Finite-strain Poynting-Thomson model: Existence and linearization

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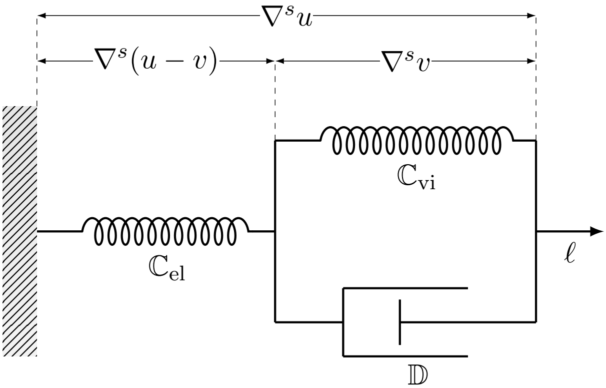
SFB PDE Taming Complexity in Partial Differential Systems

Vienna School of Mathematics

Outline

- Preliminaries
  - Assumptions
  - Energy and Dissipation
- Existence

What's our problem?

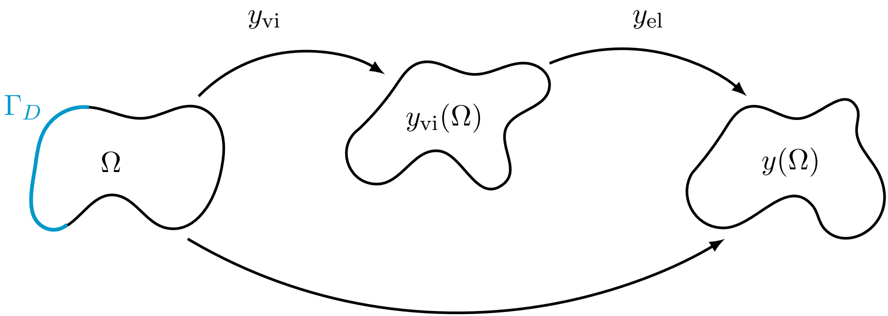


$$\begin{cases} -\operatorname{div}\left(\mathbb{C}_{\mathrm{el}} \nabla (u - v)\right) = \ell & \text{in } \Omega \\ -\operatorname{div}\left(\mathbb{D} \nabla \dot{v} + \mathbb{C}_{\mathrm{vi}} \nabla v\right) = \ell & \text{in } \Omega \end{cases}$$

Preliminaries

Domains and Deformations

$\Omega \subset \mathbb{R}^d$ non empty, Lipschitz domain



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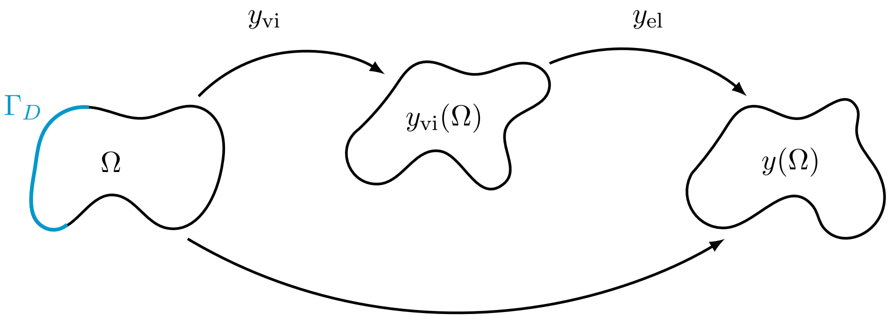

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$$\nabla y = F_{\mathrm{el}} F_{\mathrm{vi}}$$

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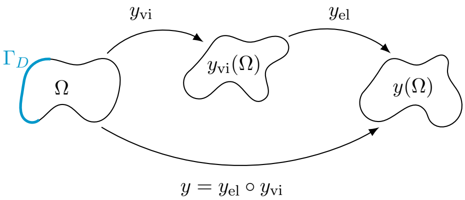

- $y_{\mathrm{vi}}(\Omega) \longrightarrow$ some regularity required:

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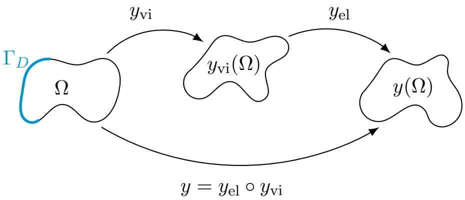

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  - Sobolev extension domains
  - closed under Hausdorff convergence (e.g. uniformly Lipschitz)

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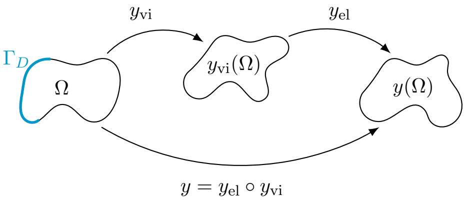

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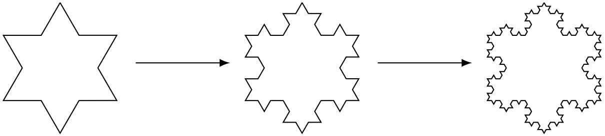

Admissible States

- y_{vi} viscous deformation:
  - $y_{\text{vi}} \in W^{1,p_{\text{vi}}}(\Omega;\mathbb{R}^d), \quad p_{\text{vi}} > d(d-1)$
  - $\det \nabla y_{\text{vi}} = 1$ a.e. in Ω (locally volume preserving)

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disappears in change of variable

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  - y_{vi} homeomorphism ($\Rightarrow$ invertible)
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Energy

Total energy of the system:

$$\mathcal{E}(t, y_{\text{el}}, y_{\text{vi}}) := \underbrace{\mathcal{W}(y_{\text{el}}, y_{\text{vi}})}_{\text{Stored energy}} - \underbrace{\langle \ell(t), y_{\text{el}} \circ y_{\text{vi}} \rangle}_{\text{work of external actions}}$$

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$$\mathcal{W}(y_{\mathrm{el}}, y_{\mathrm{vi}}) = \int_{y_{\mathrm{vi}}(\Omega)} \underbrace{W_{\mathrm{el}}(\nabla y_{\mathrm{el}}(\xi))}_{\text{stored elastic energy}} \, \mathrm{d}\xi + \int_{\Omega} \underbrace{W_{\mathrm{vi}}(\nabla y_{\mathrm{vi}}(X))}_{\text{stored viscous energy}} \, \mathrm{d}X$$

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- $W_{\mathrm{el}} : \mathbb{R}^{d \times d} \to \mathbb{R}$ polyconvex
- $c_1 |A|^{p_{\mathrm{el}}} \le W_{\mathrm{el}}(A) \le \frac{1}{c_1}(1 + |A|^{p_{\mathrm{el}}})$
- $W_{\mathrm{vi}} : \mathbb{R}^{d \times d} \to \overline{\mathbb{R}}$ polyconvex
- $W_{\mathrm{vi}}(A) \ge \begin{cases} c_2 |A|^{p_{\mathrm{vi}}} - \frac{1}{c_2} & A \in SL(d) \\ \infty & \text{otherwise} \end{cases}$

Dissipation

Istantaneous dissipation of the system:

$$\Psi(y_{\mathrm{vi}}, \dot{y}_{\mathrm{vi}}) := \int_\Omega \underbrace{\psi(\nabla \dot{y}_{\mathrm{vi}} (\nabla y_{\mathrm{vi}})^{-1})}_{\text{dissipation density}} \, \mathrm{d}X$$

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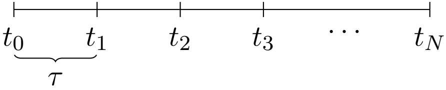
- $\psi : \mathbb{R}^{d\times d} \to \mathbb{R}$ convex, differentiable at 0
- $\psi(A) \geq c_3 |A|^{p_\psi}$
- $\psi(\lambda A) = |\lambda|^{p_\psi} \psi(A)$

$\Big\} \rightsquigarrow \psi(A) := \dfrac{|A|^{p_\psi}}{p_\psi}$

Existence

The Time Discretization Scheme

- $\Pi_\tau := \{0 = t_0 < t_1 \cdots < t_N = T\}\,, \quad t_i - t_{i-1} = \tau = T/N$



- $(y_{\mathrm{el}}^0, y_{\mathrm{vi}}^0)$ compatible initial condition

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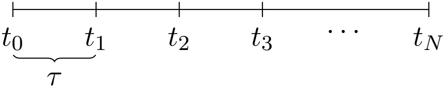

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Incremental minimization problem, $i = 1, \ldots, N$:

$$\inf_{(y_{\mathrm{el}}, y_{\mathrm{vi}}) \in \mathcal{A}} \left\{ \mathcal{E}(t_i, y_{\mathrm{el}}, y_{\mathrm{vi}}) + \int_\Omega \tau \psi \left(\frac{\nabla (y_{\mathrm{vi}} - y_{\mathrm{vi}}^{i-1})}{\tau} \left(\nabla y_{\mathrm{vi}}^{i-1}\right)^{-1} \right) \right\} \qquad (IP)$$

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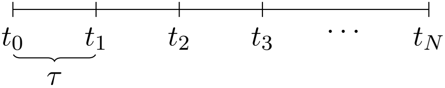

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Theorem

Problem (IP) *admits minimizers (not unique).*

Existence Result

Theorem (C.-Kružík-Stefanelli)

Given $(y^0_{\mathrm{el}}, y^0_{\mathrm{vi}}) \in \mathcal{A}$, *for any sequence* $(\Pi_\tau)_\tau$ *of partitions of the interval* $[0,T]$ *with mesh sizes* $\tau \to 0$, *there exist a (not relabeled) subsequence and functions* $(y_{\mathrm{el}}, y_{\mathrm{vi}}) : [0,T] \to \mathcal{A}$ *such that, for a.e.* $t \in [0,T]$,

• [Approximation]

$$(\overline{y}_{\mathrm{el},\tau}(t), \overline{y}_{\mathrm{vi},\tau}(t)) \rightharpoonup (y_{\mathrm{el}}(t), y_{\mathrm{vi}}(t)) \quad \text{in } \mathcal{A},$$

• [Energy inequality]

$$\mathcal{E}(t, y_{\mathrm{el}}(t), y_{\mathrm{vi}}(t)) + p_\psi \int_0^t \Psi\left(s, y_{\mathrm{vi}}(s)\right) \mathrm{d}s \le \mathcal{E}(0, y^0_{\mathrm{el}}, y^0_{\mathrm{vi}}) - \int_0^t \langle \dot{\ell}(s), y(s) \rangle \, \mathrm{d}s,$$

• [Semistability]

$$\mathcal{E}\left(t, y_{\mathrm{el}}(t), y_{\mathrm{vi}}(t)\right) \le \mathcal{E}\left(t, \tilde{y}_{\mathrm{el}}, y_{\mathrm{vi}}(t)\right) \quad \forall \tilde{y}_{\mathrm{el}} \text{ with } (\tilde{y}_{\mathrm{el}}, y_{\mathrm{vi}}(t)) \in \mathcal{A}.$$

“Naive” Energy Inequality

$$\mathcal{E}(t, y_{\rm el}(t), y_{\rm vi}(t)) + 1\int_0^t \Psi\left(s, y_{\rm vi}(s)\right)\, \mathrm{d}s \leq \mathcal{E}(0, y_{\rm el}^0, y_{\rm vi}^0) - \int_0^t \langle \dot{\ell}(s), y(s)\rangle\, \mathrm{d}s$$

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$$\mathcal{E}(t_i, y_{\mathrm{el}}^i, y_{\mathrm{vi}}^i) + \int_\Omega \tau\psi\left(\frac{\nabla(y_{\mathrm{vi}}^i - y_{\mathrm{vi}}^{i-1})}{\tau}\left(\nabla y_{\mathrm{vi}}^{i-1}\right)^{-1}\right) \overset{\text{min}}{\le} \mathcal{E}(t_i, y_{\mathrm{el}}^{i-1}, y_{\mathrm{vi}}^{i-1})$$

$$= \mathcal{E}(t_{\boldsymbol{i-1}}, y_{\mathrm{el}}^{i-1}, y_{\mathrm{vi}}^{i-1}) - \int_{t_{i-1}}^{t_i} \langle \dot{\ell}, y^{i-1}\rangle$$

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Summing up $\Downarrow$ $i = 1, \ldots, n$

$$\mathcal{E}(t_n, y_{\rm el}^n, y_{\rm vi}^n) + \sum_{i=1}^n \int_\Omega \tau\psi\left(\frac{\nabla(y_{\rm vi}^i - y_{\rm vi}^{i-1})}{\tau}\left(\nabla y_{\rm vi}^{i-1}\right)^{-1}\right)$$

$$\le \mathcal{E}(0, y_{\rm el}^0, y_{\rm vi}^0) - \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \langle \dot\ell, y^{i-1}\rangle$$

Sharp Energy Inequality

$$\mathcal{E}(t, y_{\text{el}}(t), y_{\text{vi}}(t)) + p_\psi \int_0^t \Psi\left(s, y_{\text{vi}}(s)\right)\, \mathrm{d}s \leq \mathcal{E}(0, y_{\text{el}}^0, y_{\text{vi}}^0) - \int_0^t \langle \dot{\ell}(s), y(s) \rangle\, \mathrm{d}s$$

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$$\Phi(\tau; y_{\text{old}}, y_{\text{el}}, y_{\text{vi}}) := \mathcal{E}(t_i, y_{\text{el}}, y_{\text{vi}}) + \tau\Psi\left(y_{\text{old}}, \frac{y_{\text{vi}} - y_{\text{old}}}{\tau}\right)$$

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$$\mathcal{E}(t_i, y_{\text{el},\tau}, y_{\text{vi},\tau}) + \tau\Psi\left(y_{\text{old}}, \frac{y_{\text{vi},\tau} - y_{\text{old}}}{\tau}\right) - \mathcal{E}(t_i, y_{\text{el}}, y_{\text{old}}) = -\int_0^\tau \Psi_r(y_{\text{old}})\,\mathrm{d}r$$

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Main reference:

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Thank you for your attention!