Finite-strain Poynting-Thomson model: Existence and linearization

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Joint work with Martin Kružík (Czech Academy of Sciences) and Ulisse Stefanelli (University of Vienna)

2^{nd} Austrian Calculus of Variations Day

universität wien

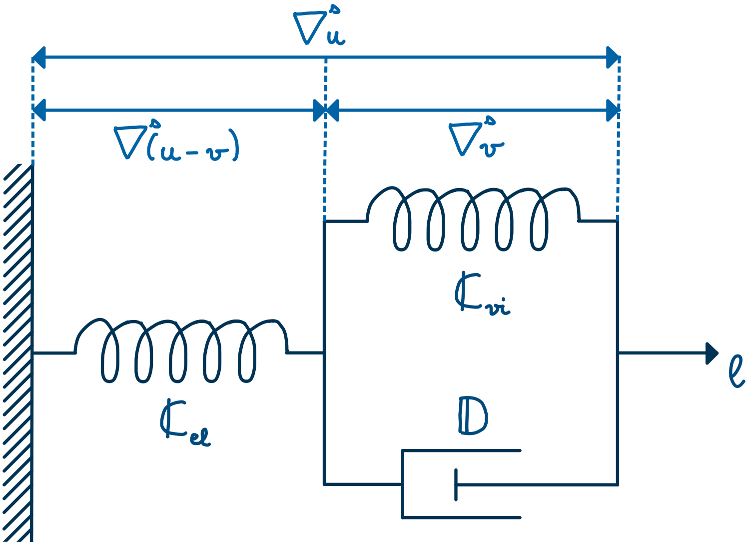
SFB PDE Taming Complexity in Partial Differential Systems

Vienna School of Mathematics

Outline

- Preliminaries
    - Assumptions
    - Energy and Dissipation
- Existence
- Linearization

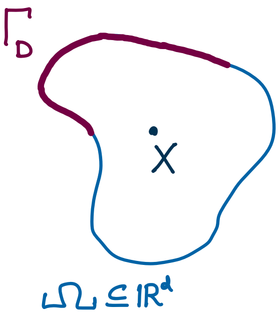
What's our problem?



$$\begin{cases} -\operatorname{div}\left(\mathbb{D}\nabla^s \dot{v} + \mathbb{C}_{vi}\nabla^s v\right) = \ell \\ -\operatorname{div}\left(\mathbb{C}_{el}\nabla^s (u-v)\right) = \ell \end{cases}$$

Domains and Deformations

$\Omega \subseteq \mathbb{R}^d$ ($\rightarrow d \geq 2$) non empty, open, connected, Lipschitz



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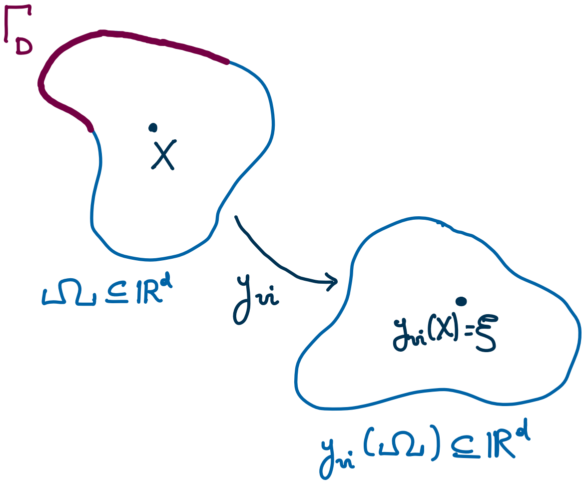
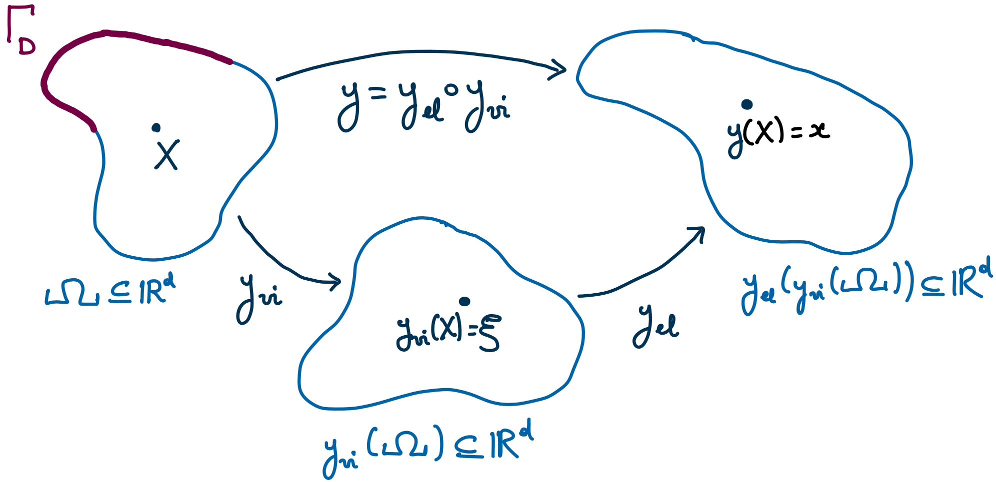
$d \geq 2$



Domains and Deformations

$\Omega \subseteq \mathbb{R}^d$ non empty, open, connected, Lipschitz ($d \geq 2$)



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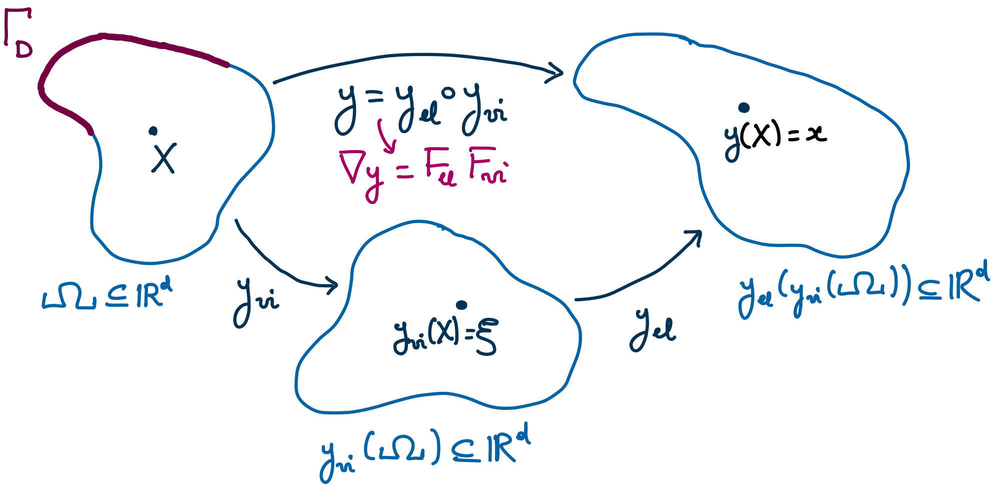

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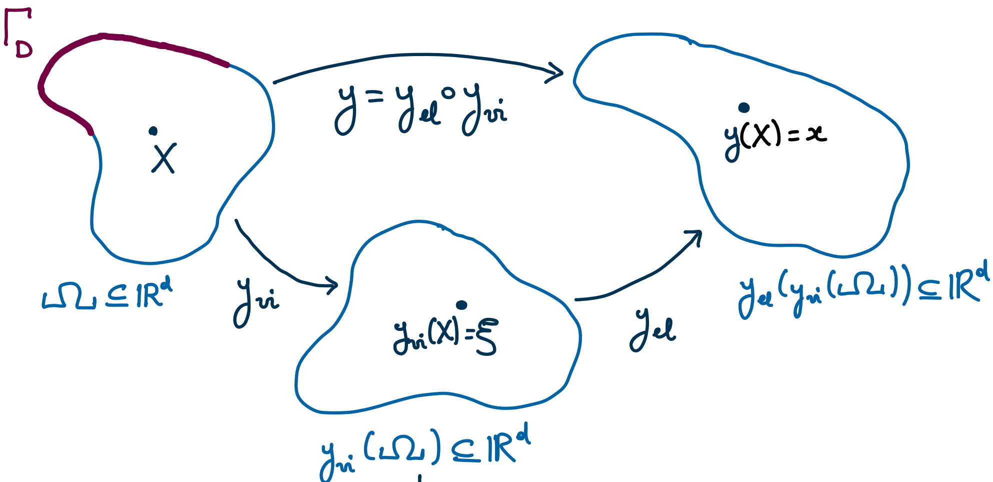
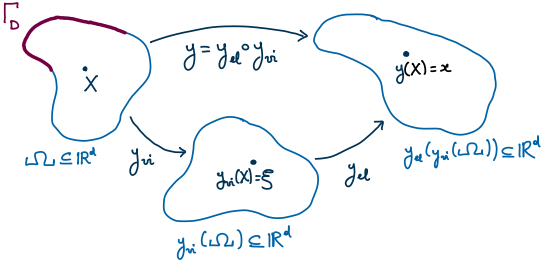


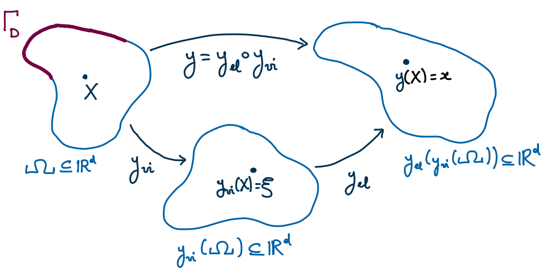
Intermediate configuration

- $y_{vi}(\Omega) \rightarrow$ some regularity required



Domains and Deformations

- $y_{vi}(\Omega)$ → some regularity required
  - Sobolev extension domains
  - closed under Hausdorff convergence



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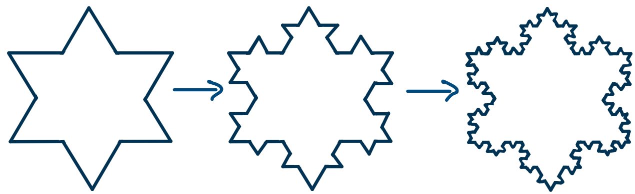

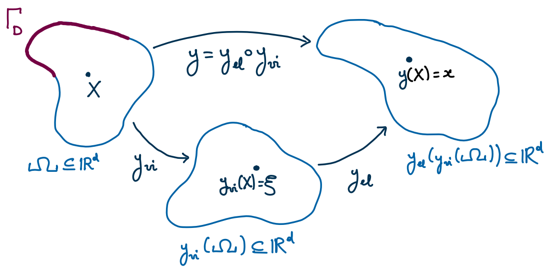

Domains and Deformations

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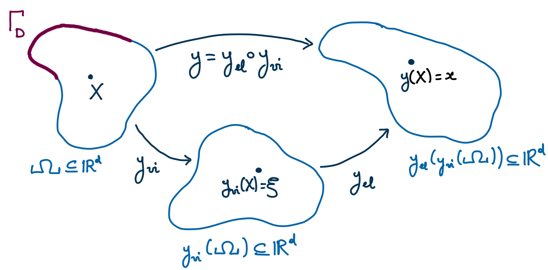
e.g.) Uniformly Lipschitz or (ε,δ)-domains



Admissible states: viscous deformations

y_{vi} viscous deformation

- $y_{vi} \in W^{1,p_{vi}}(\Omega;\mathbb{R}^d)$, $\quad p_{vi} > d(d-1)$
- $\det \nabla y_{vi} = 1$ a.e in Ω (locally volume preserving)

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disappears in change of variables

we cannot ask it for the elastic deformation

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$$\Downarrow$$

- y_{vi} homeomorphism
- y_{vi} locally invertible
- Chain rule

$$\nabla y(X) = \nabla y_{el}(y_{vi}(X))\nabla y_{vi}(X)$$

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$$\nabla y(X) = \nabla y_{el}(y_{vi}(X))\,\nabla y_{vi}(X)$$

$\hookrightarrow \nabla y = F_{el} F_{vi}$

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- $\det \nabla y_{vi} = 1$ a.e in Ω (locally volume preserving)
- other assumptions

Energy

Total energy of the system:

$$\mathcal{E}(t, y_{el}, y_{vi}) := \mathcal{W}(y_{el}, y_{vi}) - \langle \ell(t), y_{el} \circ y_{vi} \rangle$$

$\mathcal{W}(y_{el}, y_{vi})$: Stored energy

$\langle \ell(t), y_{el} \circ y_{vi} \rangle$: work of external actions

- $\ell \in W^{1,1}(0,T;(W^{1,q}(\Omega;\mathbb{R}^d))^*)$

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$$\mathcal{W}(y_{el}, y_{vi}) := \int_{y_{vi}(\Omega_i)} W_{el}(\nabla y_{el}(\xi))\, d\xi + \int_{\Omega_i} W_{vi}(\nabla y_{vi}(X))\, dX$$

First integral: stored elastic energy; second integral: stored viscous energy

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NO second gradient $\nabla^2 y_{vi}$

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First integral: stored elastic energy; second integral: stored viscous energy

- $W_{el} : \mathbb{R}^{d\times d} \to \mathbb{R}$ polyconvex
- $0 \leq c|A|^{p_{el}} \leq W_{el}(A) \leq \frac{1}{c}(1 + |A|^{p_{el}})$

- $W_{vi} : \mathbb{R}^{d\times d} \to \overline{\mathbb{R}}$ polyconvex
- $W_{vi}(A) \geq \begin{cases} c|A|^{p_{vi}} - \frac{1}{c} & A \in SL(d) \\ \infty & \text{otherwise} \end{cases}$

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  $\to$ weakly lower semicontinuous

Energy

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- $0 \leq c|A|^{p_{el}} \leq W_{el}(A) \leq \frac{1}{c}(1 + |A|^{p_{el}})$

we cannot impose $W_{el} = +\infty$ when $\det A \leq 0$!

- $W_{vi} : \mathbb{R}^{d\times d} \to \overline{\mathbb{R}}$ polyconvex, closed
- $W_{vi}(A) \geq \begin{cases} c|A|^{p_{vi}} - \frac{1}{c} & A \in SL(d) \\ \infty & \text{otherwise} \end{cases}$

W_{vi}: weakly lower semicontinuous

Dissipation

Dissipation of the system

$$\Psi(t, \nabla y_{vi}, \nabla \dot{y}_{vi}) := \int_\Omega \psi(\nabla \dot{y}_{vi} \nabla y_{vi}^{-1})\, dX$$

$$\psi(A) := \frac{|A|^{p_\psi}}{p_\psi}$$

- convex
- $\int \psi$ w.l.s.
- $\psi \geq \psi(0) = 0$

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$$\downarrow$$

$$\psi(A) := \frac{|A|^{p_\psi}}{p_\psi}$$

→ • convex
• $\int \psi$ w.l.s.
• $\psi \geq \psi(0) = 0$

$$\Downarrow$$

• $|\nabla \dot{y}_{vi}| \leq |\nabla \dot{y}_{vi} \nabla y_{vi}^{-1}| \, |\nabla y_{vi}^{-1}| \in L^{p_r}$

($\nabla \dot{y}_{vi} \nabla y_{vi}^{-1}$: L^{p_ψ}; ∇y_{vi}^{-1}: $\in L^{p_{vi}}$)

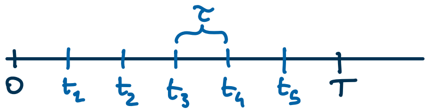
$$\frac{1}{p_r} = \frac{1}{p_\psi} + \frac{1}{p_{vi}} < 1$$

↑ p_ψ must be "large enough"

Existence

Time-discretization scheme

- $\Pi_\tau := \{0 = t_0 < t_1 < \ldots < t_N = T\}$ $\quad t_i - t_{i-1} = \tau = \frac{T}{N}$



- $(y^0_{el}, y^0_{vi}) \in \mathcal{A}$ compatible initial condition

  $\mathcal{E}(0, y^0_{el}, y^0_{vi}) < \infty$

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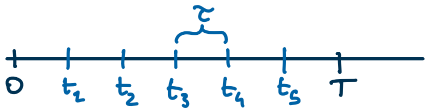

- $(y^0_{el}, y^0_{vi}) \in \mathcal{A}$ compatible initial condition

Incremental minimization problem, $i = 1, \ldots, N$

$$\inf_{(y_{el}, y_{vi}) \in \mathcal{A}} \left\{ \mathcal{E}(t_i, y_{el}, y_{vi}) + \int_{\Omega_1} \tau \psi \left(\frac{\nabla (y_{vi} - y^{i-1}_{vi})}{\tau} (\nabla y^{i-1}_{vi})^{-1} \right) \right\} \quad \text{(IP)}$$

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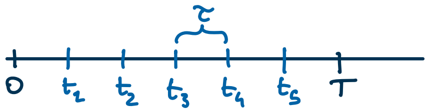

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finite difference instead of $\dot{\nabla y_{vi}}$

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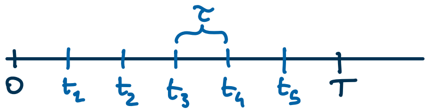

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Theorem (IP) admits minimizers (not unique)

A "weaker" solution for the continuum problem

Theorem ("Existence" for the continuum problem)

Given $(y_{el}^0, y_{vi}^0) \in \mathcal{A}$, for any sequence $(\Pi_\tau)_\tau$ of partitions of the interval $[0,T]$ with mesh sizes $\tau \longrightarrow 0$, there exist a subsequence $(\Pi_{\tau_k})_{k\in\mathbb{N}}$ and functions $(y_{el}, y_{vi}) : [0,T] \longrightarrow \mathcal{A}$ such that, for a.e. $t \in [0,T]$,

- **Approximability**

  $$(y_{el}^{\tau_k}(t), y_{vi}^{\tau_k}(t)) \xrightarrow[k\to+\infty]{} (y_{el}(t), y_{vi}(t)) \quad \text{in } \mathcal{A}$$

  ↳ constant interpolant of the incremental solutions corresponding to Π_{τ_k}

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  $$(y^{\tau_k}_{el}(t), y^{\tau_k}_{vi}(t)) \xrightarrow[k\to+\infty]{} (y_{el}(t), y_{vi}(t)) \quad \text{in } \mathcal{A}$$
- **Energy inequality**
  $$\mathcal{E}(t, y_{el}(t), y_{vi}(t)) + \int_0^t \Psi(s, y_{vi}(s))\,ds \leq \mathcal{E}(0) - \int_0^t \langle \dot{\ell}(s), y(s) \rangle\,ds$$

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- **Semistability**
  $\mathcal{E}(t, y_{el}(t), y_{vi}(t)) \le \mathcal{E}(t, \tilde{y}_{el}, y_{vi}(t)) \quad \forall \tilde{y}_{el}$ s.t. $(\tilde{y}_{el}, y_{vi}(t)) \in \mathcal{A}$

Linearization

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Define:

$$v := \frac{y_{vi} - id_{\Omega}}{\varepsilon} \qquad u := \frac{y_{el} - id_{\Omega}}{\varepsilon}$$

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By Taylor expansion we expect:

$$\frac{1}{\varepsilon^2}\int_\Omega W_{el}\big(\underbrace{(I+\varepsilon\nabla u)}_{\nabla y}\underbrace{(I+\varepsilon\nabla v)^{-1}}_{(\nabla y_{vi})^{-1}}\big) \longrightarrow \frac{1}{2}\int_\Omega \mathbb{C}_{el}\nabla(u-v):\nabla(u-v)$$

rescaling (of $\frac{1}{\varepsilon^2}$)

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rescaling

$$\frac{1}{\varepsilon^2}\int_\Omega W_{vi}(I+\varepsilon\nabla v) \longrightarrow \frac{1}{2}\int_\Omega \mathbb{C}_{vi}\nabla v:\nabla v$$

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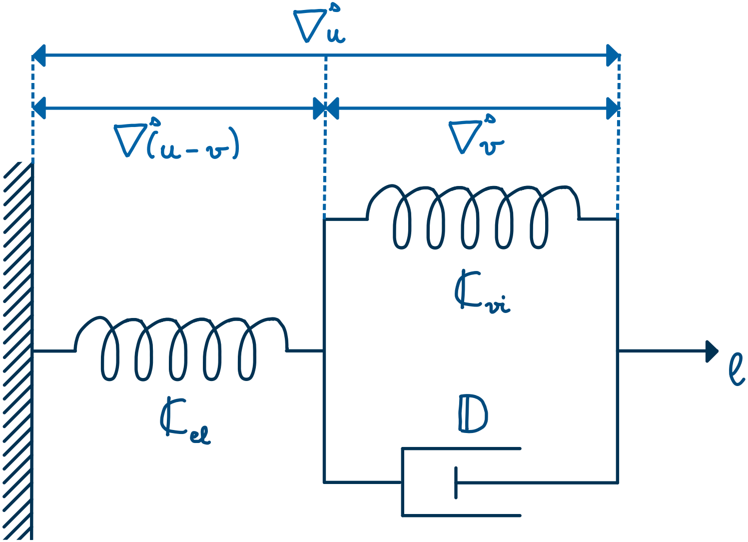
$$\frac{1}{\varepsilon^2}\int_\Omega W_{vi}(I+\varepsilon\nabla v) \longrightarrow \frac{1}{2}\int_\Omega \mathbb{C}_{vi}\nabla v:\nabla v$$

$$\frac{1}{\varepsilon^2}\int_\Omega \Psi\big(\varepsilon\nabla\dot{v}(I+\varepsilon\nabla v)^{-1}\big) \longrightarrow \frac{1}{2}\int_\Omega \mathbb{D}\nabla\dot{v}:\nabla\dot{v}$$

Poynting-Thomson



$$
\begin{cases}
-\operatorname{div}\left(\mathbb{D}\nabla^s \dot{v} + \mathbb{C}_{vi}\nabla^s v\right) = \ell \\
-\operatorname{div}\left(\mathbb{C}_{el}\nabla^s (u-v)\right) = \ell
\end{cases}
$$

Linearization result

Theorem (Linearization) Let $(u_\varepsilon, v_\varepsilon)_\varepsilon$ be a sequence of approximable solutions. → of the rescaled functional

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← average zero

- $u_\varepsilon(t) \longrightarrow u(t)$, $v_\varepsilon(t) \rightarrow v(t)$ in $H^1(\Omega;\mathbb{R}^d)$

  $\nabla \dot{v}_\varepsilon(t) \rightarrow \nabla \dot{v}(t)$ in $L^2(\Omega;\mathbb{R}^{d\times d})$

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- $u_\varepsilon(t) \longrightarrow u(t)$, $v_\varepsilon(t) \to v(t)$ in $H^1(\Omega;\mathbb{R}^d)$
  $\nabla \dot{v}_\varepsilon(t) \longrightarrow \nabla \dot{v}(t)$ in $L^2(\Omega;\mathbb{R}^{d\times d})$
- Energy inequality

$$\int_\Omega \mathbb{C}_{el} \nabla(u(t)-v(t)) : \nabla(u(t)-v(t)) + \int_\Omega \mathbb{C}_{vi} \nabla v(t) : \nabla v(t) - \langle \ell^\circ(t), u(t) \rangle +$$
$$+ \int_0^t \int_\Omega \mathbb{D} \nabla \dot{v}(t) : \nabla \dot{v}(t) \leq \mathcal{E}^\circ(0) - \int_0^t \langle \dot{\ell}^\circ(s), u(s) \rangle$$

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- $u_\varepsilon(t) \longrightarrow u(t)$, $v_\varepsilon(t) \to v(t)$ in $H^1(\Omega;\mathbb{R}^d)$
  $\nabla \dot{v}_\varepsilon(t) \longrightarrow \nabla \dot{v}(t)$ in $L^2(\Omega;\mathbb{R}^{d\times d})$
- **Energy inequality**
  $$\int_\Omega \mathbb{C}_{el} \nabla(u(t)-v(t)) : \nabla(u(t)-v(t)) + \int_\Omega \mathbb{C}_{vi} \nabla v(t) : \nabla v(t) - \langle \ell^0(t), u(t) \rangle + \int_0^t \int_\Omega \mathbb{D} \nabla \dot{v}(t) : \nabla \dot{v}(t) \leq \mathcal{E}^0(0) - \int_0^t \langle \dot{\ell}^0(\lambda), u(\lambda) \rangle$$
- **Semistability**
  $$\int_\Omega \mathbb{C}_{el} \nabla(u(t)-v(t)) : \nabla(u(t)-v(t)) - \langle \ell^0(t), u(t) \rangle \leq \int_\Omega \mathbb{C}_{el} \nabla(\tilde{u}-v(t)) : \nabla(\tilde{u}-v(t)) - \langle \ell^0(t), \tilde{u} \rangle \quad \forall \tilde{u} \in H^1_{\Gamma_D}(\Omega;\mathbb{R}^d)$$

Linearization result

Theorem (Linearization) Let $(u_\varepsilon, v_\varepsilon)_\varepsilon$ be a sequence of approximable solutions. Under suitable assumptions, there exist functions $(u,v):[0,T]\longrightarrow H^1_{\Gamma_D}(\Omega;\mathbb{R}^d)\times H^1_{\#}(\Omega;\mathbb{R}^d)$ such that, ← average zero
for every $t\in[0,T]$, up to a non relabeled subsequence,

- $u_\varepsilon(t)\longrightarrow u(t)$, $v_\varepsilon(t)\to v(t)$ in $H^1(\Omega;\mathbb{R}^d)$

  $\nabla \dot v_\varepsilon(t)\longrightarrow \nabla \dot v(t)$ in $L^2(\Omega;\mathbb{R}^{d\times d})$

- **Energy inequality**

$$\int_\Omega \mathbb{C}_{el}\nabla(u(t)-v(t)):\nabla(u(t)-v(t)) + \int_\Omega \mathbb{C}_{vi}\nabla v(t):\nabla v(t) - \langle \ell^\circ(t),u(t)\rangle +$$
$$+\int_0^t\int_\Omega \mathbb{D}\nabla\dot v(t):\nabla \dot v(t) \leq \mathcal{E}^\circ(0) - \int_0^t \langle \dot\ell^\circ(\lambda),u(\lambda)\rangle$$

- **Semistability** ⟶ $u(t)$ unique given $v(t)$

$$\int_\Omega \mathbb{C}_{el}\nabla(u(t)-v(t)):\nabla(u(t)-v(t)) - \langle \ell^\circ(t),u(t)\rangle \leq$$
$$\leq \int_\Omega \mathbb{C}_{el}\nabla(\tilde u-v(t)):\nabla(\tilde u-v(t)) - \langle \ell^\circ(t),\tilde u\rangle \qquad \forall \tilde u\in H^1_{\Gamma_D}(\Omega;\mathbb{R}^d)$$

Theorem (Linearization) Let $(u_\varepsilon, v_\varepsilon)_\varepsilon$ be a sequence of approximable solutions. Under suitable assumptions, there exist functions $(u,v) : [0,T] \longrightarrow H^1_{\Gamma_D}(\Omega;\mathbb{R}^d) \times H^1_\#(\Omega;\mathbb{R}^d)$ such that, for every $t \in [0,T]$, up to a non relabeled subsequence,

← average zero

- $u_\varepsilon(t) \longrightarrow u(t)$, $v_\varepsilon(t) \to v(t)$ in $H^1(\Omega;\mathbb{R}^d)$

  $\nabla \dot{v}_\varepsilon(t) \longrightarrow \nabla \dot{v}(t)$ in $L^2(\Omega;\mathbb{R}^{d\times d})$

- **Energy inequality**

$$\int_\Omega \mathbb{C}_{el}\nabla(u(t)-v(t)) : \nabla(u(t)-v(t)) + \int_\Omega \mathbb{C}_{vi}\nabla v(t) : \nabla v(t) - \langle \ell^\circ(t), u(t)\rangle + \int_0^t\int_\Omega \mathbb{D}\nabla\dot{v}(t) : \nabla\dot{v}(t) \leq \mathcal{E}^\circ(0) - \int_0^t \langle \dot{\ell}^\circ(s), u(s)\rangle$$

- **Semistability** → $u(t)$ unique given $v(t)$ → uniqueness of v?

$$\int_\Omega \mathbb{C}_{el}\nabla(u(t)-v(t)) : \nabla(u(t)-v(t)) - \langle \ell^\circ(t), u(t)\rangle \leq$$
$$\leq \int_\Omega \mathbb{C}_{el}\nabla(\tilde{u}-v(t)) : \nabla(\tilde{u}-v(t)) - \langle \ell^\circ(t), \tilde{u}\rangle \qquad \forall \tilde{u} \in H^1_{\Gamma_D}(\Omega;\mathbb{R}^d)$$

Thank you for your attention!