Finite-strain Poynting-Thomson model: Existence and linearization

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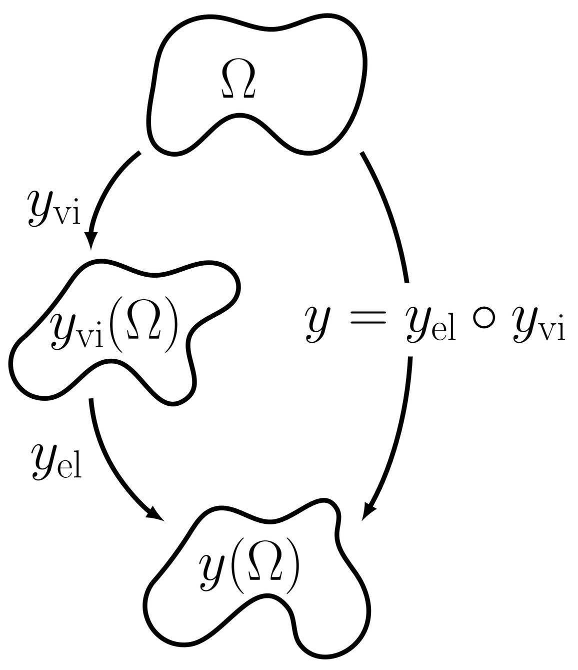
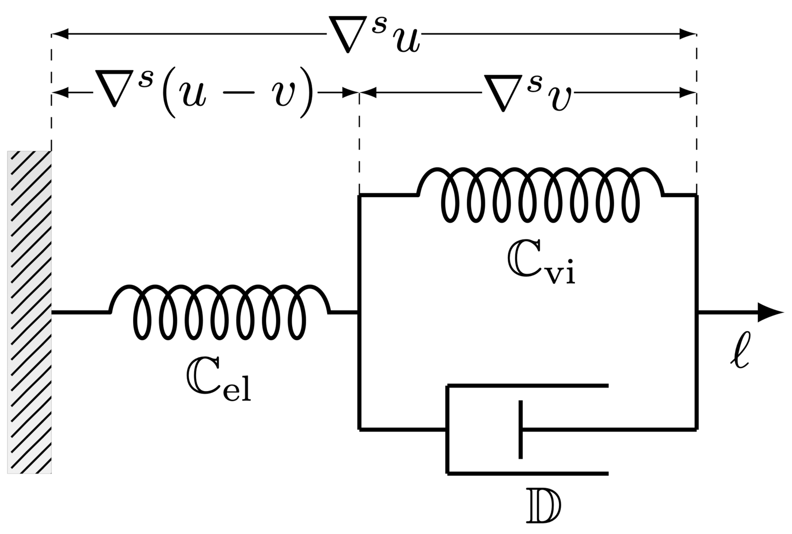
Aim





$$-\operatorname{div}(\mathbb{D}\nabla^s\dot{v} + \mathbb{C}_{\text{vi}}\nabla^s v) = \ell$$
$$-\operatorname{div}(\mathbb{C}_{\text{el}}\nabla^s(u-v)) = \ell$$

Admissible deformations

$\Omega \subset \mathbb{R}^d$ Lipschitz domain, $d \geq 2$.

$\mathcal{A} :=$ set of admissible deformations, satisfying the following

Viscous deformation y_{vi}

- $y_{\text{vi}} \in W^{1,p_{\text{vi}}}(\Omega;\mathbb{R}^d)$, with $p_{\text{vi}} > d(d-1)$;
- locally volume preserving, $\det \nabla y_{\text{vi}} = 1$ a.e. in Ω;
- a.e. injective, $|\Omega| = |y_{\text{vi}}(\Omega)| = 1$;
- $y_{\text{vi}}(\Omega)$ uniformly Lipschitz;
- $\int_\Omega y_{\text{vi}} = 0$.

$\Rightarrow$

- y_{vi} homeomorphism;
- $y_{\text{vi}}(\Omega)$ open;
- change of variables formula.

Elastic deformation y_{el}

- $y_{\text{el}} \in W^{1,p_{\text{el}}}(y_{\text{vi}}(\Omega);\mathbb{R}^d)$, with $p_{\text{el}} > d$. $\Rightarrow$ $y_{\text{el}} \in C(y_{\text{vi}}(\Omega);\mathbb{R}^d)$.

Total deformation y

- $y := y_{\text{el}} \circ y_{\text{vi}} = \text{id}$ on $\Gamma_D \subset \partial\Omega$. $\Rightarrow$ $y \in W^{1,q}(\Omega;\mathbb{R}^d)$.

Energy and Dissipation

Total Energy $\mathcal{E}$

$$\mathcal{E}(t, y_{\text{el}}, y_{\text{vi}}) := \underbrace{\int_{y_{\text{vi}}(\Omega)} W_{\text{el}}(\nabla y_{\text{el}})}_{\text{Stored elastic energy}} + \underbrace{\int_\Omega \overbrace{W_{\text{vi}}(\nabla y_{\text{vi}})}^{\text{No second gradients}}}_{\text{Stored viscous energy}} - \underbrace{\langle \ell(t), y_{\text{el}} \circ y_{\text{vi}} \rangle}_{\text{Work of external mechanical actions}}$$

- $W_{\text{el}} : \mathbb{R}^{d\times d} \to \mathbb{R}$ polyconvex;
- $c_1|A|^{p_{\text{el}}} \leq W_{\text{el}}(A) \leq \frac{1}{c_1}(1 + |A|^{p_{\text{el}}})$, $\forall A \in GL(d)$.
- $W_{\text{vi}} : \mathbb{R}^{d\times d} \to \overline{\mathbb{R}}$ polyconvex;
- $W_{\text{vi}}(A) \geq c_2|A|^{p_{\text{vi}}} - \frac{1}{c_2}$ on $SL(d)$, $W_{\text{vi}} = \infty$ otherwise.

Dissipation Ψ

$$\Psi(y_{\text{vi}}, \dot{y}_{\text{vi}}) := \int_\Omega \psi(\nabla\dot{y}_{\text{vi}}(\nabla y_{\text{vi}})^{-1})$$

- $\psi : \mathbb{R}^{d\times d} \to [0,\infty)$ convex, differentiable at 0 with $0 = \psi(0)$;
- $\psi(\xi) \geq c_3|A|^{p_\psi}$ for every $A \in \mathbb{R}^{d\times d}$.

Time-discretization scheme

Let $\Pi_\tau := \{0{=}\tau{<}2\tau{<}...{<}N\tau{=}T\}$ be a uniform partition of the time interval $[0,T]$.

Theorem: Existence for the incremental minimization problem

Given $(y^0_{\text{el}}, y^0_{\text{vi}}) \in \mathcal{A}$, the incremental minimization problem

$$\min_{(y_{\text{el}},y_{\text{vi}})\in\mathcal{A}} \left\{ \mathcal{E}(i\tau, y_{\text{el}}, y_{\text{vi}}) + \tau\Psi\left(y^{i-1}_{\text{vi}}, \frac{y_{\text{vi}} - y^{i-1}_{\text{vi}}}{\tau}\right)\right\} \qquad (m_\tau)$$

admits a solution for every $i = 1, ..., N$.

Incrementally approximable solutions

Theorem: Existence of incrementally approximable solutions

There exists an incrementally approximable solution, i.e., a map $(y_{\text{el}}, y_{\text{vi}}) : [0,T] \to \mathcal{A}$ such that:

- there exists a sequence of partitions $(\Pi_\tau)_\tau$ of the interval $[0,T]$ with mesh size $\tau \to 0$, such that the sequence of backward-constant interpolants $(\overline{y}_{\text{el},\tau}, \overline{y}_{\text{vi},\tau})$ of solutions to the incremental minimization problems (m_τ) satisfy

$$(\overline{y}_{\text{el},\tau}(t), \overline{y}_{\text{vi},\tau}(t)) \rightharpoonup (y_{\text{el}}(t), y_{\text{vi}}(t)) \text{ in } W^{1,p_{\text{el}}}(y_{\text{vi}}(t,\Omega);\mathbb{R}^d) \times W^{1,p_{\text{vi}}}(\Omega;\mathbb{R}^d)$$

and, for a.e. $t \in [0,T]$, $(y_{\text{el}}(t), y_{\text{vi}}(t))$ satisfies

- the energy inequality

$$\mathcal{E}(t, y_{\text{el}}(t), y_{\text{vi}}(t)) + \int_0^t \Psi(y_{\text{vi}}(s), \dot{y}_{\text{vi}}(s)) \leq \mathcal{E}(0) - \int_0^t \langle \dot{\ell}(s), y_{\text{el}} \circ y_{\text{vi}}(s)\rangle \qquad \text{(E)}$$

- and the semistability condition

$$\mathcal{E}(t, y_{\text{el}}(t), y_{\text{vi}}(t)) \leq \mathcal{E}(t, \tilde{y}_{\text{el}}, y_{\text{vi}}(t)) \text{ for every } \tilde{y}_{\text{el}} \text{ such that } (\tilde{y}_{\text{el}}, y_{\text{vi}}(t)) \in \mathcal{A}. \qquad \text{(S)}$$

Linearization

Define

$$v := \frac{y_{\text{vi}} - \text{id}_\Omega}{\varepsilon} \quad \text{and} \quad u := \frac{y - \text{id}_\Omega}{\varepsilon}.$$

Assuming sufficient regularity, by Taylor expansion we expect the following Γ-convergences:

$$\frac{1}{\varepsilon^2}\int_\Omega W_{\text{el}}\left((I{+}\varepsilon\nabla u)(I{+}\varepsilon\nabla v)^{-1}\right) \longrightarrow \frac{1}{2}\int_\Omega \mathbb{C}_{\text{el}}\nabla(u{-}v){:}\nabla(u{-}v)$$

$$\frac{1}{\varepsilon^2}\int_\Omega W_{\text{vi}}(I{+}\varepsilon\nabla v) \longrightarrow \frac{1}{2}\int_\Omega \mathbb{C}_{\text{vi}}\nabla v{:}\nabla v$$

$$\frac{1}{\varepsilon^2}\int_\Omega \psi\left(\varepsilon\nabla\dot{v}(I{+}\varepsilon\nabla v)^{-1}\right) \longrightarrow \frac{1}{2}\int_\Omega \mathbb{D}\nabla\dot{v}{:}\nabla\dot{v}$$

Theorem: Convergence of incrementally approximable solutions

For every $\varepsilon > 0$ let $(u_\varepsilon, v_\varepsilon)$ be an incrementally approximable solution at level ε. Then, there exist functions $(u,v) : [0,T] \to H^1_{\Gamma_D}(\Omega;\mathbb{R}^d) \times H^1_\sharp(\Omega;\mathbb{R}^d)$ such that, up to a non relabeled subsequence, we have

$$u_\varepsilon(t) \rightharpoonup u(t),\ v_\varepsilon(t) \rightharpoonup v(t) \text{ in } H^1(\Omega;\mathbb{R}^d) \quad \text{and} \quad \nabla\dot{v}_\varepsilon(t) \rightharpoonup \nabla\dot{v}(t) \text{ in } L^2(\Omega;\mathbb{R}^{d\times d}).$$

Moreover, for every $t \in [0,T]$, $(u(t), v(t))$ satisfy the linear energy inequality

$$\int_\Omega \mathbb{C}_{\text{vi}}\nabla v(t){:}\nabla v(t) + \int_\Omega \mathbb{C}_{\text{el}}\nabla(u(t){-}v(t)){:}\nabla(u(t){-}v(t)) - \langle \ell(t), u(t)\rangle + \int_0^t\int_\Omega \mathbb{D}\nabla\dot{v}(s){:}\nabla\dot{v}(s) \leq \text{“Initial energy”} - \int_0^t \langle\dot{\ell}(s), u(s)\rangle \qquad (\text{E}_{\text{lin}})$$

and the linear semistability condition

$$\int_\Omega \mathbb{C}_{\text{el}}\nabla(u(t){-}v(t)){:}\nabla(u(t){-}v(t)) - \langle \ell(t), u(t)\rangle \leq \int_\Omega \mathbb{C}_{\text{el}}\nabla(\tilde{u}{-}v(t)){:}\nabla(\tilde{u}{-}v(t)) - \langle \ell(t), \tilde{u}\rangle \quad \forall \tilde{u} \in H^1_{\Gamma_D}(\Omega;\mathbb{R}^d). \qquad (\text{S}_{\text{lin}})$$

References

[1] M. Kružík, D. Melching, U. Stefanelli, Quasistatic evolution for dislocation-free finite plasticity. *ESAIM Control Optim. Calc. Var.* 26 (2020), Paper No. 123.

[2] M. Kružík,, T. Roubíček, *Mathematical methods in continuum mechanics of solids*, Interaction of Mechanics and Mathematics, Springer, Cham, 2019.

[3] A. Mielke, U. Stefanelli, Linearized plasticity is the evolutionary Γ-limit of finite plasticity. *J. Eur. Math. Soc. (JEMS)*, 15 (2013), no. 3, pp. 923–948.