

WETTING AND DEWETTING DYNAMICS OF ANISOTROPIC PARTICLES



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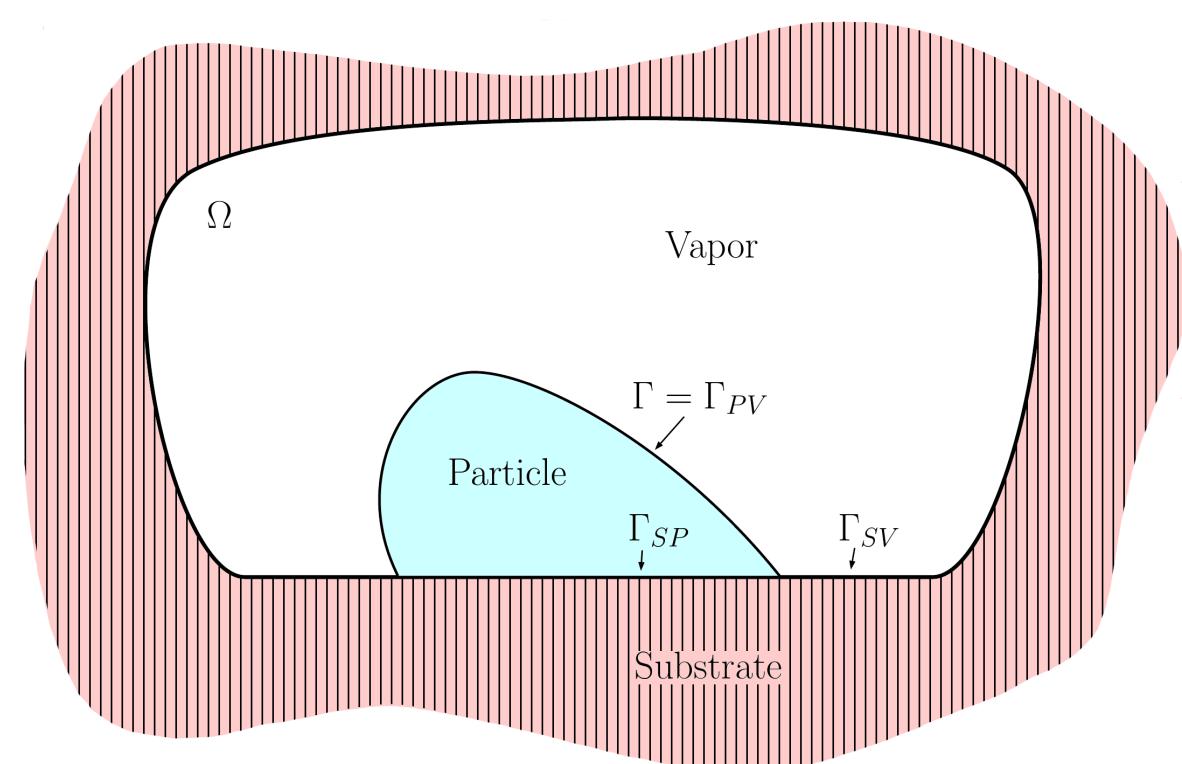
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Setting



- Particle P
- Vapor V
- Substrate S
- $\Omega := \mathbb{R}^d \setminus S$ bounded

Energy

$$E(P) = \int_{\Gamma} \gamma d\mathcal{H}^{d-1}(x) + \int_{\partial P \cap \partial S} \sigma d\mathcal{H}^{d-1}(x)$$

- $\gamma = \gamma(\nu) \in C^2(\mathbb{R}^d \setminus \{0\})$ with γ^2 convex, 1-homogeneous, and $c_\gamma |\nu| \leq \gamma(\nu) \leq C_\gamma |\nu|$
- $\sigma = \sigma(x) \in C^1(\mathbb{R}^d)$ with $\sigma := \gamma_{SP} - \gamma_{SV}$

Gradient flow

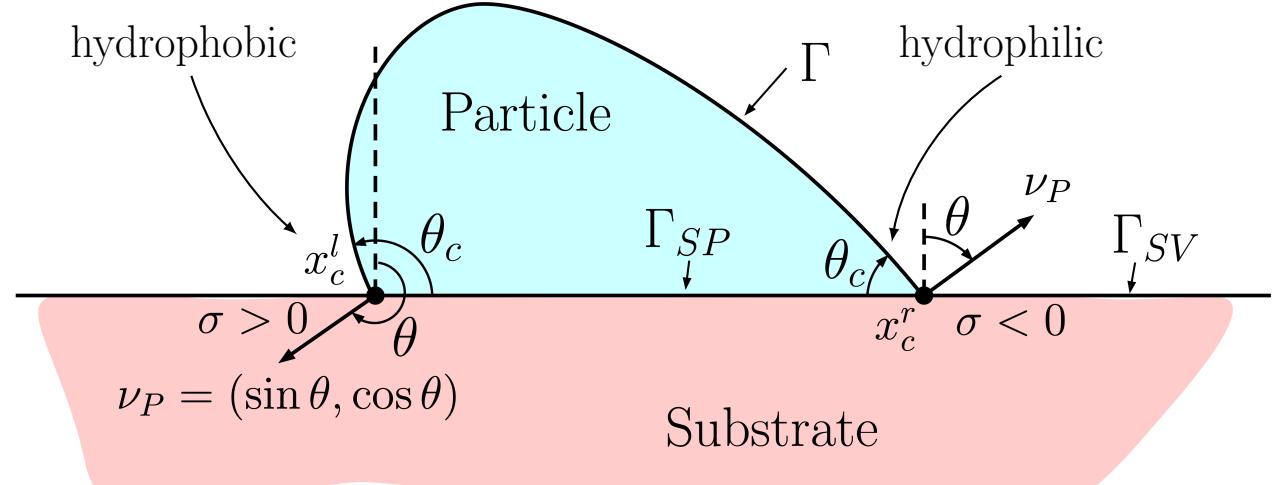
$$H_\gamma(x) = \operatorname{div}(\nabla \gamma(\nu(x)))$$
 anisotropic mean curvature

Volume preserving gradient flow equation

$$V = \begin{cases} -\mu_\gamma(\nu)(H_\gamma - \Lambda) & \text{on } \Gamma \\ 0 & \text{on } \partial P \cap \partial S \end{cases}$$

Herring angle condition

$$\gamma(\nu)b_{PS} - (\nabla_\nu \gamma(\nu) \cdot b_P)\nu_{PS} + \sigma(x)b_S = 0 \quad \text{on } \partial\Gamma$$



- In dimension $d = 2$
 $\tilde{\gamma} \cos \theta - \tilde{\gamma}' \sin \theta + \sigma = 0$
- Isotropic case (Young's equation)
 $\tilde{\gamma} \cos \theta + \sigma = 0$

Γ -convergence

Approximate energy

$$E(u) = \begin{cases} \int_{\Gamma} \gamma d\mathcal{H}^{d-1}(x) + \int_{\partial P \cap \partial S} \sigma d\mathcal{H}^{d-1}(x) & u = \mathbb{1}_P \in BV(\Omega; \{0, 1\}) \\ \infty & \text{otherwise} \end{cases}$$

$$\Downarrow$$

$$E_h(u) = \frac{1}{\sqrt{h}} \int_{\Omega} u K_h * (\mathbb{1}_\Omega - u) d\mathcal{H}^d(x) + \frac{1}{\sqrt{\pi h}} \int_{\Omega} \sigma u K_h * \mathbb{1}_S d\mathcal{H}^d(x)$$

$K \in L^1(\mathbb{R}^d)$ suitable kernel representing the anisotropy γ

$$K_h(x) := (h)^{-d/2} K\left(\frac{x}{\sqrt{h}}\right)$$

$$C_\gamma \leq \gamma_{SP} - \gamma_{SV}, \quad \gamma_{SP} \leq c_\gamma + \gamma_{SV}, \quad \gamma_{SV} \leq c_\gamma + \gamma_{SP}$$

Theorem: Γ -convergence

Let $\gamma_{SP}, \gamma_{SV} \in C^1(\mathbb{R}^d) \cap C^{0,1}(\mathbb{R}^d)$. Under suitable assumptions on the kernel K , then

$$E_h \xrightarrow[s-L^1]{\Gamma} E$$

How is our result different from the result of [Esedoḡlu–Otto]?

- The substrate energy density σ is space dependent
- We can consider $\gamma = \gamma(x, \nu)$ (but not in the MBO scheme)

The MBO scheme

Given $u_k := \mathbb{1}_{P^k}$ and $V^k := \Omega \setminus P^k$ at time step $t_k = kh$
 $u_{k+1} := \mathbb{1}_{P^{k+1}}$ and $V^{k+1} := \Omega \setminus P^{k+1}$ are obtained as follows:

The thresholding scheme

- Convolution:

$$\varphi^k(x) = \frac{1}{\sqrt{h}} K_h * (\mathbb{1}_\Omega - 2u_k) + \frac{1}{\sqrt{\pi h}} \sigma K_h * \mathbb{1}_S.$$

- Thresholding:

$$P^{k+1} = \{x \in \Omega \mid \varphi^k(x) < \delta^k\}, \quad V^{k+1} = \Omega \setminus P^{k+1}$$

where δ^k is such that the mass of P^{k+1} is equal to the one of P^k .

Define $u_h(t) = u^0 \quad \text{if } t \leq 0, \quad u_h(t) = u^k \quad \text{if } t \in [kh, (k+1)h).$

Convergence of the scheme

Theorem: Existence of a mean curvature flow

Let $u^0 = \mathbb{1}_{P^0} \in BV(\Omega; \{0, 1\})$ and $\hat{K} > 0$. Then there exists a $u : (0, T) \times \Omega \rightarrow \{0, 1\}$ such that

$$u_h \rightarrow u \quad \text{in } L^1((0, T) \times \Omega).$$

and ∇u is a bounded measure, which is equi-integrable in t . If we assume

$$\limsup_h \int_0^T E_h^K(u_h(t)) dt \leq \frac{1}{2} \int_0^T \int_{\Omega} \gamma(\nu) |\nabla u| dt,$$

then u moves by anisotropic volume-preserving mean curvature flow with obstacle S .

Namely, there exist $H_\gamma : [0, T] \times \Omega \rightarrow \mathbb{R}$ with

$$\int_0^T \int_{\Omega} |H_\gamma|^2 |\nabla u| dt < \infty,$$

mean curvature in the sense that it satisfies

$$\begin{aligned} \int_0^T \int_{\Omega} H_\gamma \xi \cdot \nu |\nabla u| dt &= \int_0^T \int_{\Omega} (\gamma(\nu) \operatorname{div} \xi - \nu \nabla \xi \cdot \nabla_\nu \gamma(\nu)) |\nabla u| dt \\ &\quad + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} (\sigma(\nu \cdot \nabla \xi \nu - \operatorname{div} \xi) - \nabla \sigma \cdot \xi) (|\nabla u| + |\nabla \mathbb{1}_\Omega| - |\nabla(\mathbb{1}_\Omega - u)|) dt \end{aligned}$$

for all $\xi \in C^\infty((0, T) \times \Omega; \mathbb{R}^d)$ with $\xi|_{\partial S} \cdot \nu_S = 0$;

there exist $V : [0, T] \times \Omega \rightarrow \mathbb{R}$ with

$$\int_0^T \int_{\Omega} |V|^2 |\nabla u| dt < \infty,$$

normal velocity in the sense that it satisfies

$$\begin{aligned} \int_{\Omega} \zeta(s, x) u(s, x) d\mathcal{H}^d(x) - \int_{\Omega} \zeta(0, x) u^0(x) d\mathcal{H}^d(x) \\ = \int_0^s \int_{\Omega} \partial_t \zeta(t, x) u(t, x) d\mathcal{H}^d(x) dt + \int_0^s \int_{\Omega} \zeta(t, x) V(t, x) |\nabla u| dt \end{aligned}$$

for all $\zeta \in C^\infty([0, T] \times \Omega)$ and $s \in (0, T]$.

Moreover, they satisfy the

the De Giorgi Inequality

$$E(u(T)) + \int_0^T \int_{\Omega} \frac{V^2}{\mu_\gamma(\nu)} + \mu_\gamma(\nu) \left(\frac{H_\gamma - \Lambda}{2} \right)^2 |\nabla u| dt \leq E(u^0)$$

for all $\xi \in C^\infty([0, T] \times \Omega)$ with $\xi|_{\partial S} \cdot \nu_S = 0$.

References

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