} ik_i}\Bigg) \\ &= \prod_{i=1}^{n} \frac{(ex_i/a_i;q)_N\,(dex_i/bc;q)_N}{(dex_i/a_ibc;q)_N\,(ex_i;q)_N} \\ \times \sum_{\substack{k_1,k_2,\ldots,k_n\geq 0\\ 0\leq|\mathbf{k}|\leq N}} \Bigg(&\prod_{1\leq i<j\leq n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right) \times \prod_{i,j=1}^{n} \frac{(a_jx_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \\ &\times \prod_{i=1}^{n} \frac{(q^{1-N}/ex_i;q)_{|\mathbf{k}|-k_i}}{(a_iq^{1-N}/ex_i;q)_{|\mathbf{k}|}} (d/b;q)_{|\mathbf{k}|}\, (q^{-N};q)_{|\mathbf{k}|} \\ &\times \prod_{i=1}^{n} \left[\frac{(dx_i/c;q)_{k_i}}{(dx_i;q)_{k_i}\,(dex_i/bc;q)_{k_i}}\right] q^{\sum_{i=1}^{n} ik_i}\Bigg). \end{aligned} \tag{5.10}$$

Proof. Set $d \mapsto aq/d$, $e \mapsto aq/e$, and then take the limit as $a \to 0$ in (3.17), and then relabel the resulting identity by replacing f_j by a_j. □

Remark 5.11. When $d = b$, the series on the right hand side of (5.10) reduces to 1, and we obtain a balanced ${}_3\phi_2$ summation theorem found by Milne, see [19].

Theorem 5.12 (An A_n non-terminating ${}_3\phi_2$ transformation)**.** *Let a_1, …, a_n, b, c, d and x_1, …, x_n be indeterminate, and let $n \geq 1$. Suppose that none of the denominators in* (5.13) *vanish, and assume that $|z/x_i| < 1$, and $|bcz/dx_i| < 1$ for $i = 1, \ldots, n$. Then,*

$$\sum_{\substack{k_i \geq 0 \\ i=1,\ldots,n}} \Bigg(\prod_{1 \leq i < j \leq n} \left(\frac{1 - q^{k_i - k_j} x_i/x_j}{1 - x_i/x_j} \right) \prod_{i,j=1}^{n} \frac{(a_j x_i/x_j; q)_{k_i}}{(q x_i/x_j; q)_{k_i}} \times \prod_{i=1}^{n} \left[\frac{(bcz/dx_i; q)_{|\mathbf{k}| - k_i} \, (bx_i; q)_{k_i}}{(a_i bcz/dx_i; q)_{|\mathbf{k}|} \, (dx_i; q)_{k_i}} \right] (c; q)_{|\mathbf{k}|} \times z^{|\mathbf{k}|} q^{\sum_{i=2}^{n} (i-1) k_i} q^{e_2(\mathbf{k})} \prod_{i=1}^{n} x_i^{-k_i} \Bigg)$$
$$= \prod_{i=1}^{n} \frac{(a_i z/x_i; q)_\infty \, (bcz/dx_i; q)_\infty}{(a_i bcz/dx_i; q)_\infty \, (z/x_i; q)_\infty}$$
$$\times \sum_{\substack{k_i \geq 0 \\ i=1,\ldots,n}} \Bigg(\prod_{1 \leq i < j \leq n} \left(\frac{1 - q^{k_i - k_j} x_i/x_j}{1 - x_i/x_j} \right) \prod_{i,j=1}^{n} \frac{(a_j x_i/x_j; q)_{k_i}}{(q x_i/x_j; q)_{k_i}} \times \prod_{i=1}^{n} \left[\frac{(z/x_i; q)_{|\mathbf{k}| - k_i} \, (dx_i/c; q)_{k_i}}{(a_i z/x_i; q)_{|\mathbf{k}|} \, (dx_i; q)_{k_i}} \right] (d/b; q)_{|\mathbf{k}|} \times \left(\frac{bcz}{d} \right)^{|\mathbf{k}|} q^{\sum_{i=2}^{n} (i-1) k_i} q^{e_2(\mathbf{k})} \prod_{i=1}^{n} x_i^{-k_i} \Bigg), \quad (5.13)$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\mathbf{k}$.

Proof. Replace e by eq^{-N}, and then take the limit as $N \to \infty$ in (5.10). In the resulting identity, replace e by q/z. It is not difficult to see that the term by term limit is given by (5.13). However, to justify the limiting process, we have to use the dominated convergence theorem. Further, to find the convergence conditions of the dominating series, and the series in (5.13), we have to invoke the multiple power series ratio test. The details of this justification are very similar to a calculation in [26], and are not repeated here. □

Remark 5.14. When $n = 1$, Theorem 5.12 reduces to an equivalent case of [11, equation (3.2.7)]. If we just take the limit as $N \to \infty$ in (5.10), we obtain another A_n extension of the same identity.

Setting $a_i = 0$, for $i = 1, \ldots, n$, and relabeling $c \mapsto a$, and $d \mapsto c$ in (5.13), gives an extension of Heine's transformation [11, equation (1.4.3)], which is a q-analogue of a transformation formula of Euler.

Theorem 5.15 (An $\boldsymbol{A_n}$ Heine's $\boldsymbol{{}_2\phi_1}$ transformation)**.** *Let a, b, c and $x_1, \ldots, x_n$ be indeterminate, and let $n \geq 1$. Suppose that none of the denominators in* (5.16) *vanish, and assume that $|z/x_i| < 1$, and $|abz/cx_i| < 1$, for $i = 1, \ldots, n$. Then,*

$$\begin{aligned}
&\sum_{\substack{k_i \geq 0 \\ i=1,\ldots,n}} \Bigg(\prod_{1\leq i<j\leq n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j} \right) \prod_{i,j=1}^{n} \frac{1}{(qx_i/x_j;q)_{k_i}} \\
&\qquad \times \prod_{i=1}^{n} \left[\frac{(abz/cx_i;q)_{|\mathbf{k}|-k_i}\,(bx_i;q)_{k_i}}{(cx_i;q)_{k_i}} \right] (a;q)_{|\mathbf{k}|} \\
&\qquad \times z^{|\mathbf{k}|} q^{\sum_{i=2}^{n}(i-1)\,k_i} q^{e_2(\mathbf{k})} \prod_{i=1}^{n} x_i^{-k_i} \Bigg) \\
&= \prod_{i=1}^{n} \frac{(abz/cx_i;q)_\infty}{(z/x_i;q)_\infty} \\
&\times \sum_{\substack{k_i \geq 0 \\ i=1,\ldots,n}} \Bigg(\prod_{1\leq i<j\leq n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j} \right) \prod_{i,j=1}^{n} \frac{1}{(qx_i/x_j;q)_{k_i}} \\
&\qquad \times \prod_{i=1}^{n} \left[\frac{(z/x_i;q)_{|\mathbf{k}|-k_i}\,(cx_i/a;q)_{k_i}}{(cx_i;q)_{k_i}} \right] (c/b;q)_{|\mathbf{k}|} \\
&\qquad \times \left(\frac{abz}{c} \right)^{|\mathbf{k}|} q^{\sum_{i=2}^{n}(i-1)\,k_i} q^{e_2(\mathbf{k})} \prod_{i=1}^{n} x_i^{-k_i} \Bigg), \qquad (5.16)
\end{aligned}$$

where $e_2(\mathbf{k})$ is the second elementary symmetric function of $\mathbf{k}$.

Finally, we note two summation theorems which follow immediately from the above two transformation formulas. While these are closely related to Milne's summation theorems presented in [19], and may be obtained from Milne's results, they appear to have been missed by earlier authors.

Next, let $d = b$ in (5.13), and set $c \mapsto b$, $z \mapsto c/a_1 \cdots a_n b$, to obtain an extension of Heine's q-Gauss summation [11, equation (1.5.1)].

Theorem 5.17 (An $\boldsymbol{A_n}$ q-Gauss summation)**.** *Let $a_1, \ldots, a_n$, b, c and $x_1, \ldots, x_n$ be indeterminate, and let $n \geq 1$. Suppose that none of the denominators in* (5.18) *vanish, and assume that $|c/a_1 \cdots a_n b x_i| < 1$,* *for* $i = 1, \ldots, n$. *Then,*

$$\sum_{\substack{k_i \geq 0 \\ i=1,\ldots,n}} \Bigg(\prod_{1 \leq i < j \leq n} \left(\frac{1 - q^{k_i - k_j} x_i/x_j}{1 - x_i/x_j} \right) \prod_{i,j=1}^{n} \frac{(a_j x_i/x_j; q)_{k_i}}{(q x_i/x_j; q)_{k_i}} \\ \times \prod_{i=1}^{n} \left[\frac{(c/a_1 \cdots a_n x_i; q)_{|\mathbf{k}| - k_i}}{(a_i c/a_1 \cdots a_n x_i; q)_{|\mathbf{k}|}} \right] (b; q)_{|\mathbf{k}|} \\ \times \left(\frac{c}{a_1 \cdots a_n b} \right)^{|\mathbf{k}|} q^{\sum_{i=2}^{n} (i-1) k_i} q^{e_2(\mathbf{k})} \prod_{i=1}^{n} x_i^{-k_i} \Bigg) \\ = \prod_{i=1}^{n} \frac{(a_i c/a_1 \cdots a_n b x_i; q)_\infty \, (c/a_1 \cdots a_n x_i; q)_\infty}{(a_i c/a_1 \cdots a_n x_i; q)_\infty \, (c/a_1 \cdots a_n b x_i; q)_\infty}, \quad (5.18)$$

where $e_2(\mathbf{k})$ *is the second elementary symmetric function of* $\mathbf{k}$.

Finally, we set $c = b$ in (5.16) to obtain an A_n extension of the q-binomial theorem [11, equation (1.3.2)].

Theorem 5.19 (An $\boldsymbol{A_n}$ q-binomial theorem)**.** *Let* a, *and* $x_1, \ldots, x_n$ *be indeterminate, and let* $n \geq 1$. *Suppose that none of the denominators in* (5.20) *vanish, and assume that* $|z/x_i| < 1$, *for* $i = 1, \ldots, n$. *Then,*

$$\sum_{\substack{k_i \geq 0 \\ i=1,2,\ldots,n}} \Bigg(\prod_{1 \leq i < j \leq n} \left(\frac{1 - q^{k_i - k_j} x_i/x_j}{1 - x_i/x_j} \right) \prod_{i,j=1}^{n} \frac{1}{(q x_i/x_j; q)_{k_i}} (a; q)_{|\mathbf{k}|} \, z^{|\mathbf{k}|} \\ \times \prod_{i=1}^{n} (az/x_i; q)_{|\mathbf{k}| - \mathbf{k_i}} \, q^{\sum_{i=2}^{n} (i-1) k_i} q^{e_2(\mathbf{k})} \prod_{i=1}^{n} x_i^{-k_i} \Bigg) \\ = \prod_{i=1}^{n} \frac{(az/x_i; q)_\infty}{(z/x_i; q)_\infty}, \quad (5.20)$$

where $e_2(\mathbf{k})$ *is the second elementary symmetric function of* $\mathbf{k}$.

Remark 5.21. If we set $c \mapsto a_1 \cdots a_n b z$, and set $b = 0$ in Theorem 5.17, we obtain a multivariable generalization of the q-binomial theorem found by Milne and Lilly [25, Theorem 4.7]. These authors call it a C_n q-binomial theorem, because it is obtained from C_n summation theorems. However, in the convention followed in this paper, we regard Theorem 4.7 of [25] as an A_n theorem. Further, note that if we set $c \mapsto a_1 \cdots a_n b z$, and set $a_i = 0$ in (5.18), we obtain Theorem 5.19.

The above theorems are only a small sample of possibilities. For instance, Milne and Newcomb [27] and Degenhardt and Milne [9] have found interesting applications of Milne and Newcomb's [26] A_n ${}_{10}\phi_9$

formula. It would be interesting to carry out analogous computations starting from some of the results of this paper.

Appendix A

Here we state a useful simplification lemma found by Milne [23] and the multidimensional extensions of Jackson's classical ${}_8\phi_7$ summation theorem which are needed for our proofs of the transformations in §2 and §3. These summation theorems include the A_n extension of Jackson's theorem found by Milne [21], a C_n summation found independently by Denis and Gustafson [10] and Milne and Lilly [25], and D_n theorems found by Bhatnagar [7] and Schlosser [29]. We begin with

Lemma A.1 (Milne)**.**

$$\prod_{1\le i<j\le n}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i,j=1}^{n}\frac{(q^{-k_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}=(-1)^{|\mathbf{k}|}\,q^{-\binom{|\mathbf{k}|}{2}}\,q^{-\sum_{i=1}^n i\,k_i}.$$

Remark A.2. Lemma A.1 is Lemma 6.11 of [23], where it is proved by some elementary manipulations.

Next we state various multivariable extensions of Theorem 1.10.

Theorem A.3 ((Milne) An $\boldsymbol{A_n}$ Jackson's sum)**.** *Let a, b, c, d and $x_1,\dots,x_n$ be indeterminate, let N_i be nonnegative integers for $i=1,2,\dots,n$ with $n\ge 1$, and suppose that none of the denominators in* (A.4) *vanish. Then*

$$\begin{aligned}\sum_{\substack{0\le k_i\le N_i\\ i=1,2,\dots,n}}&\left(\prod_{i=1}^{n}\left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right)\prod_{1\le i<j\le n}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\right.\\&\times\prod_{i,j=1}^{n}\frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}}\prod_{i=1}^{n}\frac{(ax_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+N_i};q)_{|\mathbf{k}|}}\\&\times\prod_{i=1}^{n}\frac{(dx_i;q)_{k_i}\,(a^2x_iq^{1+|\mathbf{N}|}/bcd;q)_{k_i}}{(ax_iq/b;q)_{k_i}\,(ax_iq/c;q)_{k_i}}\\&\left.\times\frac{(b;q)_{|\mathbf{k}|}\,(c;q)_{|\mathbf{k}|}}{(aq/d)_{|\mathbf{k}|}\,(bcdq^{-|\mathbf{N}|}/a;q)_{|\mathbf{k}|}}\,q^{\sum_{i=1}^n i\,k_i}\right)\\&=\frac{(aq/bd;q)_{|\mathbf{N}|}\,(aq/cd;q)_{|\mathbf{N}|}}{(aq/d;q)_{|\mathbf{N}|}\,(aq/bcd;q)_{|\mathbf{N}|}}\prod_{i=1}^{n}\frac{(ax_iq;q)_{N_i}\,(ax_iq/bc;q)_{N_i}}{(ax_iq/b;q)_{N_i}\,(ax_iq/c;q)_{N_i}}.\qquad\text{(A.4)}\end{aligned}$$

Remark A.5. Theorem A.3 appeared as Theorem 6.14 in [21] with different notation, see also [26, A12].

Theorem A.6 ((Denis–Gustafson and Milne–Lilly) A $\boldsymbol{C_n}$ Jackson's sum)**.** *Let a, b, c, d and x_1, $\ldots$, x_n be indeterminate, let N_i be nonnegative integers for $i = 1, 2, \ldots, n$ with $n \geq 1$, and suppose that none of the denominators in* (A.7) *vanish. Then*

$$\sum_{\substack{0\leq k_i\leq N_i\\ i=1,2,\ldots,n}} \Bigg(\prod_{i=1}^{n} \left(\frac{1-ax_i^2 q^{2k_i}}{1-ax_i^2} \right) \prod_{1\leq i<j\leq n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j} \, \frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j} \right)$$
$$\times \prod_{i=1}^{n} \frac{(bx_i;q)_{k_i}\,(cx_i;q)_{k_i}\,(dx_i;q)_{k_i}\,(a^2x_iq^{1+|\mathbf{N}|}/bcd;q)_{k_i}}{(ax_iq/b;q)_{k_i}\,(ax_iq/c;q)_{k_i}\,(ax_iq/d;q)_{k_i}\,(bcdx_iq^{-|\mathbf{N}|}/a;q)_{k_i}}$$
$$\times \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j;q)_{k_i}\,(ax_ix_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}\,(ax_ix_jq^{1+N_j};q)_{k_i}} \, q^{\sum_{i=1}^n ik_i} \Bigg)$$
$$= \prod_{1\leq i<j\leq n} (ax_ix_jq;q)^{-1}_{N_i+N_j} \prod_{i,j=1}^{n} (ax_ix_jq;q)_{N_i}$$
$$\times \frac{(aq/bc;q)_{|\mathbf{N}|}\,(aq/bd;q)_{|\mathbf{N}|}\,(aq/cd;q)_{|\mathbf{N}|}}{\prod_{i=1}^{n}\left[(ax_iq/b;q)_{N_i}\,(ax_iq/c;q)_{N_i}\,(ax_iq/d;q)_{N_i}\,(aq^{1+|\mathbf{N}|-\mathbf{N_i}}/bcdx_i;q)_{N_i}\right]}. \tag{A.7}$$

Remark A.8. Theorem A.6 appeared as Theorem 4.1 in [10] with different notation. In a form very similar to ours above, with relabeled parameters and slightly rearranged product side, it appeared (independently from [10]) as Theorem 6.13 in [25].

Theorem A.9 ((Bhatnagar) A $\boldsymbol{D_n}$ Jackson's sum)**.** *Let a, b, c, d and $x_1, \ldots, x_n$ be indeterminate, let N_i be nonnegative integers for $i = 1, 2, \ldots, n$ with $n \geq 1$, and suppose that none of the denominators in* (A.10) *vanish. Then*

$$\sum_{\substack{0\leq k_i\leq N_i\\ i=1,2,\ldots,n}} \Bigg(\prod_{i=1}^{n} \left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i} \right) \prod_{1\leq i<j\leq n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j} \right)$$
$$\times \prod_{1\leq i<j\leq n} (ax_ix_jq/d;q)_{k_i+k_j} \prod_{i,j=1}^{n} (ax_ix_jq/d;q)^{-1}_{k_i}$$
$$\times \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{n} \frac{(ax_i;q)_{|\mathbf{k}|}\,(d/x_i;q)_{|\mathbf{k}|}}{(ax_iq^{1+N_i};q)_{|\mathbf{k}|}\,(d/x_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}$$
$$\times \frac{\prod_{i=1}^{n}\left[(bx_i;q)_{k_i}\,(cx_i;q)_{k_i}\,(a^2x_iq^{1+|\mathbf{N}|}/bcd;q)_{k_i}\right]}{(aq/b;q)_{|\mathbf{k}|}\,(aq/c;q)_{|\mathbf{k}|}\,(bcdq^{-|\mathbf{N}|}/a;q)_{|\mathbf{k}|}} \, q^{\sum_{i=1}^n ik_i} \Bigg)$$

$$= \prod_{1\le i<j\le n} (ax_ix_jq/d;q)_{N_i+N_j} \prod_{i,j=1}^{n} (ax_ix_jq/d;q)_{N_i}^{-1}$$
$$\times \frac{\prod_{i=1}^{n} \left[(ax_iq;q)_{N_i} (ax_iq/cd;q)_{N_i} (ax_iq/bd;q)_{N_i} (aq^{1+|\mathbf{N}|-\mathbf{N_i}}/bcx_i;q)_{N_i}\right]}{(aq/b;q)_{|\mathbf{N}|} (aq/c;q)_{|\mathbf{N}|} (aq/bcd;q)_{|\mathbf{N}|}}. \tag{A.10}$$

Remark A.11. Theorem A.9 appears in [7] where it has been derived by combining a D_n balanced ${}_3\phi_2$ summation and a D_n very-well-poised ${}_6\phi_5$ summation theorem.

Theorem A.9 is also equivalent to Theorem A.12 since it follows from reversing the sum in (A.13) and relabeling parameters.

Theorem A.12 ((Schlosser) A $\boldsymbol{D_n}$ Jackson's sum)**.** *Let a, b, c, d and $x_1,\dots,x_n$ be indeterminate, let N_i be nonnegative integers for $i = 1,2,\dots,n$ with $n \ge 1$, and suppose that none of the denominators in* (A.13) *vanish. Then*

$$\sum_{\substack{0\le k_i\le N_i\\ i=1,2,\dots,n}} \Bigg(\prod_{i=1}^{n} \left(\frac{1-ax_iq^{k_i+|\mathbf{k}|}}{1-ax_i}\right) \prod_{1\le i<j\le n} \left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)$$
$$\times \prod_{1\le i<j\le n} (a^2x_ix_jq/bcd;q)_{k_i+k_j}^{-1} \prod_{i,j=1}^{n} (a^2x_ix_jq^{1+N_j}/bcd;q)_{k_i}$$
$$\times \prod_{i,j=1}^{n} \frac{(q^{-N_j}x_i/x_j;q)_{k_i}}{(qx_i/x_j;q)_{k_i}} \prod_{i=1}^{n} \frac{(ax_i;q)_{|\mathbf{k}|} (bcd/ax_i;q)_{|\mathbf{k}|-\mathbf{k_i}}}{(ax_iq^{1+N_i};q)_{|\mathbf{k}|} (bcdq^{-N_i}/ax_i;q)_{|\mathbf{k}|}}$$
$$\times \frac{(b;q)_{|\mathbf{k}|} (c;q)_{|\mathbf{k}|} (d;q)_{|\mathbf{k}|}}{\prod_{i=1}^{n} \left[(ax_iq/b;q)_{k_i} (ax_iq/c;q)_{k_i} (ax_iq/d;q)_{k_i}\right]} q^{\sum_{i=1}^{n} i k_i} \Bigg)$$
$$= \prod_{i=1}^{n} \frac{(ax_iq;q)_{N_i} (ax_iq/bc;q)_{N_i} (ax_iq/bd;q)_{N_i} (ax_iq/cd;q)_{N_i}}{(ax_iq/bcd;q)_{N_i} (ax_iq/d;q)_{N_i} (ax_iq/c;q)_{N_i} (ax_iq/b;q)_{N_i}}. \tag{A.13}$$

Remark A.14. Theorem A.12 appears in [29, Theorem 5.6] and [30] with slightly relabeled parameters. It was derived by a D_n matrix inversion combined with the C_n ${}_8\phi_7$ summation in Theorem A.6.

Theorem A.12 is also equivalent to Theorem A.9 since it follows from reversing the sum in (A.10) and relabeling parameters.

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