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Variational models for finite-strain viscoelastic materials

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Che cosa può ancora aspettarsi la gente da un uomo che dall'alba al crepuscolo si affanna a trasformare in definizione ogni assurdità esistente sotto questo cielo?

EMIL CIORAN

Bei soviel Angst soviel Symbolik!

PAUL CELAN

Abstract

Viscoelasticity is the response of materials like rubber, clay, and various polymers or metals exhibiting both elastic and viscous behavior with respect to the action of external forces. The interplay between the solid-like behavior of elasticity and the fluid-like one of viscosity allows to model several phenomena in continuum mechanics and has originated rich and interesting mathematical theories.

This dissertation aims at investigating recent developments in variational nonlinear models for the evolution of viscoelastic materials at finite-strain and focuses on two main aspects. On the one hand, we study the Poynting-Thomson model at large strains: We show the existence of solutions in a suitable weak sense without resorting to regularizing second-order terms whose physical interpretation is disputed. In addition, we perform rigorous linearization and prove that the classical small-strain model is recovered. On the other hand, we consider the interplay of viscoelastic effects with accretive growth, as occurs in crystallization, swelling of polymer gels, and solidification processes. We show the existence of solutions to the associated coupled problem for different models: We focus on diffused- and sharp-interface two-phase materials and on solids accumulating residual stresses during growth.

Table of Contents

1 Introduction

1.1 Motivation: Continuum mechanics and viscoelasticity

To provide a comprehensive description of material behavior is the aim of continuum mechanics, which lies at the intersection between physics and mathematics [51]. Over the past decades, the field has undergone significant development, driven by both theoretical advances and practical applications. New mathematical techniques and tools have been developed stemming from the rich collection of phenomena like elasticity, viscosity, plasticity, and cracks and damage formation. In particular, this dissertation focuses on viscoelastic media. These materials exhibit both elastic and viscous properties when undergoing deformations. Namely, elasticity is the characteristic of solids like springs or rubber to return to their original configuration in the small-deformation regime after an applied force is no longer active [17]. Conversely, viscosity is a liquid-like property that expresses the rate-dependent resistance to flow, dissipating mechanical energy along the motion. It can be distinctively observed in thick fluids like honey or syrup. Most materials, however, like polymers, metals, and clay [92], do not display a specific behavior but present a complex and multifaceted combination of the before-mentioned characteristics. They are hence said to be viscoelastic. For such materials, when an external loading is applied and then removed, the body dissipates energy as it goes back to its original shape [84]. There is thus a delay between the moment the force is deactivated - or, correspondingly, activated - and the response of the medium; the state of the system, hence, depends also on its history. Moreover, the dissipated energy may be rate-dependent, i.e., may depend on how fast the loading and unloading are enacted [92].

As a result, the mathematical description of viscoelasticity offers a wide variety of problems and poses several interesting analytical challenges. In this thesis, we focus on two specific aspects of this broad theory:

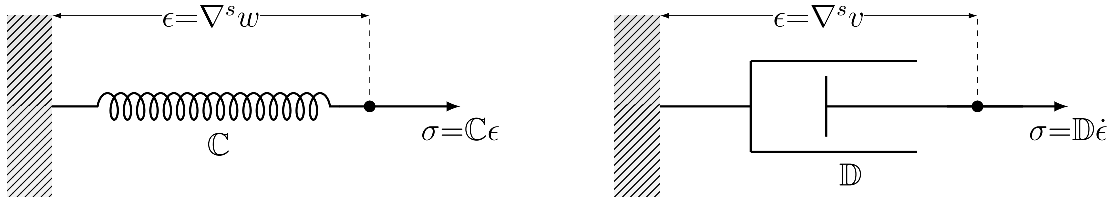
1. A finite-strain Poynting-Thomson model, expanding the linear theory of viscoelastic rheologies. In Chapter 2, we focus on the existence of suitably weak solutions without resorting to second-gradient regularizing terms, and we show a rigorous linearization result.

2. The interplay between growth and viscoelastic response of the medium in the nonlinear setting. Chapter 3 and 4 are devoted to two models for accretive phase-transition and accretive growth, respectively.

We introduce the equations and basic notions of linear elasticity in Section 1.1.1, building up for the finite-strain setting of the following chapters in Section 1.1.2. We then provide some background on different notions of solutions for viscoelastic systems in Section 1.1.3. Finally, in Section 1.2, we summarize the thesis's main results.

1.1.1 Linear viscoelasticity

We begin by discussing viscoelasticity in the linear setting. Let Ω be a nonempty, bounded, and connected open subset of $\mathbb{R}^d$ denoting the reference configuration of the material. In the simplest possible framework, we can consider two idealized mathematical elements representing elastic and viscoelastic response, respectively [51, Chap. 6], see Figure 1.1. The first is



(a) Purely elastic element (b) Purely viscous element

Figure 1.1: Schematic representation of the two basic linear rheological models, depicting elasticity and viscosity, respectively.

usually illustrated with a spring (Figure 1.1a) and is called *Hooke* element. We denote by $w\colon \Omega \to \mathbb{R}^d$ the elastic displacement map, where the actual configuration of the medium is given by $(\mathrm{id} + w)(\Omega)$. The small-strain or linearized strain tensor $\epsilon\colon \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is given by

$$\epsilon := \nabla^s w = \frac{1}{2}\left(\nabla w + \nabla w^\top\right).$$

The constitutive equation for a linear (purely) elastic material, which relates the strain with the stress tensor $\sigma\colon \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$, reads

$$\sigma = \mathbb{C}\epsilon. \tag{1.1.1}$$

Here, $\mathbb{C}\colon \Omega \to \mathbb{R}^{d\times d\times d\times d}$ is the 4-tensor specifying the elastic moduli of the medium. In the following, we assume $\mathbb{C}$ to be elliptic, namely $\sum_{i,j,k,\ell=1}^{d} \xi_{ij}\mathbb{C}_{ijk\ell}\xi_{k\ell} \geq c\sum_{i,j=1}^{d} \xi_{ij}^2$ for every $\xi \in \mathbb{R}^{d\times d}$, for some constant $c > 0$. Equation (1.1.1) is known as *Hooke's law*, where the stress is proportional to the strain. Ignoring any frictional effects, the equilibrium equations of the system under the action of an external forces with density $f\colon \Omega \to \mathbb{R}^d$ are [17, Chap. 3]

$$-\operatorname{div}\sigma \overset{(1.1.1)}{=} -\operatorname{div}\mathbb{C}\epsilon = -\operatorname{div}\mathbb{C}\nabla^s w = f \qquad \text{on } \Omega.$$

They express the local balance between the external forces and the elastic response of the medium. Notice that for $f \equiv 0$ the unique solution for homogeneous Dirichlet boundary conditions is $w \equiv 0$, and hence the material is in its original configuration, as expected. Moreover, if the loading term is bounded, i.e., $\|f\|_{L^\infty(\Omega)} \leq c$, then also w and, in turn, the actual configuration $(\mathrm{id} + w)(\Omega)$ are.

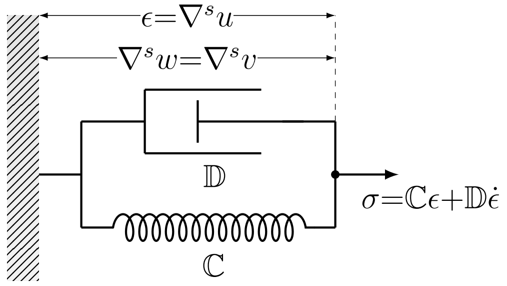
The purely viscous element, on the other hand, is usually represented by a dashpot (Figure 1.1b) and is often labeled *Stokes* element. If $v\colon \Omega \to \mathbb{R}^d$ is the viscous displacement map and its linearized stress tensor is $\epsilon := \nabla^s v$, analogously as above the constitutive equation reads

$$\sigma = \mathbb{D}\dot{\epsilon}, \tag{1.1.2}$$

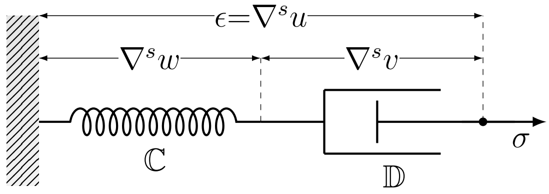
where $\dot{\epsilon}$ denotes the time derivative of the strain tensor, and $\mathbb{D} \in \mathbb{R}^{d\times d\times d\times d}$ is the viscous moduli tensor. Here, the stress is proportional to the strain rate. The equilibrium equations, neglecting inertia, then read [51]

$$-\operatorname{div}\sigma \overset{(1.1.2)}{=} -\operatorname{div}\mathbb{D}\dot{\epsilon} = -\operatorname{div}\mathbb{D}\nabla^s \dot{v} = f \qquad \text{on } [0,T]\times\Omega,$$

where $T > 0$ is some fixed final time. In this case, if f is bounded, then so is $\dot{v}$, $\|\dot{v}\|_{L^\infty([0,T]\times\Omega)} \leq c$. Hence, the best bound that can be expected on v itself is of the form $\|v(t)\|_{L^\infty(\Omega)} \leq c(1+t)$. In contrast to the purely elastic case, the configuration may escape any fixed compact set for t large enough. This behavior is a characteristic of fluid-like materials, which can present



(a) Kelvin-Voigt solid rheology



(b) Maxwell fluidic rheology

Figure 1.2: Schematic representation of combination in parallel and in series of elastic and viscous rheological elements.

arbitrarily large deformations under the action of bounded loadings, e.g., honey flowing under the effect of gravity.

As stated above, the purely elastic and viscous elements are too elementary to describe most materials already in the small-strain setting. However, they are the abstract building blocks used in the theory of rheological models to depict more complex media. The simplest representations for viscoelastic materials are the *Kelvin-Voigt* and *Maxwell* rheologies [51] illustrated in Figure 1.2. The latter is made of an elastic and a viscous element combined in series. Mathematically, this means that the total displacement $u\colon \Omega \to \mathbb{R}^d$ is the sum of the elastic and the viscous one, $u = w + v$. The strain ϵ is thus also given by the additive decomposition $\epsilon = \nabla^s w + \nabla^s v$. On the other hand, the total strain σ is the same on the two elements, i.e., $\sigma = \sigma_{\mathrm{el}} = \sigma_{\mathrm{vi}}$. The Maxwell model is, analogously to the Stokes one, of fluid type.

In this dissertation, we focus on viscoelastic solids instead and concentrate on the Kelvin-Voigt rheology, which is made of an elastic and a viscous element combined in parallel. Here, the total displacement u coincides with the elastic and the viscous one, and thus the total strain is $\epsilon = \nabla^s w = \nabla^s v$. Instead, the total stress σ is additively decomposed as

$$\sigma = \sigma_{\mathrm{el}} + \sigma_{\mathrm{vi}} = \mathbb{C}\epsilon + \mathbb{D}\dot\epsilon \tag{1.1.3}$$

and the equilibrium equations are given by

$$-\operatorname{div}\sigma \overset{(1.1.3)}{=} -\operatorname{div}(\mathbb{C}\epsilon + \mathbb{D}\dot\epsilon) = -\operatorname{div}(\mathbb{C}\nabla^s u + \mathbb{D}\nabla^s \dot u) = f \qquad \text{on } [0,T]\times\Omega. \tag{1.1.4}$$

In the present thesis, we aim at studying evolution problems of this form in the finite-strain setting, which we will describe in Section 1.1.2, through the lens of the calculus of variations. In particular, we will interpret (1.1.4) as the dissipative evolution equation associated with suitable energy and dissipation. Let us introduce this framework first for the linear case. Let u be a solution to (1.1.4). Multiplying (1.1.4) by $\dot u$ and integrating over $[0,t]\times\Omega$, $t\in(0,T]$, we get the *energy balance*

$$\begin{aligned}\frac12\int_\Omega \mathbb{C}\nabla^s u(t){:}\nabla^s u(t)\,\mathrm{d}x - \frac12\int_\Omega \mathbb{C}\nabla^s u(0){:}\nabla^s u(0)\,\mathrm{d}x + \int_0^t\int_\Omega \mathbb{D}\nabla^s\dot u(s){:}\nabla^s\dot u(s)\,\mathrm{d}x\mathrm{d}s \\ = \int_\Omega f\cdot u(t)\mathrm{d}x - \int_\Omega f\cdot u(0)\mathrm{d}x\end{aligned}$$

by the divergence theorem and the chain rule. Defining the total complementary energy $\mathcal{E}^0\colon\ H^1(\Omega;\mathbb{R}^d)\to\mathbb{R}$ and the instantaneous dissipation $\mathcal{R}^0\colon\ H^1(\Omega;\mathbb{R}^d)\to\mathbb{R}$ of the system as

$$\mathcal{E}^0(u) := \frac12\int_\Omega \mathbb{C}\nabla^s u{:}\nabla^s u - f\cdot u\,\mathrm{d}x, \qquad \mathcal{R}^0(\dot u) := \frac12\int_\Omega \mathbb{D}\nabla^s\dot u{:}\nabla^s\dot u\,\mathrm{d}x, \tag{1.1.5}$$

respectively, we can rewrite the energy balance as

$$\mathcal{E}^0(u(t)) + 2\int_0^t \mathcal{R}^0(\dot u(s))\,\mathrm{d}s = \mathcal{E}^0(u(0)) \quad \text{for every } t \in (0,T]. \tag{1.1.6}$$

Namely, the energy of the solution at time t equals the initial energy minus the energy dissipated along the evolution [68]. Hence, (1.1.4) can be seen as the dissipative evolution equation associated with $\mathcal{E}^0$ and $\mathcal{R}^0$. In Section 1.1.3, we will recall the basic notions of gradient flows and clarify this statement, as well as specify the weak notions of solutions we consider.

Before discussing the nonlinear description of viscoelasticity, we remark that also the Kelvin-Voigt rheology is still not sophisticated enough for the applications. This leads to the introduction of the so-called *Standard solid models* for the *Poynting-Thomson-Zener materials* [51, Sec. 6.5], see Figure 2.1. We anticipate that we will study a finite-strain version of such models in Chapter 2.

1.1.2 Finite strain viscoelasticity

The linear picture presented in the previous section is usually a good approximation for deformations $y\colon \Omega \to \mathbb{R}^{d\times d}$ with strains ∇y being close to the identity. In such regimes, the displacement $u := y - \mathrm{id}$ is used to characterize the system. However, this description fails for large deformations, and new *finite-strain* theories are needed to accurately depict material behavior, in contrast to the small- or infinitesimal-strain ones introduced above.

Let us start the discussion on finite-strain viscoelasticity by presenting the equilibrium equations for a nonlinear Kelvin-Voigt rheology [4, 5, 58, 75]. In analogy to the linear case (1.1.4) and neglecting inertial effects, we consider the system

$$-\operatorname{div}\left(\mathrm{D}W(\nabla y) + \partial_{\nabla\dot y}R(\nabla y, \nabla\dot y)\right) = f \quad \text{on } [0,T]\times\Omega. \tag{1.1.7}$$

Here, $W\colon \mathbb{R}^{d\times d} \to [0,\infty]$ is the elastic energy density, whereas $R\colon \mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d} \to [0,\infty)$ is the instantaneous viscous dissipation potential density. The term $\mathrm{D}W(\nabla y)$ is the elastic part of the *Piola-Kirchhoff stress tensor*. Notice that equations (1.1.7) reduce to the linear case (1.1.4) if the energy and dissipation densities are quadratic, namely $W(F) = \frac{1}{2}\mathbb{C}F{:}F$ and $R(F,\dot F) = \frac{1}{2}\mathbb{D}\dot F{:}\dot F$, so that their derivatives are linear in F and $\dot F$, respectively. This is however not acceptable from the mechanical standpoint.

In general, proving the existence of solutions to (1.1.7) is challenging since nonlinearities do not behave well under weak convergence. Indeed, variational existence theories hinge on the construction of suitable minimizing sequences, for which only weak compactness is available, to certain time-discretized functionals (see Section 1.1.3 for more details). It is thus customary to consider the case of *nonsimple materials* [4, 32, 72], where the energy is assumed to depend also on the deformation's second gradient $\nabla^2 y$. The system of equations (1.1.7) then takes the form

$$-\operatorname{div}\big(\mathrm{D}W(\nabla y) - \operatorname{div}(\mathrm{D}H(\nabla^2 y)) + \partial_{\nabla\dot y}R(\nabla y,\nabla\dot y)\big) = f \quad \text{on } [0,T]\times\Omega. \tag{1.1.8}$$

Here, the density $H\colon \mathbb{R}^{d\times d\times d} \to [0,\infty)$ introduces a second-order term and provides more compactness to the model. The prototype for this term is $H(G) = |G|^p$, where $p > d$. We remark that the introduction of such regularization is debatable due to their uncertain physical interpretation, despite being necessary for showing the existence of suitably strong solutions and being widely used in the mathematical literature [4, 32, 51, 75]. In Chapter 3 and 4, we will resort to the standard theory of nonsimple materials to study the accretive growth of viscoelastic materials. In Chapter 2, conversely, we investigate the existence of a weaker notion of solutions to a finite-strain Poynting-Thomson model without resorting to second gradients, analyzing a possible strategy to avoid them in the analysis.

The finite-strain theory for nonsimple materials reduces to the linear one of Section 1.1.1 for small deformations [4, 21, 33, 54]. Indeed, in regimes where the deformation y is close to being the identity, we can write define the displacement $u_\varepsilon := (y - \mathrm{id})/\varepsilon$ for $\varepsilon > 0$, so that $y = \mathrm{id} + \varepsilon u_\varepsilon$. Assuming sufficient regularity of the energy densities and L^∞ bounds on y and its gradient and Hessian, we can write the energy by Taylor expansion as

$$\begin{aligned}\mathcal{E}(\mathrm{id} + \varepsilon u_\varepsilon) = &\int_\Omega W(I) + \varepsilon \mathrm{D}W(I){:}\nabla u + \frac{\varepsilon^2}{2}\mathrm{D}^2 W(I)\nabla u{:}\nabla u \,\mathrm{d}x \\ &+ \int_\Omega H(0) + \varepsilon \mathrm{D}H(0)\,\vdots\,\nabla^2 u + \frac{\varepsilon^2}{2}\mathrm{D}^2 H(0)\nabla^2 u\,\vdots\,\nabla^2 u \,\mathrm{d}x + \mathrm{o}(\varepsilon^2),\end{aligned}$$

Let us assume $\min W = W(I) = 0$, so that $\mathrm{D}W(I) = 0$, meaning that the reference configuration satisfies elastic equilibrium, and that $H = |\cdot|^p$, $p > \min\{d, 2\}$, so that $H(0) = 0$, $\mathrm{D}H = 0$, and $\mathrm{D}^2 H = 0$. Dividing by ε^2 and sending $\varepsilon \to 0$, we formally recover (1.1.5), where $\mathbb{C} := \mathrm{D}^2 W(I)$. The calculations for the dissipation potential are analogous. For the rigorous linearization procedure of a thermoviscoelastic Kelvin-Voigt rheology, we refer to [4, 5, 33]. In Section 2.4.4, we show an analogous result for the Poynting-Thomson rheological model.

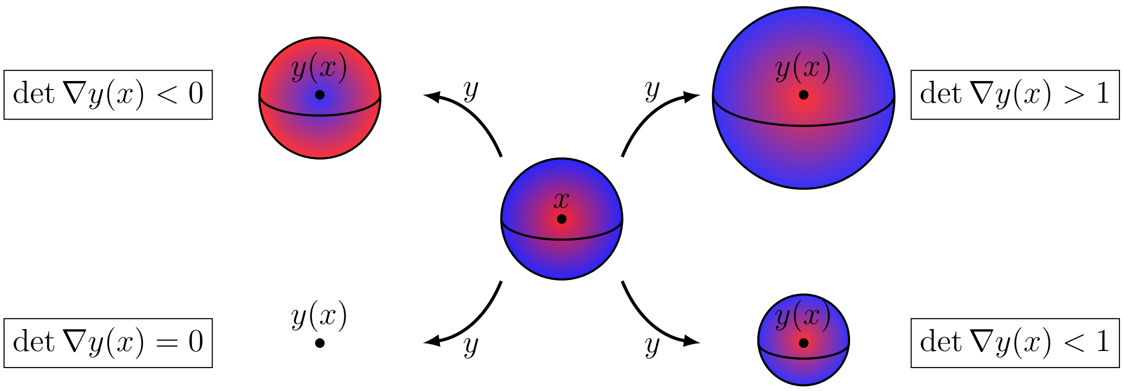


Figure 1.3: Local volumetric behavior of a neighborhood of $y(x)$ depending on the values of $\det \nabla y(x)$: expansion for $\det \nabla y(x) > 1$, compression for $\det \nabla y(x) < 1$, vanishing to a point for $\det \nabla y(x) = 0$, and orientation inversion for $\det \nabla y(x) < 0$.

Let us now record some physical requirements of the mathematical model. First, rigid motions should not change the system's energy level, i.e., the description should be invariant under rotations and translations of the body or, equivalently, of the observer's frame of reference. Notice that translational invariance is automatically satisfied since W, H, and R are functions of the deformation gradient ∇y and the Hessian $\nabla^2 y$. Rotational indifference, on the other hand, reads as

$$W(QF) = W(F), \quad H(QG) = H(G), \quad R(QF, Q\dot{F}) = R(F, \dot{F}),$$

for every $F, \dot{F} \in \mathbb{R}^{d\times d}$ and every rotation $Q \in SO(d)$. These conditions are called *frame indifference* and are usually assumed to hold for physical relevance. It has been observed [3] that the frame-indifference of the dissipation R implies that it must be a function of the *right Cauchy stress tensor* $\nabla y^\top \nabla y$ and its time derivative $\nabla \dot{y}^\top \nabla y + \nabla y^\top \nabla \dot{y}$, namely

$$R(F, \dot{F}) = \widetilde{R}(C, \dot{C}) \quad \text{for every } F, \dot{F} \in \mathbb{R}^{d\times d},$$

for some $\widetilde{R}\colon \mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d}\to[0,\infty)$, where $C=F^\top F$ and $\dot C=\dot F^\top F+F^\top\dot F$.

Moreover, the deformation y should avoid self-interpenetration and should not map sets of positive measures into null sets. This physical constraint can be locally expressed by requiring the determinant of the deformation gradient to be positive almost everywhere, $\det\nabla y>0$. Indeed, $\det\nabla y$ measures (local) volumetric change: $\det\nabla y=1$ encodes incompressibility, $\det\nabla y>1$ corresponds to the expansion of the medium, $0<\det\nabla y<1$ to compression (to a set of measure zero for $\det\nabla y=0$), whereas $\det\nabla y<0$ means that the orientation of the body has been reversed (see Figure 1.3). The positivity of the determinant is usually enforced in the analytical model by assuming the following growth condition from below on the energy

$$W(F)\gtrsim\begin{cases}\frac{1}{|\det F|^q} & \text{if } \det F>0\\ +\infty & \text{else,}\end{cases} \tag{1.1.9}$$

up to multiplicative and additive constants. Here, $q>1$ is a suitable exponent depending on the dimension and the growth conditions on H (see (3.2.5) below for the precise formulation). For nonsimple materials, this condition guarantees that weak solutions to (1.1.8) have positive determinants almost everywhere and for almost every time [49]. Notice that the constraint $\det\nabla y>0$ amounts to a local requirement. In [48], conditions for global non-self-interpenetration are studied instead. In general, injectivity of the deformation in Ω can be enforced through the so-called *Ciarlet-Nečas condition* [18]

$$|\Omega|=\int_\Omega\det\nabla y\,\mathrm{d}x=|y(\Omega)|.$$

Self-touching of $y(\partial\Omega)$ can still occur, resulting in a reaction traction term on the corresponding portion of the boundary [13, 14, 48, 83].

Another noteworthy consequence of the growth condition (1.1.9) is that the energy density W cannot be convex since the space of invertible matrices with positive determinant $\mathrm{GL}_+(d)$ is not. Indeed, let us assume for the sake of contradiction that W satisfies (1.1.9) and is convex, and fix $d=2$. The identity matrix I and $-I$ have determinant one, but

$$+\infty=W(0)=W\left(\frac{1}{2}I+\frac{1}{2}(-I)\right)\le\frac{1}{2}W(I)+\frac{1}{2}W(-I).$$

Hence, finite-strain theories for (visco)elasticity are bound to consider nonconvex energy densities, which pose several challenges to the analysis: As we will clarify in Section 1.1.3, variational existence theories hinge on the possibility of passing to the limit in the equations where suitable sequences $(y_n)_n$ converge weakly in some suitable Sobolev space. Weak convergence, however, is little informative when composed with nonlinear functions. Notice that the presence of the second-order term H allows to avoid this issue since weak convergence in $W^{2,p}(\Omega;\mathbb{R}^d)$ implies strong convergence of the gradients by Sobolev embedding, for $p>d$. When considering simple materials, on the other hand, there is a rich theory of weaker notions of convexity that can be considered, like *quasiconvexity* and *polyconvexity* [20], which still allow passing to lower limits [69]. We will follow this path in Chapter 2, cf. (E2).

1.1.3 Variational notion of solutions for viscoelasticity

In general, due to the nonlinear and nonconvex nature of finite-strain viscoelasticity, the existence of strong solutions to (1.1.8), i.e., solving the system almost everywhere, cannot be ascertained. This leads to the introduction of weaker notions of solutions, which we will present in this section.

First, we consider *weak* or *distributional solutions*. Let us pair the system (1.1.8) with boundary and initial conditions. For the sake of simplicity, we consider Dirichlet boundary conditions on y and the natural homogeneous condition on the hyperstress $\mathrm{D}H(\nabla^2 y)$, i.e.,

$$\begin{aligned} & y = \mathrm{id} \quad \text{on } [0,T]\times\partial\Omega, \\ & \mathrm{D}H(\nabla^2 y){:}(\nu\otimes\nu) = 0 \quad \text{on } [0,T]\times\partial\Omega, \\ & y(0,\cdot) = y_0(\cdot) \quad \text{on } \Omega, \end{aligned} \tag{1.1.10}$$

where ν is the outer unit normal to $\partial\Omega$. By (scalar) multiplying (1.1.8) by a smooth test function $z \in C_0^\infty(\Omega;\mathbb{R}^d)$, integrating over $[0,T]\times\Omega$, and the divergence theorem, we find

$$\int_0^T \int_\Omega \mathrm{D}W(\nabla y){:}\nabla z + \mathrm{D}H(\nabla^2 y)\vdots\nabla^2 z + \partial_{\nabla\dot y} R(\nabla y, \nabla \dot y){:}\nabla z - f\cdot z \,\mathrm{d}x = 0 \tag{1.1.11}$$

for every $z \in C_0^\infty(\Omega;\mathbb{R}^d)$. The identity (1.1.11) defines the usual notion of solution considered in finite-strain theories [4, 5, 13, 14, 33, 75].

Analogously and in the linear setting of Section 1.1.1, the system of equations (1.1.8) can also be formally interpreted as the dissipative evolution equation

$$\delta\mathcal{E}(y) + \delta_{\dot y}\mathcal{R}(y,\dot y) = 0. \tag{1.1.12}$$

Here, the elastic energy $\mathcal{E}\colon W^{2,p}(\Omega;\mathbb{R}^d) \to [0,\infty]$ and the instantaneous viscous dissipation $\mathcal{R}\colon H^1(\Omega;\mathbb{R}^d)\times H^1(\Omega;\mathbb{R}^d) \to [0,\infty)$ are defined as

$$\mathcal{E}(y) := \int_\Omega W(\nabla y) + H(\nabla^2 y) - f\cdot y \,\mathrm{d}x, \qquad \mathcal{R}(y,\dot y) := \int_\Omega R(\nabla y, \nabla\dot y)\,\mathrm{d}x,$$

respectively, and δ denotes a variational derivative, to be suitably defined. At least on the formal level, by multiplying by $\dot y$ and by applying the chain rule, (1.1.12) can be stated as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(y) = -\delta_{\dot y}\mathcal{R}(y,\dot y){:}\nabla\dot y.$$

If $R = R(F,\dot F)$ is convex in $\dot F$, then $\partial_{\dot F} R(F,\dot F){:}\dot F \geq 0$ for every $F,\dot F \in \mathbb{R}^{d\times d}$, so that the right-hand side is nonpositive and the energy decreases along the viscoelastic evolution. This interpretation provides a natural strategy, the *minimizing movement scheme*, to show the existence of weak solutions to (1.1.8) through the implicit Euler scheme. The time interval $[0,T]$ is uniformly discretized, $\{0 = t_0 < \dots < t_i = i\tau < \dots N_\tau\tau = T\}$, where $\tau = T/N_\tau > 0$ is the mesh size of the discretization, for some $N_\tau \in \mathbb{N}\setminus\{0\}$. The sequence $(y_\tau^i)_{i=0}^{N_\tau} \subset W^{2,p}(\Omega;\mathbb{R}^d)$ is iteratively defined as solutions to the incremental minimization problems

$$y_\tau^i \in \operatorname*{arg\,min}_{y\in\mathcal{A}} \left\{ \mathcal{E}(y) + \tau\mathcal{R}\left(y_\tau^{i-1}, \frac{y - y_\tau^{i-1}}{\tau}\right)\right\}, \quad i = 1,\dots,N_\tau, \tag{1.1.13}$$

where $\mathcal{A}$ is the set of admissible deformation, encoding the boundary conditions and other constraints like the positivity of the determinant of the deformation gradient. Weak solutions to (1.1.8) are then obtained by passing to the limit as the mesh size τ goes to zero. Indeed, by minimality y_τ^i satisfies the discrete *Euler-Lagrange* equations

$$\int_\Omega \mathrm{D}W(\nabla y_\tau^i){:}\nabla z + \mathrm{D}H(\nabla^2 y_\tau^i)\vdots\nabla^2 z - f\cdot z\,\mathrm{d}x + \tau\int_\Omega \partial_{\nabla\dot y} R\left(y_\tau^{i-1}, \frac{y_\tau^i - y_\tau^{i-1}}{\tau}\right){:}\nabla z\,\mathrm{d}x = 0$$

for every admissible $z \in C_0^\infty(\Omega;\mathbb{R}^d)$. This, integrated over $[0,T]$, formally correspond to (1.1.11) at the time-discretized level. We will perform the above-sketched argument in detail for the problems we study in Chapters 3 and 4 in Sections 3.4 and 4.4, respectively. This passage to the limit procedure hinges on suitable compactness estimates on y and its time derivative $\dot{y}$ in time and space. These are obtained through the following energetic estimates: By minimality, at every time step t_i, $i = 1,\dots,N_\tau$, the energy level of y_τ^i is lower than the one of y_τ^{i-1}, namely.

$$\mathcal{E}(y_\tau^i) + \tau\mathcal{R}\left(y_\tau^{i-1}, \frac{y_\tau^i - y_\tau^{i-1}}{\tau}\right) \leq \mathcal{E}(y_\tau^{i-1})$$

where we assumed that $R(\cdot,0) = 0$, i.e., no dissipation when there is no time evolution. Summing over $i = 1,\dots,n \leq N_\tau$ and telescoping, one finds

$$\mathcal{E}(y_\tau^n) + \tau\sum_{i=1}^n \mathcal{R}\left(y_\tau^{i-1}, \frac{y_\tau^i - y_\tau^{i-1}}{\tau}\right) \leq \mathcal{E}(y_0). \tag{1.1.14}$$

Assuming suitable bounds from below on the energy and dissipation densities, the uniform estimates on $(y_\tau^i)_{i=0}^{N_\tau}$ and its time-interpolants then follow.

Up to now, we have considered weak solutions. The interpretation of (1.1.8) as the dissipative evolution equation (1.1.12) and the energy inequality (1.1.14), however, hint to another possible notion of solution based on the theory of gradient flows [2]. For the sake of simplicity, let us assume in the following that $R(F,\dot{F}) = \frac{1}{2}\mathbb{D}(C)\dot{C}{:}\dot{C}$ for every $F, \dot{F} \in \mathbb{R}^{d\times d}$, where $\mathbb{D}\colon \mathbb{R}^{d\times d} \to [0,\infty)$, and we recall that $C = F^\top F$ and $\dot{C} = \dot{F}^\top F + F^\top \dot{F}$. Assuming as in [70] the existence of a global distance $D\colon \mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d} \to [0,\infty)$ such that $R(F,\dot{F}) = \lim_{\tau\to 0}\frac{1}{\tau^2}(D(F, F+\tau\dot{F}))^2$, the incremental minimization problem (1.1.13) can be rewritten as

$$y_\tau^i \in \operatorname*{arg\,min}_{y\in\mathcal{A}}\left\{\mathcal{E}(y) + \frac{1}{2\tau}d^2\left(y_\tau^i, y_\tau^{i-1}\right)\right\}, \quad i = 1,\dots,N_\tau,$$

where the distance $d\colon H^1(\Omega;\mathbb{R}^d)\times H^1(\Omega;\mathbb{R}^d) \to [0,\infty)$ is defined as

$$d^2(u,v) := \int_\Omega (D(\nabla u, \nabla v))^2\,\mathrm{d}x.$$

The viscoelastic evolution can be thus interpreted as the gradient flow of the energy $\mathcal{E}$ with respect to the dissipation distance d.

Assuming the time derivative $\dot{y}$ of a solution to (1.1.11) to be an admissible test function, the choice of $z = \dot{y}$ in (1.1.11) entails the validity of the *energy-dissipation balance*

$$\mathcal{E}(y(t)) + 2\int_0^t \mathcal{R}(y(s),\dot{y}(s))\,\mathrm{d}s = \mathcal{E}(y_0) \quad \text{for every } t \in (0,T] \tag{1.1.15}$$

in analogy to the linear case, cf. (1.1.6). For smooth energy and dissipation densities, the converse is also true. Thus, the energy equality (1.1.15) and (1.1.11) are equivalent. The equivalence holds also for nonsmooth energies provided that they satisfy the *chain rule* [70]

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(y(t)) = \langle \Xi(t), \dot{y}(t)\rangle \quad \text{for every } \Xi(t) \in \partial\mathcal{E}(y(t)) \text{ and almost every } t \in (0,T). \tag{1.1.16}$$

Here, $\partial\mathcal{E}$ denotes the *subdifferential* of $\mathcal{E}$, a generalization of the concept of derivative for nonsmooth functionals [20, Sec. 2.3.6]. Thanks to this equivalence, also called *energy-dissipation* *principle* (*EDP*) [68], the energy-dissipation balance (1.1.15) can be used to define the notion of *Energy-dissipation balance* (*EDB*) solutions to (1.1.8) [70]. The advantage of EDB solutions is that, in contrast to (1.1.11), (1.1.15) makes sense even for nonsmooth energy and dissipation densities. Moreover, notice that the discrete energy inequality (1.1.14) considered above represents the discrete counterpart of (1.1.15), up to a multiplicative factor 2 in front of the dissipation $\mathcal{R}$. Indeed, the energy-dissipation balance (1.1.15) represents an improvement of the simpler energy inequality $\mathcal{E}(y(t)) + \int_0^t \mathcal{R}(y(s), \dot{y}(s))\,\mathrm{d}s = \mathcal{E}(y_0)$, and can be obtained similarly to (1.1.14) with a more accurate choice of a test function $\widetilde{y}^i_\tau$, see [2, Sec. 3.2]. We perform the proof of the energy-dissipation inequality, see below, following this strategy for the Poynting-Thomson rheological model in Section 2.6.2. In Section 3.3, conversely, we show it for a viscoelastic accretive model making use of the generalized chain rule.

In numerous situations, however, e.g., for simple materials, the balance (1.1.15) is often beyond the reach of the current mathematical theories. Without the additional compactness granted by the regularizing second-order terms, it is in general only possible to pass to the $\liminf$ in the energy and dissipation under weak convergence [69]. This leads to considering a weaker notion of solution with respect to EDB ones, namely satisfying the *energy-dissipation inequality* (*EDI*)

$$\mathcal{E}(y(t_2)) + 2\int_{t_1}^{t_2} \mathcal{R}(y(s), \dot{y}(s))\,\mathrm{d}s \le \mathcal{E}(y(t_1)) \quad \text{for almost every } 0 \le t_1 < t_2 \le T. \qquad (1.1.17)$$

The deformations y for which (1.1.17) holds are called *EDI solutions*, and they coincide with EDB solutions and weak solutions when the chain rule (1.1.16) holds [72]. We show the existence of EDI-type solutions for a simple material Poynting-Thomson model in Chapter 2.

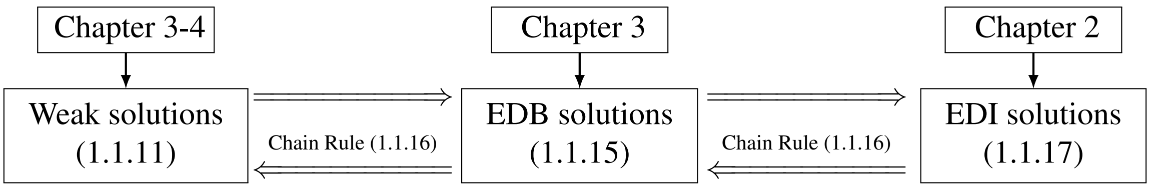


Figure 1.4: Schematic representation of the relations between the different notions of solutions used in the different chapters.

1.2 Outline and main results

In this section, we summarize the main results and outline the structure of the thesis. We refer to the corresponding introductory sections of each chapter for a detailed discussion of the setting and the state of the art.

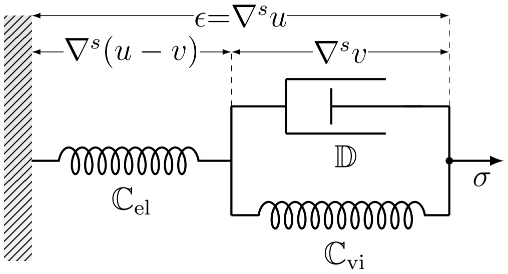
In Chapter 2, we show the existence of EDI-type solutions to a Poynting-Thomson rheological model for finite-strain simple materials. We additionally perform linearization and recover the small-strain counterpart of the nonlinear description. This chapter consists of my paper [15] with MARTIN KRŮŽIK and ULISSE STEFANELLI that appeared in the journal *Mathematics and Mechanics of Solids*.

Chapter 3 is devoted to showing the existence of weak/viscosity solutions to a bi-phase viscoelastic medium, in which one phase grows by accretion at the expense of the other. We consider both a sharp and a diffuse interface model and show that the solutions satisfy the energy-dissipation balance. This chapter consists of my paper [16] with ULISSE STEFANELLI, which appeared in the journal *Zeitschrift für angewandte Mathematik und Physik*.

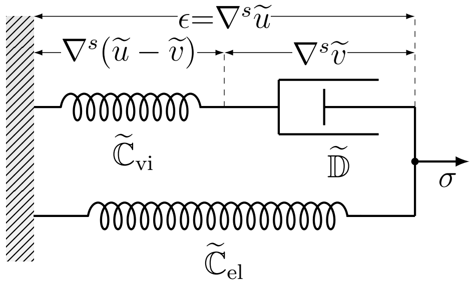
In chapter 4, we focus on an accretive growth model for viscoelastic solids, where the material is unstressed at the time and place of deposition. We prove the existence of weak/viscosity solutions to the system, assuming the presence of a highly compliant Ersatzmaterial surrounding the growing body. This chapter is based on a paper with ULISSE STEFANELLI, which is currently in preparation.

1.2.1 Chapter 2: Finite-strain Poynting-Thomson model

In [15], we consider a finite-strain Poynting-Thomson model for viscoelastic solids. As remarked in Section 1.1.1, the Kelvin-Voigt rheology is not sophisticated enough to model many physical situations correctly. In particular, under so-called *hard-device* loading, i.e., when the deformation is prescribed independently of what force is required to realize it, such model features unbounded and hence unphysical stress response when there is a jump in the strain, see [48, Chap. 6.4 and Fig. 6.9]. Consequently, more complex rheological models are considered, namely the Poynting-Thomson and Zener ones. These are made of an elastic element combined in series with a Kelvin-Voigt one or in parallel with a Maxwell one, respectively, see Figure 1.5. Since the two models are equivalent in the linear setting [51, Chap. 6.5], we focus on the Poynting-Thomson one.



(a) Poynting-Thomson standard solid rheology



(b) Zener standard solid rheology

Figure 1.5: Schematic representation of the linear Poynting-Thomson and Zener rheological models

Let $u\colon \Omega \to \mathbb{R}^d$ be the total displacement. As described above for the combination in series, the total strain $\epsilon = \nabla^s u$ is additively decomposed as

$$\epsilon = \nabla^s v + \nabla^s(u-v), \tag{1.2.1}$$

where $v\colon \Omega \to \mathbb{R}^d$ is the displacement of the Kelvin-Voigt element, whereas $(u-v) =: w\colon \Omega \to \mathbb{R}^d$ is the elastic displacement of the Hooke one. The total stress σ, on the other hand, coincides with the ones of the two components so that

$$\sigma = \mathbb{C}_{\mathrm{el}}\nabla^s(u-v) = \mathbb{D}\nabla^s\dot{v} + \mathbb{C}_{\mathrm{vi}}\nabla^s v,$$

and the equilibrium equation $-\operatorname{div}\sigma = f$ can be equivalently written as

$$-\operatorname{div}(\mathbb{C}_{\mathrm{el}}\nabla^s(u-v)) = f, \quad \text{or} \quad -\operatorname{div}(\mathbb{D}\nabla^s\dot{v} + \mathbb{C}_{\mathrm{vi}}\nabla^s v) = f. \tag{1.2.2}$$

In the finite-strain setting, it is natural [55, 101] to consider a *multiplicative decomposition* for the total strain instead of the additive one: Letting $y\colon \Omega \to \mathbb{R}^d$ be the total deformation, consider ∇y to be of the form

$$\nabla y = F_{\mathrm{el}}F_{\mathrm{vi}}, \tag{1.2.3}$$

where F_{el} and F_{vi} are the elastic and viscous (actually viscoelastic) strain tensors, respectively. Such decomposition is standard in finite-strain elastoplasticity [50, 55, 96], where it is called *Kröner-Lee decomposition*. In [15], we additionally assume that the viscous strain is *compatible*, namely, that there exists $y_{\mathrm{vi}} \colon \Omega \to \mathbb{R}^d$ such that $F_{\mathrm{vi}} = \nabla y_{\mathrm{vi}}$. In particular, from (1.2.3) follows that also the elastic strain is compatible, $F_{\mathrm{el}} = \nabla y_{\mathrm{el}}$ for some $y_{\mathrm{el}} \colon y_{\mathrm{vi}}(\Omega) \to \mathbb{R}^d$. Thus, (1.2.3) corresponds to the composition of maps

$$y = y_{\mathrm{el}} \circ y_{\mathrm{vi}} \colon \Omega \xrightarrow{y_{\mathrm{vi}}} y_{\mathrm{vi}}(\Omega) \xrightarrow{y_{\mathrm{el}}} \mathbb{R}^d, \tag{1.2.4}$$

which offers a nonlinear analog to the additive decomposition (1.2.1). Here, $y_{\mathrm{vi}}(\Omega)$ represents a (mathematical) intermediate configuration between the reference Ω and actual one $y(\Omega)$. In this Section, and correspondingly in Chapter 2, we denote by $X \in \Omega$ the Lagrangian variable in the reference configuration Ω and by $\xi \in y_{\mathrm{vi}}(\Omega)$ the Eulerian one in the intermediate configuration $y_{\mathrm{vi}}(\Omega)$.

The composition structure (1.2.4) of the total deformation y indeed leads to considering a mixed Lagrangian-Eulerian problem where the stored elastic energy and the work of external forces are

$$\mathcal{E}(t, y_{\mathrm{el}}, y_{\mathrm{vi}}) := \int_{y_{\mathrm{vi}}(\Omega)} W_{\mathrm{el}}(\nabla y_{\mathrm{el}}(\xi))\,\mathrm{d}\xi + \int_{\Omega} W_{\mathrm{vi}}(\nabla y_{\mathrm{vi}}(X)) - f(t,X)\cdot y_{\mathrm{el}}(y_{\mathrm{vi}}(X))\,\mathrm{d}X,$$

for some suitable energy densities W_{el} and W_{vi}, cf. (E1)–(E2) and (L1)–(L6). As remarked in Section 1.1.2 for simple materials, we assume W_{el} and W_{vi} to be polyconvex, see (E2). Due to the Eulerian nature of the first term, we will assume local incompressibility of the viscous deformation, $\det \nabla y_{\mathrm{vi}} = 1$. Indeed, we will often resort to the change of variable formula $\int_{y_{\mathrm{vi}}(\Omega)} W_{\mathrm{el}}(\nabla y_{\mathrm{el}})\,\mathrm{d}\xi = \int_\Omega W_{\mathrm{el}}(\nabla y (\nabla y_{\mathrm{vi}})^{-1}) \det \nabla y_{\mathrm{vi}}\,\mathrm{d}X$ to transform integrals over the intermediate configuration $y_{\mathrm{vi}}(\Omega)$ into integrals over the fixed reference one Ω. The incompressibility of y_{vi} simplifies the above expression and allows us to show the existence of solutions in the weak setting we consider.

The (instantaneous) dissipation instead is defined as

$$\Psi(y_{\mathrm{vi}}, \dot y_{\mathrm{vi}}) := \int_\Omega \psi(\nabla \dot y_{\mathrm{vi}} (\nabla y_{\mathrm{vi}})^{-1})\,\mathrm{d}X,$$

where ψ is a suitable density, cf. (E4)–(E6) and (L8)–(L10). Most notably, ψ is p_ψ-homogeneous for some $p_\psi \geq 2$, i.e., $\psi(\lambda F) = \lambda^{p_\psi}\psi(F)$ for every $F \in \mathbb{R}^{d\times d}, \lambda \geq 0$, see (E6). This is a crucial assumption to show the (sharp)-energy inequality (1.2.5). Moreover, let us remark that the dissipation we consider does not satisfy frame-indifference as it would be desirable [3] as observed in 1.1.2. However, we believe this to be not necessarily problematic, as the viscous deformation takes values in the intermediate, purely mathematical, configuration $y_{\mathrm{vi}}(\Omega)$, where this physical constraint can be neglected.

The formal equilibrium equations for the finite-strain Poynting-Thomson model we are considering take the form (2.2.6) below, corresponding to the linear equations (1.2.2), up to the incompressibility. See (2.2.2) and the associated discussion for more details. Notice that we contemplate the case of simple materials without considering second gradient regularization terms, cf. 1.1.7. Hence, as already observed in Section 1.1.3, we resort to EDI-type solutions. We define *approximable solutions* in Definition 2.4.1 as trajectories $(y_{\mathrm{el}}, y_{\mathrm{vi}}) \colon t \in [0,T] \mapsto W^{1,p_{\mathrm{el}}}(y_{\mathrm{vi}}(t,\Omega);\mathbb{R}^d) \times W^{1,p_{\mathrm{vi}}}(\Omega;\mathbb{R}^d)$, for some $p_{\mathrm{el}}, p_{\mathrm{vi}} > d$, with given initial datum $(y_{\mathrm{el},0}, y_{\mathrm{vi},0})$

satisfying, for every $t \in [0,T]$, the energy-dissipation inequality, cf. (1.1.17),

$$\begin{aligned} &\mathcal{E}(t, y_{\mathrm{el}}(t), y_{\mathrm{vi}}(t)) + p_\psi \int_0^t \Psi(y_{\mathrm{vi}}(s), \dot{y}_{\mathrm{vi}}(s))\,\mathrm{d}s \\ &\quad \le \mathcal{E}(0, y_{\mathrm{el},0}, y_{\mathrm{vi},0}) - \int_0^t \int_\Omega \partial_s f(s)\cdot y(s)\,\mathrm{d}X\mathrm{d}s, \end{aligned} \tag{1.2.5}$$

as well as the (elastic) *semistability*

$$\mathcal{E}(t, y_{\mathrm{el}}(t), y_{\mathrm{vi}}(t)) \le \mathcal{E}(t, \widetilde{y}_{\mathrm{el}}, y_{\mathrm{vi}}(t)) \quad \text{for every } \widetilde{y}_{\mathrm{el}} \text{ such that } (\widetilde{y}_{\mathrm{el}}, y_{\mathrm{vi}}(t,\cdot)) \text{ is admissible,}$$

and *approximability*, i.e., the trajectories $(y_{\mathrm{el}}, y_{\mathrm{vi}})$ can be obtained as limits of the piecewise-in-time interpolants of the solutions to the incremental minimization problems (2.2.9) below, cf. (1.1.13). Semistability specifies that the elastic deformation minimizes the elastic energy at all times. The approximability condition allows to better characterizes the solutions. Indeed, the energy inequality and semistability alone do not exclude the unphysical constant-in-time solutions $(y_{\mathrm{el}}(t), y_{\mathrm{vi}}(t)) \equiv (y_{\mathrm{el},0}, y_{\mathrm{vi},0})$ when for example the external force does not depend on time. Instead, approximability ensures that viscous dissipation occurs even when no external loading is applied, see Section 2.4.2 and Figure 2.2. Furthermore, notice that the energy-dissipation inequality (1.2.5) is sharp, since the dissipation is multiplied by the correct prefactor p_ψ, which is related to the p_ψ-homogeneity of the dissipation density ψ: When ψ is quadratic as in Section 1.1.3, then $p_\psi = 2$ and (1.2.5) reduces to (1.1.17), up to the work of time-dependent external forces. The proof of (1.2.5) heavily relies on the metric interpretation of gradient flows [2] in order to properly characterize the dissipated energy and obtain the energy inequality in its sharp version, see Section 2.6 and in particular Subsection 2.6.2 for the detailed argument.

Finally, in Section 2.7, we consider the linearization of the finite-strain model and show that we recover the classical linear Poynting-Thomson model. Let us formally define the linearized energy and instantaneous dissipation as $\mathcal{E}^0(u,v) := \int_\Omega W^0(\nabla u, \nabla v) - f\cdot u\,\mathrm{d}X$ and $\Psi^0(v) := \int_\Omega \psi^0(\nabla v, \nabla \dot{v})\,\mathrm{d}X$, respectively, where

$$W^0(F, F_{\mathrm{vi}}) := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} W^\varepsilon(F, F_{\mathrm{vi}}), \quad W^\varepsilon(F, F_{\mathrm{vi}}) := W_{\mathrm{el}}\left((I+\varepsilon F)(I+\varepsilon F_{\mathrm{el}})^{-1}\right) + W_{\mathrm{vi}}(I+\varepsilon F_{\mathrm{vi}}),$$

$$\psi^0(F_{\mathrm{vi}}, \dot{F}_{\mathrm{vi}}) := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \psi^\varepsilon(F_{\mathrm{vi}}, \dot{F}_{\mathrm{vi}}), \quad \psi^\varepsilon(F_{\mathrm{vi}}, \dot{F}_{\mathrm{vi}}) := \psi\left(\varepsilon \dot{F}_{\mathrm{vi}}(I+F_{\mathrm{vi}})^{-1}\right).$$

As described above, for every $\varepsilon > 0$ there exist approximable solutions $(y_{\mathrm{el},\varepsilon}, y_{\mathrm{vi},\varepsilon})$ to the dissipative evolution associated to the rescaled energy $\mathcal{E}^\varepsilon = \frac{1}{\varepsilon^2}\mathcal{E}$ (where we consider with abuse of notation $f^\varepsilon = \varepsilon f$) and dissipation $\Psi^\varepsilon = \frac{1}{\varepsilon^2}\Psi$. Letting

$$u_\varepsilon := \frac{y_\varepsilon - \mathrm{id}_\Omega}{\varepsilon} \quad \text{and} \quad v_\varepsilon := \frac{y_{\mathrm{vi},\varepsilon} - \mathrm{id}_\Omega}{\varepsilon},$$

we make rigorous the Taylor expansion argument of Section 1.1.2 and show that, up to subsequences, $(u_\varepsilon, v_\varepsilon)$ converge weakly in $H^1(\Omega;\mathbb{R}^d) \times H^1(\Omega;\mathbb{R}^d)$ for every time $t \in [0,T]$ to $(u,v)\colon [0,T] \to H^1(\Omega;\mathbb{R}^d) \times H^1(\Omega;\mathbb{R}^d)$. The limit (u,v) satisfies the linearized energy inequality

$$\mathcal{E}^0(t, u(t), v(t)) + 2\int_0^t \Psi^0(\dot{v}(s))\,\mathrm{d}s \le \mathcal{E}^0(0, u^0, v^0) - \int_0^t \dot{f}(s)\cdot u(s)\,\mathrm{d}s$$

and semistability

$$\mathcal{E}^0(t, u(t), v(t)) \le \mathcal{E}^0(t, \widehat{u}, v(t)) \quad \text{for every admissible } \widehat{u}.$$

As a consequence of the EDI-type solution at the finite-strain setting, (u, v) are EDI-type solutions in the linear setting, where the existence of solutions is known. In particular, thanks to semistability, for given viscoelastic evolution v, the elastic one u is unique. However, the linearized energy inequality is insufficient to determine v uniquely. It would be possible to recover the existence of the classical unique solution to the linear system considering a second-gradient regularization for the finite-strain model, which disappears in the limit [33].

1.2.2 Chapter 3: Viscoelasticity and accretive phase-change at finite strains

In [16], we analyze the interplay between viscoelasticity and accretive growth for a two-phase material. More precisely, we consider a viscoelastic medium with reference configuration $U \subset \mathbb{R}^d$ made up of two components, whose reference configurations at time $t \in [0, T]$ are $\Omega(t) \subset\subset U$ and $U \setminus \overline{\Omega(t)}$, respectively, see Figure 3.1.

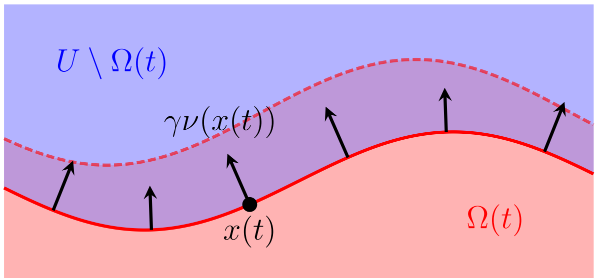


Figure 1.6: Accretive growth of the reference configuration $\Omega(t)$.

The map $t \mapsto \Omega(t)$ specifies the evolution of the corresponding phase, which we assume to grow *accretively* at the expense of the other one. By accretive growth, we mean an expansion that can be observed in numerous applications, like plant and shell growth, as well as solidification and 3D printing (see Section 3.1 for more examples and the relevant references), and which takes place in the normal direction. A point $x(t) \in \partial\Omega(t)$ on the boundary of the growing reference configuration $\Omega(t)$ at time $t \in [0, T]$ then moves according to the ODE flow

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \gamma\nu(x(t)), \tag{1.2.6}$$

where $\nu(x(t))$ denotes the outer unit normal to $\Omega(t)$ at $x(t)$ and $\gamma > 0$ is the *growth rate*. The strict positivity of the growth rate γ ensures that the material can only grow and does not shrink or halt the evolution. As clarified later, the growth rate γ is assumed to be a function of the deformation y.

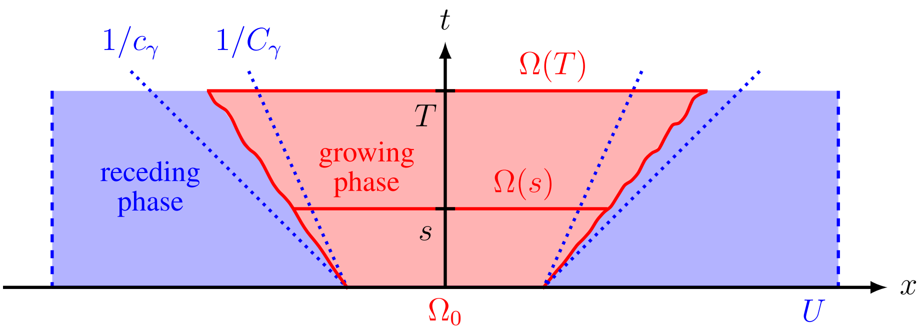


Figure 1.7: Time-space diagram of the growing and receding phase for $d = 1$.

In [16], we postulate that the growing reference configuration $\Omega(t)$ is the sublevel set of a suitable function. Namely, we assume there exists a function $\theta\colon U \to [0,\infty)$ such that

$$\Omega(t) := \{x \in U \mid \theta(x) \le t\}, \qquad \text{for all } t \in [0,T]. \tag{1.2.7}$$

The map θ is called *time-of-attachment function* and expresses the time $\theta(x)$ at which the material point x is added to $\Omega(t)$, i.e. $\theta(x(t)) = t$. Assuming the smoothness of the involved quantities, by differentiating this identity with respect to time, we find

$$1 = \nabla\theta(x(t))\cdot\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \gamma\nabla\theta(x(t))\cdot\nu(x(t)) = \gamma\nabla\theta(x(t))\cdot\frac{\nabla\theta(x(t))}{|\nabla\theta(x(t))|} = \gamma|\nabla\theta(x(t))|,$$

where we used the flow rule (1.2.6) and the fact that the normal ν to $\Omega(t)$, which is a sublevel set of θ, is given by $\nabla\theta(x(t))/|\nabla\theta(x(t))|$. We have thus obtained that θ satisfies a (generalized) eikonal equation. More precisely, given the viscoelastic deformation evolution $y\colon [0,T]\times U \to \mathbb{R}^d$, θ satisfies

$$\begin{cases} \gamma\big(y(\theta(x),x),\nabla y(\theta(x),x)\big)|\nabla\theta(x)| = 1 \ \text{ in } \ U\setminus\overline{\Omega_0} \\ \theta = 0 \ \text{ in } \ \Omega_0 \subset\subset U, \end{cases} \tag{1.2.8}$$

where Ω_0 is the given initial reference configuration of the accretive phase. The growth rate γ depends on the position of the body through y, modeling the effects of the presence of nutrients or catalysts, and on the strain through ∇y, since the local mechanical state of the body may indeed influence its growth [38]. Moreover, notice that the unknown θ appears in the eikonal equation (1.2.8) inside the coefficient γ through its dependence on the values of y and ∇y traced on the hypersurface $\{(t,x) \in [0,T]\times U \mid t = \theta(x)\}$.

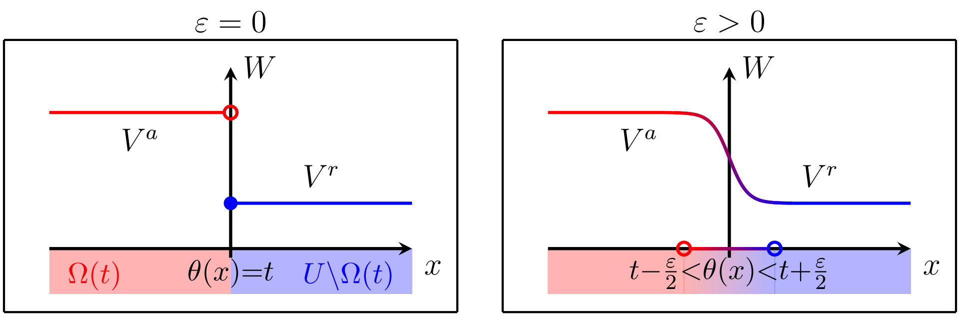


Figure 1.8: Illustration of the 1-dimensional sharp ($\varepsilon = 0$) and diffused ($\varepsilon > 0$) interpolation of the energy densities V^a and V^r.

Thus, the growth model depends on the viscoelastic evolution y of the medium, for which we assume that the two phases are finite-strain nonsimple Kelvin-Voigt materials. The equilibrium equation, cf. (1.1.8), is

$$-\operatorname{div}\big(\partial_{\nabla y}W_\varepsilon(\theta(x)-t,\nabla y)+\partial_{\nabla\dot y}R_\varepsilon(\theta(x)-t,\nabla y,\nabla\dot y)-\operatorname{div}\mathrm{D}H(\nabla^2 y)\big)=f(\theta(x)-t,x) \tag{1.2.9}$$

in $[0,T]\times U$. Here, the value $\theta(x)-t$ determines the phase: At time $t\in[0,T]$, by the definition of $\Omega(t)$ as the t-sublevel set of θ, if $\theta(x)-t<0$ or $\theta(x)-t>0$ then $x\in U$ belongs to the accreting set $\Omega(t)$ or the receding $U\setminus\overline{\Omega(t)}$ one, respectively. The dependence of f on $\theta(x)-t$ is meant to cover the case of the gravitational force, which depends on the material properties of the phases, namely, on their densities. The energy and dissipation densities $W_\varepsilon : \mathbb{R} \times \mathbb{R}^{d\times d} \to [0,\infty]$ and $R_\varepsilon : \mathbb{R} \times \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \to [0,\infty)$ are defined as

$$W_\varepsilon(\sigma, F) := (1 - h_\varepsilon(\sigma))V^a(F) + h_\varepsilon(\sigma)V^r(F) + V^J(F)$$

$$R_\varepsilon(\sigma, F, \dot{F}) := (1 - h_\varepsilon(\sigma))R^a(F, \dot{F}) + h_\varepsilon(\sigma)R^r(F, \dot{F}), \quad R^i(F, \dot{F}) := \frac{1}{2}\mathbb{D}^a(C)\dot{C}{:}C,\ i = a, r,$$

where h_0 is the discontinuous Heaviside-like function defined as $h_0(\sigma) = 0$ if $\sigma < 0$ and $h_0(\sigma) = 1$ if $\sigma \geq 0$,

$$h_\varepsilon(\sigma) = \begin{cases} 0 & \text{if } \sigma \leq -\varepsilon/2, \\ 1 & \text{if } \sigma \geq \varepsilon/2, \end{cases} \qquad \text{with } h_\varepsilon \to h_0 \text{ in } \mathbb{R} \setminus \{0\} \text{ as } \varepsilon \to 0,$$

and V^a, V^r, V^J, $\mathbb{D}^a$, $\mathbb{D}^r$ are specified in Section 3.2. Here, σ is a placeholder for the value $\theta(x) - t$, and h_ε defines the transition between the energy and dissipation densities $V^a + V^J$ and R^a of the accreting phase and the ones $V^r + V^J$ and R^r of the receding phase, respectively. For $\varepsilon = 0$, we consider a *sharp-interface* model, where the energy and dissipation densities jump from one phase to the other, whereas, for $\varepsilon > 0$, h_ε interpolates between the two values and mixing of the two phases is allowed in a tubular neighborhood of width of order ε of the interface giving a *diffuse-interface* model, see Figure 1.8. Notice that we are considering the case of nonsimple materials with a second-order term H in (1.2.9), which we assume to be independent of the phase. Thus, the viscoelastic deformation $y(t)$ belongs for almost all times to the Sobolev space $W^{2,p}(\Omega; \mathbb{R}^d)$, for some $p > d$, cf. (3.2.9), and, by Sobolev embedding, to $C^1(\Omega; \mathbb{R}^d)$. The sharp-interface terminology is, hence, to be understood as referring to the energy and dissipation densities rather than to the deformations.

In Sections 3.4 and 3.5, we show that the coupled problem (1.2.8)–(1.2.9) equipped with the boundary and initial conditions, cf. (1.1.10),

$$\begin{aligned}
&y = \mathrm{id} \ \text{ on } \ [0, T] \times \Gamma_D, \\
&\mathrm{D}H(\nabla^2 y){:}(\nu \otimes \nu) = 0 \ \text{ on } \ [0, T] \times \partial U, \\
&(\partial_{\nabla y} W(\theta(x){-}t, \nabla y){+}\partial_{\nabla \dot{y}} R_\varepsilon(\theta(x){-}t, \nabla y, \nabla \dot{y}))\, \nu \\
&\quad - \mathrm{div}_S\, (\mathrm{D}H(\nabla^2 y)\nu) = 0 \ \text{ on } \ [0, T] \times \Gamma_N, \\
&y(0, \cdot) = y_0 \ \text{ on } \ U
\end{aligned}$$

admits solutions in the weak/viscosity sense, see Definition 3.2.1, for both the diffuse- ($\varepsilon > 0$) and the sharp-interface case ($\varepsilon = 0$). Namely, we consider (1.2.9) to be solved weakly, as in (1.1.11) in Section 1.1.3. On the other hand, for the generalized eikonal equation (1.2.8), we consider the case of viscosity solutions. Indeed, eikonal-type equations admit an overabundance of solutions in the almost-everywhere sense [7], whereas (nonnegative) viscosity solutions are unique in $\mathbb{R}^d \setminus \Omega_0$, see Proposition 3.4.2. The proof of the existence of weak/viscosity solutions for $\varepsilon \geq 0$ hinges on an iterative procedure: First, (1.2.8) and (1.2.9) are solved separately for given y and θ in Proposition 3.4.2 and 3.4.1, respectively. Thus, solving alternately the two problems provides a sequence (y_k, θ_k), which we show to converge to a solution to the coupled problem, see Section 3.4. Moreover, it also holds that sequences $(y_\varepsilon, \theta_\varepsilon)_{\varepsilon>0}$ of solutions to the diffuse-interface model converge uniformly, up to subsequences, to a solution (y, θ) to the sharp-interface one, see Section 3.5.

In Section 3.3, we prove that the weak/viscosity solutions additionally satisfy the energy-dissipation balance for the diffuse- and sharp-interface case, thanks to the validity of a suitable chain rule, cf. (1.1.16). For $\varepsilon \geq 0$, let us define the energy $\mathcal{E}_\varepsilon\colon C^{0,1}([0, T] \times \Omega) \times$

$W^{2,p}(U;\mathbb{R}^d) \to \mathbb{R}$ and instantaneous dissipation $\mathcal{R}_\varepsilon \colon C^{0,1}([0,T]\times\Omega)\times H^1(U)\times H^1(U) \to [0,\infty)$ as

$$\mathcal{E}_\varepsilon(\sigma,y) := \int_U W_\varepsilon(\sigma,\nabla y) + H(\nabla^2 y) - f(\sigma)\cdot y \,\mathrm{d}x,$$
$$\mathcal{R}_\varepsilon(\sigma,y,\dot{y}) := \int_U R_\varepsilon(\sigma,\nabla y,\nabla\dot{y})\,\mathrm{d}x.$$

The energy-dissipation balance for $\varepsilon > 0$ is

$$\begin{aligned}\mathcal{E}_\varepsilon(\theta{-}t,y) &+ 2\int_0^t \mathcal{R}_\varepsilon(\theta{-}s,y(s),\dot{y}(s))\,\mathrm{d}s \\ &= \mathcal{E}_\varepsilon(\theta,y_0) - \int_0^t\int_U \partial_\sigma f(\theta{-}s)\cdot y(s)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_U \partial_\sigma W_\varepsilon(\theta{-}s,\nabla y(s))\,\mathrm{d}x\,\mathrm{d}s,\end{aligned} \tag{1.2.10}$$

whereas, for $\varepsilon = 0$, takes the form

$$\begin{aligned}\mathcal{E}_0(\theta{-}t,y) &+ 2\int_0^t \mathcal{R}_0(\theta{-}s,y(s),\dot{y}(s))\,\mathrm{d}s \\ &= \mathcal{E}_0(\theta,y_0) - \int_0^t\int_U \partial_\sigma f(\theta{-}s)\cdot y(s)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_{\{\theta=s\}} \frac{V^r(\nabla y(s)){-}V^a(\nabla y(s))}{|\nabla\theta|}\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}s.\end{aligned} \tag{1.2.11}$$

In the identities above, cf. (1.1.15), the second term of the right-hand side is the work of the external (varying in time) forces. The third characterizes the energy stored or dissipated by the growth along the evolution. In particular, considering the sharp-interface case $\varepsilon = 0$, if $V^a \le V^r$, i.e., the accretive phase is softer than the receding one, then $-(V^r - V^a) \le 0$ and the third term in the right-hand side encodes the expected energy dissipated by the material as it relaxes, thanks to the growth of the soft phase into the stiffer one. On the other hand, if $V^a \ge V^r$, then some energy is stored in the system due to the accretion of the stiffer phase.

1.2.3 Chapter 4: Viscoelastic surface growth at finite strains with Ersatzmaterial

In Chapter 4, we revise a model proposed by [108] and for which an existence theory in the setting of linearized elasticity has been proposed in [24].

We focus on the evolution of a viscoelastic solid with reference configuration $\Omega(t)$ at time $t \in [0,T]$ growing by accretive growth. As in Chapter 3, we assume the existence of a time-of-attachment function $\theta\colon \mathbb{R}^d \to [0,\infty)$ such that $\Omega(t)$ is the t-sublevel set of θ, see (1.2.7), and θ satisfies the generalized eikonal equation

$$\begin{cases}\gamma\big(y(\theta(x),x),\nabla y(\theta(x),x)\big)|\nabla\theta(x)| = 1 \ \text{ in } \ U\setminus\overline{\Omega_0}\\ \theta = 0 \ \text{ in } \ \Omega_0 \subset\subset U,\end{cases} \tag{1.2.12}$$

Regarding the material, we study a finite-strain nonsimple Kelvin-Voigt medium with energy density $W\colon \mathbb{R}^{d\times d} \to [0,\infty]$, higher-order density $H\colon \mathbb{R}^{d\times d\times d} \to [0,\infty)$, and dissipation density $R\colon \mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d} \to [0,\infty)$. We further regularize the problem by additionally considering the presence of a (highly compliant) *Ersatzmaterial* with reference configuration $U\setminus\overline{\Omega(t)}$ surrounding the body. The open, connected, and bounded set $U \subset \mathbb{R}^d$ is assumed to be large enough so that $\Omega(T) \subset\subset U$, cf. (H15) and (4.4.4). The Ersatzmaterial is assumed to have elastic energy and instantaneous dissipation densities to be a small rescaling of the accreting material ones, namely $\frac{\delta}{1+\delta}W$ and $\frac{\delta}{1+\delta}R$, respectively, where $\delta \in (0,1)$. Defining $h\colon \mathbb{R} \to [0,1]$ as $h(\sigma) = 1$ if $\sigma \leq 0$ and $h(\sigma) = \frac{\delta}{1+\delta}$ if $\sigma > 0$, the viscoelastic equilibrium takes the form

$$\begin{aligned} -\operatorname{div}\big(h(\theta(x){-}t)\mathrm{D}W(\nabla y A^{-1})A^{-\top}{+}V^J(\nabla y){+}h(\theta(x){-}t)\partial_{\nabla\dot{y}}R(\nabla y,\nabla\dot{y}){-}\operatorname{div}\mathrm{D}H(\nabla^2 y)\big) \\ = h(\theta(x){-}t)f(t,x) \end{aligned} \tag{1.2.13}$$

in $[0,T] \times U$. Analogously to (1.2.9), the second-order term H is assumed to be the same for both the medium and the Ersatzmaterial. This is also the case for the term $V^J\colon \mathbb{R}^{d\times d} \to [0,\infty]$, which satisfies a growth condition of the form (1.1.9) and penalizes self-interpenetration of the material.

The viscoelastic equilibrium (1.2.9) features the *backstrain* tensor $A\colon [0,T] \times U \to \mathbb{R}^{d\times d}$ defined as

$$A(t,x) := \begin{cases} A_0 & \text{if } x \in \Omega_0, \\ \nabla y(\theta(x),x) & \text{if } x \in \Omega(t) \setminus \Omega_0, \\ I & \text{if } x \in U \setminus \Omega(t). \end{cases} \tag{1.2.14}$$

As already remarked in Section 1.2.2 in the energy-dissipation balance equations (1.2.10) and (1.2.11), growth contributes at dissipating energy along the evolution. Its effects are, however, usually also visible in the material, where residual stresses accumulate due to accretion [90, 93, 108]. Hence, the elastic energy density of a deformation y at a point $x \in \Omega(t)$ is $W(\nabla y(x)A^{-1})$, where $A \in \mathbb{R}^{d\times d}$ encodes the effect of the corresponding accumulated strains and depends in general on y itself. As in [24], we follow [108] and assume that $A(x) = \nabla y(\theta(x),x)$ for $t \in [0,T]$ and $x \in \Omega(t)$, so that the material is unstressed at the time and place of deposition, i.e., $W(\nabla y(\theta(x),x)A^{-1}) = W(I)$. For $x \in U \setminus \Omega(t)$, instead, we define A as the identity since the presence of nontrivial backstrain is related to growth. For $x \in U \setminus \Omega_0$ we set it to be some given initial backstrain $A_0 \in L^\infty(\Omega_0)$, cf. (1.2.14).

In Section 4.4, we show that the system of equations (1.2.12)–(1.2.13) admits weak/viscosity solution, similarly as in Chapter 3. Here, we consider the natural boundary condition on the hyperstress and initial condition

$$\begin{aligned} &\mathrm{D}H(\nabla^2 y){:}(\nu\otimes\nu) = 0 \ \text{ on } \ [0,T]\times\partial U, \\ &y(0,\cdot) = y_0 \ \text{ on } \ U \end{aligned} \tag{1.2.15}$$

Notice that we do not assume Dirichlet nor Neumann boundary conditions, though this would be possible, since they concern ∂U, the external boundary of the artificial Ersatzmaterial. On the other hand, we fix the position of the material along the evolution on a portion of the starting configuration Ω_0, however, imposing the *docking condition* [24]

$$y \equiv \mathrm{id} \ \text{ on } \ [0,T]\times\omega, \tag{1.2.16}$$

where $\omega \subset\subset \Omega_0$. This condition implies the validity of the following Poincaré-type inequality

$$\|y\|_{W^{2,p}(U;\mathbb{R}^d)} \leq c\left(1 + \|\nabla^2 y\|_{L^p(U;\mathbb{R}^{d\times d\times d})}\right) \quad \forall y \in W^{2,p}_\omega(U;\mathbb{R}^d),$$

which is crucial in the existence proof to compensate for the loss of compactness due to the presence of the backstrain. The existence proof provided in Section 4.4 then follows the iterative strategy illustrated in the previous section.

1.3 Notation

We devote this section to introduce the notation used in the following chapters. We denote by $\mathbb{R}^{d\times d}$ the Euclidean space of $d\times d$ real matrices, $d\geq 2$, by $\mathbb{R}^{d\times d}_{\rm sym}$ the subspace of symmetric matrices, and by I the identity matrix. Given $A\in\mathbb{R}^{d\times d}$ we indicate its transpose by $A^\top$ and its Frobenius norm by $|A|^2 := A{:}A$, where the contraction product between 2-tensors is defined as $A{:}B := A_{ij}B_{ij}$ (we use the summation convention over repeated indices). Analogously, let $\mathbb{R}^{d\times d\times d}$ be the set of real 3-tensors, and define their contraction product as $A\vdots B := A_{ijk}B_{ijk}$ for $A, B\in\mathbb{R}^{d\times d\times d}$. A 4-tensor $\mathbb{C}\in\mathbb{R}^{d\times d\times d\times d}$ is said to be *major symmetric* if $\mathbb{C}_{ijk\ell} = \mathbb{C}_{k\ell ij}$ and *minor symmetric* if $\mathbb{C}_{ijk\ell} = \mathbb{C}_{ij\ell k} = \mathbb{C}_{jik\ell}$. Given a major and minor symmetric positive definite 4-tensor $\mathbb{C}\in\mathbb{R}^{d\times d\times d\times d}$ and the matrix $A\in\mathbb{R}^{d\times d}$ we indicate by $\mathbb{C}A\in\mathbb{R}^{d\times d}$ and $A\mathbb{C}\in\mathbb{R}^{d\times d}$ the matrices given in components by $(\mathbb{C}A)_{ij} = \mathbb{C}_{ijk\ell}A_{k\ell}$ and $(A\mathbb{C})_{ij} = A_{k\ell}C_{k\ell ij}$, respectively. Moreover, given a symmetric positive definite 4-tensor $\mathbb{C}\in\mathbb{R}^{d\times d\times d\times d}$, the corresponding induced matrix norm is defined as $|A|^2_{\mathbb{C}} := \mathbb{C}A : A/2$. We denote by $A^{\rm sym} := (A+A^\top)/2$ the symmetric part of a matrix $A\in\mathbb{R}^{d\times d}$. We shall use the following matrix sets

$$
\begin{aligned}
\mathrm{SL}(d) &:= \{A\in\mathbb{R}^{d\times d} \mid \det A = 1\},\\
\mathrm{SO}(d) &:= \{A\in\mathrm{SL}(d) \mid AA^\top = I\},\\
\mathrm{GL}(d) &:= \{A\in\mathbb{R}^{d\times d} \mid \det A \neq 0\},\\
\mathrm{GL}_+(d) &:= \{A\in\mathbb{R}^{d\times d} \mid \det A > 0\}.
\end{aligned}
$$

The scalar product of two vectors $a,\ b\in\mathbb{R}^d$ is classically indicated by $a{\cdot}b$. The symbol $B_r^{\mathbb{R}^{d\times d}}(A)\subset\mathbb{R}^{d\times d}$ denotes the open ball of radius $r>0$ and center $A\in\mathbb{R}^{d\times d}$, whereas $B_R\subset\mathbb{R}^d$ denotes the open ball of radius $R>0$ and center $0\in\mathbb{R}^d$. We make use of the function spaces

$$
\begin{aligned}
H^1_\sharp(\Omega;\mathbb{R}^d) &:= \Big\{u\in H^1(\Omega;\mathbb{R}^d) \ \Big|\ \int_\Omega u\,\mathrm{d}X = 0\Big\},\\
H^1_\Gamma(\Omega;\mathbb{R}^d) &:= \{u\in H^1(\Omega;\mathbb{R}^d) \mid u = 0 \text{ on } \Gamma\subset\partial\Omega\},
\end{aligned}
$$

where Γ is nonempty, open in the relative topology of $\partial\Omega$, and a measurable subset of $\partial\Omega$.

Moreover, we denote by $\mathcal{H}^{d-1}$ the $(d-1)$-dimensional Hausdorff measure, by $|\omega|$ the d-dimensional Lebesgue measure of the measurable set ω, and by $\mathbb{1}_\omega$ the corresponding characteristic function, namely, $\mathbb{1}_\omega(x) = 1$ for $x\in\omega$ and $\mathbb{1}_\omega(x) = 0$ otherwise. For $E\subset\mathbb{R}^d$ nonempty and $x\in\mathbb{R}^d$ we define $\mathrm{dist}(x,E) := \inf_{e\in E}|x-e|$. We define $x\wedge y := \min\{x;y\}$ for all $x\,,y\in\mathbb{R}$. In the following, we use the symbol $\dot{u}$ for the partial time derivative of the generic time-dependent function u, whereas $\frac{\mathrm{d}}{\mathrm{d}t}$ stands for the total time derivative, in case u depends on time only.

Finally, we henceforth indicate by c a generic positive constant possibly depending on data but independent of the discretization step τ. Note that the value of c may change even within the same line.

2 Finite-strain Poynting-Thomson model

This chapter consists of my paper [15] with Martin Krúžik and Ulisse Stefanelli.

Abstract

We analyze the finite-strain Poynting-Thomson viscoelastic model. In its linearized small-deformation limit, this corresponds to the serial connection of an elastic spring and a Kelvin-Voigt viscoelastic element. In the finite-strain case, the total deformation of the body results from the composition of two maps, describing the deformation of the viscoelastic element and the elastic one, respectively. We prove the existence of suitably weak solutions by a time-discretization approach based on incremental minimization. Moreover, we prove a rigorous linearization result, showing that the corresponding small-strain model is indeed recovered in the small-loading limit.

2.1 Introduction

Viscoelastic solids appear ubiquitously in applications. Polymers, rubber, biomaterials, wood, clay, and soft solids, including metals at close-to-melting temperatures, behave viscoelastically. The mechanical response of viscoelastic solids is governed by the interplay between elastic and viscous dynamics: by applying stresses both strains and strain rates ensue [84]. This is at the basis of different effects, from viscoelastic creep, to viscous relaxation, to rate-dependence in material response, to dissipation of mechanical energy [92].

The modelization of viscoelastic solid response dates back to the early days of Continuum Mechanics. In the linearized, infinitesimal-strain setting of the standard-solid rheology, two basic models are the *Maxwell* and the *Kelvin-Voigt* one, where an elastic spring is connected to a viscous dashpot in series or in parallel, respectively. These models offer only a simplified description of actual viscoelastic behavior. More accurate descriptions necessarily call for more complex models. A first option in this direction is the *Poynting-Thomson* model, resulting from the combination in series of an elastic and a Kelvin-Voigt component, see Figure 2.1. A second option would be the *Zener* model, which consists of an elastic and a Maxwell element in parallel. Note however, that Poynting-Thomson and Zener can be proved to be equivalent in the linearized setting, see [51, Remark 6.5.4].

The aim of this paper is to investigate the Poynting-Thomson model in the finite-strain setting. From the modeling viewpoint, extending the model beyond the small-strain case is crucial, for viscoelastic materials commonly experience large deformations. In fact, finite-strain versions of the Poynting-Thomson model have already been considered. The reader is referred to [54], where a comparison between Poynting-Thomson and Zener models at finite strains is discussed, and to [64], focusing on the anisothermal version the Poynting-Thomson model.

To the best of our knowledge, mathematical results on the finite-strain Poynting-Thomson model are still not available. The focus of this paper is to fill this gap by presenting

- an **existence theory** for solutions of the finite-strain Poynting-Thomson model, as well as a convergence result for time-discretizations (Theorem 2.4.1);

- a rigorous **linearization result**, proving that finite-strain solutions converge (up to subsequences) to solutions of the linearized system in the limit of small loadings and, correspondingly, small strains (Theorem 2.4.2).

Our analysis is variational in nature. The convergence result provides a rigorous counterpart to the classical heuristic arguments based on Taylor expansions [54].

We postpone to Section 2.2 both the detailed discussion of the model and a first presentation of our main results. We anticipate however here that the theory requires no second-gradient terms but rather relies on a decomposition of the total deformation in terms of an elastic and a viscous deformation, see (2.2.3) below. Correspondingly, the variational formulation of the problem features both Lagrangian and Eulerian terms. Note moreover that the viscous dissipation is here assumed to be p_ψ-homogeneous, with superlinear homogeneity $p_\psi \geq 2$.

Our notion of solution (see Definition 2.4.1) hinges on the validity of an energy inequality, an elastic semistability inequality, and an approximability property via time-discrete problems. Albeit very weak, this notion replicates the important features of viscoelastic evolution, including elastic equilibrium, energy dissipation, and viscous relaxation.

Before moving on, let us put our results in context with respect to the available literature. In the purely PDE setting, existence results for viscoelastic dissipative systems are classical. The reader is referred to the recent monograph [51] for a comprehensive collection of references. As it is well known, the PDE setting is local in nature and, as such, does not allow considering global constraints such as injectivity of deformations, i.e., noninterpenetration of matter. Variational theories for viscoelastic evolution offer a remedy in this respect. By making use of the underlying gradient-flow structure of viscoelastic evolution, existence results for variational solutions have been obtained in the one-dimensional [70] and in the multi-dimensional case [33]. The latter paper also delivers a rigorous evolutive Γ-convergence linearization result. See also [48] for the case of self-contact and [4, 75] for some extension to nonisothermal situations. With respect to these contributions, we deal here with an internal-variable formulation, where the elastic variable does not dissipate. From the technical viewpoint, the novelty of our approach resides in avoiding the second-gradient theory by virtue of the composition assumption (2.2.3). This impacts on the functional setting, as well as on the required mathematical techniques.

In the different but related frame of activated inelastic deformations, the closest contributions to ours are [72] and [86], both dealing with rate-dependent viscoplasticity ($p_\psi > 1$) under the multiplicative-decomposition setting. In both papers, existence of solutions is discussed, by taking into account additional gradient-type terms for the viscous strain. In particular, the full gradient is considered in [72], whereas in [86] only its curl is penalized. The approach in [86] analogous to ours in terms of solution notion, despite the differences in the model. In contrast with these papers, viscous evolution is here not activated. In addition, by not considering here additional gradient terms, we avoid introducing a second length scale in the model and thus tackle so-called *simple materials*. Moreover, we investigate here linearization, which was not discussed in [72, 86].

In the fully rate-independent setting $p_\psi = 1$ of activated elastoplasticity, the papers [50, 96] and [77], contribute an existence and linearization theory which is parallel the current viscoelastic one. More precisely, [50, 96] deal with a decomposition of deformations in the same spirit of (2.2.3) below, avoiding the use of second gradients, whereas [77] features no gradients, but is a pure convergence result, in a setting where existence is not known. With respect to these contributions, the superlinear, non activated nature of the dissipation of the viscous setting calls for using a different set of analytical tools from gradient-flow theory [71].

Note that, also in the rate-independent setting, by including a gradient term of the plastic strain, hence resorting to so-called *strain-gradient* finite plasticity, one obtains stronger results. In particular, the existence of energetic solutions in strain-gradient finite plasticity is in [59] and the linearization in some symmetrized case is in [40]. Under the mere penalization of the curl of the gradient of the plastic strain, existence of incremental solutions is proved in [69] and linearization is in [89].

The paper is organized as follows. In Section 2.2, we provide an illustration of the finite-strain Poynting-Thomson model under consideration, as well as an introduction to our main results. Some preliminary material and comment on the functional setting is provided in Section 2.3. In particular, we discuss the set of admissible deformations in Subsection 2.3.1. In Subsections 2.4.1 and 2.4.3 we list and comment the assumptions, whereas the statements of our main results, Theorem 2.4.1 and Theorem 2.4.2 are presented in Subsections 2.4.2 and 2.4.4, respectively. The solvability of the time-discrete incremental problems is discussed in Section 2.5, whereas the proofs of Theorems 2.4.1 and 2.4.2 are given in Sections 2.6 and 2.7, respectively.

2.2 The finite-strain Poynting-Thomson model

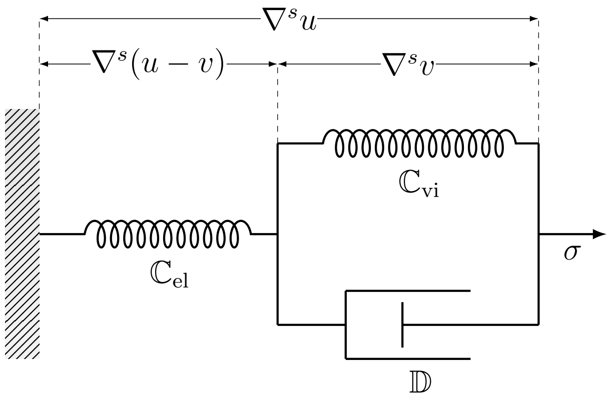


Figure 2.1: The Poynting-Thomson rheological model (linearized setting).

In order to illustrate our results, we start by recalling the classical Poynting-Thomson in the linearized setting of infinitesimal strains. By indicating by $u:\Omega\to\mathbb{R}^d$ the *infinitesimal displacement* from the *reference configuration* $\Omega\subset\mathbb{R}^d$, the total strain $\nabla^s u$ (here ∇^s denotes the symmetrized gradient $\nabla^s u=(\nabla u+\nabla u^\top)/2$) is additively decomposed in its elastic and its viscous parts as $\nabla^s u=\mathbb{C}_{\rm el}^{-1}\sigma+\nabla^s v$. In its quasistatic approximation, the evolution of the body results from the combination of the equilibrium system and the constitutive relation, namely,

$$
\begin{aligned}
-\operatorname{div}\left(\mathbb{C}_{\rm el}\nabla^s(u-v)\right)=f \quad \text{in } \Omega\times(0,T),\\
\mathbb{D}\nabla^s\dot v+(\mathbb{C}_{\rm vi}+\mathbb{C}_{\rm el})\nabla^s v=\mathbb{C}_{\rm el}\nabla^s u \quad \text{in } \Omega\times(0,T),
\end{aligned}
$$

where f stands for a given body force and $\dot v$ denotes the time-derivative of v. The reader is referred to the monographs [28, 51, 101] for a comprehensive collection of analytical results. Let us remark that in this paper we will specifically consider the case of incompressible viscosity, i.e., in the linearized setting $\operatorname{tr} v=0$. Hence, the evolution of the system considered is actually determined by the following equations

$$-\operatorname{div}\left(\mathbb{C}_{\rm el}\nabla^s(u-v)\right)=f \quad \text{in } \Omega\times(0,T), \tag{2.2.1}$$
$$\mathbb{D}\nabla^s\dot v+(\mathbb{C}_{\rm vi}+\mathbb{C}_{\rm el})\nabla^s v=\operatorname{dev}(\mathbb{C}_{\rm el}\nabla^s u) \quad \text{in } \Omega\times(0,T), \tag{2.2.2}$$

where dev denotes the deviatoric part of a tensor. Restricting to the incompressible case would call for accordingly specifying the rheological diagram from Figure 2.1 by distinguishing the volumetric and the deviatoric components.

In the finite-strain Poynting-Thomson model [54, 64], the state of the viscoelastic system is specified in terms of its *deformation* $y:\Omega\to\mathbb{R}^d$. As it is common in finite-strain theories [55], the deformation gradient ∇y is *multiplicatively decomposed* as $\nabla y = F_{\rm el}F_{\rm vi}$, where $F_{\rm el}$ and $F_{\rm vi}$ are the *elastic* and *viscous strain* tensors, representing the elastic and viscous response of the medium, respectively.

A distinctive feature of our approach is that we assume the viscous strain to be *compatible*: we identify $F_{\rm vi}$ with the gradient $\nabla y_{\rm vi}$ of a *viscous deformation* $y_{\rm vi}:\Omega\to y_{\rm vi}(\Omega)\subset\mathbb{R}^d$, mapping the *reference* configuration Ω to the *intermediate* one $y_{\rm vi}(\Omega)$. Correspondingly, the elastic strain is compatible as well and there exists an elastic deformation $y_{\rm el}\colon y_{\rm vi}(\Omega)\to\mathbb{R}^d$ with $F_{\rm el}=\nabla y_{\rm el}$ mapping the intermediate configuration to the *actual* one. As such, the multiplicative decomposition $\nabla y = F_{\rm el}F_{\rm vi}$ ensues from an application of the classical chain rule to the composition

$$y:=y_{\rm el}\circ y_{\rm vi}:\Omega\to\mathbb{R}^d. \tag{2.2.3}$$

Moving from this position, the state of the medium is described by the pair $(y_{\rm vi},y_{\rm el})$, effectively distinguishing viscous and elastic responses.

Before moving on, let us stress that the compatibility assumption on $F_{\rm vi}$, whence the composition assumption (2.2.3), realistically describes a variety of viscoelastic evolution settings and refer to [50, 96] for some parallel theory in the frame of finite-strain plasticity. In particular, position (2.2.3) is flexible enough to cover both limiting cases of a purely elastic ($y_{\rm vi}=\mathrm{id}$) and of a plain Kelvin-Voigt ($y_{\rm el}=\mathrm{id}$) materials. In the linearized setting, these would formally correspond to the cases $\mathbb{C}_{\rm vi}\to\infty$ and $\mathbb{C}_{\rm el}\to\infty$, respectively. Let us note that by choosing $\mathbb{C}_{\rm vi}=0$ the linearized system (2.2.1)-(2.2.2) reduces to the *Maxwell* fluidic rheological model. By assuming (2.2.3) we exclude the onset of defects, such as dislocations and disclinations. Albeit this could limit the application of the theory in some specific cases, it is to remark that viscous materials are often amorphous, so that the relevance of strictly crystallographic descriptions may be questionable. From the more analytical viewpoint, assumption (2.2.3) allows us to present a comprehensive mathematical theory within the setting of so-called *simple materials*, i.e., without resorting to second-gradient theories. The alternative path of including second-order deformation gradients, is also viable and, as far as existence is concerned, has been considered in [72] in the activated case of viscoplasticity.

A first consequence of the composition (2.2.3) is that the elastic deformation $y_{\rm el}$ is defined on the a-priori unknown intermediate configuration $y_{\rm vi}(\Omega)$, making the analysis delicate. In particular, the variational description of the viscoelastic behavior results in a mixed Lagrangian-Eulerian variational problem. This mixed nature of the problem will be tamed by means of change-of-variables techniques, which in turn ask for some specification on the class of admissible intermediate configurations. Let us anticipate that $y_{\rm vi}$ will be required to be an incompressible ($\det\nabla y_{\rm vi}=1$) homeomorphism throughout. We refer to [41, 103] for models of incompressible viscoelastic solids. As it is mentioned in [25], incompressibility is a somewhat standard assumption in the setting of biological applications. See also [10] for modeling of brain tissues. Our interest in the incompressible case is also motivated by the prospects of devising a sound existence theory. Assuming incompressibility has the net effect of simplifying change-of-variable formulas, ultimately allowing the mathematical treatment.

The *stored energy* of the medium is assumed to be of the form

$$\mathcal{W}(y_{\rm el}, y_{\rm vi}) := \int_{y_{\rm vi}(\Omega)} W_{\rm el}(\nabla y_{\rm el}(\xi))\mathrm{d}\xi + \int_{\Omega} W_{\rm vi}(\nabla y_{\rm vi}(X))\mathrm{d}X. \tag{2.2.4}$$

Here and in the following, we indicate by X the variable in the reference configuration Ω and by ξ the variable in the intermediate configuration $y_{\rm vi}(\Omega)$. The first integral above corresponds to the *stored elastic energy* and the given function $W_{\rm el}$ is the stored elastic energy density. Its argument $\nabla y_{\rm el}(\xi)$ can be equivalently rewritten in Lagrangian variables as the usual product $\nabla y(X)\nabla y_{\rm vi}^{-1}(X)$. By comparing these two expressions, the advantage of working in Eulerian variables is apparent, for $\nabla y_{\rm el}(\xi)$ is linear in $y_{\rm el}$. The function $W_{\rm vi}$ is the *stored viscous energy density* instead and the corresponding integral is Lagrangian.

The *instantaneous dissipation* of the system is given by

$$\Psi(y_{\rm vi}, \dot{y}_{\rm vi}) := \int_{\Omega} \psi(\nabla \dot{y}_{\rm vi}(\nabla y_{\rm vi})^{-1})\mathrm{d}X \tag{2.2.5}$$

where $\psi(\cdot)$ models the instantaneous dissipation density and is assumed to be p_ψ-positively homogeneous for $p_\psi \geq 2$.

By formally taking variations of the above introduced functionals, we obtain the quasistatic equilibrium system [74]

$$\begin{aligned}
&-\operatorname{div} \mathrm{D}W_{\rm el}\,(\nabla y_{\rm el}) = f \circ y_{\rm vi}^{-1} \qquad \text{in } y_{\rm vi}(\Omega) \times (0,T)\\
&\mathrm{D}W_{\rm el}\,(\nabla y_{\rm el}(y_{\rm vi}) : \mathrm{D}^2 y_{\rm el}\,(y_{\rm vi})) + \operatorname{div} \mathrm{D}W_{\rm vi}\,(\nabla y_{\rm vi})\\
&\qquad - \nabla y_{\rm el}(y_{\rm vi})^\top f = -\operatorname{div}\left(\mathrm{D}\psi\,(\nabla \dot{y}_{\rm vi}(\nabla y_{\rm vi}))^{-1}\,(\nabla y_{\rm vi})^{-\top}\right) \qquad \text{in } \Omega \times (0,T).
\end{aligned} \tag{2.2.6}$$

The highly nonlinear character of this system, combined with the absence of higher-order gradients in the viscous variable, forces us to consider a suitable weak-solution notion.

Inspired by [21, Def. 2.12] and [86, Def. 2.2], in our first main result, Theorem 2.4.1, we prove the existence of *approximable solutions* (see Definition 2.4.1). These are everywhere defined trajectories $(y_{\rm el}, y_{\rm vi}) : [0,T] \to W^{1,p_{\rm el}}(y_{\rm vi}(\Omega);\mathbb{R}^d) \times W^{1,p_{\rm vi}}(\Omega;\mathbb{R}^d)$ starting from some given initial datum $(y_{\rm el,0}, y_{\rm vi,0})$ and satisfying for all $t \in [0,T]$

Energy inequality:

$$\int_{y_{\rm vi}(t,\Omega)} W_{\rm el}(\nabla y_{\rm el}(t,\xi)){\rm d}\xi + \int_\Omega W_{\rm vi}(\nabla y_{\rm vi}(t,X)){\rm d}X - \int_\Omega f(t,X)\cdot y_{\rm el}(t,y_{\rm vi}(t,X)){\rm d}X$$
$$+ p_\psi \int_0^t\int_\Omega \psi(\nabla \dot y_{\rm vi}(s,X)(\nabla y_{\rm vi}(s,X))^{-1}){\rm d}X{\rm d}s$$
$$\leq \int_{y_{\rm vi,0}(\Omega)} W_{\rm el}(\nabla y_{\rm el,0}(\xi)){\rm d}\xi + \int_\Omega W_{\rm vi}(\nabla y_{\rm vi,0}(X)){\rm d}X - \int_\Omega f(0,X)\cdot y_{\rm el,0}(y_{\rm vi,0}(X)){\rm d}X$$
$$- \int_0^t \int_\Omega \partial_s f(s,X)\cdot y_{\rm el}(s,y_{\rm vi}(s,X)){\rm d}X{\rm d}s \qquad (2.2.7)$$

Semistability condition:

$$\int_{y_{\rm vi}(t,\Omega)} W_{\rm el}(\nabla y_{\rm el}(t,\xi)){\rm d}\xi - \int_\Omega f(t,X)\cdot y_{\rm el}(t,y_{\rm vi}(t,X)){\rm d}X$$
$$\leq \int_{y_{\rm vi}(t,\Omega)} W_{\rm el}(\nabla \tilde y_{\rm el}(\xi)){\rm d}\xi - \int_\Omega f(t,X)\cdot \tilde y_{\rm el}(y_{\rm vi}(t,X)){\rm d}X$$
$$\forall \tilde y_{\rm el} \text{ with } (\tilde y_{\rm el}, y_{\rm vi}(t,\cdot))\in\mathcal{A} \qquad (2.2.8)$$

where $\mathcal{A}$ is the set of admissible deformations, introduced in Section 2.3.1 below. The first line of inequality (2.2.7) corresponds to the *total energy* of the medium at time t and state $(y_{\rm el}(t,\cdot), y_{\rm vi}(t,\cdot))$. In particular, the term $-\int_\Omega f\cdot(y_{\rm el}\circ y_{\rm vi}){\rm d}X$ is the work of the (external) force f (later, a boundary traction will be considered, as well). Solutions $t\mapsto (y_{\rm el}(t), y_{\rm vi}(t))$ are moreover required to be *approximable*, namely, to ensue as limit of time discretizations. In this respect, we consider the *incremental minimization problems*, for $i=1,\dots,N$,

$$\min_{(y_{\rm el},y_{\rm vi})\in\mathcal{A}} \left\{ \int_{y_{\rm vi}(\Omega)} W_{\rm el}(\nabla y_{\rm el}(\xi)){\rm d}\xi + \int_\Omega W_{\rm vi}(\nabla y_{\rm vi}(X)){\rm d}X - \int_\Omega f(i\tau,X)\cdot y_{\rm el}(y_{\rm vi}(X)){\rm d}X \right.$$
$$\left. + \tau\int_\Omega \psi\left(\frac{\nabla y_{\rm vi}(X) - \nabla y_{\rm vi}^{i-1}(X)}{\tau}(\nabla y_{\rm vi}^{i-1}(X))^{-1}\right){\rm d}X \right\} \quad \text{for } y_{\rm vi}^{i-1} \text{ given} \qquad (2.2.9)$$

on a given uniform time-partition $\{0=t_0<t_1<\cdots<t_N=T\}$, where the set of admissible states $\mathcal{A}$ is defined in Subsection 2.3.1 below.

The notion of approximable solution is capable of reproducing the main features of viscoelastic evolution. First of all, the semistability condition (2.2.8) implies that $y_{\rm el}$ solves the elastic equilibrium at all times, given the viscous-state evolution. Correspondingly, the description of the purely elastic response of the material is complete. Secondly, the energy inequality (2.2.7) is sharp, in the sense that it may indeed hold as equality in specific smooth situations. In other words, all dissipative contributions are correctly taken into account in (2.2.7). Note in this respect the presence of the factor p_ψ multiplying the dissipation term in (2.2.7). Eventually, the approximation property ensures that viscous evolution actually occurs, even in absence of applied loads. We give an illustration of this fact in Section 2.4.2 below, see Figure 2.2.

Under suitable assumptions, the incremental minimization problems (2.2.9) are proved to admit solutions in Proposition 2.4.1 below. These time-discrete solutions fulfill a discrete energy inequality and a discrete semistability inequality. The existence of approximable solutions

(Theorem 2.4.1) follows by passing to the limit in the time-discrete problems. In order to pass from the time-discrete to the time-continuous energy inequality (2.2.7), lower semicontinuity of the energy and dissipation functionals is necessary, which translates in our setting in asking for the polyconvexity of the respective densities. In order to obtain the specific form (2.2.7) we need to resort to the notion of De Giorgi variational interpolant [2, Def. 3.2.1, p. 66], and adapt this tool from its original metric-space application to the current one.

For establishing the elastic semistability (2.2.8), a suitable recovery-sequence construction is required. This calls for the extension of the elastic deformations from the intermediate configurations to the whole $\mathbb{R}^d$. The possibility of performing this extension requires some regularity of the boundary of the intermediate configurations, which we ask to be *Jones domains* (see Definition 2.3.1).

The second main focus of the paper is on the rigorous linearization of the system through evolutionary Γ-convergence [76] in the case of quadratic dissipations, namely for $p_\psi = 2$. Moving from the seminal paper [22] in the stationary, hyperelastic case, the application of Γ-convergence to inelastic evolutive problems has been started in [77] and has been applied to different settings. In particular, linearization in the incompressible case has been discussed in [45, 60, 61]. The goal is to provide a rigorous formalization of heuristic Taylor expansion arguments which for the finite-strain Poynting-Thomson model were already presented in [54]. At first, let us review this heuristic by assuming sufficient regularity of all ingredients. Consider the functions u, v, w defined as

$$u := \frac{y - \mathrm{id}_\Omega}{\varepsilon}, \quad v := \frac{y_{\mathrm{vi}} - \mathrm{id}_\Omega}{\varepsilon}, \quad \text{and} \quad w := \frac{y_{\mathrm{el}} - \mathrm{id}_{y_{\mathrm{vi}}(\Omega)}}{\varepsilon},$$

so that u, v, w actually correspond to the ε-rescaled displacements of y, y_{vi}, y_{el}, respectively. To compute the linearization it will be more convenient to work with the pair (u, v) corresponding to the total and viscous deformations (y, y_{vi}). In particular, we replace $\nabla y = I + \varepsilon\nabla u$ and $\nabla y_{\mathrm{vi}} = I + \varepsilon\nabla v$ in the stored energy and $\nabla \dot y_{\mathrm{vi}} = \varepsilon\nabla\dot v$ in the dissipation. By formally Taylor expanding the (rescaled) energy terms and taking $\varepsilon \to 0$ we find

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_\Omega W_{\mathrm{el}}\left((I+\varepsilon\nabla u)(I+\varepsilon\nabla v)^{-1}\right)\mathrm{d}X = \int_\Omega \frac{1}{2}\mathrm{D}^2 W_{\mathrm{el}}(I)\nabla(u-v) : \nabla(u-v)\mathrm{d}X + \mathrm{o}(\varepsilon)\\
&\qquad \to \frac{1}{2}\int_\Omega \nabla(u-v) : \mathbb{C}_{\mathrm{el}}\nabla(u-v)\mathrm{d}X,\\
&\frac{1}{\varepsilon^2}\int_\Omega W_{\mathrm{vi}}(I+\varepsilon\nabla v)\mathrm{d}X = \int_\Omega \frac{1}{2}\,\mathrm{D}^2 W_{\mathrm{vi}}(I)\nabla v : \nabla v\mathrm{d}X + \mathrm{o}(\varepsilon) \to \frac{1}{2}\int_\Omega \nabla v : \mathbb{C}_{\mathrm{vi}}\nabla v\mathrm{d}X,\\
&\frac{1}{\varepsilon^2}\int_\Omega \psi\left(\varepsilon\nabla\dot v(I+\varepsilon\nabla v)^{-1}\right)\mathrm{d}X = \int_\Omega \frac{1}{2}\,\mathrm{D}^2\psi(0)\nabla\dot v : \nabla\dot v\mathrm{d}X + \mathrm{o}(\varepsilon) \to \frac{1}{2}\int_\Omega \mathbb{D}\nabla\dot v : \nabla\dot v\mathrm{d}X.
\end{aligned}$$

Here, we have assumed $W_{\mathrm{el}}(I) = W_{\mathrm{vi}}(I) = 0$, $\mathrm{D}W_{\mathrm{el}}(I) = \mathrm{D}W_{\mathrm{vi}}(I) = 0$, and have defined $\mathbb{C}_{\mathrm{el}} := \mathrm{D}^2 W_{\mathrm{el}}(I)$, $\mathbb{C}_{\mathrm{vi}} := \mathrm{D}^2 W_{\mathrm{vi}}(I)$, and $\mathbb{D} := \mathrm{D}^2\psi(0)$. Moreover, we assume that the force f is small, i.e., $f = f^\varepsilon = \varepsilon f^0$. Hence, by neglecting the term $f^0 \cdot \mathrm{id}_\Omega$, which is independent of the displacement, the rescaled loading term reads

$$-\frac{1}{\varepsilon^2}\int_\Omega f^\varepsilon \cdot y_{\mathrm{el}} \circ y_{\mathrm{vi}}\mathrm{d}X = -\frac{1}{\varepsilon^2}\int_\Omega \varepsilon f^0 \cdot \varepsilon u\mathrm{d}X = -\int_\Omega f^0 \cdot u\mathrm{d}X.$$

The above pointwise convergences are the classical heuristic linearization procedure. Still, one is left with actually checking that the finite-strain trajectories indeed converge to a solution of the linearized system. This is the aim of our second main result, Theorem 2.4.2, where we prove that, given a sequence of approximable solutions $(y_{\mathrm{vi},\varepsilon}, y_{\mathrm{el},\varepsilon})_\varepsilon$ and upon defining $y_\varepsilon = y_{\mathrm{el},\varepsilon} \circ y_{\mathrm{vi},\varepsilon}$ and the corresponding rescaled displacements $u_\varepsilon = (y_\varepsilon - \mathrm{id}_\Omega)/\varepsilon$ and $v_\varepsilon = (y_{\mathrm{vi},\varepsilon} - \mathrm{id}_\Omega)/\varepsilon$, the sequence $(u_\varepsilon, v_\varepsilon)_\varepsilon$ converges pointwise in time (up to subsequences) to $(u,v) : [0,T] \to H^1(\Omega;\mathbb{R}^d) \times H^1(\Omega;\mathbb{R}^d)$ with $(u(0), v(0)) = (u^0, v^0) := \lim_{\varepsilon\to 0}(u_\varepsilon(0), v_\varepsilon(0))$ and satisfying, for all $t \in [0,T]$,

Linearized energy inequality:

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \nabla(u(t)-v(t)) : \mathbb{C}_{\mathrm{el}}\nabla(u(t)-v(t))\mathrm{d}X + \frac{1}{2}\int_\Omega \nabla v(t) : \mathbb{C}_{\mathrm{vi}}\nabla v(t)\mathrm{d}X - \int_\Omega f^0(t)\cdot u(t)\mathrm{d}X \\
&\quad + \int_0^t\!\!\int_\Omega \mathbb{D}\nabla\dot v(s) : \nabla\dot v(s)\mathrm{d}X\mathrm{d}s \\
&\quad \le \frac{1}{2}\int_\Omega \nabla(u_0-v_0) : \mathbb{C}_{\mathrm{el}}\nabla(u_0-v_0)\mathrm{d}X + \frac{1}{2}\int_\Omega \nabla v_0 : \mathbb{C}_{\mathrm{vi}}\nabla v_0\mathrm{d}X - \int_\Omega f^0(0)\cdot u_0\mathrm{d}X \\
&\quad - \int_0^t\!\!\int_\Omega \partial_s f^0(s)\cdot u(s)\ \mathrm{d}X\mathrm{d}s,
\end{aligned} \tag{2.2.10}
$$

Linearized semistability:

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \nabla(u(t)-v(t)) : \mathbb{C}_{\mathrm{el}}\nabla(u(t)-v(t))\mathrm{d}X - \int_\Omega f^0(t)\cdot u(t)\mathrm{d}X \\
&\quad \le \frac{1}{2}\int_\Omega \nabla(\hat u-v(t)) : \mathbb{C}_{\mathrm{el}}\nabla(\hat u-v(t))\mathrm{d}X - \int_\Omega f^0(t)\cdot \hat u\mathrm{d}X \quad \forall \hat u \text{ admissible.}
\end{aligned} \tag{2.2.11}
$$

The linearized energy inequality and the linearized semistability deliver a weak notion of solution for the linearized problem (2.2.1)-(2.2.2). Albeit (2.2.10)-(2.2.11) are too weak to fully characterize the unique solution of linearized Poynting-Thomson system (2.2.1)-(2.2.2), the equilibrium system (2.2.1) is fully recovered. In particular, u is uniquely determined at all times, given v. Moreover, the linearized energy equality (2.2.10) is sharp and turns out to be an equality in specific cases.

To conclude, let us note that one could alternatively perform the linearization at the time-discrete level and then pass to the time-continuous limit. This way one recovers the unique strong solution of the linearized Poynting-Thomson system (2.2.1)- (2.2.2). This fact provides some additional justification of the finite-strain model. Still, we do not follow hier this alternative path, which could be easily treated along the lines of the analysis in of Sections 2.6 and 2.7.

2.3 Preliminaries

We devote this section to presenting some preliminary results.

2.3.1 Deformations and admissible states

Let fix the reference configuration Ω of the body to be a nonempty, open, bounded, and connected Lipschitz subset of $\mathbb{R}^d$. We assume without loss of generality that Ω is such that $\int_\Omega X\mathrm{d}X = 0$. We let Γ_D, Γ_N be open subsets of $\partial\Omega$ (in the relative topology of $\partial\Omega$) such that $\overline{\Gamma}_D \cup \overline{\Gamma}_N = \partial\Omega$, $\mathring{\Gamma}_D \cap \mathring{\Gamma}_N = \emptyset$, and $\mathcal{H}^{d-1}(\Gamma_D) > 0$.

The viscous deformation is required to fulfill

$$
y_{\mathrm{vi}} \in W^{1,p_{\mathrm{vi}}}(\Omega;\mathbb{R}^d) \quad \text{for some} \quad p_{\mathrm{vi}} > d(d-1)
$$

and to be locally volume-preserving, i.e., $\det \nabla y_{\mathrm{vi}} = 1$ almost everywhere in Ω. In the following, y_{vi} is tacitly identified with its Hölder-continuous representative. More precisely, $y_{\mathrm{vi}} \in C^{0,1-d/p_{\mathrm{vi}}}(\Omega;\mathbb{R}^d)$ and is almost everywhere differentiable (see [30]). In addition, since we will use the change-of-variables formula to pass from Lagrangian to Eulerian variables, we require y_{vi} to be injective almost everywhere. Equivalently, we ask for the *Ciarlet-Nečas condition* [18]

$$|\Omega| = \int_\Omega \det \nabla y_{\mathrm{vi}} \mathrm{d}X = |y_{\mathrm{vi}}(\Omega)| \tag{2.3.1}$$

to hold. As a consequence we have the change-of-variables formula

$$\int_\omega \varphi(y_{\mathrm{vi}}(X))\mathrm{d}X = \int_{y_{\mathrm{vi}}(\omega)} \varphi(\xi)\mathrm{d}\xi$$

for every measurable set $\omega \subseteq \Omega$ and every measurable function $\varphi : y_{\mathrm{vi}}(\omega) \to \mathbb{R}^d$. Note that $y_{\mathrm{vi}} \in W^{1,p}(\Omega;\mathbb{R}^d)$ has *distortion* $K := |\nabla y_{\mathrm{vi}}|^d / \det \nabla y_{\mathrm{vi}} = |\nabla y_{\mathrm{vi}}|^d \in L^{p_{\mathrm{vi}}/d}(\Omega;\mathbb{R})$, since it is locally volume preserving. As $p_{\mathrm{vi}}/d > d-1$, this bound on the distortion K implies that y_{vi} is either constant or open [43, Theorem 3.4]. By the Ciarlet- Nečas condition (2.3.1), y_{vi} cannot be constant, and hence y_{vi} is open. In particular $y_{\mathrm{vi}}(\Omega)$ is an open set. Moreover, we also have that y_{vi} is (globally) injective [39, Lemma 3.3], and that y_{vi} is actually a homeomorphism with inverse $y_{\mathrm{vi}}^{-1} \in W^{1,p_{\mathrm{vi}}/(d-1)}(y_{\mathrm{vi}}(\Omega);\mathbb{R}^d)$ (see [30]).

In order to make the statement of the model precise, we need to require some regularity of the intermediate configuration $y_{\mathrm{vi}}(\Omega)$. We recall the following definition.

Definition 2.3.1 ((η_1,η_2)-Jones domain [46]). *Let $\eta_1,\eta_2 > 0$. A bounded open set $\omega \subset \mathbb{R}^d$ is said to be a (η_1,η_2)-*Jones domain*, if for every $x,y \in \omega$ with $|x-y| < \eta_2$ there exists a Lipschitz curve $\gamma \in W^{1,\infty}([0,1];\omega)$ with $\gamma(0) = x$ and $\gamma(1) = y$ satisfying the following two conditions:*

$$l(\gamma) := \int_0^1 |\dot\gamma(s)|\mathrm{d}s \le \frac{1}{\eta_1}|x-y|$$

and

$$d(\gamma(t),\partial\omega) \ge \eta_1 \frac{|x-\gamma(t)||\gamma(t)-y|}{|x-y|} \quad \text{for every} \quad t \in [0,1].$$

The set of (η_1,η_2)-Jones domains will be denoted by $\mathcal{J}_{\eta_1,\eta_2}$.

In the following, we will exploit the fact that (η_1,η_2)-Jones domains are *Sobolev extension domains*: for all $\eta_1,\eta_2 > 0$, $p \in [1,\infty)$, and all $\omega \in \mathcal{J}_{\eta_1,\eta_2}$ there exists a positive constant $C = C(\eta_1,\eta_2,p,\omega,d)$ and a linear operator $E : W^{1,p}(\omega;\mathbb{R}^d) \to W^{1,p}(\mathbb{R}^d;\mathbb{R}^d)$ such that $Ey = y$ on ω and

$$\|Ey\|_{W^{1,p}(\mathbb{R}^d)} \le C\|y\|_{W^{1,p}(\omega)} \quad \text{for every } y \in W^{1,p}(\omega;\mathbb{R}^d).$$

Note that the class of (η_1,η_2)-Jones domains is closed under Hausdorff convergence [50]. In the following, we will need to consider extensions and we then ask for the regularity

$$y_{\mathrm{vi}}(\Omega) \in \mathcal{J}_{\eta_1,\eta_2}.$$

Finally, since the problem will be formulated only in terms of the gradient of y_{vi}, we impose the normalisation condition

$$\int_\Omega y_{\mathrm{vi}} \mathrm{d}X = 0. \tag{2.3.2}$$

Given a viscous deformation $y_{\rm vi}$, we assume the elastic deformation to fulfill

$$y_{\rm el} \in W^{1,p_{\rm el}}(y_{\rm vi}(\Omega);\mathbb{R}^d) \quad \text{for some} \quad p_{\rm el} > d$$

and we tacitly identify $y_{\rm el}$ with its Hölder-continuous representative.

For all given viscous deformation $y_{\rm vi} : \Omega \to \mathbb{R}^d$ and elastic deformation $y_{\rm el} : y_{\rm vi}(\Omega)^\circ \to \mathbb{R}^d$, we define the total deformation as the composition of the two, i.e.,

$$y := y_{\rm el} \circ y_{\rm vi} : \Omega \to \mathbb{R}^d.$$

We assume that y satisfies a Dirichlet boundary condition on Γ_D, namely,

$$y = \mathrm{id} \quad \text{on } \Gamma_D. \tag{2.3.3}$$

Since $y_{\rm vi}$ is invertible and both $y_{\rm vi}$ and $y_{\rm el}$ are almost everywhere differentiable, the following chain rule

$$\nabla y(X) = \nabla y_{\rm el}(y_{\rm vi}(X))\nabla y_{\rm vi}(X)$$

holds for almost every $X \in \Omega$. Hence, y satisfies

$$\|\nabla y\|_{L^q(\Omega)} \leq \|\nabla y_{\rm el}\|_{L^{p_{\rm el}}(y_{\rm vi}(\Omega))}\|\nabla y_{\rm vi}\|_{L^{p_{\rm vi}}(\Omega)} \quad \text{where} \quad \frac{1}{q} := \frac{1}{p_{\rm el}} + \frac{1}{p_{\rm vi}},$$

as can be readily checked by a change of variables and by the Hölder inequality. In particular, the boundary condition (2.3.3) should be understood in the classical trace sense.

To sum up, the set of *admissible states* is defined as

$$\mathcal{A} := \Bigg\{ (y_{\rm el}, y_{\rm vi}) \in W^{1,p_{\rm el}}(y_{\rm vi}(\Omega);\mathbb{R}^d) \times W^{1,p_{\rm vi}}(\Omega;\mathbb{R}^d) \;\Bigg|\; \det \nabla y_{\rm vi} = 1 \text{ a.e. in } \Omega,$$
$$\int_\Omega y_{\rm vi} \mathrm{d}X = 0,\ |\Omega| = |y_{\rm vi}(\Omega)|,\ y_{\rm vi}(\Omega) \in \mathcal{J}_{\eta_1,\eta_2},\ y = y_{\rm el} \circ y_{\rm vi} = \mathrm{id} \text{ on } \Gamma_D \Bigg\}.$$

Viscoelastic states are naturally depending on time. From now on, we are hence interested in *trajectories* $(y_{\rm el}, y_{\rm vi}) : [0,T] \to \mathcal{A}$ in the set of admissible states.

2.4 Main results

We devote this section to the statements of our assumptions and our main results.

2.4.1 Assumptions for the existence theory

In this section we specify the assumptions needed for the existence results, namely, Proposition 2.4.1 and Theorem 2.4.1.

The *total energy* of the system at time $t \in [0,T]$ and state $(y_{\rm el}, y_{\rm vi}) \in \mathcal{A}$ is given by

$$\mathcal{E}(t, y_{\rm el}, y_{\rm vi}) := \mathcal{W}(y_{\rm el}, y_{\rm vi}) - \langle \ell(t), y_{\rm el} \circ y_{\rm vi}\rangle,$$

where $\mathcal{W}(y_{\rm el}, y_{\rm vi})$ is the *stored energy* and the pairing $\langle \ell(t), y_{\rm el} \circ y_{\rm vi}\rangle$ represents the work of *external mechanical actions*.

More precisely, the stored energy is defined as

$$\mathcal{W}(y_{\rm el}, y_{\rm vi}) := \int_{y_{\rm vi}(\Omega)} W_{\rm el}(\nabla y_{\rm el}(\xi))\mathrm{d}\xi + \int_\Omega W_{\rm vi}(\nabla y_{\rm vi}(X))\mathrm{d}X$$

where $W_{\rm el} : \mathbb{R}^{d\times d} \to \mathbb{R}$ and $W_{\rm vi} : \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{\infty\}$ are the stored *elastic* and the stored *viscous* energy densities, respectively. On the energy densities we assume that:

(E1) there exist positive constants c_1, c_2 such that

$$c_1|A|^{p_{\rm el}} \le W_{\rm el}(A) \le \frac{1}{c_1}(1+|A|^{p_{\rm el}}) \quad \text{for every } A \in {\rm GL}(d), \tag{2.4.1}$$

$$W_{\rm vi}(A) \ge \begin{cases} c_2|A|^{p_{\rm vi}} - \frac{1}{c_2} & \text{for every } A \in {\rm SL}(d) \\ \infty & \text{otherwise,} \end{cases} \tag{2.4.2}$$

for $p_{\rm el} > d$ and $p_{\rm vi} > d(d-1)$.

(E2) $W_{\rm el}, W_{\rm vi}$ are polyconvex, i.e., there exist two convex functions $\hat{W}_{\rm el}, \hat{W}_{\rm vi} : \mathbb{R}^{\zeta(d)} \to \mathbb{R} \cup \{\infty\}$ such that

$$W_{\rm el}(A) = \hat{W}_{\rm el}(T(A)) \quad \text{and} \quad W_{\rm vi}(A) = \hat{W}_{\rm vi}(T(A))$$

where the minors $T(A)$ of A are given by $T : \mathbb{R}^{d\times d} \to \mathbb{R}^{\zeta(d)}$

$$T(A) := (A, {\rm adj}_2 A, \cdots, {\rm adj}_d A).$$

Here, ${\rm adj}_s A$ denotes the matrix of all minors $s \times s$ of the matrix $A \in \mathbb{R}^{d\times d}$, for $s = 2, \cdots, d$ and $\zeta(d) := \sum_{s=1}^{d} \binom{s}{d}^2$.

Notice that, since $p_{\rm vi} > d$, the mapping $y_{\rm vi} \mapsto {\rm adj}_s \nabla y_{\rm vi}$ is $(W^{1,p_{\rm vi}}, L^{p_{\rm vi}/s})$-weakly sequentially continuous. Hence, given $y_{{\rm vi},n} \rightharpoonup y_{\rm vi}$ in $W^{1,p_{\rm vi}}(\Omega;\mathbb{R}^d)$ with $\det \nabla y_{{\rm vi},n} = 1$ almost everywhere in Ω, we have that

$$1 = \det \nabla y_{{\rm vi},n} \rightharpoonup \det \nabla y_{\rm vi} = 1 \quad \text{in } L^{p_{\rm vi}/d}(\Omega).$$

As $\nabla y_{\rm vi}(X) \in {\rm SL}(d)$ a.e. in Ω, we have that

$$\int_\Omega W_{\rm vi}(\nabla y_{\rm vi}(X)){\rm d}X \le \liminf_n \int_\Omega W_{\rm vi}(\nabla y_{{\rm vi},n}(X)){\rm d}X$$

by polyconvexity of $W_{\rm vi}$. In particular, $y_{\rm vi} \mapsto \int_\Omega W_{\rm vi}(\nabla y_{\rm vi}(X)){\rm d}X$ is weakly lower semicontinuous in $W^{1,p_{\rm vi}}(\Omega;\mathbb{R}^d)$.

The growth condition (2.4.2) ensures that all viscous deformations $y_{\rm vi}$ of finite energy are incompressible. Local elastic incompressibility $\det \nabla y_{\rm el} = 1$ or even the weaker $\det \nabla y_{\rm el} > 0$ cannot be required, however. This is due to the fact that we later need to consider the Sobolev extension of $y_{\rm el}$ from the moving domain $y_{\rm vi}(\Omega)$ to $\mathbb{R}^d$ in order to compute the limit of an infimizing sequence. As it is well-known, such extensions may not preserve the positivity of $\det \nabla y_{\rm el}$.

On the other hand, our assumptions on the elastic energy density are compatible with *frame indifference*. In particular, we could ask $W_{\rm el}(RA) = W_{\rm el}(A)$ for every rotation $R \in {\rm SO}(d)$ and every $A \in \mathbb{R}^{d\times d}$. Note nonetheless that this property, although fundamental from the mechanical standpoint, is actually not needed for the analysis. The above assumptions would be compatible with requiring that $W_{\rm vi}$ is invariant by left multiplication with special rotations, as well. Still, such an invariance would be little relevant from the modeling viewpoint, for the viscous energy density is defined on viscous deformations, which take values in the intermediate configuration.

Eventually, the work of external mechanical actions is assumed to result from a given time-dependent *body force* $f : [0,T] \times \Omega \to \mathbb{R}^d$ and a given time-dependent boundary *traction* $g : [0,T] \times \Gamma_N \to \mathbb{R}^d$ as follows

$$\langle \ell(t), y\rangle := \int_\Omega f(t,X)\cdot y(X){\rm d}X + \int_{\Gamma_N} g(t,X)\cdot y(X){\rm D}\mathcal{H}^{d-1}(X). \tag{2.4.3}$$

We assume

(E3) $f \in W^{1,\infty}(0,T;L^{(q^*)'}(\Omega;\mathbb{R}^d))$ and $g \in W^{1,\infty}(0,T;L^{(q^\#)'}(\Gamma_N;\mathbb{R}^d))$ where q^* and $q^\#$ are the Sobolev and trace exponent related to $W^{1,q}(\Omega;\mathbb{R}^d)$, respectively (see [88]) and the prime denotes conjugation.

Consequently, we have

$$\ell \in W^{1,\infty}\big(0,T;(W^{1,q}(\Omega;\mathbb{R}^d))^*\big),$$

where $(W^{1,q}(\Omega;\mathbb{R}^d))^*$ is the dual space of $W^{1,q}(\Omega;\mathbb{R}^d)$.

Given a time-dependent viscous trajectory $y_{\rm vi}:[0,T] \to W^{1,p_{\rm vi}}(\Omega;\mathbb{R}^d)$, we define the *total instantaneous dissipation* of the system [72] as

$$\Psi(y_{\rm vi},\dot y_{\rm vi}) := \int_\Omega \psi(\nabla \dot y_{\rm vi}(\nabla y_{\rm vi})^{-1})\mathrm{d}X. \tag{2.4.4}$$

Here and in the following, the dot represent a partial derivative with respect to time. Above, the dissipation density $\psi:\mathbb{R}^{d\times d} \to [0,\infty)$ is assumed to be:

(E4) convex and differentiable at 0 with $\psi(0)=0$;

(E5) fulfilling

$$\psi(A) \geq c_3|A|^{p_\psi} \quad \text{for every } A \in \mathbb{R}^{d\times d} \tag{2.4.5}$$

for some positive constant c_3;

(E6) positively p_ψ-homogeneous, namely

$$\psi(\lambda A) = \lambda^{p_\psi}\psi(A) \quad \text{for every } A \in \mathbb{R}^{d\times d}, \lambda \geq 0. \tag{2.4.6}$$

The form of the instantaneous dissipation is parallel to the analogous definition in elastoplasticity, where nonetheless ψ is assumed to be positively 1-homogeneous, namely $p_\psi = 1$ [66, 67]. In particular, let us explicitly point out that it does not fall within the frame-indifferent setting from [3]. Indeed, in this case viscous deformations take values in the intermediate configuration only and frame-indifference should not necessarily be imposed there.

In the following, we ask

$$p_\psi \geq 2 \geq \frac{d(d-1)}{d(d-1)-1} \tag{2.4.7}$$

where we have used $d \geq 2$. In particular, we have that $p'_\psi < p_\psi$ and, by defining p_r by $1/p_r := 1/p_\psi + 1/p_{\rm vi}$, one has that $p_r > 1$. Again by Hölder's Inequality, this entails that

$$\|\nabla \dot y_{\rm vi}\|_{L^{p_r}(\Omega)} \leq \|\nabla \dot y_{\rm vi}(\nabla y_{\rm vi})^{-1}\|_{L^{p_\psi}(\Omega)}\|\nabla y_{\rm vi}\|_{L^{p_{\rm vi}}(\Omega)} \leq c\Psi(y_{\rm vi},\dot y_{\rm vi})\left(\mathcal{W}(y_{\rm el},y_{\rm vi})^{1/p_{\rm vi}}+1\right).$$

In particular, $\nabla \dot y_{\rm vi}$ belongs to $L^{p_r}(\Omega;\mathbb{R}^{d\times d})$ with $p_r > 1$ whenever energy and dissipation are finite.

Here and in the following, the symbol c denotes a generic positive constant, possibly depending on data and changing from line to line.

2.4.2 Existence results

Before presenting the statements of our main results, we make the notion of solution to the problem precise. To this aim, let $\Pi_\tau := \{0 = t_0 < t_1 < \ldots < t_N = T\}$ denote the uniform partition of the time interval $[0,T]$ with time step $t_i - t_{i-1} = \tau > 0$ for every $i = 1,\ldots,N_\tau := T/\tau$. From now on, let $(y_{\rm el,0},y_{\rm vi,0})$ be a compatible initial condition, i.e.,

$$(y_{\rm el,0},y_{\rm vi,0}) \in \mathcal{A} \text{ with } \mathcal{E}(0,y_{\rm el,0},y_{\rm vi,0}) < \infty. \tag{2.4.8}$$

Given $(y_{\mathrm{el}}^0, y_{\mathrm{vi}}^0) := (y_{\mathrm{el},0}, y_{\mathrm{vi},0})$, for all $i = 1, ..., N$ we define the incremental minimization problems

$$\min_{(y_{\mathrm{el}}, y_{\mathrm{vi}}) \in \mathcal{A}} \left\{ \mathcal{E}(t_i, y_{\mathrm{el}}, y_{\mathrm{vi}}) + \tau \Psi \left(y_{\mathrm{vi}}^{i-1}, \frac{y_{\mathrm{vi}} - y_{\mathrm{vi}}^{i-1}}{\tau} \right) \right\}. \tag{2.4.9}$$

We call a sequence of minimizers $(y_{\mathrm{el}}^i, y_{\mathrm{vi}}^i)_{i=0}^N$ of (2.4.9) an *incremental solution* of the problem corresponding to time step τ.

Note that incremental solutions exist. In particular, we have the following.

Proposition 2.4.1 (Existence of incremental solutions)**.** *Under assumptions (E1), (E2), (E3), (E4), and (E5) of Section 2.4.1 and* (2.4.8) *the incremental minimization problem* (2.4.9) *admits an incremental solution* $(y_{\mathrm{el}}^i, y_{\mathrm{vi}}^i)_{i=0}^N \subset \mathcal{A}$.

The proof of Proposition 2.4.1 is given in Section 2.5.

In the following, we make use of the following notation for interpolations. Given a vector $(u_0, ..., u_N)$, we define its backward-constant interpolant $\overline{u}_\tau$, its forward-constant interpolant $\underline{u}_\tau$, and its piecewise-affine interpolant $\hat{u}_\tau$ on the partition Π_τ as

$$\begin{aligned}
&\overline{u}_\tau(0) := u_0, \quad \overline{u}_\tau(t) := u_i && \text{if } t \in (t_{i-1}, t_i] \text{ for } i = 1, \ldots, N,\\
&\underline{u}_\tau(T) := u_N, \quad \underline{u}_\tau(t) := u_{i-1} && \text{if } t \in [t_{i-1}, t_i) \text{ for } i = 1, \ldots, N,\\
&\hat{u}_\tau(0) := u_0, \quad \hat{u}_\tau(t) := \frac{u_i - u_{i-1}}{t_i - t_{i-1}}(t - t_{i-1}) + u_{i-1} && \text{if } t \in (t_{i-1}, t_i] \text{ for } i = 1, \ldots, N.
\end{aligned}$$

We are now in the position of introducing our notion of solution to the large-strain Poynting-Thomson model.

Definition 2.4.1 (Approximable solution)**.** *We call* $(y_{\mathrm{el}}, y_{\mathrm{vi}}) : [0, T] \to \mathcal{A}$ *an* approximable solution *if there exist a sequence of uniform partitions of the interval* $[0, T]$ *with mesh size* $\tau \to 0$*, corresponding incremental solutions* $(y_{\mathrm{el}}^i, y_{\mathrm{vi}}^i)_{i=0}^N$*, and a nondecreasing function* $\delta : [0, T] \to [0, \infty)$ *such that, for every* $0 \le s \le t \le T$*,*

Approximation:

$$(\overline{y}_{el,\tau}(t), \overline{y}_{vi,\tau}(t)) \rightharpoonup (y_{\mathrm{el}}(t), y_{vi}(t)) \text{ in } W^{1,p_{\mathrm{el}}}_{loc}(y_{\mathrm{vi}}(t, \Omega); \mathbb{R}^d) \times W^{1,p_{\mathrm{vi}}}(\Omega; \mathbb{R}^d),$$

$$\int_0^t \Psi \left(\underline{y}_{\mathrm{vi},\tau}, \dot{\hat{y}}_{\mathrm{vi},\tau} \right) \to \delta(t),$$

$$\int_s^t \Psi \left(\underline{y}_{\mathrm{vi},\tau}, \dot{\hat{y}}_{\mathrm{vi},\tau} \right) \le \delta(t) - \delta(s),$$

Energy inequality:

$$\mathcal{E}(t, y_{\mathrm{el}}, y_{\mathrm{vi}}) + p_\psi \delta(t) \le \mathcal{E}(0, y_{\mathrm{el},0}, y_{\mathrm{vi},0}) - \int_0^t \langle \dot{\ell}(s), y \rangle, \tag{2.4.10}$$

Semistability:

$$\mathcal{E}\left(t, y_{\mathrm{el}}(t), y_{\mathrm{vi}}(t)\right) \le \mathcal{E}\left(t, \tilde{y}_{\mathrm{el}}, y_{\mathrm{vi}}(t)\right) \quad \forall \tilde{y}_{\mathrm{el}} \text{ with } (\tilde{y}_{\mathrm{el}}, y_{\mathrm{vi}}(t)) \in \mathcal{A}. \tag{2.4.11}$$

Our first main result concerns the existence of approximable solutions.

Theorem 2.4.1 (Existence of approximable solutions)**.** *Under the assumptions (E1), (E2), (E3), (E4), (E5), and (E6) of Section 2.4.1 and* (2.4.8) *there exists an approximable solution* $(y_{\mathrm{el}}, y_{\mathrm{vi}}) : [0, T] \to \mathcal{A}$.

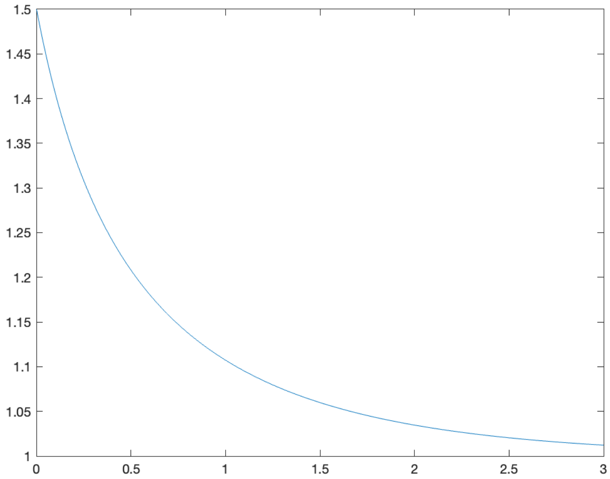


Figure 2.2: Evolution of the viscous strain $t \in [0,3] \mapsto F_{\rm vi}(t)$ in the limit $\tau \to 0$ from problem (2.4.12), starting from $F^0_{\rm vi} = 1.5$.

The proof of Theorem 2.4.1 is detailed in Section 2.6.

As already mentioned in the introduction, the fact that solutions are *approximable* ensures that viscous evolution actually occurs, even in absence of applied loads. We show this fact by resorting to the simplest, scalar model at a single material point. We consider the energy densities, the dissipation to be quadratic, and that no loading is present. More precisely, we let $F \in \mathbb{R}$ and $F_{\rm vi} > 0$ represent the total and viscous (scalar) strains, respectively, we define $W_{\rm el}(F_{\rm el}) = W_{\rm el}(FF^{-1}_{\rm vi}) := \frac{1}{2}|FF^{-1}_{\rm vi} - 1|^2$, $W_{\rm vi}(F_{\rm vi}) := \frac{1}{2}|F_{\rm vi} - 1|^2$, $\psi(\dot F_{\rm vi}F^{-1}_{\rm vi}) := \frac{1}{2}|\dot F_{\rm vi}F^{-1}_{\rm vi}|^2$, and we let $\ell(t) \equiv 0$ for every $t \in [0,T]$. In this setting, the discrete incremental problem (2.4.9) is specified as

$$\min_{F\in\mathbb{R}, F_{\rm vi}>0} \left(\frac{1}{2}\left|FF^{-1}_{\rm vi} - 1\right|^2 + \frac{1}{2}|F_{\rm vi} - 1|^2 + \frac{1}{2\tau}\left|(F_{\rm vi} - F^{i-1}_{\rm vi})(F^{i-1}_{\rm vi})^{-1}\right|^2 \right) \qquad \text{for } i = 1,\dots,N, \tag{2.4.12}$$

Take now initial values $(F^0, F^0_{\rm vi})$ with $F^0_{\rm vi} \neq 1$, so that some nonvanishing viscous stress present at time 0. In this case, it is easy to check that the constant in time solution $(F^0, F^0_{\rm vi})$ satisfies the energy inequality and semistability, but it is not approximable. This implies that the viscous strain $F_{\rm vi}$ corresponding to an approximable solution must evolve with time, see Figure 2.2. In this simple setting, asking the solution of the continuous problem to be approximable indeed implies uniqueness, as all discrete trajectories converge to the unique solution of the limiting differential problem.

2.4.3 Assumptions for the linearization theory

In addition to the assumptions stated in Section 2.4.1, we will require the following conditions in order to prove the linearization result.

On the stored elastic energy density $W_{\rm el}$ we assume that:

(L1) $W_{\rm el}$ is locally Lipschitz;

(L2) $W_{\rm el}$ satisfies the growth condition

$$W_{\rm el}(A) \geq c_4 \,\mathrm{dist}^2(A, \mathrm{SO}(d)) \tag{2.4.13}$$

for some $c_4 > 0$;

(L3) there exists a positive definite tensor $\mathbb{C}_{\rm el}$ such that, for every $\delta > 0$, there exists $c_{\rm el}(\delta) > 0$ satisfying

$$\big|W_{\rm el}(I+A) - |A|^2_{\mathbb{C}_{\rm el}}\big| \le \delta |A|^2_{\mathbb{C}_{\rm el}} \quad \text{for every } A \in B^{\mathbb{R}^{d\times d}}_{c_{\rm el}(\delta)}(0). \tag{2.4.14}$$

In particular these conditions imply that $\mathbb{C}_{\rm el}$ is symmetric and

$$c_4 |A^{\,\rm sym}|^2 \le |A|^2_{\mathbb{C}_{\rm el}} \quad \text{for every } A \in \mathbb{R}^{d\times d}.$$

We can also equivalently state inequality (2.4.14) as follows:

$$(1-\delta)|A|^2_{\mathbb{C}_{\rm el}} \le W_{\rm el}(I+A) \le (1+\delta)|A|^2_{\mathbb{C}_{\rm el}} \quad \text{for every } A \in B^{\mathbb{R}^{d\times d}}_{c_{\rm el}(\delta)}(0). \tag{2.4.15}$$

Concerning the viscous stored energy density $W_{\rm vi}$ we ask that

(L4)

$$W_{\rm vi}(A) = \begin{cases} \widetilde{W}_{\rm vi}(A) & \text{if } A \in K \\ \infty & \text{otherwise,} \end{cases}$$

where $K \subset\subset \mathrm{SL}(d)$ contains a neighbourhood of the identity;

(L5) $\widetilde{W}_{\rm vi}$ is locally Lipschitz continuous in a neighbourhood of the identity and

$$\widetilde{W}_{\rm vi}(I+A) \ge c_5 |A|^2 \quad \text{for every } A \in \mathbb{R}^{d\times d} \text{ with } I + A \in K \tag{2.4.16}$$

for some $c_5 > 0$;

(L6) there exists a positive definite tensor $\mathbb{C}_{\rm vi}$ such that, for every $\delta > 0$, there exists $c_{\rm vi}(\delta) > 0$ satisfying

$$\Big|\widetilde{W}_{\rm vi}(I+A) - |A|^2_{\mathbb{C}_{\rm vi}}\Big| \le \delta |A|^2_{\mathbb{C}_{\rm vi}} \quad \text{for every } A \in B^{\mathbb{R}^{d\times d}}_{c_{\rm vi}(\delta)}(0),$$

or, equivalently,

$$(1-\delta)|A|^2_{\mathbb{C}_{\rm vi}} \le \widetilde{W}_{\rm vi}(I+A) \le (1+\delta)|A|^2_{\mathbb{C}_{\rm vi}} \quad \text{for every } A \in B^{\mathbb{R}^{d\times d}}_{c_{\rm vi}(\delta)}(0). \tag{2.4.17}$$

As above we have that

$$c_5 |A^{\,\rm sym}|^2 \le |A|^2_{\mathbb{C}_{\rm vi}} \quad \text{for every } A \in \mathbb{R}^{d\times d}.$$

Moreover, there exists a constant $c_K > 0$ (depending only on the compact set K) such that

$$|A| + |A^{-1}| \le c_K \quad \text{for every } A \in K \tag{2.4.18}$$

and

$$|A - I| \ge \frac{1}{c_K} \quad \text{for every } A \in \mathrm{SL}(d) \setminus K.$$

These last two inequalities will provide L^∞-bounds on the terms $\varepsilon\nabla v$ and $(I+\varepsilon\nabla v)^{-1}$ later on. Note however that the effect of the constraint K will disappear as $\varepsilon \to 0$. In particular, the limiting linearized problem is independent of K.

On the forcing term ℓ^0 we assume that

(L7) $\ell^0 \in W^{1,1}\big(0,T;(H^1(\Omega;\mathbb{R}^d))^*\big)$.

Finally, on the dissipation density ψ we assume that

(L8) ψ satisfies the growth condition

$$\psi(A) \geq c_6|A|^2 \quad \text{for every } A \in \mathbb{R}^{d\times d} \tag{2.4.19}$$

for some $c_6 > 0$;

(L9) there exists a positive definite tensor $\mathbb{D}$ such that, for every $\delta > 0$, there exists $c_\psi(\delta) > 0$ satisfying

$$\big|\psi(A) - |A|^2_{\mathbb{D}}\big| \leq \delta|A|^2_{\mathbb{D}} \quad \text{for every } A \in B^{\mathbb{R}^{d\times d}}_{c_\psi(\delta)}(0); \tag{2.4.20}$$

(L10) ψ is positively 2-homogeneous, i.e.,

$$\psi(\lambda A) = \lambda^2\psi(A) \quad \text{for every } A \in \mathbb{R}^{d\times d}, \lambda \geq 0.$$

The specification $p_\psi = 2$ of assumption (L10) (compare with the more general $p_\psi \geq 2$ from (E6)) is just needed in the linearization setting to recover the linearized energy inequality (2.4.24) below.

2.4.4 Linearization result

Before moving on, let us reformulate the setting and the existence results of Proposition 2.4.1 and Theorem 2.4.1 in terms of the linearization variables u and v. For all $\varepsilon > 0$ fixed, the admissible set $\mathcal{A}$ is equivalently rewritten as

$$\widetilde{\mathcal{A}}_\varepsilon := \Bigg\{ (u,v) \in W^{1,q}(\Omega;\mathbb{R}^d) \times W^{1,p_{\text{vi}}}(\Omega;\mathbb{R}^d) \;\Bigg|\; u = 0 \text{ on } \Gamma_D,\ \det(I+\varepsilon\nabla v) = 1,$$
$$\int_\Omega v \,\mathrm{d}X = 0,\ |\Omega| = |(\mathrm{id}+\varepsilon v)(\Omega)|,\ (\mathrm{id}+\varepsilon v)(\Omega) \in \mathcal{J}_{\eta_1,\eta_2} \Bigg\},$$

where we recall that Ω is chosen to be such that $\int_\Omega \mathrm{X}\,\mathrm{d}X = 0$ so that

$$0 \overset{(2.3.2)}{=} \int_\Omega y_{\text{vi}}\mathrm{d}X = \int_\Omega (\mathrm{id}+\varepsilon v)\mathrm{d}X = \varepsilon\int_\Omega v\mathrm{d}X.$$

We use the following notation for the rescaled energies and dissipation

$$\mathcal{W}^\varepsilon_{\text{el}}(u,v) := \frac{1}{\varepsilon^2}\int_\Omega W_{\text{el}}\left((I+\varepsilon\nabla u)(I+\varepsilon\nabla v)^{-1}\right),$$
$$\mathcal{W}^\varepsilon_{\text{vi}}(v) := \frac{1}{\varepsilon^2}\int_\Omega W_{\text{vi}}(I+\varepsilon\nabla v),$$
$$\Psi^\varepsilon(v,\dot{v}) := \frac{1}{\varepsilon^2}\int_\Omega \psi\left(\varepsilon\nabla\dot{v}(I+\varepsilon\nabla v)^{-1}\right).$$

Their corresponding linearized counterparts read

$$\mathcal{W}^0_{\text{el}}(u,v) := \frac{1}{2}\int_\Omega \nabla(u-v) : \mathbb{C}_{\text{el}}\nabla(u-v),$$
$$\mathcal{W}^0_{\text{vi}}(v) := \frac{1}{2}\int_\Omega \nabla v : \mathbb{C}_{\text{vi}}\nabla v,$$
$$\Psi^0(\dot{v}) := \frac{1}{2}\int_\Omega \mathbb{D}\nabla\dot{v} : \nabla\dot{v}.$$

We also define for brevity

$$\mathcal{E}^\varepsilon(u,v) := \mathcal{W}^\varepsilon_{\rm vi}(v) + \mathcal{W}^\varepsilon_{\rm el}(u,v) - \langle \ell^0, u\rangle \quad \text{and} \quad \mathcal{E}^0(u,v) := \mathcal{W}^0_{\rm vi}(v) + \mathcal{W}^0_{\rm el}(u,v) - \langle \ell^0, u\rangle.$$

Finally, let $(u^0_\varepsilon, v^0_\varepsilon) \in \widetilde{\mathcal{A}}_\varepsilon$ be a *well-prepared* sequence of initial data, namely

$$(u^0_\varepsilon, v^0_\varepsilon) \rightharpoonup (u^0, v^0) \text{ in } H^1(\Omega)\times H^1(\Omega) \quad \text{and} \quad \lim_{\varepsilon\to 0} \mathcal{E}^\varepsilon(u^0_\varepsilon, v^0_\varepsilon) = \mathcal{E}^0(u^0, v^0) \tag{2.4.21}$$

Proposition 2.4.1 and Theorem 2.4.1 can therefore be rewritten in terms of the new variables (u,v) and in the presence of the rescaling prefactor $1/\varepsilon^2$ as follows.

Corollary 2.4.1 (Existence in terms of $(u_\varepsilon, v_\varepsilon)$)**.** *Under the assumptions (E1), (E2), (E3), (E4), (E5), and (L10) of Section 2.4.1 and* (2.4.21) *for every $\varepsilon > 0$ there exist a sequence of partitions $(\Pi_{\tau^\varepsilon})_{\tau^\varepsilon}$ of the interval $[0,T]$ with mesh size $\tau^\varepsilon \to 0$ and functions $(u_\varepsilon, v_\varepsilon) : [0,T] \to \widetilde{\mathcal{A}}_\varepsilon$ such that for every $t \in [0,T]$*

Approximation:

$$(\overline{u}_{\tau^\varepsilon}(t), \overline{v}_{\tau^\varepsilon}(t)) \rightharpoonup (u_\varepsilon(t), v_\varepsilon(t)) \textit{ in } W^{1,q}(\Omega;\mathbb{R}^d)\times W^{1,p_{\rm vi}}(\Omega;\mathbb{R}^d)$$

Energy inequality:

$$\begin{aligned}&\mathcal{W}^\varepsilon_{\rm vi}(v_\varepsilon(t)) + \mathcal{W}^\varepsilon_{\rm el}(u_\varepsilon(t), v_\varepsilon(t)) - \langle \ell^0, u_\varepsilon(t)\rangle + 2\int_0^t \Psi^\varepsilon(v_\varepsilon, \dot v_\varepsilon)\\ &\quad \le \mathcal{W}^\varepsilon_{\rm vi}(v^0_\varepsilon) + \mathcal{W}^\varepsilon_{\rm el}(u^0_\varepsilon, v^0_\varepsilon) - \int_0^t \langle \dot\ell^0, u_\varepsilon\rangle \end{aligned}\tag{2.4.22}$$

Semistability:

$$\begin{aligned}&\mathcal{W}^\varepsilon_{\rm el}(u_\varepsilon(t), v_\varepsilon(t)) - \langle \ell^\varepsilon(t), u_\varepsilon(t)\rangle \le \mathcal{W}^\varepsilon_{\rm el}(\tilde u_\varepsilon, v_\varepsilon(t)) - \langle \ell^\varepsilon(t), \tilde u_\varepsilon(t)\rangle\\ &\qquad \forall \tilde u_\varepsilon \textit{ with } (\tilde u_\varepsilon, v_\varepsilon(t)) \in \widetilde{\mathcal{A}}_\varepsilon. \end{aligned}\tag{2.4.23}$$

In the following result, we show that a sequence $(u_\varepsilon, v_\varepsilon)_\varepsilon$ of approximable solutions at level ε converges weakly to (u,v) satisfying the linearized energy and the linearized semistability inequalities.

Theorem 2.4.2 (Linearization)**.** *For every $\varepsilon > 0$ let $(u_\varepsilon, v_\varepsilon)$ be an approximable solutions given as in Corollary 2.4.1. Then, under the assumptions (L1), (L2), (L3), (L4), (L5), (L6), (L7), (L8), (L9), and (L10) of Section 2.4.3 and* (2.4.21) *there exist functions $(u,v) : [0,T] \to H^1_{\Gamma_D}(\Omega;\mathbb{R}^d) \times H^1_\sharp(\Omega;\mathbb{R}^d)$ such that, for every $t \in [0,T]$, up to a not relabeled subsequence,*

$$\begin{aligned} u_\varepsilon(t) \rightharpoonup u(t),\ v_\varepsilon(t) \rightharpoonup v(t) &\ \textit{ weakly in } H^1(\Omega;\mathbb{R}^d),\\ \nabla \dot v_\varepsilon(t) \rightharpoonup \nabla \dot v(t) &\ \textit{ weakly in } L^2(\Omega;\mathbb{R}^{d\times d}). \end{aligned}$$

Moreover, for every $t \in [0,T]$, we have

Linearized energy inequality:

$$\begin{aligned}&\mathcal{W}^0_{\rm vi}(v(t)) + \mathcal{W}^0_{\rm el}(u(t), v(t)) - \langle \ell^0(t), u(t)\rangle + 2\int_0^t \Psi^0(\dot v(s))\\ &\quad \le \mathcal{W}^0_{\rm vi}(v^0) + \mathcal{W}^0_{\rm el}(u^0, v^0) - \langle \ell^0(0), u^0\rangle - \int_0^t \langle \dot\ell^0(s), u(s)\rangle, \end{aligned}\tag{2.4.24}$$

Linearized semistability:

$$\mathcal{W}^0_{\rm el}(u(t), v(t)) - \langle \ell^0(t), u(t)\rangle \le \mathcal{W}^0_{\rm el}(\hat u, v(t)) - \langle \ell^0(t), \hat u\rangle \quad \forall \hat u \in H^1_{\Gamma_D}(\Omega;\mathbb{R}^d). \tag{2.4.25}$$

The proof of Theorem 2.4.2 is to be found in Section 2.7 below.

Before moving on, let us remark that the linearized energy inequality (2.4.24) and the linearized semistability (2.4.25) cannot be expected to uniquely determine solutions of the linearized problem (2.2.1)-(2.2.2). On the other hand, inequalities (2.4.24)-(2.4.25) would uniquely characterize solutions (u, v) to (2.2.1)-(2.2.2) if in addition one assumes that (u, v) are *approximable*, namely, are limits of time discretizations of (2.2.1)-(2.2.2). Although the trajectories (u, v) are limits of approximable solutions $(u_\varepsilon, v_\varepsilon)$, we are not able to prove that (u, v) are approximable themselves, for the property of being approximable seems not guaranteed to pass to the linearization limit.

2.5 Time-discretization scheme: Proof of Proposition 2.4.1

To start with, notice that the infimum in the incremental problems (2.4.9) is finite for every $i = 1, \dots, N_\tau$. Indeed, since the initial condition satisfies $\mathcal{E}(0, y_{\rm el}^0, y_{\rm vi}^0) < \infty$, by arguing by induction and choosing $(y_{\rm el}, y_{\rm vi}) = (y_{\rm el}^{i-1}, y_{\rm vi}^{i-1})$, we get that

$$\mathcal{E}(t_i, y_{\rm el}, y_{\rm vi}) + \tau\Psi\left(y_{\rm vi}^{i-1}, \frac{y_{\rm vi} - y_{\rm vi}^{i-1}}{\tau}\right) = \mathcal{E}(t_i, y_{\rm el}^{i-1}, y_{\rm vi}^{i-1}) < \infty.$$

Fix now $1 \le i \le N$ and let $(y_{{\rm el},m}^i, y_{{\rm vi},m}^i)_{m\in\mathbb{N}} = (y_{{\rm el},m}, y_{{\rm vi},m})_{m\in\mathbb{N}} \subset \mathcal{A}$ be an infimizing sequence for problem (2.4.9) at time step i.

2.5.1 Coercivity

Let us first show that $(y_{{\rm el},m}, y_{{\rm vi},m})_{m\in\mathbb{N}}$ is bounded in $W^{1,p_{\rm el}}(y_{{\rm vi},m}(\Omega);\mathbb{R}^d) \times W^{1,p_{\rm vi}}(\Omega;\mathbb{R}^d)$. This requires some care since $y_{{\rm el},m}$ is defined on the moving domain $y_{{\rm vi},m}(\Omega)$. Since the infimum is finite, we have by (2.4.1) and (2.4.2)

$$c_1\int_{y_{{\rm vi},m}(\Omega)} |\nabla y_{{\rm el},m}|^{p_{\rm el}} + c_2\int_\Omega |\nabla y_{{\rm vi},m}|^{p_{\rm vi}} - \frac{|\Omega|}{c_2} \le \mathcal{W}(y_{{\rm el},m}, y_{{\rm vi},m}) \le c - \langle \ell(t_i), y_m\rangle$$

where we have posed $y_m := y_{{\rm el},m}\circ y_{{\rm vi},m}$. The loading term can be controlled as follows

$$\begin{aligned}|\langle \ell(t_i), y_m\rangle| &\le \|\ell(t_i)\|_{(W^{1,q}(\Omega))^*}\|y_m\|_{W^{1,q}(\Omega)} \le c\|\ell(t_i)\|_{(W^{1,q}(\Omega))^*}\|\nabla y_m\|_{L^q(\Omega)}\\ &\overset{\text{Hölder}}{\le} c\|\ell(t_i)\|_{(W^{1,q}(\Omega))^*}\|\nabla y_{{\rm el},m}\|_{L^{p_{\rm el}}(y_{{\rm vi},m}(\Omega))}\|\nabla y_{{\rm vi},m}\|_{L^{p_{\rm vi}}(\Omega)}\\ &\overset{\text{Young}}{\le} c\|\ell(t_i)\|^{1/q'}_{(W^{1,q}(\Omega))^*} + \frac{c_1}{2}\|\nabla y_{{\rm el},m}\|^{p_{\rm el}}_{L^{p_{\rm el}}(y_{{\rm vi},m}(\Omega))} + \frac{c_2}{2}\|\nabla y_{{\rm vi},m}\|^{p_{\rm vi}}_{L^{p_{\rm vi}}(\Omega)}.\end{aligned}$$

This entails that

$$\|\nabla y_{{\rm el},m}\|^{p_{\rm el}}_{L^{p_{\rm el}}(y_{{\rm vi},m}(\Omega))} + \|\nabla y_{{\rm vi},m}\|^{p_{\rm vi}}_{L^{p_{\rm vi}}(\Omega)} \le c,$$

which in turn guarantees that

$$\|\nabla y_m\|^q_{L^q(\Omega)} \le c.$$

Now, using the growth condition (2.4.2) and the Poincaré-Wirtinger inequality, recalling that $y_{\rm vi}$ has zero mean, we have that

$$\|y_{{\rm vi},m}\|_{W^{1,p_{\rm vi}}(\Omega)} \le c.$$

Recalling that y_m satisfies the Dirichlet boundary condition (2.3.3), by the Poincaré inequality we obtain

$$\|y_m\|_{W^{1,q}(\Omega)} \le c.$$

A change of variables ensures that

$$\int_{y_{\mathrm{vi},m}(\Omega)} |y_{\mathrm{el},m}|^q \mathrm{d}\xi = \int_\Omega |y_m|^q \mathrm{d}X \le c$$

so that $\|y_{\mathrm{el},m}\|_{L^q(y_{\mathrm{vi},m}(\Omega))} \le c$, as well. Again the Poincaré inequality guarantees that

$$\|y_{\mathrm{el},m}\|_{W^{1,p_{\mathrm{el}}}(y_{\mathrm{vi},m}(\Omega))} \le c. \tag{2.5.1}$$

Up to a not relabeled subsequence we hence have that

$$\begin{aligned} y_{\mathrm{vi},m} = y^i_{\mathrm{vi},m} &\rightharpoonup y^i_{\mathrm{vi}} \quad \text{in } W^{1,p_{\mathrm{vi}}}(\Omega;\mathbb{R}^d) \\ y_m = y^i_m = y^i_{\mathrm{vi},m} \circ y^i_{\mathrm{el},m} &\rightharpoonup y^i \quad \text{in } W^{1,q}(\Omega;\mathbb{R}^d). \end{aligned} \tag{2.5.2}$$

We now want to extract a converging subsequence from the elastic deformations $y_{\mathrm{el},m}$, which are however defined on the moving domains $y_{\mathrm{vi},m}(\Omega)$. Consider the trivial extensions $\overline{y_{\mathrm{el},m}}$ and $\overline{\nabla y_{\mathrm{el},m}}$ on the whole $\mathbb{R}^d$ by setting $y_{\mathrm{el},m}$ and $\nabla y_{\mathrm{el},m}$ to be zero on $\mathbb{R}^d \setminus y_{\mathrm{vi},m}(\Omega)$, respectively. Recalling the bound (2.5.1), we have (up to a subsequence)

$$\begin{aligned} \overline{y_{\mathrm{el},m}} &\rightharpoonup y^i_{\mathrm{el}} \quad \text{in } L^{p_{\mathrm{el}}}(\mathbb{R}^d;\mathbb{R}^d) \\ \overline{\nabla y_{\mathrm{el},m}} &\rightharpoonup G \quad \text{in } L^{p_{\mathrm{el}}}(\mathbb{R}^d;\mathbb{R}^{d\times d}). \end{aligned} \tag{2.5.3}$$

We want to show that $G = \nabla y^i_{\mathrm{el}}$ on the limiting set $y^i_{\mathrm{vi}}(\Omega)$. By Sobolev embedding, possibly by extracting a further subsequence, we have that $y_{\mathrm{vi},m} \to y^i_{\mathrm{vi}}$ uniformly. Letting $\omega \subset\subset y^i_{\mathrm{vi}}(\Omega)$, for m large enough we eventually have that $\omega \subset\subset y_{\mathrm{vi},m}(\Omega)$. By uniqueness of the limit we have $\overline{y_{\mathrm{el},m}} \rightharpoonup y^i_{\mathrm{el}}$ in $L^{p_{\mathrm{el}}}(\omega;\mathbb{R}^d)$ and $\overline{\nabla y_{\mathrm{el},m}} \rightharpoonup \nabla y^i_{\mathrm{el}} = G$ in $L^{p_{\mathrm{el}}}(\omega;\mathbb{R}^{d\times d})$. Hence $G = \nabla y^i_{\mathrm{el}}$ in every $\omega \subset\subset y^i_{\mathrm{vi}}(\Omega)$. An exhaustion argument ensures that $G = \nabla y^i_{\mathrm{el}}$ in $y^i_{\mathrm{vi}}(\Omega)$.

2.5.2 Closure of the set of admissible deformations

Let us now check that the weak limit $(y^i_{\mathrm{el}}, y^i_{\mathrm{vi}})$ belongs to the admissible set $\mathcal{A}$. First of all, since $p_{\mathrm{vi}} > d$ we have that

$$1 = \det \nabla y^i_{\mathrm{vi},m} \rightharpoonup \det \nabla y^i_{\mathrm{vi}} \quad \text{in } L^{p_{\mathrm{vi}}/d}(\Omega)$$

and hence $\det \nabla y^i_{\mathrm{vi}} = 1$ almost everywhere. On the other hand, [50, Lemmas 3.1-3.2] imply that $y^i_{\mathrm{vi}}(\Omega) \in \mathcal{J}_{\eta_1,\eta_2}$. By the linearity of the mean and trace operators and by the weak convergence of $y^i_{\mathrm{vi},m}$, we find $\int_\Omega y^i_{\mathrm{vi}} \mathrm{d}X = 0$ and $y^i = \mathrm{id}$ on Γ_D. Moreover, by [39, Lemma 5.2 (i)] we have that

$$|y^i_{\mathrm{vi},m}(\Omega) \Delta y^i_{\mathrm{vi}}(\Omega)| \to 0,$$

where the symbol Δ denotes the symmetric difference, and, for every $\omega \subset \Omega$, that $\mathbb{1}_{y^i_{\mathrm{vi},m}(\omega)} \to \mathbb{1}_{y^i_{\mathrm{vi}}(\omega)}$ almost everywhere in Ω. This implies that y^i_{vi} satisfies the Ciarlet-Nečas condition, since

$$|\Omega| = |y^i_{\mathrm{vi},m}(\Omega)| \to |y^i_{\mathrm{vi}}(\Omega)|.$$

It remains to show that $y^i = y^i_{\mathrm{el}} \circ y^i_{\mathrm{vi}}$. Let us take any measurable $\omega \subset \Omega$ and consider, by changing variables,

$$\begin{aligned} \int_\omega y^i(X)\mathrm{d}X \leftarrow \int_\omega y_m(X)\mathrm{d}X &= \int_\omega y_{\mathrm{el},m}(y_{\mathrm{vi},m}(X))\mathrm{d}X = \int_{y_{\mathrm{vi},m}(\omega)} y_{\mathrm{el},m}(\xi)\mathrm{d}\xi \\ &= \int_{\mathbb{R}^d} y_{\mathrm{el},m}(\xi)\mathbb{1}_{y_{\mathrm{vi},m}(\omega)}(\xi)\mathrm{d}\xi \to \int_{\mathbb{R}^d} y^i_{\mathrm{el}}(\xi)\mathbb{1}_{y^i_{\mathrm{vi}}(\omega)}(\xi)\mathrm{d}\xi = \int_\omega y^i_{\mathrm{el}}(y^i_{\mathrm{vi}}(X))\mathrm{d}X, \end{aligned}$$

where in the last limit we used the weak convergence of $y_{\mathrm{el},m}$ and the strong convergence of $\mathbb{1}_{y^i_{\mathrm{vi}}(\omega)}$. Since $\omega \subset \Omega$ is arbitrary we conclude that $y^i = y^i_{\mathrm{el}} \circ y^i_{\mathrm{vi}}$. In particular, we have that $(y^i_{\mathrm{el}}, y^i_{\mathrm{vi}}) \in \mathcal{A}$.

2.5.3 Weak lower semicontinuity

We aim to show that the functional in (2.4.9) is weakly lower semicontinuous with respect to the above convergences.

By polyconvexity of the viscous energy density $W_{\rm vi}$ and (2.5.2), we have

$$\int_\Omega W_{\rm vi}(\nabla y^i_{\rm vi})\mathrm{d}X \le \liminf_{m\to\infty}\int_\Omega W_{\rm vi}(\nabla y_{{\rm vi},m})\mathrm{d}X.$$

For what concerns the dissipation, from the weak convergence of $y_{{\rm vi},m}$ in $W^{1,p_{\rm vi}}(\Omega)$, we also have

$$\frac{\nabla(y_{{\rm vi},m}-y^{i-1}_{\rm vi})}{\tau}(\nabla y^{i-1}_{\rm vi})^{-1} \rightharpoonup \frac{\nabla(y^i_{\rm vi}-y^{i-1}_{\rm vi})}{\tau}(\nabla y^{i-1}_{\rm vi})^{-1} \quad \text{in } L^{p_\psi}(\Omega;\mathbb{R}^{d\times d}).$$

Hence, by the weak lower semicontinuity of Ψ, it follows that

$$\Psi\left(y^{i-1}_{\rm vi},\frac{y^i_{\rm vi}-y^{i-1}_{\rm vi}}{\tau}\right) \le \liminf_{m\to\infty}\Psi\left(y^{i-1}_{\rm vi},\frac{y_{{\rm vi},m}-y^{i-1}_{\rm vi}}{\tau}\right).$$

As the loading term is linear, we have

$$\langle \ell(t_i), y^i\rangle = \lim_{m\to\infty}\langle \ell(t_i), y_m\rangle$$

by weak convergence of y_m.

Finally, for any $\omega \subset\subset y^i_{\rm vi}(\Omega)$ we can treat the elastic energy as follows

$$\int_\omega W_{\rm el}(\nabla y^i_{\rm el})\mathrm{d}\xi \le \liminf_{m\to\infty}\int_\omega W_{\rm el}(\nabla y_{{\rm el},m})\mathrm{d}\xi \overset{(2.4.1)}{\le} \liminf_{m\to\infty}\int_{y_{{\rm vi},m}(\Omega)} W_{\rm el}(\nabla y_{{\rm el},m})\mathrm{d}\xi,$$

where we have used the polyconvexity of $W_{\rm el}$ and convergence (2.5.3). Taking the supremum over $\omega \subset\subset y^i_{\rm vi}(\Omega)$ we conclude via an exhaustion argument that

$$\int_{y_{{\rm vi},m}(\Omega)} W_{\rm el}(\nabla y_{\rm el})\mathrm{d}\xi \le \liminf_{m\to\infty}\int_{y_{{\rm vi},m}(\Omega)} W_{\rm el}(\nabla y_{{\rm el},m})\mathrm{d}\xi.$$

All in all, we have proved that $(y^i_{\rm el}, y^i_{\rm vi}) \in \mathcal{A}$ and

$$\mathcal{E}(t_i, y^i_{\rm el}, y^i_{\rm vi}) + \tau\Psi\left(y^{i-1}_{\rm vi},\frac{y^i_{\rm vi}-y^{i-1}_{\rm vi}}{\tau}\right) = \min_{(y_{\rm el},y_{\rm vi})\in\mathcal{A}}\left\{\mathcal{E}(t_i,y_{\rm el},y_{\rm vi}) + \tau\Psi\left(y^{i-1}_{\rm vi},\frac{y_{\rm vi}-y^{i-1}_{\rm vi}}{\tau}\right)\right\}$$

so that the assertion of Proposition 2.4.1 follows.

2.6 Existence of approximable solutions: Proof of Theorem 2.4.1

We split the proof in subsequent steps. The basic energy estimate and its consequences are presented in Subsection 2.6.1. The energy estimate is then sharpened in Subsection 2.6.2, leading to the discrete energy inequality. By taking limits as the time step τ goes to 0, the time-continuous energy inequality (2.4.10) and the time-continuous semistability (2.4.11) are proved in Subsections 2.6.3 and 2.6.4, respectively.

2.6.1 Energy estimate and its consequences

Let $(y^i_{\rm el} y^i_{\rm vi})_{i=0}^N$ be a solution to (2.4.9). By minimality we have, for every $i = 1, ..., N$,

$$\begin{aligned}\mathcal{E}(t_i, y^i_{\rm el}, y^i_{\rm vi}) + \tau\Psi\left(y^{i-1}_{\rm vi}, \frac{y^i_{\rm vi} - y^{i-1}_{\rm vi}}{\tau}\right) &\le \mathcal{E}(t_i, y^{i-1}_{\rm el}, y^{i-1}_{\rm vi}) \\ &= \mathcal{E}(t_{i-1}, y^{i-1}_{\rm el}, y^{i-1}_{\rm vi}) - \int_{t_{i-1}}^{t_i} \langle \dot\ell, y^{i-1}\rangle.\end{aligned}$$

Summing up over $i = 1, ..., n \le N$ we get

$$\mathcal{E}(t_n, y^n_{\rm el}, y^n_{\rm vi}) + \sum_{i=1}^n \tau\Psi\left(y^{i-1}_{\rm vi}, \frac{y^i_{\rm vi} - y^{i-1}_{\rm vi}}{\tau}\right) \le \mathcal{E}(0, y_{\rm el,0}, y_{\rm vi,0}) - \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \langle \dot\ell, y^{i-1}\rangle. \tag{2.6.1}$$

By using the notation for the interpolants, we have, for all $t \in [0, T]$,

$$\begin{aligned}&\mathcal{E}(\bar t_\tau(t), \overline{y}_{\rm el,\tau}(t), \overline{y}_{\rm vi,\tau}(t)) + \int_0^{\bar t_\tau(t)} \Psi\left(\underline{y}_{\rm vi,\tau}, \dot{\hat y}_{\rm vi,\tau}\right) \le \mathcal{E}(0, y_{\rm el,0}, y_{\rm vi,0}) - \int_0^{\bar t_\tau(t)} \langle \dot\ell, \underline{y}_\tau\rangle \\ &\le \mathcal{E}(0, y_{\rm el,0}, y_{\rm vi,0}) + \int_0^{\bar t_\tau(t)} \|\dot\ell\|_{(W^{1,q}(\Omega))^*} \|\underline{y}_\tau\|_{W^{1,q}(\Omega)} \\ &\overset{\text{Poincaré}}{\le} \mathcal{E}(0, y_{\rm el,0}, y_{\rm vi,0}) + c\int_0^{\bar t_\tau(t)} \|\dot\ell\|_{(W^{1,q}(\Omega))^*} \|\nabla\underline{y}_\tau\|_{L^q(\Omega)} \\ &\le \mathcal{E}(0, y_{\rm el,0}, y_{\rm vi,0}) + c\int_0^{\bar t_\tau} \|\nabla\underline{y}_{\rm el,\tau}\|_{L^{p_{\rm el}}(\underline{y}_{\rm vi,\tau}(t,\Omega))} \|\nabla\underline{y}_{\rm vi,\tau}\|_{L^{p_{\rm vi}}(\Omega)}.\end{aligned}$$

On the other hand, by the growth assumptions (2.4.1), (2.4.2), and (2.4.5), we also have

$$\begin{aligned}\mathcal{E}(\bar t_\tau(t), &\overline{y}_{\rm el,\tau}(t), \overline{y}_{\rm vi,\tau}(t)) + \int_0^{\bar t_\tau(t)} \Psi\left(\underline{y}_{\rm vi,\tau}, \dot{\hat y}_{\rm vi,\tau}\right) \ge c\|\nabla\overline{y}_{\rm vi,\tau}(t)\|^{p_{\rm vi}}_{L^{p_{\rm vi}}(\Omega)} \\ &+ c\|\nabla\overline{y}_{\rm el,\tau}(t)\|^{p_{\rm el}}_{L^{p_{\rm el}}(\overline{y}_{\rm vi,\tau}(t,\Omega))} + c\int_0^{\bar t_\tau(t)} \|\nabla\dot{\hat y}_{\rm vi,\tau}(\nabla\underline{y}_{\rm vi,\tau})^{-1}\|^{p_\psi}_{L^{p_\psi}(\Omega)} - c.\end{aligned} \tag{2.6.2}$$

In particular, combining the two inequalities above we get

$$\begin{aligned}&c\|\nabla\overline{y}_{\rm vi,\tau}(t)\|^{p_{\rm vi}}_{L^{p_{\rm vi}}(\Omega)} + c\|\nabla\overline{y}_{\rm el,\tau}(t)\|^{p_{\rm el}}_{L^{p_{\rm el}}(\overline{y}_{\rm vi,\tau}(t,\Omega))} + \int_0^{\bar t_\tau(t)} \|\nabla\dot{\hat y}_{\rm vi,\tau}(\nabla\underline{y}_{\rm vi,\tau})^{-1}\|^{p_\psi}_{L^{p_\psi}(\Omega)} \\ &\le \mathcal{E}(0, y_{\rm el,0}, y_{\rm vi,0}) + c\int_0^{\bar t_\tau(t)} \|\nabla\underline{y}_{\rm el,\tau}\|_{L^{p_{\rm el}}(\underline{y}_{\rm vi,\tau}(t,\Omega))} \|\nabla\underline{y}_{\rm vi,\tau}\|_{L^{p_{\rm vi}}(\Omega)} + c \\ &\le c + c\int_0^{\bar t_\tau(t)} \left(\|\nabla\underline{y}_{\rm el,\tau}\|^{p_{\rm el}}_{L^{p_{\rm el}}(\underline{y}_{\rm vi,\tau}(\Omega))} + \|\nabla\underline{y}_{\rm vi,\tau}\|^{p_{\rm vi}}_{L^{p_{\rm vi}}(\Omega)}\right).\end{aligned}$$

We can apply the Discrete Gronwall Lemma [51, (C.2.6), p. 534] to find

$$\|\nabla\overline{y}_{\rm vi,\tau}(t)\|_{L^{p_{\rm vi}}(\Omega)} + \|\nabla\overline{y}_{\rm el,\tau}(t)\|_{L^{p_{\rm el}}(\overline{y}_{\rm vi,\tau}(\Omega))} \le c.$$

Thus, for every $t \in [0, T]$ we have

$$\|\overline{y}_{\rm vi,\tau}(t)\|_{W^{1,p_{\rm vi}}(\Omega)} \le c \quad \text{and} \quad \|\nabla\overline{y}_{\rm el,\tau}(t)\|_{L^{p_{\rm el}}(\overline{y}_{\rm vi,\tau}(t,\Omega))} \le c.$$

Then, using Poincaré inequality on the total deformation, we find $\|\overline{y}_\tau(t)\|_{W^{1,q}(\Omega)} \leq c$ and hence, as before, for every $t \in [0,T]$

$$\|\overline{y}_{\mathrm{el},\tau}(t)\|_{W^{1,p_{\mathrm{el}}}(\overline{y}_{\mathrm{vi},\tau}(t,\Omega))} \leq c.$$

Moreover, thanks to (2.6.2), we also have for every $t \in [0,T]$

$$\int_0^{\bar{t}_\tau(t)} \|\nabla \dot{\hat{y}}_{\mathrm{vi},\tau} (\nabla \underline{y}_{\mathrm{vi},\tau})^{-1}\|^{p_\psi}_{L^{p_\psi}(\Omega)} \leq c. \tag{2.6.3}$$

By recalling that $1/p_r = 1/p_\psi + 1/p_{\mathrm{vi}}$ this implies that

$$\int_0^T \|\nabla \dot{\hat{y}}_{\mathrm{vi},\tau}\|_{L^{p_r}(\Omega)} \leq \int_0^T \|\nabla \dot{\hat{y}}_{\mathrm{vi},\tau} (\nabla \underline{y}_{\mathrm{vi},\tau})^{-1}\|_{L^{p_\psi}(\Omega)} \|\nabla \underline{y}_{\mathrm{vi},\tau}\|_{L^{p_{\mathrm{vi}}}(\Omega)} \leq c.$$

2.6.2 Energy inequality, sharp version

In the previous section, we have found the energy estimate (2.6.1), which features the dissipation with a prefactor 1. In order to prove the sharp version of the energy inequality (2.4.10) with the prefactor p_ψ, we need a finer argument, mutated from [2].

First, we introduce some notation. Let

$$\mathcal{V} = \left\{ y_{\mathrm{vi}} \in W^{1,p_{\mathrm{vi}}}(\Omega;\mathbb{R}^d) \;\middle|\; \det \nabla y_{\mathrm{vi}} = 1 \text{ a.e. in } \Omega \right\}$$

and, for all $i = 1, \ldots, N$, define the functionals $\Phi^i : [0,T] \times \mathcal{V} \times \mathcal{A} \to \mathbb{R}$ as

$$\Phi^i(\tau; y_{\mathrm{old}}, y_{\mathrm{el}}, y_{\mathrm{vi}}) := \mathcal{E}(t_i, y_{\mathrm{el}}, y_{\mathrm{vi}}) + \tau \Psi \left(y_{\mathrm{old}}, \frac{y_{\mathrm{vi}} - y_{\mathrm{old}}}{\tau} \right).$$

Recall that, by definition (2.4.4) of Ψ and by the p_ψ-homogeneity (2.4.6), we have

$$\tau \Psi \left(y_{\mathrm{old}}, \frac{y_{\mathrm{vi}} - y_{\mathrm{old}}}{\tau} \right) = \frac{1}{\tau^{p_\psi - 1}} \int_\Omega \psi \left((\nabla y_{\mathrm{old}})^{-1} (\nabla y_{\mathrm{vi}} - \nabla y_{\mathrm{old}}) \right). \tag{2.6.4}$$

For all $(t, y_{\mathrm{old}}) \in [0,T] \times \mathcal{V}$ we also define the minimal value of the latter functional as

$$\phi^i_\tau(y_{\mathrm{old}}) := \inf_{(y_{\mathrm{el}}, y_{\mathrm{vi}}) \in \mathcal{A}} \Phi^i(\tau; y_{\mathrm{old}}, y_{\mathrm{el}}, y_{\mathrm{vi}})$$

and denote the set of minimizers by $J^i_\tau(y_{\mathrm{old}}) := \arg\min\{\Phi^i(\tau; y_{\mathrm{old}}, y_{\mathrm{el}}, y_{\mathrm{vi}}) \,|\, (y_{\mathrm{el}}, y_{\mathrm{vi}}) \in \mathcal{A}\}$, which is nonempty by Proposition 2.4.1. Finally, introduce

$$\Psi^{+,i}_\tau(y_{\mathrm{old}}) := \sup_{(y_{\mathrm{el},\tau}, y_{\mathrm{vi},\tau}) \in J^i_\tau(y_{\mathrm{old}})} \int_\Omega \psi \left((\nabla y_{\mathrm{old}})^{-1} \frac{(\nabla y_{\mathrm{vi},\tau} - \nabla y_{\mathrm{old}})}{\tau} \right),$$

$$\Psi^{-,i}_\tau(y_{\mathrm{old}}) := \inf_{(y_{\mathrm{el},\tau}, y_{\mathrm{vi},\tau}) \in J^i_\tau(y_{\mathrm{old}})} \int_\Omega \psi \left((\nabla y_{\mathrm{old}})^{-1} \frac{(\nabla y_{\mathrm{vi},\tau} - \nabla y_{\mathrm{old}})}{\tau} \right).$$

We start by stating an auxiliary result, providing the continuity property of the map $\tau \mapsto \phi^i_\tau(y_{\mathrm{old}})$ in 0 and the monotonicity of $\tau \mapsto \Psi\left(y_{\mathrm{old}}, y_{\mathrm{vi},\tau} - y_{\mathrm{old}}\right)$.

Lemma 2.6.1. *For every* $i = 1, \dots, N$ *and every* $y_{\text{old}} \in \mathcal{V}$*, we have*

$$\lim_{\tau \searrow 0} \phi^i_\tau(y_{\text{old}}) = \mathcal{E}(t_i, y_{\text{el}}, y_{\text{old}}) \tag{2.6.5}$$

where $y_{\text{el}} \in \arg\min\{\mathcal{E}(t_i, \widetilde{y}_{\text{el}}, y_{\text{old}}) \,|\, \widetilde{y}_{\text{el}} \in W^{1,p_{\text{el}}}(y_{\text{old}}(\Omega); \mathbb{R}^d)\}$.

Moreover, if $0 < \tau_0 < \tau_1$*, then*

$$\Psi\left(y_{\text{old}}, y_{\text{vi},\tau_0} - y_{\text{old}}\right) \le \Psi\left(y_{\text{old}}, y_{\text{vi},\tau_1} - y_{\text{old}}\right) \textit{ for every } (y_{\text{el},\tau_j}, y_{\text{vi},\tau_j}) \in J^i_{\tau_j}(y_{\text{old}}),\ j = 0, 1. \tag{2.6.6}$$

Proof. We start by proving the continuity property of $\tau \mapsto \phi^i_\tau(y_{\text{old}})$. Let $(y_{\text{el},\tau}, y_{\text{vi},\tau}) \in J^i_\tau(y_{\text{old}})$. By the growth condition (2.4.5), the p_ψ-homogeneity (2.4.6), and coercivity, we have

$$\|(\nabla y_{\text{old}})^{-1}(\nabla y_{\text{vi},\tau} - \nabla y_{\text{old}})\|_{L^{p_{\text{vi}}}(\Omega)} \le c\Psi\left(y_{\text{old}}, y_{\text{vi},\tau} - y_{\text{old}}\right) = c\tau^{p_\psi}\Psi\left(y_{\text{old}}, \frac{y_{\text{vi},\tau} - y_{\text{old}}}{\tau}\right) \le c\tau^{p_\psi - 1}.$$

The proves that $\nabla y_{\text{vi},\tau} \to \nabla y_{\text{old}}$ in $L^{p_{\text{vi}}}(\Omega; \mathbb{R}^{d\times d})$ as $\tau \to 0$. Moreover, by (2.5.1), we have $y_{\text{el},\tau} \rightharpoonup y_{\text{el}}$ weakly in $W^{1,p_{\text{el}}}(\Omega; \mathbb{R}^d)$. Thus, by weak lower semicontinuity, we have

$$\lim_{\tau \searrow 0} \phi^i_\tau(y_{\text{old}}) = \lim_{\tau \searrow 0} \Phi^i(\tau; y_{\text{old}}, y_{\text{el},\tau}, y_{\text{vi},\tau}) \ge \liminf_{\tau \searrow 0} \mathcal{E}(t_i, y_{\text{el},\tau}, y_{\text{vi},\tau}) \ge \mathcal{E}(t_i, y_{\text{el}}, y_{\text{old}}).$$

On the other hand, from minimality we get $\mathcal{E}(t_i, y_{\text{el}}, y_{\text{old}}) \ge \phi^i_\tau(y_{\text{old}})$. This implies that

$$\lim_{\tau \searrow 0} \phi^i_\tau(y_{\text{old}}) = \mathcal{E}(t_i, y_{\text{el}}, y_{\text{old}}).$$

The fact that $y_{\text{el}} \in \arg\min\{\mathcal{E}(t_i, \widetilde{y}_{\text{el}}, y_{\text{old}}) \,|\, \widetilde{y}_{\text{el}} \in W^{1,p_{\text{el}}}(y_{\text{old}}(\Omega); \mathbb{R}^d)\}$ follows from minimality since

$$\mathcal{E}(t_i, y_{\text{el}}, y_{\text{old}}) = \lim_{\tau \searrow 0} \phi^i_\tau(y_{\text{old}}) \le \lim_{\tau \searrow 0} \Phi^i(\tau; y_{\text{old}}, \widetilde{y}_{\text{el}}, y_{\text{old}}) = \mathcal{E}(t_i, \widetilde{y}_{\text{el}}, y_{\text{old}})$$

for every $\widetilde{y}_{\text{el}} \in W^{1,p_{\text{el}}}(y_{\text{old}}(\Omega); \mathbb{R}^d)$.

Let us now prove the monotonicity of $\tau \mapsto \Psi\left(y_{\text{old}}, y_{\text{vi},\tau} - y_{\text{old}}\right)$. Let $0 < \tau_0 < \tau_1$ and $y_{\text{el},\tau_j}, y_{\text{vi},\tau_j} \in J^i_{\tau_j}(y_{\text{old}})$, $j = 0, 1$. From minimality, we have that

$$\begin{aligned}
\phi^i_{\tau_0} &= \mathcal{E}(t_i, y_{\text{el},\tau_0}, y_{\text{vi},\tau_0}) + \frac{1}{\tau_0^{p_\psi - 1}}\Psi\left(y_{\text{old}}, y_{\text{el},\tau_0} - y_{\text{old}}\right) \\
&\le \mathcal{E}(t_i, y_{\text{el},\tau_1}, y_{\text{vi},\tau_1}) + \frac{1}{\tau_0^{p_\psi - 1}}\Psi\left(y_{\text{old}}, y_{\text{el},\tau_1} - y_{\text{old}}\right) \\
&= \mathcal{E}(t_i, y_{\text{el},\tau_1}, y_{\text{vi},\tau_1}) + \frac{1}{\tau_1^{p_\psi - 1}}\Psi\left(y_{\text{old}}, y_{\text{el},\tau_1} - y_{\text{old}}\right) + \left(\frac{1}{\tau_0^{p_\psi - 1}} - \frac{1}{\tau_1^{p_\psi - 1}}\right)\Psi\left(y_{\text{old}}, y_{\text{el},\tau_1} - y_{\text{old}}\right) \\
&\le \mathcal{E}(t_i, y_{\text{el},\tau_0}, y_{\text{vi},\tau_0}) + \frac{1}{\tau_1^{p_\psi - 1}}\Psi\left(y_{\text{old}}, y_{\text{el},\tau_0} - y_{\text{old}}\right) + \left(\frac{1}{\tau_0^{p_\psi - 1}} - \frac{1}{\tau_1^{p_\psi - 1}}\right)\Psi\left(y_{\text{old}}, y_{\text{el},\tau_1} - y_{\text{old}}\right).
\end{aligned}$$

This implies that

$$\left(\frac{1}{\tau_0^{p_\psi - 1}} - \frac{1}{\tau_1^{p_\psi - 1}}\right)\Psi\left(y_{\text{old}}, y_{\text{el},\tau_1} - y_{\text{old}}\right) \le \left(\frac{1}{\tau_0^{p_\psi - 1}} - \frac{1}{\tau_1^{p_\psi - 1}}\right)\Psi\left(y_{\text{old}}, y_{\text{el},\tau_1} - y_{\text{old}}\right),$$

which concludes the proof. □

In the following Lemma, we calculate the derivative with respect to τ of the minimal incremental energy ϕ^i_τ and provide a crucial estimate.

Lemma 2.6.2. *For every $y_{\rm old} \in \mathcal{V}$ and $i = 1, \dots N$, the map $\tau \mapsto \phi^i_\tau(y_{\rm old})$ is locally Lipschitz on $(0,1)$. Moreover, we have*

$$\frac{d}{d\tau}\phi^i_\tau(y_{\rm old}) = -(p_\psi - 1)\Psi^{\pm,i}_\tau(y_{\rm old}) \tag{2.6.7}$$

for almost every $\tau \in (0,1)$. In particular, for almost every $\tau \in (0,1)$ we have

$$\tau\Psi\left(y_{\rm old}, \frac{y_{{\rm vi},\tau} - y_{\rm old}}{\tau}\right) + (p_\psi - 1)\int_0^\tau \Psi^{\pm,i}_r(y_{\rm old}){\rm d}r = \mathcal{E}(t_i, y_{\rm el}, y_{\rm old}) - \mathcal{E}(t_i, y_{{\rm el},\tau}, y_{{\rm vi},\tau}) \tag{2.6.8}$$

for every $(y_{{\rm el},\tau}, y_{{\rm vi},\tau}) \in J^i_\tau(y_{\rm old})$, for some $y_{\rm el} = \arg\min\{\mathcal{E}(t_i, \widetilde{y}_{\rm el}, y_{\rm old}) \,|\, \widetilde{y}_{\rm el} \in W^{1,p_{\rm el}}(y_{\rm old}(\Omega);\mathbb{R}^d)\}$.

Proof. For every $\tau_0 \neq \tau_1$ and $(y_{{\rm el},\tau_j}, y_{{\rm vi},\tau_j}) \in J^i_{\tau_j}(y_{\rm old})$, $j = 0,1$, by minimality we have

$$\begin{aligned}\phi_{\tau_0}(y_{\rm old}) - \phi_{\tau_1}(y_{\rm old}) &\le \Phi^i(\tau_0; y_{\rm old}, y_{{\rm el},\tau_1}, y_{{\rm vi},\tau_1}) - \Phi^i(\tau_1; y_{\rm old}, y_{{\rm el},\tau_1}, y_{{\rm vi},\tau_1})\\ &= \frac{1}{\tau_0^{p_\psi-1}}\Psi\left(y_{\rm old}, y_{{\rm vi},\tau_1} - y_{\rm old}\right) - \frac{1}{\tau_1^{p_\psi-1}}\Psi\left(y_{\rm old}, y_{{\rm vi},\tau_1} - y_{\rm old}\right)\\ &= \frac{\tau_1^{p_\psi-1} - \tau_0^{p_\psi-1}}{(\tau_1\tau_0)^{p_\psi-1}}\int_\Omega \psi\left((\nabla y_{\rm old})^{-1}(\nabla y_{{\rm vi},\tau_1} - \nabla y_{\rm old})\right),\end{aligned}$$

where we used (2.6.4). We can perform an analogous calculation for

$$\phi_{\tau_0}(y_{\rm old}) - \phi_{\tau_1}(y_{\rm old}) \ge \Phi^i(\tau_0; y_{\rm old}, y_{{\rm el},\tau_0}, y_{{\rm vi},\tau_0}) - \Phi^i(\tau_1; y_{\rm old}, y_{{\rm el},\tau_0}, y_{{\rm vi},\tau_0})$$

so that, by combining the two above inequalities, for $\tau_0 < \tau_1$ we find

$$\begin{aligned}\frac{\tau_1^{p_\psi-1} - \tau_0^{p_\psi-1}}{(\tau_1\tau_0)^{p_\psi-1}(\tau_1-\tau_0)}\int_\Omega \psi\left((\nabla y_{\rm old})^{-1}(\nabla y_{{\rm vi},\tau_0} - \nabla y_{\rm old})\right) &\le \frac{\phi_{\tau_0}(y_{\rm old}) - \phi_{\tau_1}(y_{\rm old})}{\tau_1 - \tau_0}\\ &\le \frac{\tau_1^{p_\psi-1} - \tau_0^{p_\psi-1}}{(\tau_1\tau_0)^{p_\psi-1}(\tau_1-\tau_0)}\int_\Omega \psi\left((\nabla y_{\rm old})^{-1}(\nabla y_{{\rm vi},\tau_1} - \nabla y_{\rm old})\right).\end{aligned}$$

Taking the supremum over $(y_{{\rm el},\tau_0}, y_{{\rm vi},\tau_0}) \in J^i_{\tau_0}(y_{\rm old})$ in the left-hand side and the infimum over $(y_{{\rm el},\tau_1}, y_{{\rm vi},\tau_1}) \in J^i_{\tau_1}(y_{\rm old})$ in the right hand side, we find

$$\frac{\tau_0(\tau_1^{p_\psi-1} - \tau_0^{p_\psi-1})}{\tau_1^{p_\psi-1}(\tau_1-\tau_0)}\Psi^{+,i}_{\tau_0}(y_{\rm old}) \le \frac{\phi_{\tau_0}(y_{\rm old}) - \phi_{\tau_1}(y_{\rm old})}{\tau_1 - \tau_0} \le \frac{\tau_1(\tau_1^{p_\psi-1} - \tau_0^{p_\psi-1})}{\tau_0^{p_\psi-1}(\tau_1-\tau_0)}\Psi^{-,i}_{\tau_1}(y_{\rm old}),$$

which implies that $\tau \mapsto \phi^i_\tau(y_{\rm old})$ is locally Lipschitz. Then, passing to the limit for $\tau_1 \searrow \tau$ and $\tau_0 \nearrow \tau$, we get (2.6.7).

Integrating (2.6.7) from $\tau_0 > 0$ to τ, we have

$$\phi^i_\tau(y_{\rm old}) - \phi^i_{\tau_0}(y_{\rm old}) = -(p_\psi - 1)\int_{\tau_0}^\tau \Psi^{\pm,i}_r(y_{\rm old}){\rm d}r.$$

Letting $\tau_0 \searrow 0$, recalling (2.6.5), and the definition of $(y_{{\rm el},\tau}, y_{{\rm vi},\tau}) \in J^i_\tau(y_{\rm old})$, we get (2.6.8). □

We now state the definition of *De Giorgi variational interpolation* [2, Definition 3.2.1], which in our setting refers to the viscous deformation $y_{\rm vi}$ only.

Definition 2.6.1 (De Giorgi variational interpolation). *Let* $(y^i_{\rm el,\tau}, y^i_{\rm vi,\tau})_{i=0}^N$ *be an* incremental solution *of the problem of* (2.4.9). *We call* De Giorgi variational interpolation *of* $(y^i_{vi,\tau})_{i=0}^N$ *any interpolation* $\widetilde{y}_{vi,\tau}$ *of the discrete values with* $(\overline{y}_{\rm el,\tau}, \widetilde{y}_{vi,\tau}) : [0,T] \to \mathcal{A}$ *that satisfies*

$$\widetilde{y}_{\rm vi,\tau}(t) = \widetilde{y}_{\rm vi,\tau}(t_{i-1}+r) \in J_r(\widetilde{y}^{i-1}_{\rm vi,\tau}) \quad \textit{if } t_{i-1}+r \in (t_{i-1}, t_i]$$

for every $i = 1, \dots, N$.

The following Proposition provides the sharp energy estimate on the discrete level, providing an equality instead of an inequality.

Proposition 2.6.1 (Discrete energy equality). *Let* $(y^i_{\rm el,\tau}, y^i_{\rm vi,\tau})_{i=0}^N$ *be an* incremental solution *of the problem of* (2.4.9). *Then, for every* $1 \le n \le N$ *we have*

$$\tau \sum_{i=1}^n \psi\left(\left(\nabla y^{i-1}_{\rm vi,\tau}\right)^{-1}\left(\nabla y^i_{\rm vi,\tau} - \nabla y^{i-1}_{\rm vi,\tau}\right)\right) + (p_\psi - 1)\sum_{i=1}^n \int_{t_{i-1}}^{t_i} G_\tau^{p_\psi}(r){\rm d}r + \mathcal{E}\left(t_i, y^n_{\rm el,\tau}, y^n_{\rm vi,\tau}\right)$$
$$= \mathcal{E}\left(0, y_{\rm el,0}, y_{\rm vi,0}\right) - \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \langle \dot{\ell}, y^{i-1}_{\rm el,\tau} \circ y^{i-1}_{\rm el,\tau}\rangle \tag{2.6.9}$$

where

$$G_\tau(t) := \left(\Psi_r^{\pm,i}(y^{i-1}_{\rm vi,\tau})\right)^{1/p_\psi} \quad \textit{for } t = t_{i-1}+r \in (t_{i-1}, t_i].$$

Proof. By (2.6.8) for $y_{\rm old} = y^{i-1}_{\rm vi,\tau}$, $y_{\rm vi,\tau} = y^i_{\rm vi,\tau}$, and $y_{\rm el,\tau} = y^i_{\rm el,\tau}$, we find

$$\tau\Psi\left(y^{i-1}_{\rm vi,\tau}, \frac{y^i_{\rm vi,\tau} - y^{i-1}_{\rm vi,\tau}}{\tau}\right) + (p_\psi - 1)\int_0^\tau |G_\tau(r)|^{p_\psi}{\rm d}r + \mathcal{E}(t_i, y^i_{\rm el,\tau}, y^i_{\rm vi,\tau}) = \mathcal{E}(t_i, y^{i-1}_{\rm el,\tau}, y^{i-1}_{\rm vi,\tau})$$
$$= \mathcal{E}(t_{i-1}, y^{i-1}_{\rm el,\tau}, y^{i-1}_{\rm vi,\tau}) - \int_{t_{i-1}}^{t_i} \langle \dot{\ell}, y^{i-1}_{\rm el,\tau} \circ y^{i-1}_{\rm el,\tau}\rangle,$$

where we used the definition of G_τ and of De Giorgi variational interpolation. Then, summing from $i = 1$ to $i = n$ we get (2.6.9). □

Before passing to the limit for $\tau \to 0$ in the energy equality (2.6.9), we need to characterize the limit of the De Giorgi variational interpolation. In the following Lemma we show that such limit coincides with that of the backward interpolants.

Lemma 2.6.3. *If* $\overline{y}_{\rm vi,\tau}(t) \rightharpoonup y_{\rm vi}(t)$ *in* $W^{1,p_{\rm vi}}(\Omega;\mathbb{R}^d)$*, then* $\widetilde{y}_{\rm vi,\tau}(t) \rightharpoonup y_{\rm vi}(t)$ *in* $W^{1,p_{\rm vi}}(\Omega;\mathbb{R}^d)$.

Proof. First, let us show that, for $\tau > 0$ and $t \in (t^\tau_{i-1}, t^\tau_i]$ fixed, $\|\widetilde{y}_{\rm vi,\tau}(t) - \overline{y}_{\rm vi,\tau}(t)\|_{L^1(\Omega)} \le c\tau^{p_\psi - 1}$. We have, by definition of $\nabla\overline{y}_{\rm vi,\tau}$ and Hölder inequality,

$$\|\nabla\widetilde{y}_{\rm vi,\tau}(t) - \nabla\overline{y}_{\rm vi,\tau}(t)\|_{L^1(\Omega)} \le \|\nabla y^{i-1}_{\rm vi,\tau}(\nabla y^{i-1}_{\rm vi,\tau})^{-1}\left(\nabla\widetilde{y}_{\rm vi,\tau}(t) - \nabla y^i_{\rm vi,\tau}\right)\|_{L^1(\Omega)}$$
$$\le \|\nabla y^{i-1}_{\rm vi,\tau}\|^{p'_\psi}_{L^{p'_\psi}(\Omega)}\|(\nabla y^{i-1}_{\rm vi,\tau})^{-1}\left(\nabla\widetilde{y}_{\rm vi,\tau}(t) - \nabla y^i_{\rm vi,\tau}\right)\|^{p_\psi}_{L^{p_\psi}(\Omega)}.$$

Since $p_\psi \geq 2$ by (2.4.7) we have that $p'_\psi \leq p_\psi$. Hence, by the boundedness of $\nabla y^{i-1}_{\mathrm{vi},\tau}$ in $L^{p_\psi}(\Omega;\mathbb{R}^{d\times d})$ and the fact that Ω is bounded, we have that $\|\nabla y^{i-1}_{\mathrm{vi},\tau}\|^{p'_\psi}_{L^{p'_\psi}(\Omega)} \leq c$ uniformly in i and τ. Thus, by growth condition (2.4.5), we have

$$\begin{aligned}\|\nabla\widetilde{y}_{\mathrm{vi},\tau}(t) - \nabla\overline{y}_{\mathrm{vi},\tau}(t)\|_{L^1(\Omega)} &\leq c\|(\nabla y^{i-1}_{\mathrm{vi},\tau})^{-1}\left(\nabla y^{i}_{\mathrm{vi},\tau} - \nabla y^{i-1}_{\mathrm{vi},\tau}\right)\|^{p_\psi}_{L^{p_\psi}(\Omega)}\\ &\quad + c\|(\nabla y^{i-1}_{\mathrm{vi},\tau})^{-1}\left(\nabla\widetilde{y}_{\mathrm{vi},\tau}(t) - \nabla y^{i-1}_{\mathrm{vi},\tau}\right)\|^{p_\psi}_{L^{p_\psi}(\Omega)}\\ &\leq c\Psi\left(y^{i-1}_{\mathrm{vi},\tau}, y^{i}_{\mathrm{vi},\tau} - y^{i-1}_{\mathrm{vi},\tau}\right) + c\Psi\left(y^{i-1}_{\mathrm{vi},\tau}, \widetilde{y}_{\mathrm{vi},\tau} - y^{i-1}_{\mathrm{vi},\tau}\right)\\ &\leq c\Psi\left(y^{i-1}_{\mathrm{vi},\tau}, y^{i}_{\mathrm{vi},\tau} - y^{i-1}_{\mathrm{vi},\tau}\right),\end{aligned}$$

where in the last inequality we used the definition of $\widetilde{y}_{\mathrm{vi},\tau}$ and the monotonicity property (2.6.6). Using the p_ψ-homogeneity (2.4.6) and the boundedness of the dissipation, we get

$$\|\nabla\widetilde{y}_{\mathrm{vi},\tau}(t) - \nabla\overline{y}_{\mathrm{vi},\tau}(t)\|_{L^1(\Omega)} \leq c\tau^{p_\psi}\Psi\left(y^{i-1}_{\mathrm{vi},\tau}, \frac{y^{i}_{\mathrm{vi},\tau} - y^{i-1}_{\mathrm{vi},\tau}}{\tau}\right) \leq c\tau^{p_\psi - 1}.$$

Then $\|\widetilde{y}_{\mathrm{vi},\tau}(t) - \overline{y}_{\mathrm{vi},\tau}(t)\|_{L^1(\Omega)} \leq c\tau^{p_\psi-1}$ follows since $\widetilde{y}_{\mathrm{vi},\tau}(t)$ and $\overline{y}_{\mathrm{vi},\tau}(t)$ have zero mean.

The assertion follows as Ω is bounded, by assumption $\overline{y}_{\mathrm{vi},\tau}(t) \rightharpoonup y_{\mathrm{vi}}(t)$ in $W^{1,p_{\mathrm{vi}}}(\Omega;\mathbb{R}^d)$, and $\overline{y}_{\mathrm{vi},\tau}(t)$ is bounded in $W^{1,p_{\mathrm{vi}}}(\Omega;\mathbb{R}^d)$ by coercivity, as shown in Section 2.6.1 for $\overline{y}_{\mathrm{vi},\tau}(t)$. □

2.6.3 Proof of the energy inequality

In the following, we extract further subsequences without relabeling whenever necessary.

Assume to be given a sequence of partitions $(\Pi_\tau)_\tau$ with $\tau \to 0$ and denote by $(y^i_{\mathrm{el}}, y^i_{\mathrm{vi}})^N_{i=0}$ the corresponding incremental solutions. The estimates in Section 2.6.1 and Lemma 2.6.3 ensure that for every $t \in [0,T]$

$$\begin{aligned}\overline{y}_{\mathrm{vi},\tau}(t) &\rightharpoonup y_{\mathrm{vi}}(t) \quad \text{in } W^{1,p_{\mathrm{vi}}}(\Omega;\mathbb{R}^d),\\ \widetilde{y}_{\mathrm{vi},\tau}(t) &\rightharpoonup y_{\mathrm{vi}}(t) \quad \text{in } W^{1,p_{\mathrm{vi}}}(\Omega;\mathbb{R}^d),\\ \overline{y}_\tau(t) &\rightharpoonup y(t) \quad \text{in } W^{1,q}(\Omega;\mathbb{R}^d).\end{aligned}$$

Moreover, by Sobolev embedding we have that $\hat{y}_{\mathrm{vi},\tau} \rightharpoonup y_{\mathrm{vi}}$ weakly in $C([0,T];W^{1,p_r}(\Omega;\mathbb{R}^d))$.

As regards the elastic deformation, given $t \in [0,T]$ by extracting a subsequence $(\tau^t_k)_{k\in\mathbb{N}}$ possibly depending on t we get

$$\overline{y}_{\mathrm{el},\tau^t_k}(t) \rightharpoonup y_{\mathrm{el}}(t) \quad \text{in } W^{1,p_{\mathrm{el}}}(y_{\mathrm{vi}}(t,\Omega);\mathbb{R}^d).$$

Note that here we have to implement an exhaustion argument for dealing with the moving domains $\overline{y}_{\mathrm{vi}}(t,\Omega)$, exactly as in Section 2.5. Moreover, the total deformation y can be proved to fulfill $y = y_{\mathrm{el}} \circ y_{\mathrm{vi}}$ by arguing as in Section 2.5.2.

We aim at passing to the limit in the energy equality (2.6.9), which can be rewritten, thanks to the definition of $G_\tau(t)$ and of $\Psi^{+,i}_\tau$, in the weaker form

$$\begin{aligned}\mathcal{E}(\bar{t}_\tau(t), \overline{y}_{\mathrm{el},\tau}(t), \overline{y}_{\mathrm{vi},\tau}(t)) + \int_0^{\bar{t}_\tau(t)} \Psi\left(\underline{y}_{\mathrm{vi},\tau}, \dot{\hat{y}}_{\mathrm{vi},\tau}\right) + (p_\psi - 1)\int_0^{\bar{t}_\tau(t)} \Psi\left(\underline{y}_{\mathrm{vi},\tau}, \frac{\widetilde{y}_{\mathrm{vi},\tau} - \underline{y}_{\mathrm{vi},\tau}}{\tau}\right)\\ \leq \mathcal{E}(0, y_{\mathrm{el},0}, y_{\mathrm{vi},0}) - \int_0^{\bar{t}_\tau(t)} \langle\dot{\ell}, \underline{y}_\tau\rangle.\end{aligned} \tag{2.6.10}$$

Passing to the $\liminf$ in the left-hand side of inequality (2.6.10), we find by lower semicontinuity

$$\mathcal{E}(t, y_{\mathrm{el}}(t), y_{\mathrm{vi}}(t)) \leq \liminf_{\tau\to 0} \mathcal{E}(\bar{t}_\tau(t), \overline{y}_{\mathrm{el},\tau}(t), \overline{y}_{\mathrm{vi},\tau}(t)).$$

Let us now study the first dissipation term in (2.6.10). The calculations for the second one are analogous by Lemma 2.6.3. Recalling that by definition $\bar{t}_\tau(t) \geq t$ and that $\psi \geq 0$, we have that

$$\liminf_{\tau\to 0} \int_0^{\bar{t}_\tau(t)} \Psi\left(\underline{y}_{\mathrm{vi},\tau}, \dot{\hat{y}}_{\mathrm{vi},\tau}\right) \geq \liminf_{\tau\to 0} \int_0^t \Psi\left(\underline{y}_{\mathrm{vi},\tau}, \dot{\hat{y}}_{\mathrm{vi},\tau}\right).$$

Moreover, up to a subsequence, $\nabla\dot{\hat{y}}_{\mathrm{vi},\tau}(\nabla\underline{y}_{\mathrm{vi},\tau})^{-1} \rightharpoonup l$ weakly in $L^{p_\psi}(\Omega;\mathbb{R}^{d\times d})$ by (2.6.3). It hence remains to identify the limit l. To this end, let us define

$$(t,\xi) \in [0,T] \times \underline{y}_{\mathrm{vi},\tau}(t,\Omega) \mapsto v_\tau(t,\xi) := \dot{\hat{y}}_{\mathrm{vi},\tau}(t, \underline{y}_{\mathrm{vi},\tau}^{-1}(\xi)) \in \mathbb{R}^d.$$

By a pointwise-in-time change of variables we have

$$\int_0^T \int_{\underline{y}_{\mathrm{vi},\tau}(t,\Omega)} |v_\tau(t,\xi)|^{p_\psi} \mathrm{d}\xi \mathrm{d}t = \int_0^T \int_\Omega |\dot{\hat{y}}_{\mathrm{vi},\tau}(t,X)|^{p_\psi} \mathrm{d}X \mathrm{d}t \leq c.$$

In order to obtain a bound on the gradient ∇v_τ, let us consider

$$\int_0^T \int_{\underline{y}_{\mathrm{vi},\tau}(t,\Omega)} |\nabla v_\tau(t,\xi)|^{p_\psi} \mathrm{d}\xi \mathrm{d}t = \int_0^T \int_\Omega \left|\nabla\dot{\hat{y}}_{\mathrm{vi},\tau}(\nabla\underline{y}_{\mathrm{vi},\tau})^{-1}(t,X)\right|^{p_\psi} \mathrm{d}X \mathrm{d}t$$
$$\overset{(2.4.5)}{\leq} \int_0^T \Psi\left(\underline{y}_{\mathrm{vi},\tau}, \dot{\hat{y}}_{\mathrm{vi},\tau}\right) \leq c.$$

For given $t_0 \in (0,T)$, let us show that $\cap_{t\in[t_0,t_0+\delta]} y_{\mathrm{vi}}(t,\Omega)$ is not empty for small $\delta > 0$. Notice that, by Sobolev embedding, $\underline{y}_{\mathrm{vi},\tau} \to y_{\mathrm{vi}}$ in $C([0,T]\times\overline{\Omega})$. Hence, for every $\epsilon > 0$, there exists $\overline{\tau} = \overline{\tau}(\epsilon)$ such that, for every $\tau \leq \overline{\tau}$, we have

$$\sup_{X\in\overline{\Omega}} \left|\underline{y}_{\mathrm{vi},\tau}(t,X) - y_{\mathrm{vi}}(t,X)\right| \leq \frac{\epsilon}{2}.$$

Moreover, since y_{vi} is absolutely continuous in time, for $|t-s| < \nu$ and $\nu > 0$ small we also have

$$\sup_{X\in\overline{\Omega}} |y_{\mathrm{vi}}(t,X) - y_{\mathrm{vi}}(s,X)| \leq \frac{\epsilon}{2}.$$

Combining these two inequalities we get

$$\sup_{X\in\overline{\Omega}} \left|\underline{y}_{\mathrm{vi},\tau}(t,X) - y_{\mathrm{vi}}(s,X)\right| \leq \epsilon$$

for τ and ν small enough. We can hence fix $\omega \subset\subset \cap_{t\in[t_0,t_0+\nu]} \underline{y}_{\mathrm{vi}}(t,\Omega)$ and trivially extend v_τ on $\mathbb{R}^d\backslash\omega$. Then, thanks to the bounds above we have that $v_\tau \rightharpoonup v$ weakly in $L^{p_\psi}([0,T]; W^{1,p_\psi}(\omega))$. We have to show that $v = \dot{y}_{\mathrm{vi}} \circ y_{\mathrm{vi}}^{-1}$. In fact, we have

$$\int_{t_0}^{t_0+\nu}\int_\omega v(t,\xi)\mathrm{d}\xi\mathrm{d}t \leftarrow \int_{t_0}^{t_0+\nu}\int_\omega v_\tau(t,\xi)\mathrm{d}\xi\mathrm{d}t = \int_{t_0}^{t_0+\nu}\int_{\underline{y}_{\mathrm{vi},\tau}^{-1}(t,\omega)} \dot{\hat{y}}_{\mathrm{vi},\tau}(t,X)\mathrm{d}X\mathrm{d}t$$
$$= \int_{t_0}^{t_0+\nu}\int_{\mathbb{R}^d} \dot{\hat{y}}_{\mathrm{vi},\tau}(t,X)\mathbb{1}_{\underline{y}_{\mathrm{vi},\tau}^{-1}(t,\omega)}(t,X)\mathrm{d}X\mathrm{d}t \to \int_{t_0}^{t_0+\nu}\int_{\mathbb{R}^d} \dot{y}_{\mathrm{vi}}(t,X)\mathbb{1}_{y_{\mathrm{vi}}^{-1}(t,\omega)}(t,X)\mathrm{d}X\mathrm{d}t$$
$$= \int_{t_0}^{t_0+\nu}\int_\omega \dot{y}_{\mathrm{vi}}(t, y_{\mathrm{vi}}^{-1}(t,\xi))\mathrm{d}\xi\mathrm{d}t, \quad (2.6.11)$$

where we have used that $\hat{y}_{\mathrm{vi},\tau} \rightharpoonup y_{\mathrm{vi}}$ weakly in $C([0,T];W^{1,p_{\mathrm{vi}}}(\Omega))$, $\mathbb{1}_{\underline{y}_{\mathrm{vi},\tau}^{-1}(\omega)} \to \mathbb{1}_{y_{\mathrm{vi}}^{-1}(\omega)}$ strongly in $L^1(\omega)$, and the fact that $\mathbb{1}_{\underline{y}_{\mathrm{vi},\tau}^{-1}(t,\omega)}$ is bounded. Since in (2.6.11) t_0, ν, and ω are arbitrary, we have that $v = \dot{y}_{\mathrm{vi}} \circ y_{\mathrm{vi}}^{-1}$ and we have hence identified $l = \nabla v$. By weak lower semicontinuity, we thus have that

$$\begin{aligned}\liminf_{\tau\to 0}\int_0^{\bar{t}_\tau(t)} \Psi\left(\underline{y}_{\mathrm{vi},\tau}, \dot{\hat{y}}_{\mathrm{vi},\tau}\right) &= \liminf_{\tau\to 0}\int_0^{\bar{t}_\tau(t)}\int_{\overline{y}_{\mathrm{vi},\tau}(s,\Omega)} \psi(\nabla v_\tau(s,\xi))\mathrm{d}\xi\mathrm{d}s \\ &\geq \int_0^t\int_{y_{\mathrm{vi}}(s,\Omega)} \psi(\nabla v(s,\xi))\mathrm{d}\xi\mathrm{d}s = \int_0^t \Psi\left(y_{\mathrm{vi},\tau}, \dot{y}_{\mathrm{vi},\tau}\right).\end{aligned}$$

Thanks to the boundedness and to the weak lower semicontinuity of the energy and of the dissipation we can apply Helly's Selection Principle [74, Thm. B.5.13, p. 611] and find a nondecreasing function $\delta : [0,T] \to [0,\infty)$ such that

$$\int_0^t \Psi\left(\underline{y}_{\mathrm{vi},\tau}, \dot{\hat{y}}_{\mathrm{vi},\tau}\right) \to \delta(t), \tag{2.6.12a}$$

$$\int_s^t \Psi\left(\underline{y}_{\mathrm{vi},\tau}, \dot{\hat{y}}_{\mathrm{vi},\tau}\right) \leq \delta(t) - \delta(s) \tag{2.6.12b}$$

for every $s,t \in [0,T]$. Then, fixing $t \in [0,T]$, we have

$$\delta(t) \overset{(2.6.12a)}{=} \lim_{k\to\infty}\int_0^t \Psi\left(\underline{y}_{\mathrm{vi},\tau}, \dot{\hat{y}}_{\mathrm{vi},\tau}\right) \leq \liminf_{k\to\infty}\int_0^{\underline{t}_\tau(t)} \Psi\left(\underline{y}_{\mathrm{vi},\tau}, \dot{\hat{y}}_{\mathrm{vi},\tau}\right).$$

Setting $\theta_\tau(s) := -\langle\dot{\ell}(s), \underline{y}_\tau(s)\rangle$, by the regularity of ℓ and the boundedness of $(\underline{y}_\tau(t))_\tau$ for almost every $t \in [0,T]$ in $W^{1,q}(\Omega;\mathbb{R}^d)$, we have that $(\theta_\tau)_\tau$ is equiintegrable. Hence, we can apply the Dunford-Pettis Theorem (see, e.g., [74, Thm. B.3.8, p. 598]) to find a subsequence such that

$$\theta_\tau \rightharpoonup \theta \quad \text{in } L^1(0,T).$$

Furthermore, thanks to the boundedness of the energy and the dissipation, we are able to find further t-dependent subsequences $(\tau_k^t)_{k\in\mathbb{N}}$ such that

$$\theta_{\tau_k^t} \to \limsup_{\tau\to 0}\theta_\tau(t) =: \overline{\theta}(t),$$

and, by regularity of ℓ, that

$$\overline{\theta}(t) := \lim_{k\to\infty}\theta_{\tau_k^t} = \lim_{k\to\infty}\langle\dot{\ell}(t), \underline{y}_{\tau_k^t}(t)\rangle = \langle\dot{\ell}(t), y(t)\rangle.$$

In conclusion, passing to the $\liminf$ in the left-hand side and to the limit in the right-hand side of (2.6.10) we retrieve energy inequality (2.4.10).

2.6.4 Proof of the semistability condition

Fix now $t \in [0,T]$ and recall that $\overline{y}_{\mathrm{el},\tau_k^t}(t) \rightharpoonup y_{\mathrm{el}}(t)$ in $W^{1,p_{\mathrm{el}}}(y_{\mathrm{vi}}(t,\Omega);\mathbb{R}^d)$.

By minimality of the incremental solution we have

$$\mathcal{E}\left(\bar{t}_\tau(t), \overline{y}_{\mathrm{el},\tau}(t), \overline{y}_{\mathrm{vi},\tau}(t)\right) \leq \mathcal{E}\left(\bar{t}_\tau(t), \tilde{\overline{y}}_{\mathrm{el}}, \overline{y}_{\mathrm{vi},\tau}(t)\right)$$

for every $\tilde{\overline{y}}_{\rm el}$ with $\big(\tilde{\overline{y}}_{\rm el}, \overline{y}_{\rm vi}(t)\big) \in \mathcal{A}$. Let $(\tilde{y}_{\rm el}, y_{\rm vi}(t)) \in \mathcal{A}$ be given. We want to show that one can choose $\tilde{\overline{y}}_{\rm el,\tau}$ with $\big(\tilde{\overline{y}}_{\rm el,\tau}, \overline{y}_{\rm vi}(t)\big) \in \mathcal{A}$ in such a way that

$$\begin{aligned} 0 \leq \limsup_{\tau\to 0} &\Big(\mathcal{E}\left(\bar{t}_\tau(t), \tilde{\overline{y}}_{\rm el,\tau}, \overline{y}_{\rm vi,\tau}(t)\right) - \mathcal{E}\left(\bar{t}_\tau(t), \overline{y}_{\rm el,\tau}(t), \overline{y}_{\rm vi,\tau}(t)\right)\Big) \\ &\leq \mathcal{E}\left(t, \tilde{y}_{\rm el}, y_{\rm vi}(t)\right) - \mathcal{E}\left(t, y_{\rm el}(t), y_{\rm vi}(t)\right), \end{aligned} \tag{2.6.13}$$

which would then imply (2.4.11).

Since $(\tilde{y}_{\rm el}, y_{\rm vi}(t)) \in \mathcal{A}$, we have that $y_{\rm vi}(\Omega) \in \mathcal{J}_{\eta_1,\eta_2}$ and $y_{\rm vi}(\Omega)$ is a Sobolev extension domain. Hence, there exists a linear and bounded extension operator $E : W^{1,p_{\rm el}}(y_{\rm vi}(\Omega);\mathbb{R}^d) \to W^{1,p_{\rm el}}(\mathbb{R}^d;\mathbb{R}^d)$. We thus define $\tilde{\overline{y}}_{\rm el,\tau} \in W^{1,p_{\rm el}}(\overline{y}_{\rm vi,\tau}(\Omega);\mathbb{R}^d)$ as the restriction to $\overline{y}_{\rm vi,\tau}(\Omega)$ of the extension $E\tilde{y}_{\rm el}$, namely,

$$\tilde{\overline{y}}_{\rm el,\tau} := E\tilde{y}_{\rm el}\Big|_{\overline{y}_{\rm vi,\tau}(\Omega)}.$$

In the following, we just concentrate our attention on the stored elastic energy part, since the treatment of the loading term is immediate. We write

$$\begin{aligned} \int_{\overline{y}_{\rm vi,\tau}(t,\Omega)} W_{\rm el}(\nabla\tilde{\overline{y}}_{\rm el,\tau})\mathrm{d}\xi - \int_{\overline{y}_{\rm vi,\tau}(t,\Omega)} W_{\rm el}(\nabla\overline{y}_{\rm el,\tau})\mathrm{d}\xi &= \int_{\overline{y}_{\rm vi,\tau}(t,\Omega)\cap y_{\rm vi}(t,\Omega)} W_{\rm el}(\nabla\tilde{\overline{y}}_{\rm el,\tau})\mathrm{d}\xi \\ + \int_{\overline{y}_{\rm vi,\tau}(t,\Omega)\setminus y_{\rm vi}(t,\Omega)} W_{\rm el}(\nabla\tilde{\overline{y}}_{\rm el,\tau})\mathrm{d}\xi &- \int_{\overline{y}_{\rm vi,\tau}(t,\Omega)} W_{\rm el}(\nabla\overline{y}_{\rm el,\tau})\mathrm{d}\xi \end{aligned} \tag{2.6.14}$$

By the growth condition (2.4.1) on $W_{\rm el}$ and the fact that on the set $\overline{y}_{\rm vi,\tau}(t,\Omega)$ we have $\tilde{\overline{y}}_{\rm el,\tau} = \tilde{y}_{\rm el,\tau}$, which is uniformly bounded in $W^{1,p_{\rm el}}(y_{\rm vi,\tau}(t,\Omega);\mathbb{R}^d)$, we find

$$\int_{\overline{y}_{\rm vi,\tau}(t,\Omega)\setminus y_{\rm vi}(t,\Omega)} W_{\rm el}(\nabla\tilde{\overline{y}}_{\rm el,\tau})\mathrm{d}\xi = \int_{\tilde{y}_{\rm vi,\tau}(t,\Omega)\setminus y_{\rm vi}(t,\Omega)} W_{\rm el}(\nabla\tilde{y}_{\rm el,\tau})\mathrm{d}\xi \overset{(2.4.1)}{\leq} c\left|\overline{y}_{\rm vi,\tau}(t,\Omega)\setminus y_{\rm vi}(t,\Omega)\right|.$$

Since the measure of the set $\overline{y}_{\rm vi,\tau}(t,\Omega)\setminus y_{\rm vi}(t,\Omega)$ vanishes as τ goes to 0 by the uniform convergence of $\overline{y}_{\rm vi,\tau}$ to $y_{\rm vi}$, we have

$$\lim_{\tau\to 0}\int_{\overline{y}_{\rm vi,\tau}(t,\Omega)} W_{\rm el}(\nabla\tilde{\overline{y}}_{\rm el,\tau})\mathrm{d}\xi = \int_{y_{\rm vi}(t,\Omega)} W_{\rm el}(\nabla\tilde{y}_{\rm el})\mathrm{d}\xi.$$

We can hence pass to the $\limsup$ in inequality (2.6.14) as $\tau \to 0$ and obtain (2.6.13), which is nothing but the semistability (2.4.11).

2.7 Linearization: Proof of Theorem 2.4.2

We first prove in Subsection 2.7.1 some coercivity results, uniform with respect to the linearization parameter ε, which in turn provide a priori estimates on the sequence of approximable solutions $(u_\varepsilon, v_\varepsilon)_\varepsilon$. Then, we check in Subsection 2.7.2 some Γ-$\liminf$ inequalities for the energy and the dissipation. Eventually, in Subsection 2.7.3 we show that the approximable solutions $(u_\varepsilon, v_\varepsilon)_\varepsilon$ converge, up to subsequences, to solutions of the linearized problem in the sense of Theorem 2.4.2.

In the following, we use the notation

$$W^\varepsilon_{\rm el}(A) := \frac{1}{\varepsilon^2}W_{\rm el}(I+\varepsilon A), \quad \widetilde{W}^\varepsilon_{\rm vi}(A) := \frac{1}{\varepsilon^2}\widetilde{W}_{\rm vi}(I+\varepsilon A), \quad \psi^\varepsilon(A) := \frac{1}{\varepsilon^2}\psi(\varepsilon A)$$

for the rescaled energy and dissipation densities.

2.7.1 Coercivity

We devote this subsection to the proof of the following.

Lemma 2.7.1 (Coercivity). *For every $(u,v) \in \widetilde{\mathcal{A}}_\varepsilon$, it holds*

$$\|u\|^2_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)} + \|\nabla \dot v\|^2_{L^2(\Omega)} + \|\varepsilon \nabla v\|_{L^\infty(\Omega)} \leq c(1 + \mathcal{W}^\varepsilon_{\mathrm{vi}}(v) + \mathcal{W}^\varepsilon_{\mathrm{el}}(u,v) + \Psi^\varepsilon(v)).$$

Notice the bound on the term $\|\varepsilon \nabla v\|_{L^\infty(\Omega)}$, which follows from assumption (L4). This bound will play an important role in passing to the limit for $\varepsilon \to 0$.

Proof of Lemma 2.7.1. With no loss of generality we can assume $\mathcal{W}^\varepsilon_{\mathrm{vi}}(v) + \mathcal{W}^\varepsilon_{\mathrm{el}}(u,v) + \Psi^\varepsilon(v) < \infty$

By assumption (L4) we have that $I + \varepsilon \nabla v \in K \subset\subset \mathrm{SL}(d)$ almost everywhere in Ω. By using (2.4.18) we get that $|I + \varepsilon\nabla v| \leq c_K$, hence

$$\|\varepsilon \nabla v\|_{L^\infty(\Omega)} \leq c.$$

Since v has zero mean by assumption, by applying the Poincaré-Wirtinger inequality and by taking into account the growth condition (2.4.16) we get

$$\|v\|^2_{H^1(\Omega)} \leq c\|\nabla v\|^2_{L^2(\Omega)} = \frac{c}{\varepsilon^2}\|\varepsilon \nabla v\|^2_{L^2(\Omega)} \leq \frac{c}{\varepsilon^2}\int_\Omega W_{\mathrm{vi}}(I + \varepsilon \nabla v)\mathrm{d}X = c\mathcal{W}^\varepsilon_{\mathrm{vi}}(v).$$

Using condition (2.4.19) and the fact that $|I + \varepsilon\nabla v|$ is bounded in L^∞ we get

$$\begin{aligned}\|\nabla \dot v\|^2_{L^2(\Omega)} = \frac{1}{\varepsilon^2}\int_\Omega |\varepsilon \nabla \dot v|^2 \mathrm{d}X &\leq \frac{1}{\varepsilon^2}\int_\Omega |\varepsilon\nabla \dot v|^2 |I + \varepsilon\nabla v|^{-2}|I + \varepsilon \nabla v|^2 \mathrm{d}X \\ &\leq \frac{c}{\varepsilon^2}\int_\Omega \psi\left(\varepsilon \nabla \dot v (I + \varepsilon \nabla v)^{-1}\right)\mathrm{d}X = c\Psi^\varepsilon(v).\end{aligned}$$

In order to obtain the H^1-bound on u, we start by fixing $Q \in \mathrm{SO}(d)$ and define $F_{\mathrm{el}} := \nabla y (I + \varepsilon \nabla v)^{-1}$, where we recall that $y = \mathrm{id} + \varepsilon u$. We have

$$\begin{aligned}|\nabla y - Q|^2 &= |\nabla y - Q(I + \varepsilon \nabla v) + \varepsilon Q \nabla v|^2 = |(F_{\mathrm{el}} - Q)(I + \varepsilon\nabla v) + \varepsilon Q \nabla v|^2 \\ &\leq c(|F_{\mathrm{el}} - Q|^2 |I + \varepsilon \nabla v|^2 + \varepsilon^2|\nabla v|^2) \leq c(|F_{\mathrm{el}} - Q|^2 + \varepsilon^2 |\nabla v|^2).\end{aligned}$$

Taking the infimum over $Q \in \mathrm{SO}(d)$ we get

$$\mathrm{dist}^2(\nabla y, \mathrm{SO}(d)) \leq c(\mathrm{dist}^2(F_{\mathrm{el}}, \mathrm{SO}(d)) + \varepsilon^2|\nabla v|^2).$$

We now integrate over Ω and, thanks to assumption (2.4.13) and the estimate on $\|v\|^2_{H^1(\Omega)}$, we find

$$\int_\Omega \mathrm{dist}^2(\nabla y, \mathrm{SO}(d))\mathrm{d}X \leq c\int_\Omega W_{\mathrm{el}}(F_{\mathrm{el}})\mathrm{d}X + c\varepsilon^2\|\nabla v\|^2_{L^2(\Omega)} \leq c\varepsilon^2\left(\mathcal{W}^\varepsilon_{\mathrm{el}}(u,v) + \mathcal{W}^\varepsilon_{\mathrm{vi}}(v)\right).$$

The classical Rigidity Estimate [34, Theorem 3.1] implies that there exists a constant rotation $\hat Q \in \mathrm{SO}(d)$ such that

$$\|\nabla y - \hat Q\|^2_{L^2(\Omega)} \leq c\|\,\mathrm{dist}(\nabla y, \mathrm{SO}(d))\|^2_{L^2(\Omega)}.$$

We hence have that

$$\|\nabla y - \hat Q\|^2_{L^2(\Omega)} \leq c\varepsilon^2\left(\mathcal{W}^\varepsilon_{\mathrm{el}}(u,v) + \mathcal{W}^\varepsilon_{\mathrm{vi}}(v)\right).$$

Recalling that $y = \mathrm{id}$ on Γ_D, by [22, (3.14)] we also deduce

$$\|I - \hat{Q}\|^2_{L^2(\Omega)} \le c\varepsilon^2 \left(\mathcal{W}^\varepsilon_{\mathrm{el}}(u,v) + \mathcal{W}^\varepsilon_{\mathrm{vi}}(v)\right).$$

In conclusion, we get that

$$\begin{aligned}\|\nabla u\|^2_{L^2(\Omega)} &= \frac{1}{\varepsilon^2}\|\nabla y - I\|^2_{L^2(\Omega)} \le \frac{2}{\varepsilon^2}\|\nabla y - \hat{Q}\|^2_{L^2(\Omega)} + \frac{2}{\varepsilon^2}\|\hat{Q} - I\|^2_{L^2(\Omega)}\\ &\le c\left(\mathcal{W}^\varepsilon_{\mathrm{el}}(u,v) + \mathcal{W}^\varepsilon_{\mathrm{vi}}(v)\right)\end{aligned}$$

whence the assertion follows. □

2.7.2 Γ-$\liminf$ inequalities

In order to proceed with the linearization, we need to establish Γ-$\liminf$ inequalities. At first, we prove the following Lemma on the convergence of the densities.

Lemma 2.7.2 (Convergence of the densities)**.** *Assume conditions (L3), (L6), and (L9). Then, we have*

$$W^\varepsilon_{\mathrm{el}} \to |\cdot|^2_{\mathbb{C}_{\mathrm{el}}}, \qquad \widetilde{W}^\varepsilon_{\mathrm{vi}} \to |\cdot|^2_{\mathbb{C}_{\mathrm{vi}}}, \qquad \psi^\varepsilon \to |\cdot|^2_{\mathbb{D}}$$

locally uniformly. Moreover, we have

$$|z|^2_{\mathbb{C}_{\mathrm{el}}} \le \inf\left\{\liminf_{\varepsilon\to 0} W^\varepsilon_{\mathrm{el}}(z_\varepsilon) \mid z_\varepsilon \to z \textit{ in } \mathbb{R}^{d\times d}\right\}, \tag{2.7.1}$$

$$|z|^2_{\mathbb{C}_{\mathrm{vi}}} \le \inf\left\{\liminf_{\varepsilon\to 0} \widetilde{W}^\varepsilon_{\mathrm{vi}}(z_\varepsilon) \mid z_\varepsilon \to z \textit{ in } \mathbb{R}^{d\times d}\right\}, \tag{2.7.2}$$

$$|z|^2_{\mathbb{D}} \le \inf\left\{\liminf_{\varepsilon\to 0} \psi^\varepsilon(z_\varepsilon) \mid z_\varepsilon \to z \textit{ in } \mathbb{R}^{d\times d}\right\}. \tag{2.7.3}$$

Proof. Let $K_0 \subset\subset \mathbb{R}^{d\times d}$ be given. Fix $\delta > 0$ and let $c_{\mathrm{el}}(\delta)$ be the corresponding constant from assumption (2.4.15). Then, for sufficiently small ε, we have that $\varepsilon K_0 \subset B^{\mathbb{R}^{d\times d}}_{c_{\mathrm{el}}(\delta)}(0)$. Hence, by (2.4.15) we find

$$\limsup_{\varepsilon\to 0}\sup_{K_0}\left|W^\varepsilon_{\mathrm{el}}(\cdot) - |\cdot|^2_{\mathbb{C}_{\mathrm{el}}}\right| \le \delta \sup_{K_0}|\cdot|^2_{\mathbb{C}_{\mathrm{el}}} \le \delta c.$$

Since δ is arbitrary, we get local uniform convergence for $W^\varepsilon_{\mathrm{el}}$. For $\widetilde{W}^\varepsilon_{\mathrm{vi}}$ and ψ^ε proof of convergence is analogous, using the corresponding conditions (2.4.17) and (2.4.20), respectively.

For the Γ-$\liminf$ inequalities (2.7.1)-(2.7.3), let $(z_\varepsilon)_\varepsilon \subset \mathbb{R}^{d\times d}$ be such that $z_\varepsilon \to z$ in $\mathbb{R}^{d\times d}$. Assume without loss of generality that $\sup_\varepsilon W^\varepsilon_{\mathrm{el}}(z_\varepsilon) < \infty$. Then, the inequality follows from local uniform convergence. The same applies to $\widetilde{W}^\varepsilon_{\mathrm{vi}}$ and ψ^ε. □

We are now in the position of proving the Γ-$\liminf$ inequalities for the functionals.

Lemma 2.7.3 (Γ-$\liminf$ inequalities)**.** *For every $(u,v) \in \widetilde{\mathcal{A}}_\varepsilon$, we have*

$$\begin{aligned}\mathcal{W}^0_{\mathrm{el}}(u,v) + \mathcal{W}^0_{\mathrm{vi}}(v) \le \inf\Big\{ &\liminf_{\varepsilon\to 0}\left(\mathcal{W}^\varepsilon_{\mathrm{el}}(u_\varepsilon, v_\varepsilon) + \mathcal{W}^\varepsilon_{\mathrm{vi}}(v_\varepsilon)\right)\\ &\mid (u_\varepsilon, v_\varepsilon) \rightharpoonup (u,v) \textit{ weakly in } H^1(\Omega;\mathbb{R}^d)^2\Big\},\end{aligned}$$

$$\int_0^t \Psi^0(\dot{v}) \le \inf\left\{\liminf_{\varepsilon\to 0}\int_0^t \Psi^\varepsilon(v_\varepsilon, \dot{v}_\varepsilon) \mid v_\varepsilon \rightharpoonup v \textit{ weakly in } H^1([0,t]; L^2(\Omega;\mathbb{R}^d))\right\}.$$

Proof. Let $(u_\varepsilon, v_\varepsilon) \rightharpoonup (u, v)$ weakly in $H^1(\Omega;\mathbb{R}^d)^2$ and assume without loss of generality that

$$\sup_\varepsilon \left(\mathcal{W}^\varepsilon_{\rm el}(u_\varepsilon, v_\varepsilon) + \mathcal{W}^\varepsilon_{\rm vi}(v_\varepsilon)\right) < \infty.$$

Thanks to inequality (2.7.2) and [77, Lemma 4.2] we immediately handle the stored viscous energy terms as

$$\int_\Omega |\nabla v|^2_{\mathbb{C}_{\rm vi}} {\rm d}X \leq \liminf_{\varepsilon\to 0} \int_\Omega W^\varepsilon_{\rm vi}(\nabla v_\varepsilon){\rm d}X = \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_\Omega W_{\rm vi}(I+\varepsilon\nabla v_\varepsilon){\rm d}X.$$

The treatment of the stored elastic energy term requires some steps. First, notice that, since $\sup_\varepsilon \mathcal{W}^\varepsilon_{\rm vi}(v_\varepsilon) < \infty$, we have that $I + \varepsilon\nabla v_\varepsilon \in K$ almost everywhere in Ω. Hence, $\|\varepsilon\nabla v_\varepsilon\|_{L^\infty(\Omega)} \leq c$ uniformly in ε and $(I+\varepsilon\nabla v_\varepsilon)^{-1}$ is bounded in $L^\infty(\Omega;\mathbb{R}^{d\times d})$ by (2.4.18) as well.

Let us then define the auxiliary tensor Z_ε as

$$Z_\varepsilon := \frac{1}{\varepsilon}\left((I+\varepsilon\nabla v_\varepsilon)^{-1} - I + \varepsilon\nabla v_\varepsilon\right) = \varepsilon(I+\varepsilon\nabla v_\varepsilon)^{-1}(\nabla v_\varepsilon)^2$$

so that $(I+\varepsilon\nabla v_\varepsilon)^{-1} = I - \varepsilon\nabla v_\varepsilon + \varepsilon Z_\varepsilon$. Notice that $\|\varepsilon Z_\varepsilon\|_{L^\infty(\Omega)} \leq c$ since $\|\varepsilon\nabla v_\varepsilon\|_{L^\infty(\Omega)} \leq c$. Furthermore,

$$\|Z_\varepsilon\|_{L^1(\Omega)} \leq \varepsilon\|(I+\varepsilon\nabla v_\varepsilon)^{-1}(\nabla v_\varepsilon)^2\|_{L^1(\Omega)} \leq c\varepsilon\|\nabla v_\varepsilon\|_{L^2(\Omega)} \leq c\varepsilon.$$

Hence, Z_ε is bounded in $L^2(\Omega;\mathbb{R}^{d\times d})$ by interpolation, namely,

$$\|Z_\varepsilon\|_{L^2(\Omega)} \leq \|\varepsilon Z_\varepsilon\|^{1/2}_{L^\infty(\Omega)} \frac{\|Z_\varepsilon\|^{1/2}_{L^1(\Omega)}}{\varepsilon^{1/2}} \leq c.$$

We therefore conclude that $Z_\varepsilon \rightharpoonup 0$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$.

Define now $F^\varepsilon_{\rm el} := (I+\varepsilon\nabla u_\varepsilon)(I+\varepsilon\nabla v_\varepsilon)^{-1}$ and

$$A^\varepsilon := \frac{F^\varepsilon_{\rm el} - I}{\varepsilon} = \frac{1}{\varepsilon}\left((I+\varepsilon\nabla u_\varepsilon)(I+\varepsilon\nabla v_\varepsilon)^{-1} - I\right).$$

We want to show that $A^\varepsilon \rightharpoonup \nabla u - \nabla v$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$. Let us compute

$$A^\varepsilon = \frac{1}{\varepsilon}\left((I+\varepsilon\nabla u_\varepsilon)(I-\varepsilon\nabla v_\varepsilon+\varepsilon Z_\varepsilon) - I\right) = \nabla u_\varepsilon - \nabla v_\varepsilon + Z_\varepsilon - \varepsilon(\nabla u_\varepsilon\nabla v_\varepsilon - \nabla u_\varepsilon Z_\varepsilon).$$

Since $\nabla u_\varepsilon - \nabla v_\varepsilon \rightharpoonup \nabla u - \nabla v$ and $Z_\varepsilon \rightharpoonup 0$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$, it remains to show that $H_\varepsilon := \varepsilon(\nabla u_\varepsilon\nabla v_\varepsilon - \nabla u_\varepsilon Z_\varepsilon) \rightharpoonup 0$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$. Notice that $\|H_\varepsilon\|_{L^2(\Omega)} \leq c$ since ∇u_ε is bounded in $L^2(\Omega;\mathbb{R}^{d\times d})$ and $\varepsilon\nabla v_\varepsilon$ and $\varepsilon Z_\varepsilon$ are bounded in $L^\infty(\Omega;\mathbb{R}^{d\times d})$. Moreover, since ∇v_ε and Z_ε are bounded in $L^2(\Omega;\mathbb{R}^{d\times d})$, then $\|H_\varepsilon\|_{L^1(\Omega)} \leq c\varepsilon$ so that $H_\varepsilon \rightharpoonup 0$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$.

Hence, we have by (2.7.1) and [77, Lemma 4.2] that

$$\begin{aligned}\int_\Omega |\nabla u - \nabla v|^2_{\mathbb{C}_{\rm el}} {\rm d}X &\leq \liminf_{\varepsilon\to 0}\int_\Omega W^\varepsilon_{\rm el}(A^\varepsilon)\ {\rm d}X \\ &= \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega W_{\rm el}\left((I+\varepsilon\nabla u_\varepsilon)(I+\varepsilon\nabla v_\varepsilon)^{-1}\right){\rm d}X.\end{aligned}$$

Let $(v_\varepsilon)_\varepsilon$ be such that $v_\varepsilon \rightharpoonup v$ weakly in $H^1([0,t];L^2(\Omega;\mathbb{R}^d))$, and $\sup_\varepsilon \Psi^\varepsilon(v_\varepsilon, \dot v_\varepsilon) < \infty$. By coercivity we have that (up to a non relabeled subsequence) $\nabla\dot v_\varepsilon(I+\varepsilon\nabla v_\varepsilon)^{-1} \rightharpoonup Y$ weakly in $L^2(\Omega)$. We want to identify the limit as $Y = \nabla\dot v$. First, notice that

$$\nabla\dot v_\varepsilon(I+\varepsilon\nabla v_\varepsilon)^{-1} - \nabla\dot v_\varepsilon = -\varepsilon\nabla\dot v_\varepsilon\nabla v_\varepsilon(I+\varepsilon\nabla v_\varepsilon)^{-1} =: Y_\varepsilon \rightharpoonup 0 \ \text{ weakly in } L^2(\Omega;\mathbb{R}^{d\times d}).$$

Indeed, $\|Y_\varepsilon\|_{L^1(\Omega)} \leq c\varepsilon$ and $\|Y_\varepsilon\|_{L^2(\Omega)} \leq c$. Then the Γ-$\liminf$ inequality for the dissipation term follows from (2.7.3) and [77, Lemma 4.2] applied on the domain $[0,t]\times\Omega$. □

2.7.3 Convergence of approximable solutions

Thanks to Lemma 2.7.1 and the energy inequality (2.4.22) we have

$$\begin{aligned}\|u_\varepsilon(t)\|^2_{H^1(\Omega)} &\le c\,(1+\mathcal{E}^\varepsilon(t,u_\varepsilon,v_\varepsilon)) \le c\left(1+\mathcal{E}^\varepsilon(t,u_\varepsilon,v_\varepsilon)+\int_0^t \Psi^\varepsilon(v_\varepsilon,\dot v_\varepsilon)\right)\\ &\le c\left(1+\mathcal{E}^\varepsilon(u^0_\varepsilon,v^0_\varepsilon)+\int_0^t\langle\dot\ell^\varepsilon,u_\varepsilon\rangle\right). \qquad (2.7.4)\end{aligned}$$

By the Gronwall Lemma [51, Lemma C.2.1, p. 534] this implies that $\|u_\varepsilon(t)\|_{H^1(\Omega)}\le c$ for every $t\in[0,T]$.

Concerning v_ε we similarly deduce from Lemma 2.7.1 that

$$\|v_\varepsilon(t)\|^2_{H^1(\Omega)} \le c\,(1+\mathcal{E}^\varepsilon(t,u_\varepsilon,v_\varepsilon)) \le c\left(1+\mathcal{E}^\varepsilon(u^0_\varepsilon,v^0_\varepsilon)+\int_0^t\langle\dot\ell^\varepsilon,u_\varepsilon\rangle\right)$$

so that $\|v_\varepsilon(t)\|_{H^1(\Omega)}\le c$ for every $t\in[0,T]$, as well.

Again, by Lemma 2.7.1 we have $\|\nabla\dot v_\varepsilon(t)\|^2_{L^2(\Omega)}\le c\Psi^\varepsilon(v_\varepsilon(t),\dot v_\varepsilon(t))$ for every $t\in[0,T]$. This yields

$$\int_0^t\|\nabla\dot v_\varepsilon\|^2_{L^2(\Omega)}\le c\int_0^t\Psi^\varepsilon(v_\varepsilon,\dot v_\varepsilon)\le c\left(1+\mathcal{E}^\varepsilon(u^0_\varepsilon,v^0_\varepsilon)+\int_0^t\langle\dot\ell^\varepsilon,u_\varepsilon\rangle\right),$$

hence, $\|\nabla\dot v_\varepsilon\|_{L^2(0,T;L^2(\Omega))}\le c$. Therefore, up to a non relabeled subsequence, we find

$$v_\varepsilon(t)\rightharpoonup v(t)\text{ in }H^1(\Omega;\mathbb{R}^d),\qquad \nabla\dot v_\varepsilon(t)\rightharpoonup\nabla\dot v(t)\text{ in }L^2(\Omega;\mathbb{R}^{d\times d})$$

for almost every $t\in[0,T]$. Notice that, since $\mathcal{E}^\varepsilon(t,u_\varepsilon(t),v_\varepsilon(t))<\infty$ for every $t\in[0,T]$, from assumption (L4) it follows $I+\varepsilon\nabla v_\varepsilon\in K$ for almost every $x\in\Omega$ and for every $t\in[0,T]$. In particular, $\varepsilon\nabla v_\varepsilon$ are uniformly bounded. Since $v_\varepsilon\in\mathcal{A}_\varepsilon$ by developing the determinant as a third-order polynomial we get

$$1=\det(I+\varepsilon\nabla v_\varepsilon)=1+\varepsilon\operatorname{tr}\nabla v_\varepsilon+\varepsilon^2\operatorname{tr}(\operatorname{cof}\nabla v_\varepsilon)+\varepsilon^3\det\nabla v_\varepsilon+\operatorname{o}(\varepsilon^4).$$

By using $\|\nabla v_\varepsilon(t)\|_{L^2(\Omega)}\le c$ and $\varepsilon\|\nabla v_\varepsilon(t)\|_{L^\infty(\Omega)}\le c$ for a.e. $t\in(0,T)$ we hence conclude that

$$\|\operatorname{tr}\nabla v_\varepsilon(t)\|_{L^1(\Omega)}\le\varepsilon\|\operatorname{tr}(\operatorname{cof}\nabla v_\varepsilon(t))\|_{L^1(\Omega)}+\varepsilon^2\|\det\nabla v_\varepsilon(t)\|_{L^1(\Omega)}\le c\varepsilon$$

for a.e. $t\in(0,T)$. By passing to the limit as $\varepsilon\to0$, this ensures that $\operatorname{tr}\nabla v=0$ a.e.

Fix now $t\in[0,T]$. By (2.7.4) we have

$$u_\varepsilon(t)\rightharpoonup u(t)\text{ in }H^1(\Omega;\mathbb{R}^d), \qquad (2.7.5)$$

where at this point the subsequence above may in general depend on t. However, we shall see that this is not the case by uniqueness of the limit (see below).

The linearized energy inequality (2.4.24) follows immediately from the energy inequality (2.4.22) at level ε, thanks to the $\liminf$-inequalities in Lemma 2.7.3 and to the continuity of $\dot\ell$.

The linearized semistability condition (2.4.25) on the other hand is more delicate, since it requires passing to the $\limsup$ on the right-hand side of the semistability condition (2.4.23) by choosing a suitable recovery sequence $\widetilde u_\varepsilon$. In the following, we will drop the indication of the time dependence (note that time is fixed in this statement) and simply denote $u_\varepsilon(t) = u_\varepsilon, v_\varepsilon(t) = v_\varepsilon, u(t) = u$, and $v(t) = v$, to simplify notation.

We start by showing that, for all fixed $\hat{u} \in H^1_{\Gamma_D}(\Omega; \mathbb{R}^d)$ one can choose a recovery sequence $(\tilde{u}_\varepsilon)_\varepsilon$ such that

$$0 \overset{(2.4.23)}{\leq} \limsup_{\varepsilon\to 0} \left(\mathcal{W}^\varepsilon_{\rm el}(\widetilde{u}_\varepsilon, v_\varepsilon) - \mathcal{W}^\varepsilon_{\rm el}(u_\varepsilon, v_\varepsilon)\right) \leq \mathcal{W}^0_{\rm el}(\hat{u}, v) - \mathcal{W}^0_{\rm el}(u, v). \tag{2.7.6}$$

With no loss of generality we can assume by density that $\hat{u}$ has the form

$$\hat{u} := u + \widetilde{u} \quad \text{ where } \widetilde{u} \in C^\infty_c(\Omega; \mathbb{R}^d).$$

As inequality (2.4.23) holds for every $\widetilde{u}_\varepsilon$ such that $(\widetilde{u}_\varepsilon, v_\varepsilon) \in \widetilde{A}_\varepsilon$, i.e., $\widetilde{u}_\varepsilon \in H^1_{\Gamma_D}(\Omega; \mathbb{R}^d)$, we can choose

$$\widetilde{u}_\varepsilon := \hat{u} + u_\varepsilon - u = \widetilde{u} + u_\varepsilon.$$

Notice that we have

$$\widetilde{u}_\varepsilon - u_\varepsilon = \widetilde{u} \quad \text{ and } \quad \widetilde{u}_\varepsilon + u_\varepsilon = \widetilde{u} + 2u_\varepsilon \rightharpoonup \widetilde{u} + 2u \quad \text{ in } H^1(\Omega; \mathbb{R}^d). \tag{2.7.7}$$

To check inequality (2.7.6) we need to show that

$$\begin{aligned} \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^2} & \left(\int_\Omega \left(W_{\rm el}((I + \varepsilon\nabla\widetilde{u}_\varepsilon)(I + \varepsilon\nabla v_\varepsilon)^{-1}) - W_{\rm el}((I + \varepsilon\nabla u_\varepsilon)(I + \varepsilon\nabla v_\varepsilon)^{-1})\right) {\rm d}X \right) \\ & \leq \int_\Omega \left(|\nabla(\hat{u} - v)|^2_{\mathbb{C}_{\rm el}} - |\nabla(u - v)|^2_{\mathbb{C}_{\rm el}}\right) {\rm d}X. \end{aligned} \tag{2.7.8}$$

Let us first study the limiting behaviour of the arguments of these energy densities. We define $(I + \varepsilon\nabla\widetilde{u}_\varepsilon)(I + \varepsilon\nabla v_\varepsilon)^{-1} = I + \varepsilon A_\varepsilon$, namely

$$\begin{aligned} A_\varepsilon &:= \frac{1}{\varepsilon}\left((I + \varepsilon\nabla\widetilde{u}_\varepsilon)(I + \varepsilon\nabla v_\varepsilon)^{-1} - I\right) \\ &= (\nabla\widetilde{u}_\varepsilon - \nabla v_\varepsilon) - \varepsilon\nabla\widetilde{u}_\varepsilon\nabla v_\varepsilon + \varepsilon(I + \varepsilon\nabla\widetilde{u}_\varepsilon)(\nabla v_\varepsilon)^2(I + \varepsilon\nabla v_\varepsilon)^{-1} \\ &= (\nabla\widetilde{u}_\varepsilon - \nabla v_\varepsilon) - \varepsilon\nabla\widetilde{u}_\varepsilon\nabla v_\varepsilon + M_\varepsilon + \varepsilon\nabla\widetilde{u}_\varepsilon M_\varepsilon, \end{aligned}$$

where we have set $M_\varepsilon := \varepsilon(\nabla v_\varepsilon)^2(I + \varepsilon\nabla v_\varepsilon)^{-1}$. Similarly, we can write $(I + \varepsilon\nabla u_\varepsilon)(I + \varepsilon\nabla v_\varepsilon)^{-1} = I + \varepsilon B_\varepsilon$ by letting

$$\begin{aligned} B_\varepsilon &:= \frac{1}{\varepsilon}\left((I + \varepsilon\nabla u_\varepsilon)(I + \varepsilon\nabla v_\varepsilon)^{-1} - I\right) \\ &= (\nabla u_\varepsilon - \nabla v_\varepsilon) - \varepsilon\nabla u_\varepsilon\nabla v_\varepsilon + M_\varepsilon + \varepsilon\nabla u_\varepsilon M_\varepsilon. \end{aligned}$$

Notice that by definition of M_ε and the fact that $I + \varepsilon\nabla v_\varepsilon \in K$ we have

$$\|\varepsilon M_\varepsilon\|_{L^\infty(\Omega)} \leq c, \quad \|M_\varepsilon\|_{L^1(\Omega)} \leq c\varepsilon\|(\nabla v_\varepsilon)\|_{L^2(\Omega)} \leq c\varepsilon.$$

This implies by interpolation that M_ε is also bounded in $L^2(\Omega; \mathbb{R}^{d\times d})$, hence $M_\varepsilon \rightharpoonup 0$ weakly in $L^2(\Omega; \mathbb{R}^{d\times d})$. Then, we have

$$A_\varepsilon - B_\varepsilon = (\nabla\widetilde{u}_\varepsilon - u_\varepsilon)(I - \varepsilon\nabla v_\varepsilon + \varepsilon M_\varepsilon) \overset{(2.7.7)}{=} \nabla\widetilde{u} + \nabla\widetilde{u}(-\varepsilon\nabla v_\varepsilon + \varepsilon M_\varepsilon) \to \nabla\widetilde{u} \text{ strongly in } L^2(\Omega; \mathbb{R}^{d\times d}),$$

since $\nabla\widetilde{u} \in C^\infty_c(\Omega; \mathbb{R}^{d\times d})$ is bounded in $L^\infty(\Omega; \mathbb{R}^{d\times d})$ and $(-\varepsilon\nabla v_\varepsilon + \varepsilon M_\varepsilon) \to 0$ strongly in $L^2(\Omega; \mathbb{R}^{d\times d})$. Moreover, by recalling (2.7.7) we have that

$$A_\varepsilon + B_\varepsilon = (\nabla\widetilde{u}_\varepsilon + u_\varepsilon)(I - \varepsilon\nabla v_\varepsilon + \varepsilon M_\varepsilon) - 2(\nabla v_\varepsilon - M_\varepsilon) \rightharpoonup \nabla\widetilde{u} + 2\nabla u - 2\nabla v \text{ weakly in } L^2(\Omega; \mathbb{R}^{d\times d}).$$

Fix now $\delta > 0$ and let $c_{\mathrm{el}}(\delta)$ be as in assumption (2.4.15). Let us define the set

$$\Omega_\varepsilon^\delta := \{x \in \Omega \mid \varepsilon|A_\varepsilon| + \varepsilon|B_\varepsilon| \le c_{\mathrm{el}}(\delta)\}$$

containing all points where $\varepsilon|A_\varepsilon|$ and $\varepsilon|B_\varepsilon|$ are small. Notice that

$$|\Omega \setminus \Omega_\varepsilon^\delta| = \int_{\Omega\setminus\Omega_\varepsilon^\delta} 1\mathrm{d}X \le \frac{\varepsilon^2}{c_{\mathrm{el}}^2(\delta)} \int_\Omega (|A_\varepsilon| + |B_\varepsilon|)^2 \,\mathrm{d}X \le c\frac{\varepsilon^2}{c_{\mathrm{el}}^2(\delta)} \to 0 \quad \text{as } \varepsilon \to 0 \tag{2.7.9}$$

since A_ε and B_ε are bounded in $L^2(\Omega;\mathbb{R}^{d\times d})$. We split the integrals in the left-hand side of (2.7.8) in the sum of the integrals on the sets $\Omega_\varepsilon^\delta$ and on the complementary sets $\Omega \setminus \Omega_\varepsilon^\delta$. By using assumption (2.4.15), on the sets $\Omega_\varepsilon^\delta$ we have

$$\begin{aligned}\frac{1}{\varepsilon^2}&\int_{\Omega_\varepsilon^\delta} (W_{\mathrm{el}}(I+\varepsilon A_\varepsilon) - W_{\mathrm{el}}(I+\varepsilon B_\varepsilon))\,\mathrm{d}X \\ &\le \int_\Omega \left(|A_\varepsilon|^2_{\mathbb{C}_{\mathrm{el}}} - |B_\varepsilon|^2_{\mathbb{C}_{\mathrm{el}}} + \delta(|A_\varepsilon|^2_{\mathbb{C}_{\mathrm{el}}} + |B_\varepsilon|^2_{\mathbb{C}_{\mathrm{el}}})\right)\mathrm{d}X. \end{aligned} \tag{2.7.10}$$

The first term in the right-hand side above can be treated as follows

$$\begin{aligned}\int_\Omega \left(|A_\varepsilon|^2_{\mathbb{C}_{\mathrm{el}}} - |B_\varepsilon|^2_{\mathbb{C}_{\mathrm{el}}}\right)\mathrm{d}X &= \frac{1}{2}\int_\Omega \mathbb{C}_{\mathrm{el}}(A_\varepsilon + B_\varepsilon) : (A_\varepsilon - B_\varepsilon))\mathrm{d}X \\ &\to \frac{1}{2}\int_\Omega \mathbb{C}_{\mathrm{el}}((\nabla\hat{u}-\nabla v)+(\nabla u-\nabla v)):((\nabla\hat{u}-\nabla v)-(\nabla u-\nabla v))\mathrm{d}X \\ &= \int_\Omega \left(|\nabla(\hat{u}-v)|^2_{\mathbb{C}_{\mathrm{el}}} - |\nabla(u-v)|^2_{\mathbb{C}_{\mathrm{el}}}\right)\mathrm{d}X\end{aligned}$$

by means of the strong convergence of $A_\varepsilon - B_\varepsilon \to \nabla\widetilde{u}$ and the weak convergence of $A_\varepsilon + B_\varepsilon \rightharpoonup \nabla\widetilde{u} + 2\nabla u - 2\nabla v$ in $L^2(\Omega;\mathbb{R}^{d\times d})$. On the other hand, the second term in the right-hand side of (2.7.10) satisfies

$$\int_\Omega \delta(|A_\varepsilon|^2_{\mathbb{C}_{\mathrm{el}}} + |B_\varepsilon|^2_{\mathbb{C}_{\mathrm{el}}})\mathrm{d}X \le \delta c$$

since A_ε and B_ε are bounded in $L^2(\Omega;\mathbb{R}^{d\times d})$.

Hence, it remains to show that the integrals in (2.7.8) on the complements $\Omega \setminus \Omega_\varepsilon^\delta$ converge to 0 as $\varepsilon \to 0$. In order to do so, let us define

$$F_1 := (I+\varepsilon\nabla u_\varepsilon)(I+\varepsilon\nabla v_\varepsilon)^{-1} \quad F_2 := \nabla\widetilde{u}(I+\varepsilon\nabla v_\varepsilon)^{-1}.$$

Since by definition $\nabla\widetilde{u}_\varepsilon = \nabla\widetilde{u} + \nabla u_\varepsilon$ and W is locally Lipschitz, we can write

$$\begin{aligned}\frac{1}{\varepsilon^2}&\int_{\Omega\setminus\Omega_\varepsilon^\delta} \left(W_{\mathrm{el}}((I+\varepsilon\nabla\widetilde{u}_\varepsilon)(I+\varepsilon\nabla v_\varepsilon)^{-1}) - W_{\mathrm{el}}((I+\varepsilon\nabla u_\varepsilon)(I+\varepsilon\nabla v_\varepsilon)^{-1})\right)\mathrm{d}X \\ &= \frac{1}{\varepsilon^2}\int_{\Omega\setminus\Omega_\varepsilon^\delta} |W_{\mathrm{el}}(F_1+\varepsilon F_2) - W_{\mathrm{el}}(F_1)|\mathrm{d}X \le \frac{1}{\varepsilon^2}\int_{\Omega\setminus\Omega_\varepsilon^\delta} \varepsilon|F_2|\mathrm{d}X \\ &\overset{(2.7.9)}{\le} \frac{c}{\varepsilon^2}\frac{\varepsilon^2}{c_{\mathrm{el}}^2(\delta)}\varepsilon \to 0,\end{aligned}$$

where we used that F_2 is uniformly bounded in $L^\infty(\Omega;\mathbb{R}^{d\times d})$. This concludes the proof of inequality (2.7.8). The check of linearized semistability (2.4.25) then follows as soon as one passes to the limit in the loading terms, which is straightforward.

In particular, we have proved that u solves the linear minimization problem

$$\mathcal{W}_{\mathrm{el}}^0(u(t),v(t)) - \langle \ell^0(t), u(t)\rangle = \operatorname*{arg\,min}_{\hat u \in H^1_{\Gamma_D}(\Omega;\mathbb{R}^d)} \mathcal{W}_{\mathrm{el}}^0(\hat u, v(t)) - \langle \ell^0(t), \hat u\rangle$$

for given v, thanks to (2.4.25). Hence, the limit u is unique and measurable in time, since it is the image of v through a linear operator. We also remark that this implies that subsequences in (2.7.5) can be chosen independently of t.

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3 Viscoelasticity and accretive phase-change at finite strains

This chapter consists of my publication [16] with Ulisse Stefanelli.

Abstract

We investigate the evolution of a two-phase viscoelastic material at finite strains. The phase evolution is assumed to be irreversible: One phase accretes in time in its normal direction, at the expense of the other. Mechanical response depends on the phase. At the same time, growth is influenced by the mechanical state at the boundary of the accreting phase, making the model fully coupled. This setting is inspired by the early stage development of solid tumors, as well as by the swelling of polymer gels. We formulate the evolution problem by coupling the balance of momenta in weak form and the growth dynamics in the viscosity sense. Both a diffused- and a sharp-interface variant of the model are proved to admit solutions and the sharp-interface limit is investigated.

3.1 Introduction

This paper is concerned with the evolution of a viscoelastic compressible solid undergoing phase change. We assume that the material presents two phases, of which one grows at the expense of the other by *accretion*. In particular, the phase-transition front evolves in a *normal* direction to the accreting phase, with a growth rate depending on the deformation. This behavior is indeed common to different material systems. It may be observed in the early stage development of solid tumors [9, 52, 106], where the neoplastic tissue invades the healthy one. Swelling in polymer gels also follows a similar dynamics, with the swollen phase accreting in the dry one [57, 98] and causing a volume increase. Accretive growth can be observed in some solidification processes [87, 102], as well.

The focus of the modelization is on describing the interplay between mechanical deformation and accretion. On the one hand, the two phases are assumed to have a different mechanical response, having an effect on the viscoelastic evolution of the medium. On the other hand, the time-dependent mechanical deformation is assumed to influence the growth process, as is indeed common in biomaterials [38], polymeric gels [107], and solidification [90]. The mechanical and phase evolutions are thus fully coupled.

The state of the system is described by the pair $(y, \theta)\colon [0,T]\times U \to \mathbb{R}^d\times[0,\infty)$, where $T > 0$ is some final time and $U \subset \mathbb{R}^d$ $(d \geq 2)$ is the reference configuration of the body. Here, y is the deformation of the medium while θ determines its phase. More precisely, for all $t \in [0,T]$ the *accreting (growing) phase* is identified as the sublevel $\Omega(t) := \{x \in U \mid \theta(x) < t\}$, whereas the *receding phase* corresponds to $U \setminus \overline{\Omega(t)}$. The value $\theta(x)$ formally corresponds to the time at which the point $x \in U$ is added to the growing phase. As such, θ is usually referred to as *time-of-attachment* function. An illustration of the notation is given in Figure 3.1.

As growth processes and mechanical equilibration typically occur on very different time scales, we neglect inertial effects and assume the evolution to be viscoelastic. This calls for specifying the stored energy density $W(\theta(x){-}t, \nabla y)$ and the viscosity $R(\theta(x){-}t, \nabla y, \nabla \dot{y})$ of the medium, as well as the applied body forces $f(\theta(x){-}t, x)$. All these quantities are assumed

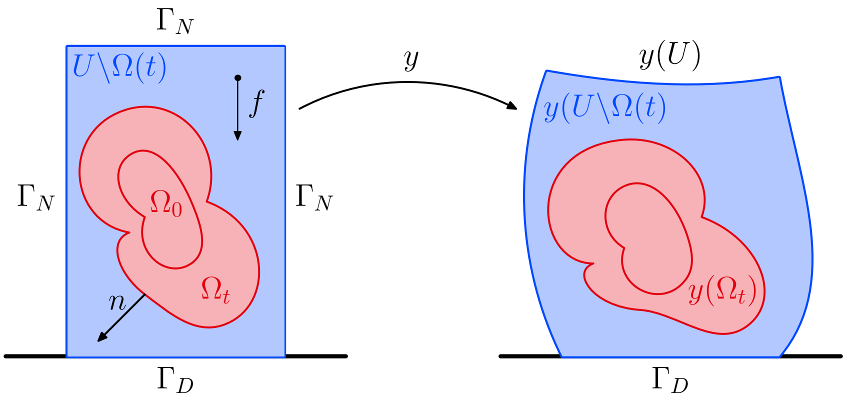

Figure 3.1: Illustration of the notation in the reference domain (left) and in the deformed one (right).

to be dependent on the phase via the *sign* of $\theta(x){-}t$, which indeed distinguishes the two phases, in the spirit of the celebrated *level-set method* [82, 91]. In addition, we include a *second-gradient* regularization term in the energy of the form $H(\nabla^2 y)$, which we take to be phase independent, for simplicity. All in all, the viscoelastic evolution system takes the form

$$-\operatorname{div}\left(\partial_{\nabla y}W(\theta(x){-}t,\nabla y)+\partial_{\nabla\dot y}R(\theta(x){-}t,\nabla y,\nabla\dot y)-\operatorname{div}\mathrm{D}H(\nabla^2 y)\right)=f(\theta(x){-}t,x). \tag{3.1.1}$$

This system is solved weakly, complemented by mixed boundary conditions on y and a homogeneous natural condition on the *hyperstress* $\mathrm{D}H(\nabla^2 y)$, namely,

$$y=\mathrm{id}\ \text{ on }\ [0,T]\times\Gamma_D, \tag{3.1.2}$$
$$\mathrm{D}H(\nabla^2 y){:}(\nu\otimes\nu)=0\ \text{ on }\ [0,T]\times\partial U, \tag{3.1.3}$$
$$\big(\partial_{\nabla y}W(\theta(x){-}t,\nabla y){+}\partial_{\nabla\dot y}R_\varepsilon(\theta(x){-}t,\nabla y,\nabla\dot y)\big)\,\nu - \operatorname{div}_S\left(\mathrm{D}H(\nabla^2 y)\nu\right)=0\ \text{ on }\ [0,T]\times\Gamma_N, \tag{3.1.4}$$

where ν is the outer unit normal to ∂U, Γ_D and Γ_N are the Dirichlet and Neumann part of the boundary ∂U, respectively, and div_S denotes the surface divergence on ∂U [75].

The viscoelastic evolution system is coupled to the phase evolution by requiring that the time-of-attachment function θ solves the generalized eikonal equation

$$\gamma\big(y(\theta(x)\wedge T,x),\nabla y(\theta(x)\wedge T,x)\big)|\nabla(-\theta)(x)|=1 \tag{3.1.5}$$

for all x in the complement of a given initial set $\Omega_0\subset\subset U$ where we set $\theta=0$. This corresponds to assuming that $\Omega(t)$ accretes in its normal direction, with *growth rate* $\gamma(\cdot)>0$. More precisely, the evolution of the generic point $x(t)$ on the boundary $\partial\Omega(t)$ follows the ODE flow

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t)=\gamma\big(y(t,x(t)),\nabla y(t,x(t))\big)\,\nu(x(t))$$

where $\nu(x(t))$ indicates the normal to $\partial\Omega(t)$ at $x(t)$. Accretive growth is paramount to a wealth of different biological models [97], including plants and trees [27, 31] and the formation of hard tissues like horns or shells [78, 93, 99]. The dependence of the growth rate γ on the actual position and strain is intended to model the possible influence of local features such as nutrient concentrations, as well as of the local mechanical state [38]. Note that accretive growth occurs in a variety of nonbiological systems, as well. These include crystallization [53, 105], sedimentation of rocks [35], glacier formation, accretion of celestial bodies [12], as well as technological applications, from epitaxial deposition [62], to coating, masonry, and 3D printing [37, 56], just to mention a few.

By assuming smoothness and differentiating the equation $\theta(x(t)) = t$ in time, one obtains the identity$\nabla\theta(x(t))\cdot\frac{\mathrm{d}}{\mathrm{d}t}x(t) = 1$. This, together with the above flow rule for $x(t)$ and the expression for the normal of the θ-sublevel sets $\nu(x(t)) = \nabla\theta(x(t))/|\nabla\theta(x(t))|$, originates the generalized eikonal equation (3.1.5). As the growth rate γ in (3.1.5) depends on the actual deformation $y(\theta(x)\wedge T, x)$ and strain $\nabla y(\theta(x)\wedge T, x)$ at the growing interface, system (3.1.1)–(3.1.5) is fully coupled.

We specify the initial conditions for the system by setting

$$\theta = 0 \ \text{ on } \Omega_0, \tag{3.1.6}$$

$$y(0,\cdot) = y_0 \ \text{ on } U, \tag{3.1.7}$$

where the initial deformation y_0 and the initial portion of the growing phase Ω_0 are given. Note that Ω_0 is not required to be connected, in order to possibly model the onset of the accreting phase at different sites. On the other hand, the evolution described by (3.1.5) does not preserve the topology and disconnected accreting regions may eventually coalesce over time.

The aim of this paper is to present an existence theory to the initial and boundary value problem (3.1.1)–(3.1.7). We tackle both a *sharp-interface* and a *diffused-interface* version of the model, by tuning the assumptions on W and R, see Sections 3.2.2–3.2.3. More precisely, in the diffused-interface model we assume that energy and dissipation densities change smoothly as functions of the phase indicator $\theta(x)-t$ across a region of width $\varepsilon > 0$, namely for $-\varepsilon/2 < \theta(x) - t < \varepsilon/2$. On the contrary, in the sharp-interface case material potentials are assumed to be discontinuous across the phase-change surface $\{\theta(x) = t\}$.

In both regimes, we prove that the fully coupled system (3.1.1)–(3.1.7) admits a weak/viscosity solution, see Definition 3.2.1 below. More precisely, the viscous evolution (3.1.1)–(3.1.4) is solved weakly, whereas the growth dynamics equation (3.1.5) is solved in the viscosity sense, see Theorem 3.2.1. We moreover prove that solutions fulfill the energy equality, where the energetic contribution of the phase transition is characterized, see Proposition 3.2.1. As a by-product, solutions of the diffused-interface model for $\varepsilon > 0$ are proved to converge up to subsequences to solutions of the sharp-interface model as the parameter ε converges to 0, see Corollary 3.2.1.

Before going on, let us mention that the engineering literature on growth mechanics is vast. Without any claim of completeness, we mention [94, 108] and [65, 79, 80, 95, 100] as examples of linearized and finite-strain theories, respectively. On the other hand, mathematical existence theories in growth mechanics are scant, and we refer to [6, 23, 36] for some recent results. To the best of our knowledge, no existence result for finite-strain accretive-growth mechanics is currently available. In the linearized case, an existence result for the model [108] has been obtained in [24].

The paper is structured as follows. Section 3.2 is devoted to the statement of the main existence result, Theorem 3.2.1. In Section 3.3, we give the proof of the energy identity. The proof of Theorem 3.2.1 is then split in Sections 3.4 and 3.5, respectively focusing on the diffused-interface and the sharp-interface setting. In the diffused-interface case, the proof relies on an iterative construction, where the mechanical and the growth problems are solved in alternation. The existence proof for the sharp-interface model is obtained by taking the limit as $\varepsilon \to 0$ in the diffused-interface model.

3.2 Main results

In this section, we specify assumptions, introduce the weak/viscosity notion of solution, and state the main results for problem (3.1.1)–(3.1.7).

3.2.1 Admissible deformations

Fix the final time $T > 0$ and let the reference configuration $U \subset \mathbb{R}^d$ $(d \geq 2)$ be nonempty, open, connected, and bounded. We assume that the boundary ∂U is Lipschitz, with Γ_D, $\Gamma_N \subset \partial U$ disjoint and open in the topology of ∂U, $\Gamma_D \neq \emptyset$ and $\overline{\Gamma_D} \cup \overline{\Gamma_N} = \partial U$, where the closure is taken in the topology of ∂U. In the following, we use the short-hand notation $Q := (0,T) \times U$ and $\Sigma_D := (0,T) \times \Gamma_D$.

Deformations are assumed to belong to the affine space

$$W^{2,p}_{\Gamma_D}(U;\mathbb{R}^d) := \left\{ y \in W^{2,p}(U;\mathbb{R}^d) \,|\, y = \text{id on } \Gamma_D \right\},$$

for almost all times and some given

$$p > d.$$

Moreover, we impose local invertibility and orientation preservation. The set of *admissible deformations* is hence defined as

$$\mathcal{A} := \left\{ y \in W^{2,p}_{\Gamma_D}(U;\mathbb{R}^d) \,\middle|\, \nabla y \in \mathrm{GL}_+(d) \text{ a.e. in } U \right\}.$$

3.2.2 Elastic energy

Let $\varepsilon \geq 0$ be given and $h_\varepsilon \in C^\infty(\mathbb{R};[0,1])$ for $\varepsilon > 0$ be nondecreasing functions such that

$$h_\varepsilon(\sigma) = \begin{cases} 0 & \text{if } \sigma \leq -\varepsilon/2, \\ 1 & \text{if } \sigma \geq \varepsilon/2, \end{cases} \qquad \|h'_\varepsilon\|_{L^\infty(\mathbb{R})} \leq \frac{2}{\varepsilon}. \tag{3.2.1}$$

Moreover, let h_0 be the discontinuous Heaviside-like function defined as $h_0(\sigma) = 0$ if $\sigma < 0$ and $h_0(\sigma) = 1$ if $\sigma \geq 0$. Note that $h_\varepsilon \to h_0$ in $\mathbb{R} \setminus \{0\}$ as $\varepsilon \to 0$.

We define the elastic energy density $W_\varepsilon : \mathbb{R} \times \mathrm{GL}_+(d) \to [0,\infty)$ of the medium as

$$W_\varepsilon(\sigma, F) := (1 - h_\varepsilon(\sigma))V^a(F) + h_\varepsilon(\sigma)V^r(F) + V^J(F). \tag{3.2.2}$$

Here, σ is a placeholder for $\theta(x) - t$, whose 0-sublevel set $\{x \in U \mid \theta(x) < t\}$ represents the accreting phase at time $t > 0$. In particular, $W_\varepsilon(\sigma, \cdot) = V^a + V^J$ for $\sigma < -\varepsilon/2$, so that $V^a + V^J$ is the elastic energy density of the accreting phase. On the other hand, $W_\varepsilon(\sigma, \cdot) = V^r + V^J$ for $\sigma > \varepsilon/2$ and $V^r + V^J$ is the elastic energy density of the receding phase.

On the elastic energy densities we require

$$V^a,\, V^r,\, V^J \in C^1(\mathrm{GL}_+(d);[0,\infty)), \tag{3.2.3}$$

$$\exists c_W > 0 : \quad V^a(F),\, V^r(F) \geq c_W |F|^p - \frac{1}{c_W},$$

$$V^a(F) - V^r(F) \leq \frac{1}{c_W}(1 + |F|^p) \quad \forall F \in \mathrm{GL}_+(d), \tag{3.2.4}$$

$$\exists q > \frac{pd}{p-d} : \quad V^J(F) \geq \frac{c_W}{(\det F)^q}. \tag{3.2.5}$$

The upper bound on $V^a - V^r$ in (3.2.4) will be instrumental in order to prove a control on the power associated with the phase transformation. In particular, if the receding phase has a higher energy density, namely, $V^r \geq V^a$, such upper bound trivially holds.

Although not strictly needed for the analysis we also require the frame indifference

$$V^a(QF) = V^a(F),\ V^r(QF) = V^r(F),\ V^J(QF) = V^J(F)\ \ \forall F \in \mathrm{GL}_+(d),\ Q \in \mathrm{SO}(d). \tag{3.2.6}$$

As regards the second-order potential H we ask for

$$H \in C^1(\mathbb{R}^{d\times d\times d}; [0,\infty)) \ \text{ convex}, \tag{3.2.7}$$

$$H(QG) = H(G) \ \text{ for all } \ G \in \mathbb{R}^{d\times d\times d},\ Q \in \mathrm{SO}(d), \tag{3.2.8}$$

$$\exists c_H > 0: \quad c_H|G|^p \leq H(G) \leq \frac{1}{c_H}(1+|G|)^p, \quad |\mathrm{D}H(G)| \leq \frac{1}{c_H}|G|^{p-1}, \tag{3.2.9}$$

$$c_H|G-\widehat{G}|^p \leq (\mathrm{D}H(G)-\mathrm{D}H(\widehat{G}))\vdots(G-\widehat{G}) \quad \forall G,\ \widehat{G} \in \mathbb{R}^{d\times d\times d}. \tag{3.2.10}$$

Again, the frame-indifference requirement (3.2.8) is not strictly needed for the analysis.

By integrating over the reference configuration U we define $\mathcal{W}_\varepsilon\colon C(\overline{U}) \times \mathcal{A} \to [0,\infty)$ and $\mathcal{H}\colon \mathcal{A} \to [0,\infty)$ as

$$\mathcal{W}_\varepsilon(\sigma,y) := \int_U W_\varepsilon(\sigma,\nabla y)\,\mathrm{d}x \quad \text{and} \quad \mathcal{H}(y) := \int_U H(\nabla^2 y)\,\mathrm{d}x.$$

3.2.3 Viscous dissipation

For $\varepsilon \geq 0$ given, set the instantaneous viscous dissipation density $R_\varepsilon : \mathbb{R} \times \mathrm{GL}_+(d) \times \mathbb{R}^{d\times d} \to [0,\infty)$ as

$$R_\varepsilon(\sigma,F,\dot{F}) := (1-h_\varepsilon(\sigma))R^a(F,\dot{F}) + h_\varepsilon(\sigma)R^r(F,\dot{F}) \tag{3.2.11}$$

Here, R^a, R^r : $\mathrm{GL}_+(d) \times \mathbb{R}^{d\times d} \to [0,\infty)$ are the instantaneous viscous dissipation densities of the accreting and of the receding phase, respectively. They are assumed to be quadratic in the rate $\dot{C} := \dot{F}^\top F + F^\top \dot{F}$ of the right Cauchy–Green tensor $C := F^\top F$, namely

$$R^a(F,\dot{F}) := \frac{1}{2}\dot{C}{:}\mathbb{D}^a(C){:}\dot{C}, \quad R^r(F,\dot{F}) := \frac{1}{2}\dot{C}{:}\mathbb{D}^r(C){:}\dot{C} \qquad \forall F \in \mathrm{GL}_+(d),\ \dot{F} \in \mathbb{R}^{d\times d}.$$

We assume that

$$\mathbb{D}^a,\ \mathbb{D}^r \in C(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d\times d\times d}) \ \text{ with } \ (\mathbb{D}^i)_{jk\ell m} = (\mathbb{D}^i)_{kj\ell m} = (\mathbb{D}^i)_{\ell m jk}$$
$$\forall j,\ k,\ \ell,\ m = 1,\dots,d, \ \text{for } i = a,\ r, \tag{3.2.12}$$

$$\exists c_{\mathbb{D}} > 0: \quad c_{\mathbb{D}}|\dot{C}|^2 \leq \dot{C}{:}\mathbb{D}^i(C){:}\dot{C} \quad \forall C,\ \dot{C} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \ \text{for } i = a,\ r. \tag{3.2.13}$$

Notice that this specific choice of R_ε ensures that

$$\begin{aligned}\partial_{\dot{F}}R_\varepsilon(\sigma,F,\dot{F}) &= 2(1-h_\varepsilon(\sigma))F\mathbb{D}^a(C){:}\dot{C} + 2h_\varepsilon(\sigma)F\mathbb{D}^r(C){:}\dot{C}\\ &= 2(1-h_\varepsilon(\sigma))F\mathbb{D}^a(F^\top F){:}(\dot{F}^\top F + F^\top\dot{F}) + 2h_\varepsilon(\sigma)F\mathbb{D}^r(F^\top F){:}(\dot{F}^\top F + F^\top\dot{F}),\end{aligned}$$

which is of course linear in $\dot{F}$. By integrating on the reference configuration U we define $\mathcal{R}_\varepsilon\colon C(\overline{U}) \times \mathcal{A} \times H^1(U;\mathbb{R}^d) \to [0,\infty)$ as

$$\mathcal{R}_\varepsilon(\sigma,y,\dot{y}) := \int_U R_\varepsilon(\sigma,\nabla y,\nabla\dot{y})\,\mathrm{d}x.$$

3.2.4 Loading and initial data

We assume that the body force density $f = f(\sigma, x)$ is (constant in time and) suitably smooth with respect to σ, namely

$$f \in W^{1,\infty}(\mathbb{R}; L^2(U; \mathbb{R}^d)), \tag{3.2.14}$$

The σ-dependence of the force density f is intended to cover the case of gravitation $f = \rho g$, where the density ρ depends on the phase, while the acceleration field g is given.

We moreover assume that the initial deformation y_0 satisfies

$$y_0 \in \mathcal{A} \ \text{ with } \int_U V^a(\nabla y_0) + V^r(\nabla y_0) + V^J(\nabla y_0) + H(\nabla^2 y_0)\, \mathrm{d}x < \infty. \tag{3.2.15}$$

3.2.5 Growth

Concerning the accretive-growth model we ask for

$$\gamma \in C^{0,1}(\mathbb{R}^d \times \mathrm{GL}_+(d)) \ \text{ with } \ c_\gamma \le \gamma(\cdot) \le C_\gamma \ \text{ for some } \ 0 < c_\gamma \le C_\gamma. \tag{3.2.16}$$

Let moreover the initial location of the accreting phase be given by

$$\emptyset \ne \Omega_0 \subset\subset U \ \text{ with } \ \Omega_0 \ \text{ open and } \ \Omega_0 + B_{C_\gamma T} \subset\subset U. \tag{3.2.17}$$

As it will be clarified later, this last requirement guarantees that the accreting phase does not touch the boundary ∂U over the time interval $[0, T]$, see (3.4.25) below.

3.2.6 Main results

Assumptions (3.2.1)–(3.2.17) will be assumed throughout the remainder of the paper. We are interested in solving (3.1.1)–(3.1.7) in the following weak/viscosity sense.

Definition 3.2.1 (Weak/viscosity solution)**.** *We say that a pair*

$$(y, \theta) \in \big(L^\infty(0, T; W^{2,p}(U; \mathbb{R}^d)) \cap H^1(0, T; H^1(U; \mathbb{R}^d))\big) \times C^{0,1}(\overline{U})$$

is a weak/viscosity solution *to the initial-boundary-value problem* (3.1.1)–(3.1.7) *if* $y(t, \cdot) \in \mathcal{A}$ *for all* $t \in (0, T)$*,* $y(0, \cdot) = y_0$*, and*

$$\int_0^T \!\! \int_U \big(\partial_F W_\varepsilon(\theta{-}t, \nabla y){:}\nabla z{+}\partial_{\dot F} R_\varepsilon(\theta{-}t, \nabla y, \nabla \dot y){:}\nabla z + \mathrm{D}H\left(\nabla^2 y\right) \vdots \nabla^2 z\big)\, \mathrm{d}x\, \mathrm{d}t$$
$$= \int_0^T \!\! \int_U f(\theta{-}t){\cdot}z\, \mathrm{d}x\, \mathrm{d}t \quad \forall z \in C^\infty(\overline{Q}; \mathbb{R}^d) \text{ with } z = 0 \text{ on } \Sigma_D, \tag{3.2.18}$$

and θ *is a viscosity solution to*

$$\gamma\big(y(\theta(x) \wedge T, x), \nabla y(\theta(x) \wedge T, x)\big)|\nabla(-\theta)(x)| = 1 \ \text{ in } \ U \setminus \overline{\Omega_0}, \tag{3.2.19}$$
$$\theta = 0 \ \text{ in } \ \Omega_0. \tag{3.2.20}$$

Namely, θ *satisfies* (3.2.20)*, and, for all* $x_0 \in U \setminus \overline{\Omega_0}$ *and any smooth function* $\varphi : U \to \mathbb{R}$ *with* $\varphi(x_0) = -\theta(x_0)$ *and* $\varphi \ge -\theta$ *(*$\varphi \le -\theta$*, respectively) in a neighborhood of* x_0*, it holds that* $\gamma(y(\theta(x_0) \wedge T, x_0), \nabla y(\theta(x_0) \wedge T, x_0))|\nabla\varphi(x_0))| \le 1 (\ge 1,$ *respectively*)*. Moreover, we ask that*

$$0 < \frac{1}{C_\gamma} \le |\nabla\theta| \le \frac{1}{c_\gamma} \ \text{ a.e. in } \ U. \tag{3.2.21}$$

Note that this weak notion of solution in Definition 3.2.1 still entails the validity of an energy equality. Namely, we have the following.

Proposition 3.2.1 (Energy equality)**.** *Under assumptions* (3.2.1)–(3.2.17)*, in the diffused-interface case* $\varepsilon > 0$ *a weak/viscosity solution* (y, θ) *fulfills for all* $t \in [0, T]$ *the energy equality*

$$\begin{aligned}\int_U \big(W_\varepsilon(\theta - t, \nabla y) + H(\nabla^2 y) - f(\theta - t)\cdot y\big)\,\mathrm{d}x - \int_U \big(W_\varepsilon(\theta, \nabla y_0) + H(\nabla^2 y_0) - f(\theta)\cdot y_0\big)\,\mathrm{d}x \\ = -2\int_0^t\int_U R_\varepsilon\,(\theta - s, \nabla y, \nabla \dot{y})\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_U \dot{f}(\theta - s)\cdot y\,\mathrm{d}x\,\mathrm{d}s \\ - \int_0^t\int_U \partial_\sigma W_\varepsilon(\theta - s, \nabla y)\,\mathrm{d}x\,\mathrm{d}s. \qquad (3.2.22)\end{aligned}$$

In the sharp-interface case $\varepsilon = 0$*, for all* $t \in [0, T]$*, one has instead*

$$\begin{aligned}\int_U \big(W_0(\theta - t, \nabla y) + H(\nabla^2 y) - f(\theta - t)\cdot y\big)\,\mathrm{d}x - \int_U \big(W_0(\theta, \nabla y_0) + H(\nabla^2 y_0) - f(\theta)\cdot y_0\big)\,\mathrm{d}x \\ = -2\int_0^t\int_U R_0\,(\theta - s, \nabla y, \nabla \dot{y})\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_U \dot{f}(\theta - s)\cdot y\,\mathrm{d}x\,\mathrm{d}s \\ - \int_0^t\int_{\{\theta = s\}} \frac{V^r(\nabla y) - V^a(\nabla y)}{|\nabla \theta|}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}s. \qquad (3.2.23)\end{aligned}$$

Relations (3.2.22)–(3.2.23) express the energy balance in the model. In particular, the difference between the actual and the initial complementary energies (left-hand side in (3.2.22)–(3.2.23)) equals the sum of the total viscous dissipation, the work of external forces, and the energy stored in the medium in connection with the phase-transition process (the three terms in the right-hand side of (3.2.22)–(3.2.23), up to signs). Proposition 3.2.1 is proved in Section 3.3.

Our main result reads as follows.

Theorem 3.2.1 (Existence)**.** *Under assumptions* (3.2.1)–(3.2.17)*, for all given* $\varepsilon \geq 0$ *there exists a weak/viscosity solution* (y, θ) *of problem* (3.1.1)–(3.1.7)*.*

A proof of Theorem 3.2.1 in the diffused-interface case of $\varepsilon > 0$ is based on an iterative strategy: for given y^k one finds a viscosity solution θ^k to (3.2.19)–(3.2.20) (with y replaced by y^k). Then, given θ^k one can find y^{k+1} satisfying (3.2.18) (with θ replaced by θ^k). Note that such y^{k+1} may be nonunique. As the set of solutions y for given θ is generally not convex, we do not proceed via a fixed-point argument for multivalued maps (see, e.g., [47]) but rather resort in directly proving the convergence of the iterative procedure. This argument is detailed in Section 3.4.

Eventually, the proof of Theorem 3.2.1 in the sharp-interface case $\varepsilon = 0$ will be obtained in Section 3.5 by passing to the limit as $\varepsilon \to 0$ along a subsequence of weak/viscosity solutions $(y_\varepsilon, \theta_\varepsilon)$ for $\varepsilon > 0$. As a by-product, we have the following.

Corollary 3.2.1 (Sharp-interface limit)**.** *Under assumptions* (3.2.1)–(3.2.17)*, let* $(y_\varepsilon, \theta_\varepsilon)$ *be weak/viscosity solutions of the diffused-interface problem* (3.1.1)–(3.1.7) *for* $\varepsilon > 0$*. Then, there exists a not relabeled subsequence such that* $(y_\varepsilon, \theta_\varepsilon) \to (y, \theta)$ *uniformly, where* (y, θ) *is a weak/viscosity solution to the sharp-interface problem for* $\varepsilon = 0$*.*

Before moving on, let us mention that the assumptions on the energy and of the instantaneous viscous dissipation density could be generalized by not requiring the specific forms (3.2.2) and (3.2.11). In fact, one could directly assume to be given $W_\varepsilon = W_\varepsilon(\sigma, F)$ and $R_\varepsilon = R_\varepsilon(\sigma, F, \dot F)$ of the form

$$R_\varepsilon(\sigma, F, \dot F) = \frac{1}{2}\dot C{:}\mathbb{D}(\sigma, C){:}\dot C$$

with $\mathbb{D} \in C(\mathbb{R} \times \mathbb{R}^{d\times d}_{\rm sym}; \mathbb{R}^{d\times d\times d\times d})$ by suitably adapting the smoothness and coercivity assumptions. Although the existence analysis could be carried out in this more general situation with no difficulties, we prefer to stick to the concrete choice of (3.2.2) and (3.2.11) as it allows a more transparent distinction of the diffused- and sharp-interface cases.

Moreover, let us point out that admissible deformations y are presently required to be solely *locally* injective, by means of the constraint $\det \nabla y > 0$. On the other hand, *global* injectivity may also be enforced, in the spirit of [48], see also [83] in the static and [13, 14] in the dynamic case. This however calls for keeping track of reaction forces due to a possible self contact at the boundary Γ_N. From the technical viewpoint, one would need to include an extra variable in the state in order to model such reaction. The existence theory of Theorem 3.2.1 can be extended to cover this case, at the price of some notational intricacies. We however prefer to avoid discussing global injectivity here, for the sake of exposition clarity.

3.3 Proof of Proposition 3.2.1: energy equalities

We firstly consider the diffused-interface setting of $\varepsilon > 0$. Let (y, θ) be a weak/viscosity solution to (3.1.1)–(3.1.7). In order to deduce the energy equality, the Euler-Lagrange equation (3.2.18) should be tested by $\dot y$. This however requires some care, as $\dot y$ is not regular enough to use it as test function. We follow the argument of [75], based on the validity of a chain rule for the functional $\mathcal{H}$. In particular, we start by checking that (3.2.18) can be equivalently rewritten as

$$\partial_2\mathcal{W}_\varepsilon(\theta{-}t, y) + \partial_3\mathcal{R}_\varepsilon(\theta{-}t, y, \dot y) + \partial\mathcal{H}(y) \ni \widehat f \quad \text{in } (H^1_{\Gamma_D}(U;\mathbb{R}^d))^*, \text{ a.e. in } (0,T). \quad (3.3.1)$$

Here, $(H^1_{\Gamma_D}(U;\mathbb{R}^d))^*$ indicates the dual of $H^1_{\Gamma_D}(U;\mathbb{R}^d) := \{z \in H^1(U;\mathbb{R}^d) \mid z = 0 \text{ on } \Gamma_D\}$, the symbol ∂ denotes the (possibly partial) subdifferential from $H^1_{\Gamma_D}(U;\mathbb{R}^d)$ to $(H^1_{\Gamma_D}(U;\mathbb{R}^d))^*$ and $\widehat f\colon (0,T) \to (H^1_{\Gamma_D}(U;\mathbb{R}^d))^*$ is given by

$$\langle \widehat f(t), z\rangle := \int_U f(\sigma{-}t){\cdot}z \,\mathrm{d}x \quad \forall z \in H^1_{\Gamma_D}(U;\mathbb{R}^d)$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between $(H^1_{\Gamma_D}(U;\mathbb{R}^d))^*$ and $H^1_{\Gamma_D}(U;\mathbb{R}^d)$. Indeed, owing to the fact that $\nabla y \in L^\infty(Q)$ and $\nabla \dot y \in L^2(Q)$ and using the regularities (3.2.3), (3.2.12), and (3.2.14) one can check that

$$\langle \partial_2\mathcal{W}_\varepsilon(\theta{-}t, y), z\rangle = \int_U \partial_F W_\varepsilon(\theta{-}t, \nabla y){:}\nabla z \quad \forall z \in H^1_{\Gamma_D}(U;\mathbb{R}^d),$$
$$\langle \partial_3\mathcal{R}_\varepsilon(\theta{-}t, y, \dot y), z\rangle = \int_U \partial_{\dot F} R_\varepsilon(\theta{-}t, \nabla y, \nabla\dot y){:}\nabla z \quad \forall z \in H^1_{\Gamma_D}(U;\mathbb{R}^d),$$

and that $\Sigma = \widehat f - \partial_2\mathcal{W}_\varepsilon(\theta{-}t, y) - \partial_3\mathcal{R}_\varepsilon(\theta{-}t, y, \dot y) \in L^2(0,T; (H^1_{\Gamma_D}(U;\mathbb{R}^d))^*)$. On the other hand, using equation (3.2.18), the fact that $y \in L^p(0,T; W^{2,p}(\Omega;\mathbb{R}^d))$, and the convexity (3.2.7) of H we get

$$\int_0^T \langle \Sigma, w - y\rangle \,\mathrm{d}t \overset{(3.2.18)}{=} \int_0^T\int_U \mathrm{D}H(\nabla^2 y)\vdots\nabla^2(w-y)\,\mathrm{d}t \le \int_0^T \big(\mathcal{H}(w) - \mathcal{H}(y)\big)\,\mathrm{d}t$$

for all $w \in L^p(0,T;W^{2,p}(\Omega;\mathbb{R}^d)) \cap L^2(0,T;H^1_{\Gamma_D}(U;\mathbb{R}^d))$. This in particular implies that $\Sigma \in \partial\mathcal{H}(y)$ a.e. in $(0,T)$, whence the abstract equation (3.3.1) follows and the chain rule [75, Prop. 3.6] entails that $\mathcal{H}(y) \in W^{1,1}(0,T)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(y) = \langle \Sigma, \dot{y}\rangle \quad \text{a.e. in } (0,T). \tag{3.3.2}$$

Note that all terms in (3.3.1) belong to $L^2(0,T;(H^1_{\Gamma_D}(U;\mathbb{R}^d))^*)$. One can hence test (3.3.1) on $\dot{y} \in L^2(0,T;H^1_{\Gamma_D}(U;\mathbb{R}^d))$ and deduce that

$$\begin{aligned}\int_0^t\!\!\int_U \partial_F W_\varepsilon(\theta - s, \nabla y){:}\nabla\dot{y}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_U \partial_{\dot{F}} R_\varepsilon(\theta - s, \nabla y, \nabla\dot{y}){:}\nabla\dot{y}\,\mathrm{d}x\,\mathrm{d}s \\ + \int_U H(\nabla^2 y(t))\,\mathrm{d}x - \int_U H(\nabla^2 y_0)\,\mathrm{d}x = \int_0^t\!\!\int_U f(\theta - s)\cdot\dot{y}\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{3.3.3}$$

We readily have that

$$\begin{aligned}\int_U W_\varepsilon(\theta - t, \nabla y)\,\mathrm{d}x - \int_U W_\varepsilon(\theta, \nabla y_0)\,\mathrm{d}x = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\int_U \partial_F W_\varepsilon(\theta - s, \nabla y)\,\mathrm{d}x\,\mathrm{d}s \\ = \int_0^t\!\!\int_U \partial_F W_\varepsilon(\theta - s, \nabla y){:}\nabla\dot{y}\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\!\!\int_U \partial_\sigma W_\varepsilon(\theta - s, \nabla y)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{3.3.4}$$

Moreover, it is a standard matter to check that $\partial_{\dot{F}} R_\varepsilon(\sigma, F, \dot{F}){:}\dot{F} = 2R_\varepsilon(\sigma, F, \dot{F})$, so that

$$\int_0^t\!\!\int_U \partial_{\dot{F}} R_\varepsilon(\theta - s, \nabla y, \nabla\dot{y}){:}\nabla\dot{y}\,\mathrm{d}x\,\mathrm{d}s = 2\int_0^t\!\!\int_U R_\varepsilon(\theta - s, \nabla y, \nabla\dot{y})\,\mathrm{d}x\,\mathrm{d}s, \tag{3.3.5}$$

whence the energy equality (3.2.22) in the diffused-interface case $\varepsilon > 0$ follows from (3.3.3).

The proof of energy equality (3.2.23) for the sharp-interface case $\varepsilon = 0$ follows the same strategy, as one can again establish (3.3.3) (for W_0 and R_0 in place of W_ε and R_ε) and (3.3.2). A notable difference is however in (3.3.4), which now requires some extra care as h_0 is discontinuous. In particular, the energy equality (3.2.23) follows as soon as we prove that

$$\begin{aligned}&\int_U W_0(\theta - t, \nabla y)\,\mathrm{d}x - \int_U W_0(\theta, \nabla y_0)\,\mathrm{d}x \\ &= \int_0^t\!\!\int_U \partial_F W_0(\theta - s, \nabla y){:}\nabla\dot{y}\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\!\!\int_{\{\theta = s\}} \frac{V^r(\nabla y) - V^a(\nabla y)}{|\nabla\theta|}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}s.\end{aligned} \tag{3.3.6}$$

The remainder of this section is devoted to check (3.3.6).

To start with, let a nonnegative and even function $\rho \in C^\infty(\mathbb{R})$ be given with support in $[-1,1]$ and with $\int_{\mathbb{R}} \rho(s)\,\mathrm{d}s = 1$. For $\varepsilon > 0$ we define $\rho_\varepsilon(t) := \rho(t/\varepsilon)/\varepsilon$ and $\eta_\varepsilon(t) := \int_{-1}^t \rho_\varepsilon(s)\,\mathrm{d}s$ for all $t \in \mathbb{R}$. As $\eta_\varepsilon \to h_0$ in $\mathbb{R} \setminus \{0\}$, by letting

$$G_\varepsilon(t) := \int_U \Big(V^a(\nabla y(t,x)) + \eta_\varepsilon(\theta(x) - t)\big(V^r(\nabla y(t,x)) - V^a(\nabla y(t,x))\big) + V^J(\nabla y(t,x))\Big)\,\mathrm{d}x$$

we readily check that

$$\lim_{\varepsilon\to 0} G_\varepsilon(t) := \int_U W_0(\theta(x) - t, \nabla y(t,x))\,\mathrm{d}x \tag{3.3.7}$$

for all $t \in [0,T]$. As $G_\varepsilon \in H^1(0,T)$ we can compute its time derivative at almost all times getting

$$\frac{\mathrm{d}}{\mathrm{d}t}G_\varepsilon(t) = \int_U \Big(\partial_F V^a(\nabla y) + \eta_\varepsilon(\theta{-}t)\big(\partial_F V^r(\nabla y){-}\partial_F V^a(\nabla y)\big) + \partial_F V^J(\nabla y)\Big){:}\nabla\dot y\,\mathrm{d}x - \int_U \rho_\varepsilon(\theta{-}t)\big(V^r(\nabla y){-}V^a(\nabla y)\big)\,\mathrm{d}x.$$

By integrating in time, taking the limit $\varepsilon \to 0$, and using (3.3.7), one hence gets

$$\begin{aligned}\int_U W_0(\theta{-}t,\nabla y)\,\mathrm{d}x - \int_U W_0(\theta,\nabla y_0)\,\mathrm{d}x &= \lim_{\varepsilon\to 0}\big(G_\varepsilon(t) - G_\varepsilon(0)\big) = \lim_{\varepsilon\to 0}\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}G_\varepsilon(s)\,\mathrm{d}s\\
&= \lim_{\varepsilon\to 0}\int_0^t\int_U \Big(\partial_F V^a(\nabla y) + \eta_\varepsilon(\theta{-}s)\big(\partial_F V^r(\nabla y){-}\partial_F V^a(\nabla y)\big) + \partial_F V^J(\nabla y)\Big){:}\nabla\dot y\,\mathrm{d}x\,\mathrm{d}s\\
&\quad - \lim_{\varepsilon\to 0}\int_0^t\int_U \rho_\varepsilon(\theta{-}s)\big(V^r(\nabla y){-}V^a(\nabla y)\big)\,\mathrm{d}x\,\mathrm{d}s\\
&= \int_0^t\int_U \partial_F W_0(\theta{-}s,\nabla y){:}\nabla\dot y\,\mathrm{d}x\,\mathrm{d}s - \lim_{\varepsilon\to 0}\int_0^t\int_U \rho_\varepsilon(\theta{-}s)\big(V^r(\nabla y){-}V^a(\nabla y)\big)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

In order to prove (3.3.6) it is hence sufficient to check that

$$\lim_{\varepsilon\to 0}\int_0^t\int_U \rho_\varepsilon(\theta{-}s)\big(V^r(\nabla y){-}V^a(\nabla y)\big)\,\mathrm{d}x\,\mathrm{d}s = \int_0^t\int_{\{\theta=s\}} \frac{V^r(\nabla y){-}V^a(\nabla y)}{|\nabla\theta|}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}s. \tag{3.3.8}$$

By introducing the short-hand notation $g = V^r(\nabla y) - V^a(\nabla y)$ and by using the coarea formula [29, Sec. 3.2.11] (recall that θ is Lipschitz continuous and $|\nabla\theta| \geq 1/C_\gamma > 0$ almost everywhere, see (3.2.21)) we can compute

$$\begin{aligned}\int_0^t\int_U \rho_\varepsilon(\theta{-}s)\big(V^r(\nabla y){-}V^a(\nabla y)\big)\,\mathrm{d}x\,\mathrm{d}s &= \int_0^t\int_{\mathbb{R}}\int_{\{\theta=r\}} \rho_\varepsilon(\theta(x){-}s)\frac{g(s,x)}{|\nabla\theta(x)|}\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}r\,\mathrm{d}s\\
&= \int_0^t\int_{\mathbb{R}}\int_{\{\theta=r\}} \rho_\varepsilon(r{-}s)\frac{g(r,x)}{|\nabla\theta(x)|}\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}r\,\mathrm{d}s\\
&\quad + \int_0^t\int_{\mathbb{R}}\int_{\{\theta=r\}} \rho_\varepsilon(r{-}s)\frac{g(s,x){-}g(r,x)}{|\nabla\theta(x)|}\mathrm{d}\mathcal{H}^{d-1}(x)\;\mathrm{d}r\,\mathrm{d}s.\end{aligned} \tag{3.3.9}$$

The coarea formula and the bound $|\nabla\theta| \leq 1/c_\gamma$ (see again (3.2.21)) ensure that $r \in \mathbb{R} \mapsto m(r) := \mathcal{H}^{d-1}(\{\theta = r\})$ is integrable. Indeed,

$$\|m\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} \mathcal{H}^{d-1}(\{\theta = r\})\,\mathrm{d}r = \int_U |\nabla\theta|\,\mathrm{d}x < \infty.$$

As $g/|\nabla\theta|$ is bounded, setting

$$r \in \mathbb{R} \mapsto \ell(r) := \int_{\{\theta=r\}} \frac{g(r,x)}{|\nabla\theta(x)|}\mathrm{d}\mathcal{H}^{d-1}(x)$$

one has that $\ell \in L^1(\mathbb{R})$, as well, since

$$\|\ell\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}}\int_{\theta=r} \frac{|g(r,x)|}{|\nabla\theta(x)|}\,\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}r \leq \sup\frac{|g|}{|\nabla\theta|}\|m\|_{L^1(\mathbb{R})} < \infty.$$

Moreover, we have that

$$\int_{\mathbb{R}}\int_{\{\theta=r\}} \rho_\varepsilon(r-s)\frac{g(r,x)}{|\nabla\theta(x)|}\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}r = \int_{\mathbb{R}} \rho_\varepsilon(s-r)\left(\int_{\{\theta=r\}}\frac{g(r,x)}{|\nabla\theta(x)|}\mathrm{d}\mathcal{H}^{d-1}(x)\right)\,\mathrm{d}r = (\rho_\varepsilon * \ell)(s)$$

where we used that ρ_ε is even and where the symbol $*$ stands for the usual convolution in $\mathbb{R}$. As $\rho_\varepsilon * \ell \to \ell$ strongly in $L^1(\mathbb{R})$ for $\varepsilon \to 0$, one can pass to the limit in the first term on the right-hand side of (3.3.9) and get

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^t\int_{\mathbb{R}}\int_{\{\theta=r\}} \rho_\varepsilon(r-s)\frac{g(r,x)}{|\nabla\theta(x)|}\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}r\,\mathrm{d}s = \lim_{\varepsilon\to 0}\int_0^t(\rho_\varepsilon*\ell)\,\mathrm{d}s = \int_0^t \ell\,\mathrm{d}s \\ = \int_0^t\int_{\{\theta=s\}}\frac{g(s,x)}{|\nabla\theta(x)|}\,\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}s = \int_0^t\int_{\{\theta=s\}}\frac{V^r(\nabla y)-V^a(\nabla y)}{|\nabla\theta|}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}s. \qquad (3.3.10)\end{aligned}$$

As regards the second term in the right-hand side of (3.3.9), notice that $\rho_\varepsilon(r-s) \neq 0$ only if $|r-s| \leq 2\varepsilon$. Hence, using the Hölder regularity of g and the boundedness of $1/|\nabla\theta|$ and $|\nabla\theta|$, we conclude that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left|\int_0^t\int_{\mathbb{R}}\int_{\{\theta=r\}} \rho_\varepsilon(r-s)\frac{g(s,x)-g(r,x)}{|\nabla\theta(x)|}\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}r\,\mathrm{d}s\right| \\ &\leq \lim_{\varepsilon\to 0} c\,\varepsilon^\alpha\int_0^t\int_{\mathbb{R}}\rho_\varepsilon(r-s)\,\mathcal{H}^{d-1}(\{\theta=r\})\,\mathrm{d}r\,\mathrm{d}s \\ &= \lim_{\varepsilon\to 0} c\,\varepsilon^\alpha\|\rho_\varepsilon * m\|_{L^1(\mathbb{R})} \leq \lim_{\varepsilon\to 0} c\,\varepsilon^\alpha\|m\|_{L^1(\mathbb{R})} = 0 \qquad (3.3.11)\end{aligned}$$

for some $\alpha \in (0,1)$. Relations (3.3.10)–(3.3.11) imply that the limit (3.3.8) holds true. This in turn proves (3.3.6) and the energy equality (3.2.23) follows.

3.4 Proof of Theorem 3.2.1: diffused-interface case

Let $\varepsilon > 0$ be fixed. We prove existence of a weak/viscous solution (y,θ) by an iterative construction. We start by proving that for all given $\theta \in C(\overline{U})$ there exists an admissible deformation y satisfying (3.2.18).

Proposition 3.4.1 (Existence of y given θ)**.** *Set $\varepsilon > 0$ and let $\theta \in C(\overline{U})$ be fixed with $\Omega(T) \subset\subset U$. Under assumptions* (3.2.1)–(3.2.15) *there exists $y \in L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d)) \cap H^1(0,T;H^1(U;\mathbb{R}^d))$ with $y(t,\cdot) \in \mathcal{A}$ for every $t \in (0,T)$ satisfying* (3.2.18)*. More precisely, there exists a positive constant c depending on data but independent of ε and θ such that*

$$\|y\|_{L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d))\cap H^1(0,T;H^1(U;\mathbb{R}^d))} \leq c. \qquad (3.4.1)$$

Proof. The assertion follows by adapting the arguments in [4] or [48]. We proceed by time discretization. Let the time step $\tau := T/N_\tau$ with $N_\tau \in \mathbb{N}$ be given and let $t_i := i\tau$, for $i = 0,\dots,N_\tau$ be the corresponding uniform partition of the time interval $[0,T]$. Within this proof, the generic constant c is always independent of the given θ, as well.

For $i = 1,\dots,N_\tau$, we define $y_\tau^i \in \mathcal{A}$ via

$$y_\tau^i \in \operatorname*{arg\,min}_{y\in\mathcal{A}}\left\{\mathcal{W}_\varepsilon(\theta-t_i,y) + \mathcal{H}(y) + \tau\mathcal{R}_\varepsilon\left(\theta-t_i, y_\tau^{i-1}, \frac{y-y_\tau^{i-1}}{\tau}\right) - \int_U f(\theta-t_i)\cdot y\,\mathrm{d}x\right\}.$$

Under the growth conditions (3.2.4)–(3.2.5), (3.2.9), and (3.2.13), and the regularity and convexity assumptions (3.2.3), (3.2.7), (3.2.12), and (3.2.14), the existence of y_τ^i for every $i = 1,\dots,N_\tau$ follows by the Direct Method of the calculus of variations. Moreover, every minimizer y_τ^i satisfies the time-discrete Euler–Lagrange equation

$$\int_U \left(\partial_F W_\varepsilon(\theta{-}t_i, \nabla y_\tau^i){+}\partial_{\dot F} R_\varepsilon\left(\theta{-}t_i, \nabla y_\tau^{i-1}, \frac{\nabla y_\tau^i{-}\nabla y_\tau^{i-1}}{\tau}\right)\right) {:}\nabla z^i \,\mathrm{d}x \\ + \int_U \mathrm{D}H\left(\nabla^2 y_\tau^i\right) \vdots \nabla^2 z^i \,\mathrm{d}x = \int_U f(\theta{-}t_i){\cdot}z\,\mathrm{d}x \tag{3.4.2}$$

for every $z^i \in \mathcal{A}$.

Let us introduce the following notation for the time interpolants on the partition: Given a vector $(u_0, ..., u_{N_\tau})$, we define its backward-constant interpolant $\overline{u}_\tau$, its forward-constant interpolant $\underline{u}_\tau$, and its piecewise-affine interpolant $\widehat{u}_\tau$ on the partition $(t_i)_{i=0}^{N_\tau}$ as

$$\begin{aligned}
&\overline{u}_\tau(0) := u_0, && \overline{u}_\tau(t) := u_i && \text{if } t \in (t_{i-1}, t_i] && \text{for } i = 1,\dots,N_\tau,\\
&\underline{u}_\tau(T) := u_{N_\tau}, && \underline{u}_\tau(t) := u_{i-1} && \text{if } t \in [t_{i-1}, t_i) && \text{for } i = 1,\dots,N_\tau,\\
&\widehat{u}_\tau(0) := u_0, && \widehat{u}_\tau(t) := \frac{u_i - u_{i-1}}{t_i - t_{i-1}}(t - t_{i-1}) + u_{i-1} && \text{if } t \in (t_{i-1}, t_i] && \text{for } i = 1,\dots,N_\tau.
\end{aligned}$$

Owing to this notation, we can take the sum in (3.4.2) for $i = 1,\dots,N_\tau$ and equivalently rewrite the discrete Euler–Lagrange equation in the compact form

$$\int_0^T\!\!\int_U \left(\partial_F W_\varepsilon(\theta{-}\bar t_\tau, \nabla \overline{y}_\tau){+}\partial_{\dot F} R_\varepsilon\left(\theta{-}\bar t_\tau, \nabla \underline{y}_\tau, \nabla \dot{\widehat{y}}_\tau\right)\right) {:}\nabla \overline{z}_\tau \,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^T\!\!\int_U \mathrm{D}H\left(\nabla^2 \overline{y}_\tau\right) \vdots \nabla^2 \overline{z}_\tau \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\!\!\int_U f(\theta{-}\bar t_\tau){\cdot}\overline{z}_\tau \,\mathrm{d}x\,\mathrm{d}t \tag{3.4.3}$$

where $\overline{z}_\tau$ is the backward-constant interpolant of $(z^i)_{i=1}^{N_\tau}$.

From the minimality of y_τ^i we get that

$$\begin{aligned}
&\int_U W_\varepsilon(\theta{-}t_i, \nabla y_\tau^i)\,\mathrm{d}x + \int_U H(\nabla^2 y_\tau^i)\,\mathrm{d}x - \int_U f(\theta{-}t_i){\cdot}y_\tau^i\,\mathrm{d}x \\
&\quad + \tau \int_U R_\varepsilon\left(\theta{-}t_i, \nabla y_\tau^{i-1}, \frac{\nabla y_\tau^i{-}\nabla y_\tau^{i-1}}{\tau}\right)\mathrm{d}x \\
&\le \int_U W_\varepsilon(\theta{-}t_{i-1}, \nabla y_\tau^{i-1})\,\mathrm{d}x + \int_U H(\nabla^2 y_\tau^{i-1})\,\mathrm{d}x - \int_U f(\theta{-}t_{i-1}){\cdot}y_\tau^{i-1}\,\mathrm{d}x \\
&\quad - \int_U \big(f(\theta{-}t_i) - f(\theta{-}t_{i-1})\big){\cdot}y_\tau^{i-1}\,\mathrm{d}x \\
&\quad - \int_U \big(W_\varepsilon(\theta{-}t_{i-1}, \nabla y_\tau^{i-1}) - W_\varepsilon(\theta{-}t_i, \nabla y_\tau^{i-1})\big)\,\mathrm{d}x.
\end{aligned} \tag{3.4.4}$$

Summing over $i = 1, \dots, n \le N_\tau$ inequality (3.4.4) we get

$$\begin{aligned}
&\int_U W_\varepsilon(\theta - t_n, \nabla y_\tau^n)\,\mathrm{d}x + \int_U H(\nabla^2 y_\tau^n)\,\mathrm{d}x - \int_U f(\theta - t_n)\cdot y_\tau^n\,\mathrm{d}x \\
&\quad + \sum_{i=1}^n \tau \int_U R_\varepsilon\left(\theta - t_i, \nabla y_\tau^{i-1}, \frac{\nabla y_\tau^i - \nabla y_\tau^{i-1}}{\tau}\right)\mathrm{d}x \\
&\le \int_U W_\varepsilon(\theta, \nabla y_0) + H(\nabla^2 y_0)\,\mathrm{d}x - \int_U f(\theta)\cdot y_0\,\mathrm{d}x \\
&\quad - \sum_{i=1}^n \int_U (f(\theta - t_i) - f(\theta - t_{i-1}))\cdot y_\tau^{i-1}\,\mathrm{d}x \\
&\quad - \sum_{i=1}^n \int_U \big(W_\varepsilon(\theta - t_{i-1}, \nabla y_\tau^{i-1}) - W_\varepsilon(\theta - t_i, \nabla y_\tau^{i-1})\big)\,\mathrm{d}x. \qquad (3.4.5)
\end{aligned}$$

By the growth conditions (3.2.4)–(3.2.5) (3.2.9), (3.2.13), and (3.2.14), we hence have that

$$\begin{aligned}
&c_W \|\nabla y_\tau^n\|^p_{L^p(U;\mathbb{R}^{d\times d})} + c_W \left\| \frac{1}{\det \nabla y_\tau^n} \right\|^q_{L^q(U)} + c_H \|\nabla^2 y_\tau^n\|^p_{L^p(U;\mathbb{R}^{d\times d\times d})} \\
&\quad + c_{\mathbb{D}} \sum_{i=1}^n \tau \int_U \left| \frac{(\nabla y_\tau^i - \nabla y_\tau^{i-1})^\top}{\tau} \nabla y_\tau^{i-1} + (\nabla y_\tau^{i-1})^\top \frac{\nabla y_\tau^i - \nabla y_\tau^{i-1}}{\tau} \right|^2 \mathrm{d}x - \frac{|U|}{c_W} \\
&\le \int_U W_\varepsilon(\theta - t_n, \nabla y_\tau^n)\,\mathrm{d}x + \int_U H(\nabla^2 y_\tau^n)\,\mathrm{d}x + \sum_{i=1}^n \tau \int_U R_\varepsilon\left(\theta - t_i, \nabla y_\tau^{i-1}, \frac{\nabla y_\tau^i - \nabla y_\tau^{i-1}}{\tau}\right)\mathrm{d}x \\
&\overset{(3.4.5)}{\le} \int_U W_\varepsilon(\theta, \nabla y_0) + H(\nabla^2 y_0)\,\mathrm{d}x - \int_U f(\theta)\cdot y_0\,\mathrm{d}x + \int_U f(\theta - t_n)\cdot y_\tau^n\,\mathrm{d}x \\
&\quad + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_U \dot f(\theta - s)\cdot y_\tau^{i-1}\,\mathrm{d}x\,\mathrm{d}s \\
&\quad - \sum_{i=1}^n \int_U \big(W_\varepsilon(\theta - t_{i-1}, \nabla y_\tau^{i-1}) - W_\varepsilon(\theta - t_i, \nabla y_\tau^{i-1})\big)\,\mathrm{d}x. \qquad (3.4.6)
\end{aligned}$$

In order to control the right-hand side above, we remark that

$$\begin{aligned}
&- \sum_{i=1}^n \int_U \big(W_\varepsilon(\theta - t_{i-1}, \nabla y_\tau^{i-1}) - W_\varepsilon(\theta - t_i, \nabla y_\tau^{i-1})\big)\,\mathrm{d}x \\
&\overset{(3.2.2)}{=} \sum_{i=1}^n \int_U \big(h_\varepsilon(\theta - t_{i-1}) - h_\varepsilon(\theta - t_i)\big)\big(V^a(\nabla y_\tau^{i-1}) - V^r(\nabla y_\tau^{i-1})\big)\,\mathrm{d}x \\
&\overset{(3.2.4)}{\le} \frac{1}{c_W} \sum_{i=1}^n \int_U \big(h_\varepsilon(\theta - t_{i-1}) - h_\varepsilon(\theta - t_i)\big)\big(1 + |\nabla y_\tau^{i-1}|^p\big)\,\mathrm{d}x
\end{aligned}$$

where we have also used that $h_\varepsilon(\theta - t_{i-1}) - h_\varepsilon(\theta - t_i) \ge 0$. As $h_\varepsilon(\theta - t_{i-1}) - h_\varepsilon(\theta - t_i) = 0$ on the complement of

$$E_i := \{x \in U \ : \ \theta(x) \in [t_{i-1} - \varepsilon/2, t_i + \varepsilon/2]\},$$

by using $\|h'_\varepsilon\|_{L^\infty(\mathbb{R})} \le 2/\varepsilon$ (recall (3.2.1)) and the embedding $L^\infty(U, \mathbb{R}^{d\times d}) \subset W^{1,p}(U, \mathbb{R}^{d\times d})$

we get

$$
\begin{aligned}
&-\sum_{i=1}^{n}\int_U \big(W_\varepsilon(\theta{-}t_{i-1},\nabla y_\tau^{i-1}) - W_\varepsilon(\theta{-}t_i,\nabla y_\tau^{i-1})\big)\,\mathrm{d}x\\
&\quad\le \frac{c\tau}{\varepsilon}\sum_{i=1}^{n}|E_i|\left(1+\|\nabla y_\tau^{i-1}\|^p_{L^\infty(U;\mathbb{R}^{d\times d})}\right)\\
&\quad\le \frac{c\tau}{\varepsilon}\sum_{i=1}^{n}|E_i|\left(1+\|\nabla y_\tau^{i-1}\|^p_{L^p(U;\mathbb{R}^{d\times d})}+\|\nabla^2 y_\tau^{i-1}\|^p_{L^p(U;\mathbb{R}^{d\times d\times d})}\right).
\end{aligned}
$$

Together with (3.2.14)–(3.2.15), this allows to deduce from inequality (3.4.6) that

$$
\begin{aligned}
&c_W\|\nabla y_\tau^n\|^p_{L^p(U;\mathbb{R}^{d\times d})} + c_W\left\|\frac{1}{\det\nabla y_\tau^n}\right\|^q_{L^q(U)} + c_H\|\nabla^2 y_\tau^n\|^p_{L^p(U;\mathbb{R}^{d\times d\times d})}\\
&\quad + c_{\mathbb{D}}\sum_{i=1}^{n}\tau\int_U\left|\frac{(\nabla y_\tau^i-\nabla y_\tau^{i-1})^\top}{\tau}\nabla y_\tau^{i-1}(\nabla y_\tau^{i-1})^\top\frac{\nabla y_\tau^i-\nabla y_\tau^{i-1}}{\tau}\right|^2\,\mathrm{d}x\\
&\le c + c\|y_\tau^n\|_{L^2(U;\mathbb{R}^d)} + c\sum_{i=1}^{n}\tau\|y_\tau^{i-1}\|_{L^2(U;\mathbb{R}^d)}\\
&\quad + \frac{c\tau}{\varepsilon}\sum_{i=1}^{n}|E_i|\left(1+\|\nabla y_\tau^{i-1}\|^p_{L^p(U;\mathbb{R}^{d\times d})}+\|\nabla^2 y_\tau^{i-1}\|^p_{L^p(U;\mathbb{R}^{d\times d\times d})}\right).
\end{aligned}
\tag{3.4.7}
$$

For $\tau<\varepsilon$ one has that $\cup_{i=1}^{N_\tau}E_i$ covers $\Omega(T)$ multiple times. In particular, we have that

$$
\sum_{i=1}^{N_\tau}|E_i| \le \left(\frac{\varepsilon+\tau}{\tau}+1\right)|\Omega(T)|. \tag{3.4.8}
$$

Hence, by the Poincaré inequality and the Discrete Gronwall Lemma [51, (C.2.6), p. 534] we find the bound

$$
\begin{aligned}
&\max_n\|y_\tau^n\|^p_{W^{2,p}(U,\mathbb{R}^d)} + \sum_{i=1}^{N_\tau}\tau\left\|\frac{(\nabla y_\tau^i{-}\nabla y_\tau^{i-1})^\top}{\tau}\nabla y_\tau^{i-1}+(\nabla y_\tau^{i-1})^\top\frac{\nabla y_\tau^i{-}\nabla y_\tau^{i-1}}{\tau}\right\|^2_{L^2(U;\mathbb{R}^{d\times d})}\\
&\le c\exp\left(\frac{c\tau}{\varepsilon}\sum_{i=i}^{N_\tau}|E_i|\right)\overset{(3.4.8)}{\le} c\exp\left(\frac{c\tau}{\varepsilon}\left(\frac{\varepsilon+\tau}{\tau}+1\right)|\Omega(T)|\right)\le c\exp(c\tau/\varepsilon),
\end{aligned}
\tag{3.4.9}
$$

where we also used the fact that $\Omega(T)\subset\subset U$.

By the Sobolev embedding of $W^{2,p}(U;\mathbb{R}^d)$ into $C^{1-d/p}(U;\mathbb{R}^d)$ and the classical result of [42, Thm. 3.1] we get

$$
\det\nabla\overline{y}_\tau \ge c_\varepsilon > 0 \ \text{ in } \ [0,T]\times\overline{U} \tag{3.4.10}
$$

where the constant c_ε depends on the bound in (3.4.9).

By the Poincaré inequality and the generalization of Korn's first inequality by [81] and [85, Thm. 2.2], also using (3.4.10) we have that

$$
\|\nabla\dot{\widehat{y}}_\tau\|^2_{L^2(0,T;L^2(Q;\mathbb{R}^{d\times d}))} \le c'_\varepsilon\int_0^T \|\nabla\dot{\widehat{y}}_\tau^\top\nabla\underline{y}_\tau + \nabla\underline{y}_\tau^\top\nabla\dot{\widehat{y}}_\tau\|^2_{L^2(U;\mathbb{R}^{d\times d})}\,\mathrm{d}s \overset{(3.4.9)}{\le} c'_\varepsilon c\exp(c\tau/\varepsilon)
$$

where the constant c'_ε depends on the bound (3.4.9) and on the constant c_ε in (3.4.10). Again by the Poincaré inequality, this time applied to $\dot{y}$, we get that

$$\|\widehat{y}_\tau\|_{H^1(0,T;H^1(U;\mathbb{R}^d))} \le c'_\varepsilon c \exp(c\tau/\varepsilon). \tag{3.4.11}$$

By using these estimates, as $\tau \to 0$, up to not relabeled subsequences we get

$$\overline{y}_\tau,\, \underline{y}_\tau \overset{*}{\rightharpoonup} y \quad \text{weakly-}* \text{ in } L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d)), \tag{3.4.12}$$

$$\nabla\dot{\widehat{y}}_\tau \rightharpoonup \nabla\dot{y} \quad \text{weakly in } L^2(Q;\mathbb{R}^d), \tag{3.4.13}$$

$$\nabla\widehat{y}_\tau \to \nabla y \quad \text{strongly in } C^{0,\alpha}(\overline{Q};\mathbb{R}^d) \tag{3.4.14}$$

for some $\alpha \in (0,1)$. In particular, from the convergences above we also get $\det \nabla\overline{y}_\tau \to \det \nabla y$ uniformly. In combination with the lower bound (3.4.10), this implies that $\nabla y \in \mathrm{GL}_+(d)$ everywhere, hence y is admissible, namely, $y(t,\cdot) \in \mathcal{A}$ for every $t \in (0,T)$.

We now pass to the limit in the time-discrete Euler–Lagrange equation (3.4.3). Let $z \in C^\infty(\overline{Q};\mathbb{R}^d)$ with $z = 0$ on Σ_D be given and let $(z^i_\tau)_{i=1}^{N_\tau} \in \mathcal{A}$ be such that $\overline{z}_\tau \to z$ strongly in $L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d))$. By (3.2.14) we have

$$\int_0^T\!\!\int_U f(\theta{-}\overline{t}_\tau)\cdot\overline{z}_\tau \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\!\!\int_U f(\theta{-}t)\cdot z \,\mathrm{d}x\,\mathrm{d}t. \tag{3.4.15}$$

As $h_\varepsilon(\theta(x){-}\overline{t}_\tau(t)) \to h_\varepsilon(\theta(x){-}t)$ for almost every $(t,x) \in Q$, the dissipation term goes to the limit as follows

$$\begin{aligned}
&\int_0^T\!\!\int_U \partial_{\dot F} R_\varepsilon\left(\theta{-}\overline{t}_\tau, \nabla\underline{y}_\tau, \nabla\dot{\widehat{y}}_\tau\right) : \nabla\overline{z}_\tau \,\mathrm{d}x\,\mathrm{d}t \\
&= 2\int_0^T\!\!\int_U (1{-}h_\varepsilon(\theta{-}\overline{t}_\tau))\nabla\underline{y}_\tau \left(\mathbb{D}^a(\nabla\underline{y}_\tau^\top\nabla\underline{y}_\tau)(\nabla\dot{\widehat{y}}_\tau^\top\nabla\underline{y}_\tau{+}\nabla\underline{y}_\tau^\top\nabla\dot{\widehat{y}}_\tau)\right) : \nabla\overline{z}_\tau \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2\int_0^T\!\!\int_U h_\varepsilon(\theta{-}\overline{t}_\tau)\nabla\underline{y}_\tau \left(\mathbb{D}^r(\nabla\underline{y}_\tau^\top\nabla\underline{y}_\tau)(\nabla\dot{\widehat{y}}_\tau^\top\nabla\underline{y}_\tau{+}\nabla\underline{y}_\tau^\top\nabla\dot{\widehat{y}}_\tau)\right) : \nabla\overline{z}_\tau \,\mathrm{d}x\,\mathrm{d}t \\
&\to 2\int_0^T\!\!\int_U (1{-}h_\varepsilon(\theta{-}t))\nabla y \left(\mathbb{D}^a(\nabla y^\top\nabla y)(\nabla\dot{y}^\top\nabla y{+}\nabla y^\top\nabla\dot{y})\right) : \nabla z \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2\int_0^T\!\!\int_U h_\varepsilon(\theta{-}t)\nabla y \left(\mathbb{D}^r(\nabla y^\top\nabla y)(\nabla\dot{y}^\top\nabla y{+}\nabla y^\top\nabla\dot{y})\right) : \nabla z \,\mathrm{d}x\,\mathrm{d}t \qquad (3.4.16) \\
&= \int_0^T\!\!\int_U \partial_{\dot F} R_\varepsilon\left(\theta{-}t, \nabla y, \nabla\dot{y}\right) : \nabla z \,\mathrm{d}x\,\mathrm{d}t \qquad (3.4.17)
\end{aligned}$$

where we used (3.2.12) and convergences (3.4.12)–(3.4.14). Moreover, we also have

$$\int_0^T\!\!\int_U \partial_F W_\varepsilon(\theta{-}\overline{t}_\tau, \nabla\overline{y}_\tau) : \nabla\overline{z}_\tau \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\!\!\int_U \partial_F W_\varepsilon(\theta{-}t, \nabla y) : \nabla z \,\mathrm{d}x\,\mathrm{d}t \tag{3.4.18}$$

by (3.2.3) and convergences (3.4.12) and (3.4.14).

For the convergence of the second-gradient term we reproduce in this setting the argument from [48]. Given the limit y, let $(w^i_\tau)_{i=1}^{N_\tau} \in \mathcal{A}$ be such that $\overline{w}_\tau \to y$ strongly in $L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d))$. We consider the test functions $\overline{z}_\tau := \overline{w}_\tau - \overline{y}_\tau$ in the time-discrete Euler–Lagrange equation (3.4.3). Convergences (3.4.12)–(3.4.13) entail that $\overline{z}_\tau \to 0$ strongly in $L^\infty(0,T;H^1(U;\mathbb{R}^d))$ and $\overline{z}_\tau \rightharpoonup 0$ weakly-$*$ in $L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d))$. Let us now compute

$$\begin{aligned}\int_0^T\!\!\int_U &(\mathrm{D}H(\nabla^2 y) - \mathrm{D}H(\nabla^2\overline{y}_\tau))\vdots(\nabla^2 y - \nabla^2\overline{y}_\tau)\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T\!\!\int_U (\mathrm{D}H(\nabla^2 y) - \mathrm{D}H(\nabla^2\overline{y}_\tau))\vdots(\nabla^2 y - \nabla^2\overline{w}_\tau)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_0^T\!\!\int_U (\mathrm{D}H(\nabla^2 y) - \mathrm{D}H(\nabla^2\overline{y}_\tau))\vdots\nabla^2\overline{z}_\tau\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.4.19}$$

As $\nabla^2\overline{w}_\tau \to \nabla^2 y$ strongly in $L^p(Q;\mathbb{R}^{d\times d\times d})$ and $\mathrm{D}H(\nabla^2\overline{y}_\tau)$ is bounded in $L^{p'}(Q;\mathbb{R}^{d\times d\times d})$ by (3.2.9), the first integral in the right-hand side above converges to 0 as $\tau \to 0$. Hence, passing to the $\limsup$ in (3.4.19), by the Euler–Lagrange equation (3.4.3) and convergences (3.4.14)–(3.4.18) we find that

$$\begin{aligned}\limsup_{\tau\to 0} &\int_0^T\!\!\int_U (\mathrm{D}H(\nabla^2 y) - \mathrm{D}H(\nabla^2\overline{y}_\tau))\vdots(\nabla^2 y - \nabla^2\overline{y}_\tau)\,\mathrm{d}x\,\mathrm{d}t\\ &= \limsup_{\tau\to 0}\int_0^T\!\!\int_U (\mathrm{D}H(\nabla^2 y) - \mathrm{D}H(\nabla^2\overline{y}_\tau))\vdots\nabla^2\overline{z}_\tau\,\mathrm{d}x\,\mathrm{d}t\\ &= \limsup_{\tau\to 0}\Bigg(\int_0^T\!\!\int_U \mathrm{D}H(\nabla^2 y)\vdots\nabla^2\overline{z}_\tau\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\!\!\int_U f(\theta{-}\overline{t}_\tau)\cdot\overline{z}_\tau\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_0^T\!\!\int_U \Big(\partial_F W_\varepsilon(\theta{-}\overline{t}_\tau,\nabla\overline{y}_\tau) + \partial_{\dot F}R_\varepsilon\Big(\theta{-}\overline{t}_\tau,\nabla\underline{y}_\tau,\nabla\dot{\widehat{y}}_\tau\Big)\Big):\nabla\overline{z}_\tau\,\mathrm{d}x\,\mathrm{d}t\Bigg) = 0 \end{aligned} \tag{3.4.20}$$

The coercivity (3.2.9) then implies that

$$\nabla^2\overline{y}_\tau \to \nabla^2 y \quad \text{strongly in } L^p(Q;\mathbb{R}^{d\times d\times d})$$

and thus

$$\mathrm{D}H(\nabla^2\overline{y}_\tau) \to \mathrm{D}H(\nabla^2 y) \quad \text{strongly in } L^{p'}(Q;\mathbb{R}^{d\times d\times d}).$$

Passing to the limit as $\tau \to 0$ in (3.4.3) we then find (3.2.18).

In order to prove the bound (3.4.1), we simply pass to the limit as $\tau \to 0$ in (3.4.9) and obtain

$$\|y\|^p_{L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d))} + \|\nabla\dot y^\top\nabla y + \nabla y^\top\nabla\dot y\|^2_{L^2(Q;\mathbb{R}^{d\times d})} \le c$$

independently of ε. Following again [42, Thm. 3.1] we have that $\det\nabla y \ge c > 0$ independently of ε. By [81] and [85, Thm. 2.2] this ensures that

$$\|\nabla\dot y\|^2_{L^2(Q;\mathbb{R}^{d\times d})} \le c\|\nabla\dot y^\top\nabla y + \nabla y^\top\nabla\dot y\|^2_{L^2(Q;\mathbb{R}^{d\times d})} \le c$$

independently of ε. Hence, (3.4.1) follows by the Poincaré inequality. □

Before moving to the proof of Theorem 3.2.1 in the diffused-interface case $\varepsilon > 0$, let us recall a well-posedness result for the growth subproblem, see [63, Thm. 3.15].

Proposition 3.4.2 (Well-posedness of the growth problem)**.** *Assume to be given $\widehat\gamma \in C(\mathbb{R}^d)$ with $c_\gamma \le \widehat\gamma(\cdot) \le C_\gamma$ for some $0 < c_\gamma \le C_\gamma$ and $\Omega_0 \subset \mathbb{R}^d$ nonempty, open, and bounded. Then, there exists a unique nonnegative viscosity solution to*

$$\widehat\gamma(x)|\nabla(-\theta)(x)| = 1 \quad \text{in } \mathbb{R}^d\setminus\overline{\Omega_0}, \tag{3.4.21}$$

$$\theta = 0 \quad \text{in } \Omega_0. \tag{3.4.22}$$

Moreover, $\theta \in C^{0,1}(\mathbb{R}^d)$ *with*

$$0 < \frac{1}{C_\gamma} \le |\nabla \theta(x)| \le \frac{1}{c_\gamma} \ \text{for a.e. } x \in \mathbb{R}^d, \tag{3.4.23}$$

and we have that

$$\frac{\operatorname{dist}(x,\Omega_0)}{C_\gamma} \le \theta(x) \le \frac{\operatorname{dist}(x,\Omega_0)}{c_\gamma} \quad \forall x \in \mathbb{R}^d \setminus \overline{\Omega_0}. \tag{3.4.24}$$

We are now ready to prove Theorem 3.2.1 in the diffused-interface case $\varepsilon > 0$. As announced, the proof hinges on an iterative construction. To start with, let us remark that y_0 from (3.2.15) is such that ∇y_0 is Hölder continuous. In particular, the mapping $\tilde{\gamma}\colon \overline{U} \to (0,\infty)$ defined by

$$\tilde{\gamma}(x) := \gamma(y_0(x), \nabla y_0(x)) \quad \forall x \in \overline{U}$$

is Hölder continuous, as well. Letting $\widehat{\gamma}$ be any continuous extension of $\tilde{\gamma}$ to $\mathbb{R}^d$ with $c_\gamma \le \widehat{\gamma}(\cdot) \le C_\gamma$, we can use Proposition 3.4.2 and find $\theta_0 \in C(\overline{U})$ solving

$$\begin{aligned} &\gamma(y_0(x), \nabla y_0(x))|\nabla(-\theta_0)(x)| = 1 \quad \text{in } U \setminus \overline{\Omega_0}, \\ &\theta_0 = 0 \quad \text{in } \Omega_0 \end{aligned}$$

in the viscosity sense, with (3.4.23) and (3.4.24) holding in $\overline{U}$. Note that (3.4.24) in particular implies that

$$\Omega^0(T) = \{x \in U \mid \theta_0(x) < T\} \subset \Omega_0 + B_{C_\gamma T} \overset{(3.2.17)}{\subset\subset} U.$$

By applying Proposition 3.4.1 for $\theta = \theta_0$ we find $y^1 \in L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d)) \cap H^1(0,T;H^1(U;\mathbb{R}^d))$.

This can be iterated as follows: For all $k \ge 1$, we define $\theta^k \in C(\overline{U})$ given $y^k \in L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d)) \cap H^1(0,T;H^1(U;\mathbb{R}^d))$ to be a viscosity solution to

$$\begin{aligned} &\gamma(y^k(\theta^k(x)\wedge T, x), \nabla y^k(\theta^k(x)\wedge T, x))|\nabla(-\theta^k)(x)| = 1 \quad \text{in } U \setminus \overline{\Omega_0}, \\ &\theta^k = 0 \quad \text{in } \Omega_0 \end{aligned}$$

with (3.4.23) and (3.4.24) holding in $\overline{U}$. The existence of such a viscosity solution follows again from Proposition 3.4.2 as the mapping on $\overline{U}$ defined as

$$x \mapsto \gamma(y^k(\theta^k(x)\wedge T, x), \nabla y^k(\theta^k(x)\wedge T, x)) \quad \forall x \in \overline{U}$$

may be extended to a continuous mapping $\widehat{\gamma}$ on $\mathbb{R}^d$ with $c_\gamma \le \widehat{\gamma}(\cdot) \le C_\gamma$. Note again that (3.4.24) implies that

$$\Omega^k(T) := \{x \in U \mid \theta^k(x) < T\} \subset \Omega_0 + B_{C_\gamma T} \overset{(3.2.17)}{\subset\subset} U. \tag{3.4.25}$$

Inclusion (3.4.25) in particular guarantees that the accreting phase defined by θ^k remains at positive distance from the boundary ∂U, independently of ε and k.

Given such θ^k, we define $y^{k+1} \in L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d)) \cap H^1(0,T;H^1(U;\mathbb{R}^d))$ by Proposition 3.4.1 applied for $\theta = \theta^k$.

Bounds (3.4.1) and (3.4.23) ensure that the sequence $(y^k, \theta^k)_{k\in\mathbb{N}}$ defined by this iterative procedure is (possibly not unique but nonetheless) uniformly bounded in

$$\big(L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d)) \cap H^1(0,T;H^1(U;\mathbb{R}^d))\big) \times C^{0,1}(\overline{U}).$$

As $(\theta^k)_{i\in\mathbb{N}}$ are uniformly Lipschitz continuous, by the Ascoli–Arzelà and the Banach–Alaoglu Theorems, possibly passing to not relabeled subsequences, one can find a pair (y,θ) such that

$$y^k \overset{*}{\rightharpoonup} y \quad \text{weakly-}* \text{ in } L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d)) \cap H^1(0,T;H^1(U;\mathbb{R}^d)), \tag{3.4.26}$$

$$y^k \to y \quad \text{strongly in } C^{1,\alpha}(\overline{Q};\mathbb{R}^d), \tag{3.4.27}$$

$$\theta^k \to \theta \quad \text{strongly in } C(\overline{U}) \tag{3.4.28}$$

for some $\alpha \in (0,1)$ and θ fulfills (3.4.23) and (3.4.24) in $\overline{U}$. As $(y^k)_{k\in\mathbb{N}}$ are uniformly Hölder continuous and γ is Lipschitz continuous, by (3.2.16) we have

$$\begin{aligned}
&|\gamma(y^k(\theta^k(x)\wedge T,x),\nabla y^k(\theta^k(x)\wedge T,x)) - \gamma(y^j(\theta^j(x)\wedge T,x),\nabla y^j(\theta^j(x)\wedge T,x))| \\
&\quad \le c|y^k(\theta^k(x)\wedge T,x) - y^j(\theta^j(x)\wedge T,x)| + c|\nabla y^k(\theta^k(x)\wedge T,x) - \nabla y^j(\theta^j(x)\wedge T,x)| \\
&\quad \le c|y^k(\theta^k(x)\wedge T,x) - y^j(\theta^k(x)\wedge T,x)| + c|\nabla y^k(\theta^k(x)\wedge T,x) - \nabla y^j(\theta^k(x)\wedge T,x)| \\
&\qquad + c|y^j(\theta^k(x)\wedge T,x) - y^j(\theta^j(x)\wedge T,x)| + c|\nabla y^j(\theta^k(x)\wedge T,x) - \nabla y^j(\theta^j(x)\wedge T,x)| \\
&\quad \le c\|y^k - y^j\|_{C^1(\overline{Q};\mathbb{R}^d)} + c\|\theta^k - \theta^j\|^\alpha_{C(\overline{U})} \quad \forall\, x \in \overline{U}.
\end{aligned}$$

Together with (3.4.27)–(3.4.28), this proves that $x \mapsto \gamma(y^k(\theta^k(x)\wedge T,x),\nabla y^k(\theta^k(x)\wedge T,x))$ converges to $x \mapsto \gamma(y(\theta(x)\wedge T,x),\nabla y(\theta(x)\wedge T,x))$ uniformly in $\overline{U}$. By the stability of the eikonal equation with respect to the uniform convergence of the data, see, e.g., [44, Prop. 1.2], θ satisfies (3.2.19)–(3.2.20) with coefficient $x \mapsto \gamma(y(\theta(x)\wedge T,x),\nabla y(\theta(x)\wedge T,x))$. Moreover, since bound (3.4.1) is independent of θ, following the argument of the proof of Proposition 3.4.1, we can pass to the limit in the Euler–Lagrange equation (3.2.18) and conclude the proof of Theorem 3.2.1 in the case $\varepsilon > 0$.

3.5 Proof of Theorem 3.2.1: sharp-interface case

The existence of weak/viscosity solutions in the sharp-interface case $\varepsilon = 0$ is obtained by passing to the limit as $\varepsilon \to 0$ in sequences of weak/viscosity solutions $(y_\varepsilon,\theta_\varepsilon)$ of the diffused-interface problem.

Notice at first that θ_ε are uniformly Lipschitz continuous, see (3.4.23). Bound (3.4.1) is independent of ε and implies that there exist not relabeled subsequences such that

$$y_\varepsilon \overset{*}{\rightharpoonup} y \quad \text{weakly-}* \text{ in } L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d)) \cap H^1(0,T;H^1(U;\mathbb{R}^d)), \tag{3.5.1}$$

$$y_\varepsilon \to y \quad \text{strongly in } C^{1,\alpha}(\overline{Q};\mathbb{R}^d), \tag{3.5.2}$$

$$\theta_\varepsilon \to \theta \quad \text{strongly in } C(\overline{U}) \tag{3.5.3}$$

for some $\alpha \in (0,1)$.

Let us now prove that we can pass to the limit $\varepsilon \to 0$ in equation (3.2.18). The convergence of the loading is straightforward. Moreover, the level sets $\{\theta(x) = t\}$ have Lebesgue measure zero by (3.4.23). Hence, by the assumptions (3.2.1) on h_ε and the uniform convergence (3.5.3) of $(\theta_\varepsilon)_\varepsilon$, we have that

$$h_\varepsilon(\theta_\varepsilon(x)-t) \to h_0(\theta(x)-t) \quad \text{for a.e. } (t,x) \in Q,$$

and that $(t,x) \mapsto h_\varepsilon(\theta_\varepsilon(x)-t)$ converges to $(t,x) \mapsto h_0(\theta(x)-t)$ strongly in $L^2(Q)$. On the other hand, by (3.2.3) and convergence (3.5.2), for all $(t,x) \in Q$ and $i = a,\ r,\ J$, we have that

$$|\partial_F V^i(\nabla y_\varepsilon)| \le c \quad \text{and} \quad \partial_F V^i(\nabla y_\varepsilon) \to \partial_F V^i(\nabla y).$$

Fix $z \in C^\infty([0,T]\times\overline{U};\mathbb{R}^d)$ with $z=0$ on Σ_D. By Lebesgue's Dominated Convergence Theorem we get

$$\begin{aligned}
&\int_0^T\!\!\int_U \partial_F W_\varepsilon(\theta_\varepsilon{-}t,\nabla y_\varepsilon){:}\nabla z\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_0^T\!\!\int_U \Big(h_\varepsilon(\theta{-}t)\partial_F V^r(\nabla y_\varepsilon)+(1-h_\varepsilon(\theta{-}t))\partial_F V^a(\nabla y_\varepsilon)+\partial_F V^J(\nabla y_\varepsilon)\Big){:}\nabla z\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\to\int_0^T\!\!\int_U \Big(h_0(\theta{-}t)\partial_F V^r(\nabla y)+(1{-}h_0(\theta{-}t))\partial_F V^a(\nabla y)+\partial_F V^J(\nabla y)\Big){:}\nabla z\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_0^T\!\!\int_U \partial_F W_0(\theta{-}t,\nabla y){:}\nabla z\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

Furthermore, by using convergence (3.5.1), we get

$$\begin{aligned}
&\int_0^T\!\!\int_U \partial_{\dot F} R_\varepsilon(\theta_\varepsilon{-}t,\nabla y_\varepsilon,\nabla\dot y_\varepsilon){:}\nabla z\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\to\int_0^T\!\!\int_U \Big(h_0(\theta{-}t)\partial_{\dot F} R^r(\nabla y,\nabla\dot y)+(1{-}h_0(\theta{-}t))\partial_{\dot F} R^a(\nabla y,\nabla\dot y)\Big){:}\nabla z\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_0^T\!\!\int_U \partial_{\dot F} R_0(\theta{-}t,\nabla y,\nabla\dot y){:}\nabla z\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

In order to prove the convergence of the second-order term, we set $z_\varepsilon = y - y_\varepsilon$ and recall that $z_\varepsilon \to 0$ strongly in $L^\infty(0,T;H^1(U;\mathbb{R}^d))$ and $z_\varepsilon \overset{*}{\rightharpoonup} 0$ weakly-$*$ in $L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d))$ in order to obtain

$$\begin{aligned}
&\limsup_{\tau\to 0}\int_0^T\!\!\int_U (\mathrm{D}H(\nabla^2 y)-\mathrm{D}H(\nabla^2 y_\varepsilon))\vdots\nabla^2 z_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\limsup_{\varepsilon\to 0}\Bigg(\int_0^T\!\!\int_U \mathrm{D}H(\nabla^2 y)\vdots\nabla^2 z_\varepsilon - f(\theta{-}t){\cdot}z_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_0^T\!\!\int_U \Big(\partial_F W_\varepsilon(\theta{-}t,\nabla y_\varepsilon){:}\nabla z_\varepsilon+\partial_{\dot F} R_\varepsilon\,(\theta{-}t,\nabla y_\varepsilon,\nabla\dot y_\varepsilon)\,{:}\nabla z_\varepsilon\Big)\,\mathrm{d}x\,\mathrm{d}t\Bigg)=0
\end{aligned}$$

Owing to (3.2.10) this proves that $\nabla^2 y_\varepsilon \to \nabla^2 y$ strongly in $L^p(Q;\mathbb{R}^{d\times d\times d})$. We hence have that $\mathrm{D}H(\nabla^2 y_\varepsilon)\to \mathrm{D}H(\nabla^2 y)$ strongly in $L^{p'}(Q;\mathbb{R}^{d\times d\times d})$, as well, and we can pass to the limit as $\varepsilon\to 0$ in (3.2.18).

In order to conclude the proof, we are left to check that θ is a viscosity solution to (3.2.19). This however readily follows as $x\mapsto \gamma(y_\varepsilon(\theta_\varepsilon(x)\wedge T,x),\nabla y_\varepsilon(\theta_\varepsilon(x)\wedge T,x))$ converges to $x\mapsto\gamma(y(\theta(x)\wedge T,x),\nabla y(\theta(x)\wedge T,x))$ uniformly and the eikonal problem is stable under uniform convergence of the data [44, Prop. 1.2].

Before closing this section, let us explicitly remark that indeed Proposition 3.2.1 actually holds in the case $\varepsilon=0$, as well. In order to check it, one would need a slightly different, and indeed simpler, a-priori estimate on the time-discrete solutions. Based on such result, one could argue as in Section 3.4 by the same iterative procedure in order to obtain an alternative proof of Theorem 3.2.1 in the sharp-interface case.

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4 Viscoelastic surface growth at finite strains with Ersatzmaterial

This chapter consists of a publication currently in preparation with Ulisse Stefanelli.

Abstract

We consider the accretive growth of a viscoelastic body, under the assumption that the accreted material is deposited in an unstressed state. We revisit a model proposed in [108] and assume that the backstrain accumulated during the evolution depends on the deformation itself. Postulating the presence of a regularizing Ersatzmaterial surrounding the growing body, we show the existence of solutions to the coupled accretion and viscoelastic equilibrium problem.

4.1 Introduction

Numerous natural and technological systems experience growth. In particular, *accretive growth* may be observed in various settings, ranging from biology, e.g., the development of plants [27, 31], shells [78], and horns [93, 99], to material sciences, e.g. solidification of metals [90], crystallization [53, 105], and 3D printing [37, 56], among others. Accretion may be descibed by assuming evolution in the (outward) normal direction ν to the boundary, so that, denoting by $\Omega(t) \subset \mathbb{R}^d$ the reference configuration of the body, a point $x(t)$ on the boundary $\partial\Omega(t)$ follows the normal flow rule

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \gamma\nu(x(t)). \tag{4.1.1}$$

Here, $\gamma(\cdot) > 0$ is the growth rate, which will be later specified to be a positive function of the deformation and deformation gradient. In the following, we assume that the reference configuration of the accreting material at time t is the t-sublevel set of a suitable function $\theta\colon \mathbb{R}^d \to [0,\infty)$, i.e.,

$$\Omega(t) := \{x \in \mathbb{R}^d \mid \theta(x) \le t\}.$$

The map θ is called *time-of-attachment* function, since $\theta(x(t)) = t$ for $x(t) \in \partial\Omega(t)$. The deformation of the viscoelastic medium at time $t \in [0,T]$ is given by $y(t)\colon \Omega(t) \to \mathbb{R}^d$. We assume that the growth is influenced by the deformation y and the deformation gradient ∇y. Specifically, letting $\gamma(x) = \gamma(y(\theta(x),x), \nabla y(\theta(x),x))$, equation (4.1.1) leads to the generalized eikonal equation for the time-of-attachment function θ [16, 24]

$$\gamma(y(\theta(x) \wedge T, x), \nabla y(\theta(x) \wedge T, x))|\nabla\theta(x)| = 1 \qquad x \in \mathbb{R}^d \setminus \overline{\Omega_0} \tag{4.1.2}$$

$$\theta(x) = 0 \qquad x \in \Omega_0. \tag{4.1.3}$$

Here, $\Omega_0 \subset \mathbb{R}^d$ is the given initial reference configuration of the growing body. The growth rate $\gamma\colon \mathbb{R}^d \times \mathbb{R}^{d\times d} \to [0,\infty)$ is assumed to be Lipschitz and such that $c_\gamma \le \gamma(\cdot) \le C_\gamma$ for some $0 < c_\gamma \le C_\gamma$.

The deformation y, on the other hand, satisfies viscoelastic equilibrium. Due to the nonlinear finite-strain setting, we regularize the problem by introducing a (soft) viscoelastic *Ersatzmaterial* (or fictitious material) surrounding the accretive medium, with reference configuration at time t given by $U \setminus \overline{\Omega(t)}$. The open and bounded container $U \subset \mathbb{R}^d$ is chosen in such a way that

$\Omega(t) \subset\subset U$ for all $t \in [0,T]$, so that the accreting set never reaches its boundary, cf. assumption (H15) and formula (4.4.4) below. If $W\colon \mathbb{R}^{d\times d} \to [0,\infty)$ and $R\colon \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \to [0,\infty)$ are the elastic energy and instantaneous dissipation density, respectively, we define the corresponding Ersatzmaterial densities as $\frac{\delta}{1+\delta}W$ and $\frac{\delta}{1+\delta}R$. The constant $\delta \in (0,1)$ is considered to be small and, correspondingly, the Ersatzmaterial to be highly compliant. The viscoelastic equilibrium equation reads as

$$\begin{aligned} &-\operatorname{div}\big(h(\theta(x){-}t)\mathrm{D}W(\nabla y A^{-1})A^{-\top}{+}V^J(\nabla y){+}h(\theta(x){-}t)\partial_{\nabla\dot y}R(\nabla y,\nabla\dot y){-}\operatorname{div}\mathrm{D}H(\nabla^2 y)\big)\\ &\qquad = h(\theta(x){-}t)f(t,x), \end{aligned} \tag{4.1.4}$$

where $V^J\colon \mathbb{R}^{d\times d} \to [0,\infty]$ is a term penalizing self-interpenetration of matter, i.e., $V^J(F) \to \infty$ as $\det F \to 0^+$ and $V^J(F) < \infty$ if and only if $\det F > 0$, $H\colon \mathbb{R}^{d\times d\times d} \to [0,\infty)$ is a second-order regularization term for nonsimple materials, $A\colon [0,T]\times U \to \mathbb{R}^{d\times d}$ with $A(t,x)$ invertible for every $t \in [0,T]$ and $x \in U$, and $f\colon [0,T]\times U \to \mathbb{R}^d$ is the density of the external force. Notice that the second-order potential density H and the term V^J are assumed to be the same for both the medium and the Ersatzmaterial.

In the viscoelastic model (4.1.4), we consider the effects of the backstrain tensor A, which describes the residual strain accumulated during growth [90, 93, 108]. We follow the constitutive assumption of [108] as in [24] and consider

$$A(t,x) = \nabla y(\theta(x),x) \quad \text{for } t \in [0,T], x \in \Omega(t). \tag{4.1.5}$$

This specifically entails that the accreting body is unstressed at the time and place where new material is added, i.e.,

$$W(\nabla y(t,x)A^{-1}(t,x)) = W(I) = 0, \quad \text{for } x \in \partial\Omega(t).$$

We prove the existence of weak/viscosity solutions to the fully coupled problem (4.1.2)–(4.1.4), see Definition 4.3.1 and Theorem 4.3.1. More precisely, we show that y satisfies (4.1.4) in the weak or distributional sense, cf. (4.3.1), when equipped with the boundary, docking, and initial conditions

$$\begin{aligned} &\mathrm{D}H(\nabla^2 y){:}(\nu\otimes\nu) = 0 \ \text{ on } \ [0,T]\times\partial U,\\ &y = \mathrm{id} \ \text{ on } \ [0,T]\times\omega\\ &y(0,\cdot) = y_0 \ \text{ on } \ U, \end{aligned}$$

where $\omega \subset\subset \Omega_0$. On the other hand, θ is required to be a viscosity solution to (4.1.4).

In Section 4.2, we introduce the assumptions on the model. In Section 4.3, we provide the definition of the weak/viscosity solutions and state the existence result in Theorem 4.3.1. Finally, we devote Section 4.4 to the proof of Theorem 4.3.1. The strategy relies on showing the existence of θ solving (4.1.2) for given y, and of y solving (4.1.4) for given θ. The existence of a solution for the coupled problem then follows by passing to the limit in a suitable sequence $(y_k,\theta_k)_{k\in\mathbb{N}}$ defined iteratively.

4.2 Setting

In this section, we specify the assumptions. Let us recall that in the following, we indicate by c a generic positive constant, possibly depending on data but independent of the time discretization step τ. In particular, here c may depend on $\delta > 0$ from (4.2.2). Note that the value of c may change from line to line.

Let $T > 0$ be a fixed final time, the reference configuration $U \subset \mathbb{R}^d$ be nonempty, open, connected, bounded, and Lipschitz, and $\Omega_0 \subset\subset U$ be nonempty and open. We define $Q := (0,T) \times U$.

Admissible deformations

The set of admissible deformations is defined as

$$\mathcal{A} := \Big\{ y \in W^{2,p}_\omega(U;\mathbb{R}^d) \mid \nabla y \in \mathrm{GL}_+(d) \text{ a.e. in } U \Big\},$$

where

$$W^{2,p}_\omega(U;\mathbb{R}^d) := \{ y \in W^{2,p}(U;\mathbb{R}^d) \mid y \equiv \mathrm{id} \text{ on } \omega \} \text{ for some fixed } p > d$$

and $\omega \subset\subset \Omega_0$ nonempty and open. Deformations y are locally invertible and orientation preserving, i.e., $\nabla y \in \mathrm{GL}_+(d)$ almost everywhere in U, and satisfy the so-called *docking condition* $y \equiv \mathrm{id}$ in ω for almost every $t \in (0,T)$. In particular, we remark that this latter condition entails the validity of the following Poincaré-type inequality

$$\|y\|_{W^{2,p}(U;\mathbb{R}^d)} \le c \left(1 + \|\nabla^2 y\|_{L^p(U;\mathbb{R}^{d\times d\times d})} \right) \quad \forall y \in W^{2,p}_\omega(U;\mathbb{R}^d). \tag{4.2.1}$$

Mechanical energy

The elastic energy density $W\colon \mathbb{R}^{d\times d} \to [0,\infty)$ of the accreting material satisfies

(H1) $W \in C^1(\mathbb{R}^{d\times d})$;

(H2) there exists $c_W > 0$ such that

$$0 = W(I) \le W(F) \le \frac{1}{c_W}(|F|^p + 1) \text{ for every } F \in \mathbb{R}^{d\times d}.$$

Even though not strictly needed for the analysis, we additionally assume

(H3) frame indifference, i.e., $W(QF) = W(F)$ for all $F \in \mathbb{R}^{d\times d}$ and $Q \in \mathrm{SO}(d)$;

(H4) isotropy, i.e., $W(FQ) = W(F)$ for all $F \in \mathbb{R}^{d\times d}$ and $Q \in \mathrm{SO}(d)$.

We remark that both (H3) and (H4) are required for the model to be frame indifferent, namely to be such that y and Qy have the same energy for every rotation $Q \in \mathrm{SO}(d)$, i.e.,

$$W(Q\nabla y(t,x)\nabla y^{-1}(\theta(x),x)Q^\top) = W(\nabla y(t,x)\nabla y^{-1}(\theta(x),x)).$$

Let the density $V^J : \mathrm{GL}_+(d) \to [0,\infty)$ be such that

(H5) $V^J \in C^1(\mathrm{GL}_+(d))$;

(H6) there exist $q > pd/(p-d)$ and $c_J > 0$ such that

$$V^J(F) \ge \frac{c_J}{|\det F|^q} - \frac{1}{c_J}.$$

Finally, let $\delta > 0$ and $h\colon \mathbb{R} \to [0,1]$ be defined as

$$h(\sigma) = \frac{\mathbb{1}_{\{\sigma \le 0\}} + \delta}{1+\delta} = \begin{cases} 1 & \text{if } \sigma \le 0, \\ \frac{\delta}{1+\delta} & \text{if } \sigma > 0. \end{cases} \tag{4.2.2}$$

The stored elastic energy $\mathcal{W}\colon C(U) \times \mathcal{A} \times L^\infty(U; \mathrm{GL}_+(d)) \to [0,\infty)$ is hence defined as

$$\mathcal{W}(\sigma, y; A) := \int_U h(\sigma) W(\nabla y A^{-1}) + V^J(\nabla y) \mathrm{d}x.$$

Here, $A \in L^\infty(U; \mathrm{GL}_+(d))$ is a placeholder for the backstrain tensor $\nabla y(\theta(\cdot), \cdot)$ and $\sigma \in C(U)$ is a placeholder for $\theta(\cdot){-}t$, whose sublevel set $\{x \in U \mid \theta(x) - t < 0\}$ represents the accreting phase at time t. In particular $W + V_J$ is the energy of the accreting material W for $\theta - t < 0$, whereas, for $\theta - t \ge 0$, the energy density is $\frac{\delta}{1+\delta} W + V_j$, where $\frac{\delta}{1+\delta} < 1$. This is meant to represent the Ersatzmaterial surrounding the growing solid, which is highly elastically compliant for small δ.

We additionally consider a second order potential $\mathcal{H}\colon W^{2,p}_\omega(U; \mathbb{R}^d) \to [0,\infty)$ given by

$$\mathcal{H}(y) := \int_U H(\nabla^2 y) \mathrm{d}x$$

where $H : \mathbb{R}^{d\times d\times d} \to [0,\infty)$ is such that

(H7) $H \in C^1(\mathbb{R}^{d\times d\times d})$ is convex;

(H8) there exists a positive constant $c_H > 0$ such that

$$c_H |G|^p - \frac{1}{c_H} \le H(G) \le \frac{1}{c_H}(1+|G|)^p, \qquad |DH(G)| \le \frac{1}{c_H}|G|^{p-1}$$

for all $G \in \mathbb{R}^{d\times d\times d}$ and

$$c_H |G - G'|^p \le (DH(G) - DH(G'))\vdots(G - G')$$

for every $G, G' \in \mathbb{R}^{d\times d\times d}$;

(H9) $H(QG) = H(G)$ for all $G \in \mathbb{R}^{d\times d\times d}, Q \in SO(d)$.

Frame indifference (H9) of H is assumed to guarantee physical consistency albeit not necessary for the analysis.

Viscous dissipation

The dissipation potential $\mathcal{R}\colon C(U) \times W^{2,p}_\omega(U; \mathbb{R}^d) \times H^1(U; \mathbb{R}^d) \to [0,\infty)$ is given by

$$\mathcal{R}(\sigma, y, \dot{y}) := \int_U h(\sigma) R(\nabla y, \nabla \dot{y}) \mathrm{d}x$$

where $R : \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \to [0,\infty)$ is defined as

$$R(F, \dot{F}) := \frac{1}{2}\dot{C}{:}\mathbb{D}(C)\dot{C} \qquad \text{for every } F, \dot{F} \in \mathbb{R}^{d\times d}$$

with $C := F^\top F$ and $\dot{C} := \dot{F}^\top F + F^\top \dot{F}$. We assume

(H10) $\mathbb{D} \in C(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d\times d\times d})$ is such that $\mathbb{D}_{ijkl} = \mathbb{D}_{jikl} = \mathbb{D}_{klij}$ for every $i, j, k, l = 1, \dots, d$;

(H11) there exists a positive constant $c_R > 0$ such that

$$c_R|\dot{C}|^2 \leq \dot{C}{:}\mathbb{D}(C)\dot{C}$$

for every $C, \dot{C} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$.

Notice that by the definition of R, we have that $\partial_{\dot{F}} R$ is linear in $\dot{F}$, namely,

$$\partial_{\dot{F}} R(F, \dot{F}) = 2F\left(\mathbb{D}(C){:}\dot{C}\right) = 2F\mathbb{D}(F^\top F){:}(\dot{F}^\top F + F^\top \dot{F}).$$

Loading and initial data

We denote by $f : [0,T]\times U \to \mathbb{R}^d$ the body force density, and we require

(H12) $f \in W^{1,\infty}\left(0,T; L^2(U;\mathbb{R}^d)\right)$.

We moreover assume on the initial backstrain A_0 and the initial deformation y_0 that

(H13) $A_0 \in L^\infty(\Omega_0; \mathrm{GL}_+(d))$, $y_0 \in \mathcal{A}$, and

$$\int_U W(\nabla y_0 A_0^{-1})\mathbb{1}_{\Omega_0} + W(\nabla y_0)\mathbb{1}_{U\setminus\Omega_0} + V^J(\nabla y_0) + H(\nabla^2 y_0)\mathrm{d}x < \infty.$$

Growth

The growth rate γ is assumed to satisfy the following assumption:

(H14) the growth rate $\gamma \in C^{0,1}(\mathbb{R}^d \times \mathrm{GL}_+(d); (0,\infty))$ is such that $c_\gamma \leq \gamma(\cdot) \leq C_\gamma$ for some $0 < c_\gamma \leq C_\gamma$.

(H15) the initial reference configuration $\Omega_0 \subset\subset U$ of the accretive material is nonempty, open, and such that $\Omega_0 + B_{C_\gamma T} \subset\subset U$.

We remark assumption (H15) guarantees that the accreting material has positive distance from the boundary of U in the time interval $[0,T]$, see (4.4.4) below.

4.3 Notion of solution and main results

4.3.1 Notion of solution

We are interested in solving (1.2.12)–(1.2.16) in the following weak/viscosity sense.

Definition 4.3.1. *We say that a pair*

$$(y,\theta) \in \left(L^\infty(0,T; W^{2,p}(U;\mathbb{R}^d)) \cap H^1(0,T; H^1(U;\mathbb{R}^d))\right) \times C^{0,1}(\overline{U})$$

is a weak/viscosity solution *to the initial-boundary-value problem* (1.2.12)–(1.2.16) *if* $y(t,\cdot) \in \mathcal{A}$ *for almost every* $t \in [0,T]$, $y(0,\cdot) = y^0(\cdot)$ *in* U, *and*

$$\int_0^T\!\!\int_U \left(h(\theta - t)\left(\partial_F W(\nabla y A^{-1})A^{-\top} + \partial_{\dot{F}} R(\nabla y, \nabla \dot{y})\right) + \partial_F V^J(\nabla y)\right){:}\nabla z + DH\left(\nabla^2 y\right)\vdots\nabla^2 z \mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_U h(\theta - t) f\cdot z\mathrm{d}x\mathrm{d}t \qquad \forall z \in C^\infty([0,T]\times\overline{U};\mathbb{R}^d) \textit{ with } z\equiv 0 \textit{ on } \omega \tag{4.3.1}$$

with backstrain tensor A defined as

$$A(t,x) := \begin{cases} A_0 & \text{if } x \in \Omega_0, \\ \nabla y(\theta(x), x) & \text{if } x \in \Omega_t \setminus \Omega_0, \\ I & \text{if } x \in U \setminus \Omega_t, \end{cases} \tag{4.3.2}$$

and θ is a viscosity solution to

$$\begin{cases} \gamma(y(\theta(x) \wedge T, x), \nabla y(\theta(x) \wedge T, x))|\nabla\theta(x)| = 1 & \text{in } U \setminus \overline{\Omega_0} \\ \theta(x) = 0 & \text{on } \Omega_0. \end{cases} \tag{4.3.3}$$

4.3.2 Main result

Our main result is the following.

Theorem 4.3.1 (Existence)**.** *Under assumptions* (H1)–(H15)*, there exists a weak/viscosity solution (y,θ) to problem* (1.2.12)–(1.2.16).

The proof of Theorem 4.3.1 is given in Section 4.4. In Proposition 4.4.2 we check that, given θ^{k-1}, there exists a solution y^k to (4.3.1). We then recall in Proposition 4.4.1 that there exists a solution θ^k to (4.3.3) for given y^k. This allows us to iteratively define a sequence $(y^k, \theta^k)_{k\in\mathbb{N}}$. We prove Theorem 4.3.1 by directly passing to the limit as $k \to \infty$.

4.4 Proof of Theorem 4.3.1

We begin by recalling a well-posedness result for the growth subproblem, see [63, Thm. 3.15].

Proposition 4.4.1 (Well-posedness of the growth problem)**.** *Assume to be given $\widehat{\gamma} \in C(\mathbb{R}^d)$ with $c_\gamma \le \widehat{\gamma}(\cdot) \le C_\gamma$ for some $0 < c_\gamma \le C_\gamma$ and $\Omega_0 \subset \mathbb{R}^d$ nonempty, open, and bounded. Then, there exists a unique nonnegative viscosity solution to*

$$\widehat{\gamma}(x)|\nabla\theta(x)| = 1 \quad \text{in } \mathbb{R}^d \setminus \overline{\Omega_0}, \tag{4.4.1}$$

$$\theta = 0 \quad \text{on } \Omega_0. \tag{4.4.2}$$

Moreover, $\theta \in C^{0,1}(\mathbb{R}^d)$ with

$$0 < \frac{1}{C_\gamma} \le |\nabla\theta(x)| \le \frac{1}{c_\gamma} \text{ for a.e. } x \in \mathbb{R}^d. \tag{4.4.3}$$

It can also be shown that the unique nonnegative viscosity solution θ to (4.4.1)–(4.4.2) also satisfies [63, Thm. 3.15]

$$\frac{\operatorname{dist}(x,\Omega_0)}{C_\gamma} \le \theta(x) \le \frac{\operatorname{dist}(x,\Omega_0)}{c_\gamma} \quad \forall x \in \mathbb{R}^d \setminus \overline{\Omega_0}.$$

Hence, by (H15) we have

$$\Omega(T) := \{x \in U \mid \theta(x) < T\} \subset\subset \Omega_0 + B_{C_\gamma T} \subset\subset U, \tag{4.4.4}$$

and thus the accretive phase does not touch the boundary of U over the time interval $[0,T]$.

Before moving to the proof of Theorem 4.3.1, let us show that, for given $\theta \in C(\overline{U})$, there exists a y with $y(t,\cdot) \in \mathcal{A}$ for almost every $t \in [0,T]$, $y(0,\cdot) = y_0(\cdot)$, and satisfying (4.3.1).

Proposition 4.4.2 (Existence of y given θ)**.** *Let $\theta \in C(\overline{U})$ and* (H1)–(H15) *hold. Then, there exists $y \in L^\infty(0,T;W^{2,p}_\omega(U;\mathbb{R}^d)) \cap H^1(0,T;H^1(U;\mathbb{R}^d))$ such that $y(t,\cdot) \in \mathcal{A}$ for almost every $t \in [0,T]$, $y(0,\cdot) = y_0(\cdot)$ in U, and satisfying* (4.3.1).

Proof. The proof of the result follows the ideas of [4] or [48] and is based on a time-discretization scheme. Let $\tau := T/N_\tau > 0$ with $N_\tau \in \mathbb{N}$ given and consider the corresponding uniform partition of the time interval $[0,T]$ $t_i := i\tau$, for $i = 0,\dots,N_\tau$. Moreover, set $A^0_\tau := A_0 \mathbb{1}_{\Omega_0} + I\mathbb{1}_{U\setminus\Omega_0}$. For $i = 1,\dots,N_\tau$, we define $y^i_\tau \in \mathcal{A}$ as

$$y^i_\tau \in \operatorname*{arg\,min}_{y\in\mathcal{A}} \left\{ \mathcal{W}(\theta - t_i, y; A^i_\tau) + \mathcal{H}(y) + \tau\mathcal{R}\left(\theta - t_i, y^{i-1}_\tau, \frac{y - y^{i-1}_\tau}{\tau}\right) - \int_U h(\theta - t_i) f\cdot y\,\mathrm{d}x \right\},$$

where

$$A^i_\tau(x) := \begin{cases} A_0(x) & \text{if } \theta(x) = 0,\\ \nabla y^k_\tau(x) & \text{if } \theta(x) \in (t_{k-1}, t_k] \text{ for some } k = 1,\dots,i-1,\\ I & \text{if } \theta(x) > t_{i-1}. \end{cases} \tag{4.4.5}$$

Notice that (H13), the definition of $\mathcal{A}$, and the fact that $p > d$, imply that $A^i_\tau \in L^\infty(U; \mathrm{GL}_+(d))$ for every $i = 0,\dots,N_\tau$.

Under the growth conditions (H6), (H8), and (H11), the regularity and convexity assumptions (H1), (H5), (H7), (H10), and (H12), and the Poincaré inequality (4.2.1), the existence of $y^i_\tau \in \mathcal{A}$ for $i = 1,\dots,N_\tau$ easily follows by the Direct Method of the calculus of variations. Moreover, every minimizer y^i_τ satisfies the time-discrete Euler–Lagrange equation

$$\begin{aligned} &\int_U h(\theta - t_i)\left(\partial_F W(\nabla y^i_\tau (A^i_\tau)^{-1})(A^i_\tau)^{-\top} + \partial_{\dot F} R\left(\nabla y^{i-1}_\tau, \frac{\nabla y^i_\tau - \nabla y^{i-1}_\tau}{\tau}\right)\right) : \nabla z^i \mathrm{d}x \\ &+ \int_U \partial_F V^J(\nabla y^i_\tau) : \nabla z^i \mathrm{d}x + \int_U DH\left(\nabla^2 y^i_\tau\right) \vdots \nabla^2 z^i \mathrm{d}x = \int_U h(\theta - t_i) f(t_i)\cdot z^i \mathrm{d}x \end{aligned} \tag{4.4.6}$$

for every $z^i \in C^\infty(U;\mathbb{R}^d)$ with $z^i \equiv 0$ on ω, and for every $i = 1,\dots,N_\tau$.

From the minimality of y^i_τ we get that

$$\begin{aligned} &\int_U h(\theta - t_i)\left(W(\nabla y^i_\tau (A^i_\tau)^{-1}) + \tau R\left(\nabla y^{i-1}_\tau, \frac{\nabla y^i_\tau - \nabla y^{i-1}_\tau}{\tau}\right) - f(t_i)\cdot y^i_\tau\right) + V^J(\nabla y^i_\tau) + H(\nabla^2 y^i_\tau)\mathrm{d}x \\ &\quad \le \int_U h(\theta - t_i)\left(W(\nabla y^{i-1}_\tau (A^i_\tau)^{-1}) - f(t_i)\cdot y^{i-1}_\tau\right) + V^J(\nabla y^{i-1}_\tau) + H(\nabla^2 y^{i-1}_\tau)\mathrm{d}x \\ &\quad = \int_U h(\theta - t_{i-1})\left(W(\nabla y^{i-1}_\tau (A^{i-1}_\tau)^{-1}) - f(t_{i-1})\cdot y^{i-1}_\tau\right) + V^J(\nabla y^{i-1}_\tau) + H(\nabla^2 y^{i-1}_\tau)\mathrm{d}x \\ &\qquad + \int_U (h(\theta - t_i) - h(\theta - t_{i-1})) W(\nabla y^{i-1}_\tau (A^{i-1}_\tau)^{-1})\mathrm{d}x \\ &\qquad + \int_U h(\theta - t_i)\left(W(\nabla y^{i-1}_\tau (A^i_\tau)^{-1}) - W(\nabla y^{i-1}_\tau (A^{i-1}_\tau)^{-1})\right)\mathrm{d}x \\ &\qquad - \int_U (h(\theta - t_i) - h(\theta - t_{i-1})) f(t_i)\cdot y^{i-1}_\tau \mathrm{d}x - \int_U h(\theta - t_{i-1})(f(t_i) - f(t_{i-1}))\cdot y^{i-1}_\tau \mathrm{d}x. \end{aligned}$$

Summing over $i = 1, \dots, n \le N_\tau$ and telescoping, we have

$$\begin{aligned}
&\int_U h(\theta-t_n)W(\nabla y_\tau^n(A_\tau^n)^{-1})+V^J(\nabla y_\tau^n)+H(\nabla^2 y_\tau^n)-h(\theta-t_n)f(t_n)\cdot y_\tau^n \mathrm{d}x\\
&\quad+\sum_{i=1}^n \tau \int_U h(\theta-t_i)R\left(\nabla y_\tau^{i-1}, \frac{\nabla y_\tau^i-\nabla y_\tau^{i-1}}{\tau}\right)\mathrm{d}x\\
&\le \int_U h(\theta)W(\nabla y_0(A_\tau^0)^{-1}) + V^J(\nabla y_0) + H(\nabla^2 y_0) - h(\theta)f(0)\cdot y_0 \mathrm{d}x\\
&\quad+\sum_{i=1}^n \int_U (h(\theta-t_i)-h(\theta-t_{i-1}))W(\nabla y_\tau^{i-1}(A_\tau^{i-1})^{-1})\mathrm{d}x\\
&\quad+\sum_{i=1}^n \int_U h(\theta-t_i)\left(W(\nabla y_\tau^{i-1}(A_\tau^i)^{-1})-W(\nabla y_\tau^{i-1}(A_\tau^{i-1})^{-1})\right)\mathrm{d}x\\
&\quad-\sum_{i=1}^n \int_U (h(\theta-t_i)-h(\theta-t_{i-1}))f(t_i)\cdot y_\tau^{i-1}+h(\theta-t_{i-1})(f(t_i)-f(t_{i-1}))\cdot y_\tau^{i-1}\mathrm{d}x.
\end{aligned}$$

By the growth conditions (H6), (H8), and (H11), and the definition (4.2.2) of h,

$$\begin{aligned}
&c_J\left\|\frac{1}{\det \nabla y_\tau^n}\right\|_{L^q(U)}^q + c_H\|\nabla^2 y_\tau^n\|_{L^p(U;\mathbb{R}^{d\times d\times d})}^p\\
&\quad+c_R\frac{\delta}{\delta+1}\sum_{i=1}^n \tau\left\|\frac{(\nabla y_\tau^i-\nabla y_\tau^{i-1})^\top}{\tau}\nabla y_\tau^{i-1}+(\nabla y_\tau^{i-1})^\top\frac{\nabla y_\tau^i-\nabla y_\tau^{i-1}}{\tau}\right\|_{L^2(U;\mathbb{R}^{d\times d})}^2\\
&\le \int_U h(\theta)W(\nabla y_0(A_\tau^0)^{-1}) + V^J(\nabla y_0) + H(\nabla^2 y_0) - h(\theta)f(0)\cdot y_0 \mathrm{d}x\\
&\quad+\sum_{i=1}^n \int_U (h(\theta-t_i)-h(\theta-t_{i-1}))W(\nabla y_\tau^{i-1}(A_\tau^{i-1})^{-1})\mathrm{d}x\\
&\quad+\sum_{i=1}^n \int_U h(\theta-t_i)\left(W(\nabla y_\tau^{i-1}(A_\tau^i)^{-1})-W(\nabla y_\tau^{i-1}(A_\tau^{i-1})^{-1})\right)\mathrm{d}x\\
&\quad-\sum_{i=1}^n \int_U (h(\theta-t_i)-h(\theta-t_{i-1}))f(t_i)\cdot y_\tau^{i-1}+h(\theta-t_{i-1})(f(t_i)-f(t_{i-1}))\cdot y_\tau^{i-1}\mathrm{d}x. \qquad (4.4.7)
\end{aligned}$$

We now control the right-hand side above. Let us start by noticing that

$$\begin{aligned}
&\sum_{i=1}^n \int_U (h(\theta-t_i)-h(\theta-t_{i-1}))W(\nabla y_\tau^{i-1}(A_\tau^{i-1})^{-1})\mathrm{d}x\\
&\overset{(4.2.2)}{=} \sum_{i=1}^n \int_U \frac{\mathbb{1}_{\{\theta\le t_i\}}-\mathbb{1}_{\{\theta\le t_{i-1}\}}}{1+\delta}W(\nabla y_\tau^{i-1}(A_\tau^{i-1})^{-1})\mathrm{d}x\\
&\le \sum_{i=1}^n \int_U \mathbb{1}_{\{t_{i-1}<\theta\le t_i\}}W(\nabla y_\tau^{i-1}(A_\tau^{i-1})^{-1})\mathrm{d}x\\
&= \sum_{i=1}^n \int_U \mathbb{1}_{\{t_{i-1}<\theta\le t_i\}}W(\nabla y_\tau^{i-1})\mathrm{d}x,
\end{aligned}$$

since $A_\tau^{i-1}(x) = I$ if $\theta(x) > t_{i-1}$. By the growth condition (H2), we then have

$$\begin{aligned}
&\sum_{i=1}^{n} \int_U (h(\theta - t_i) - h(\theta - t_{i-1})) W(\nabla y_\tau^{i-1}(A_\tau^{i-1})^{-1}) \mathrm{d}x \\
&\quad \leq \sum_{i=1}^{n} \frac{1}{c_W} (\|\nabla y_\tau^{i-1}\|^p_{L^\infty(U;\mathbb{R}^{d\times d})} + 1) \int_U \mathbb{1}_{\{t_{i-1} < \theta \leq t_i\}} \mathrm{d}x \\
&\quad \leq c \sum_{i=1}^{n} (\|\nabla^2 y_\tau^{i-1}\|^p_{L^p(U;\mathbb{R}^{d\times d})} + 1) \int_U \mathbb{1}_{\{t_{i-1} < \theta \leq t_i\}} \mathrm{d}x
\end{aligned}$$

where in the last line we used the continuous embedding of $L^\infty(U)$ into $W^{1,p}(U)$ for $p > d$, and Poincaré inequality (4.2.1).

Regarding the third term in the right-hand side of (4.4.7), we notice that, for $x \in \overline{\Omega_0}$, $A_\tau^i(x) = A_\tau^{i-1}(x) = A_0$. For $x \in U \setminus \overline{\Omega_0}$ with $\theta(x) \leq t_{i-2}$, there exists $k \in \{1, \ldots, i-2\}$ such that $\theta(x) \in (t_{k-1}, t_k]$, and thus $A_\tau^i(x) = A_\tau^{i-1}(x) = \nabla y_\tau^k$. Similarly, for $x \in U \setminus \overline{\Omega_0}$ such that $\theta(x) > t_{i-1}$ we have $A_\tau^i(x) = A_\tau^{i-1}(x) = I$. Hence, the integrand is nonzero only for $x \in U$ such that $t_{i-2} < \theta(x) \leq t_{i-1}$. For such x we have $A_\tau^i(x) = \nabla y_\tau^{i-1}$ and $A_\tau^{i-1}(x) = I$, so that

$$\begin{aligned}
&\sum_{i=1}^{n} \int_U h(\theta - t_i) \left(W(\nabla y_\tau^{i-1}(A_\tau^i)^{-1}) - W(\nabla y_\tau^{i-1}(A_\tau^{i-1})^{-1}) \right) \mathrm{d}x \\
&\quad = \sum_{i=1}^{n} \int_U \mathbb{1}_{\{t_{i-2} < \theta(x) \leq t_{i-1}\}} \left(W(I) - W(\nabla y_\tau^{i-1}) \right) \mathrm{d}x \overset{\text{(H2)}}{\leq} 0
\end{aligned}$$

Hence, by (H13), the Poincaré inequality (4.2.1), and the discrete Gronwall Lemma [51, (C.2.6), p. 534] we have the bound

$$\begin{aligned}
&\max_n \left(\|y_\tau^n\|^p_{W^{2,p}(U;\mathbb{R}^d)} + \left\| \frac{1}{\det \nabla y_\tau^n} \right\|^q_{L^q(U)} \right) \\
&\quad + \sum_{i=1}^{N_\tau} \tau \left\| \frac{(\nabla y_\tau^i - \nabla y_\tau^{i-1})^\top}{\tau} \nabla y_\tau^{i-1} + (\nabla y_\tau^{i-1})^\top \frac{\nabla y_\tau^i - \nabla y_\tau^{i-1}}{\tau} \right\|^2_{L^2(U;\mathbb{R}^{d\times d})} \\
&\quad \leq c \exp\left(\sum_{i=1}^{N_\tau} \left(\int_U \mathbb{1}_{\{t_{i-1} < \theta \leq t_i\}} \mathrm{d}x \right) \right) \leq c\,(1 + |\Omega(T)|) \exp\left(|\Omega(T)|\right). \qquad (4.4.8)
\end{aligned}$$

Let us now introduce the following notation for the time interpolants of a vector $(u_0, ..., u_{N_\tau})$ over the interval $[0, T]$: We define its backward-constant interpolant $\overline{u}_\tau$, its forward-constant interpolant $\underline{u}_\tau$, and its piecewise-affine interpolant $\hat{u}_\tau$ on the partition $(t_i)_{i=0}^{N_\tau}$ as

$$\begin{aligned}
&\overline{u}_\tau(0) := u_0, && \overline{u}_\tau(t) := u_i && \text{if } t \in (t_{i-1}, t_i] && \text{for } i = 1, \ldots, N_\tau, \\
&\underline{u}_\tau(T) := u_{N_\tau}, && \underline{u}_\tau(t) := u_{i-1} && \text{if } t \in [t_{i-1}, t_i) && \text{for } i = 1, \ldots, N_\tau, \\
&\hat{u}_\tau(0) := u_0, && \hat{u}_\tau(t) := \frac{u_i - u_{i-1}}{t_i - t_{i-1}}(t - t_{i-1}) + u_{i-1} && \text{if } t \in (t_{i-1}, t_i] && \text{for } i = 1, \ldots, N_\tau.
\end{aligned}$$

Making use of this notation, we can rewrite (4.4.8) as

$$\|\overline{y}_\tau\|^p_{L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d))} + \left\| \frac{1}{\det \nabla \overline{y}_\tau} \right\|^q_{L^\infty(0,T;L^q(U))} + \int_0^T \|\nabla \dot{\hat{y}}_\tau^\top \nabla \underline{y}_\tau + \nabla \underline{y}_\tau^\top \nabla \dot{\hat{y}}_\tau\|^2_{L^2(U;\mathbb{R}^{d\times d})} \mathrm{d}t \leq c. \qquad (4.4.9)$$

By the Sobolev embedding of $W^{2,p}(U;\mathbb{R}^d)$ into $C^{1-d/p}(U;\mathbb{R}^d)$ and the classical result of [42, Thm. 3.1], the bound (4.4.9) implies

$$\det \nabla \overline{y}_\tau \geq c > 0 \ \text{ in } \ [0,T] \times \overline{U}. \tag{4.4.10}$$

Moreover, by the Poincaré inequality (4.2.1), the generalization of Korn's first inequality by [81] and [85, Thm. 2.2], and the positivity of the determinant (4.4.10), it follows that

$$\|\nabla \dot{\widehat{y}}_\tau\|^2_{L^2(0,T;L^2(Q;\mathbb{R}^{d\times d}))} \leq c \int_0^T \|\nabla \dot{\widehat{y}}_\tau^\top \nabla \underline{y}_\tau + \nabla \underline{y}_\tau^\top \nabla \dot{\widehat{y}}_\tau\|^2_{L^2(U;\mathbb{R}^{d\times d})}\,\mathrm{d}s \overset{(4.4.9)}{\leq} c.$$

Thus, the classical Poincaré inequality applied to $\dot{y}$ proves that

$$\|\widehat{y}_\tau\|_{H^1(0,T;H^1(U;\mathbb{R}^d))} \leq c. \tag{4.4.11}$$

Hence, the estimates above yield

$$\overline{y}_\tau, \underline{y}_\tau \overset{*}{\rightharpoonup} y \quad \text{weakly-}* \text{ in } \ L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d)), \tag{4.4.12}$$
$$\nabla \dot{\widehat{y}}_\tau \rightharpoonup \nabla \dot{y} \quad \text{weakly in } \ L^2(Q;\mathbb{R}^d), \tag{4.4.13}$$
$$\nabla \widehat{y}_\tau \to \nabla y \quad \text{strongly in } \ C^{0,\alpha}(\overline{Q};\mathbb{R}^d) \tag{4.4.14}$$

for some $\alpha \in (0,1)$, as $\tau \to 0$, up to not relabeled subsequences. In particular, these convergences imply $\det \nabla \overline{y}_\tau \to \det \nabla y$ uniformly and together with the lower bound (4.4.10), that $\nabla y \in$
$GL_+(d)$ everywhere, i.e., $y(t,\cdot) \in \mathcal{A}$ for every $t \in (0,T)$.

Summing up the time-discrete Euler–Lagrange equations (4.4.6) for $i = 1,\ldots,N_\tau$ and rewriting in terms of the time interpolants, we get

$$\int_0^T \int_U h(\theta - \overline{t}_\tau) \left(\partial_F W(\nabla \overline{y}_\tau (\overline{A}_\tau)^{-1})(\overline{A}_\tau)^{-\top} + \partial_{\dot{F}} R\left(\nabla \underline{y}_\tau, \nabla \dot{\widehat{y}}_\tau\right)\right) : \nabla \overline{z}_\tau \mathrm{d}x\mathrm{d}t$$
$$+ \int_0^T \int_U \partial_F V^J(\nabla \overline{y}_\tau) : \nabla \overline{z}_\tau + DH\left(\nabla^2 \overline{y}_\tau\right) \vdots \nabla^2 \overline{z}_\tau \mathrm{d}x\mathrm{d}t = \int_0^T \int_U h(\theta - \overline{t}_\tau) f(\overline{t}_\tau) \cdot \overline{z}_\tau \mathrm{d}x\mathrm{d}t \tag{4.4.15}$$

We now pass to the limit in (4.4.15) in order to retrieve (4.3.1). Let $z \in C^\infty(\overline{Q};\mathbb{R}^d)$ with $z \equiv 0$ on $[0,T] \times \omega$ be given and let $(z_\tau^i)_{i=1}^{N_\tau} \subset W^{2,p}(U;\mathbb{R}^d)$ be such that $\nabla z_\tau^i \in \mathrm{GL}_+(d)$, $z_\tau^i \equiv 0$ on ω for every $i = 1,...,N_\tau$, and $\overline{z}_\tau \to z$ strongly in $L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d))$. First, notice that, by the coarea formula and the lipschitzianity (4.4.3) of θ,

$$\int_0^\infty \mathcal{H}^{d-1}(\partial\Omega(t))\mathrm{d}t = \int_U |\nabla \theta| \mathrm{d}x \leq \frac{|U|}{c_\gamma} < \infty.$$

Thus, $\mathcal{H}^{d-1}(\partial\Omega(t)) = 0$ and consequently $|\partial\Omega(t)| = 0$ for almost every $t \in [0,T]$. We hence have

$$\mathcal{L}^{d+1}\left(\{(t,x) \in [0,T] \times U \mid \theta(x) = t\}\right) = \int_0^T |\partial\Omega(t)|\mathrm{d}t = 0,$$

which implies $h(\theta(x) - \overline{t}_\tau(t)) \to h(\theta(x) - t)$ for almost every $(t,x) \in Q$. By (H12), it thus follows

$$\int_0^T \int_U h(\theta - \overline{t}_\tau) f(\overline{t}_\tau) \cdot \overline{z}_\tau \mathrm{d}x\mathrm{d}t \to \int_0^T \int_U h(\theta - t) f(t) \cdot z \mathrm{d}x\mathrm{d}t.$$

Similarly, for the dissipation, we find

$$\begin{aligned}
&\int_0^T\!\!\int_U h(\theta-\bar{t}_\tau)\partial_{\dot{F}}R\left(\nabla\underline{y}_\tau,\nabla\dot{\hat{y}}_\tau\right):\nabla\overline{z}_\tau\mathrm{d}x\mathrm{d}t\\
&\quad=2\int_0^T\!\!\int_U h(\theta-\bar{t}_\tau)\nabla\underline{y}_\tau\left(\mathbb{D}(\nabla\underline{y}_\tau^\top\nabla\underline{y}_\tau)(\nabla\dot{\hat{y}}_\tau^\top\nabla\underline{y}_\tau+\nabla\underline{y}_\tau^\top\nabla\dot{\hat{y}}_\tau)\right):\nabla\overline{z}_\tau\mathrm{d}x\mathrm{d}t\\
&\quad\to 2\int_0^T\!\!\int_U h(\theta-t)\nabla y\left(\mathbb{D}(\nabla y^\top\nabla y)(\nabla\dot{y}^\top\nabla y+\nabla y^\top\nabla\dot{y})\right):\nabla z\mathrm{d}x\mathrm{d}t\\
&\quad=\int_0^T\!\!\int_U h(\theta-t)\partial_{\dot{F}}R\left(\nabla y,\nabla\dot{y}\right):\nabla z\mathrm{d}x\mathrm{d}t
\end{aligned}$$

by the convergences (4.4.12)–(4.4.14), and (H10). By the continuity (H5) and the bound (4.4.10), we also have

$$\int_0^T\int_U \partial_F V^J(\nabla\overline{y}_\tau):\nabla\overline{z}_\tau\mathrm{d}x\mathrm{d}t\to\int_0^T\int_U \partial_F V^J(\nabla y):\nabla z\mathrm{d}x\mathrm{d}t$$

Moreover, notice that by (4.4.12), for almost every $(t,x)\in Q$, $\overline{A}_\tau$ converges to A given by (4.3.2). Indeed, let $(t,x)\in Q$, $(t_{i_\tau})_\tau$ such that $t\in(t_{i_\tau-1},t_{i_\tau}]$ for every $\tau>0$, and $t_{i_\tau}\to t$, as $\tau\to 0$. Thus, $\overline{A}_\tau(t,x)=A_\tau^{i_\tau}(x)$. If $x\in\overline{\Omega_0}$, then $A_\tau^{i_\tau}(x)=A_0(x)=A(x)$, whereas if $x\in U\setminus\Omega_t$, then $\theta(x)\geq t>t_{i_\tau-i}$ and thus, by definition (4.4.5), $A_\tau^{i_\tau}(x)=I=A(x)$. On the other hand, if $x\in\Omega_t\setminus\overline{\Omega_0}$, then there exists $s\in(0,t)$ such that $\theta(x)=s$ and there exist $k_\tau\in\mathbb{N}$, $k_\tau\geq 1$, for every $\tau>0$ such that $s\in(t_{k_\tau-1},t_{k_\tau}]$. Since $s<t$, we can assume $t_{k_\tau}\leq t_{i_\tau-1}$, so that $A_\tau^{i_\tau}(x)=\nabla y_\tau^{k_\tau}(x)\to\nabla y(s,x)=\nabla y(\theta(x),x)=A(x)$, by convergence (4.4.12). Hence, by the continuity (H1) and the bound (H2) of W, the convergences (4.4.12)–(4.4.14), and dominated convergence, we have

$$\begin{aligned}
&\int_0^T\int_U h(\theta-\bar{t}_\tau)\partial_F W(\nabla\overline{y}_\tau(\overline{A}_\tau)^{-1})(\overline{A}_\tau)^{-\top}:\nabla\overline{z}_\tau\mathrm{d}x\mathrm{d}t\\
&\quad\to\int_0^T\int_U h(\theta-t)\partial_F W(\theta{-}t,\nabla yA^{-1})A^{-\top}:\nabla z\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

The convergence of the second-gradient term follows by the standard argument [48], which we provide in the following for completeness. Let $(w_\tau^i)_{i=1}^{N_\tau}\subset\mathcal{A}$ approximate the limiting function y, namely such that $\overline{w}_\tau\to y$ strongly in $L^\infty(0,T;W^{2,p}_\omega(U;\mathbb{R}^d))$ as $\tau\to 0$, and define $\overline{z}_\tau:=\overline{w}_\tau-\overline{y}_\tau$. By convergences (4.4.12)–(4.4.13), it follows that $\overline{z}_\tau\to 0$ strongly in $L^\infty(0,T;H^1(U;\mathbb{R}^d))$ and $\overline{z}_\tau\overset{*}{\rightharpoonup}0$ weakly-$*$ in $L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d))$. Moreover, by the strong convergence of $\nabla^2\overline{w}_\tau$ to $\nabla^2 y$ in $L^p(Q;\mathbb{R}^{d\times d\times d})$ and the boundedness of $\mathrm{D}H(\nabla^2\overline{y}_\tau)$ is in $L^{p'}(Q;\mathbb{R}^{d\times d\times d})$ thanks to (H9), it follows

$$\begin{aligned}
&\limsup_{\tau\to 0}\int_0^T\!\!\int_U(\mathrm{D}H(\nabla^2 y)-\mathrm{D}H(\nabla^2\overline{y}_\tau))\,\vdots\,(\nabla^2 y-\nabla^2\overline{y}_\tau)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\limsup_{\tau\to 0}\int_0^T\!\!\int_U(\mathrm{D}H(\nabla^2 y)-\mathrm{D}H(\nabla^2\overline{y}_\tau))\,\vdots\,(\nabla^2 y-\nabla^2\overline{w}_\tau+\nabla^2\overline{z}_\tau)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\limsup_{\tau\to 0}\int_0^T\!\!\int_U(\mathrm{D}H(\nabla^2 y)-\mathrm{D}H(\nabla^2\overline{y}_\tau))\,\vdots\,\nabla^2\overline{z}_\tau\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Hence, the Euler–Lagrange equation (4.4.15) with test function $\overline{z}_\tau$ and convergences (4.4.12)–(4.4.14) entail

$$\begin{aligned}
&\limsup_{\tau\to 0}\int_0^T\int_U (\mathrm{D}H(\nabla^2 y)-\mathrm{D}H(\nabla^2\overline{y}_\tau))\vdots(\nabla^2 y-\nabla^2\overline{y}_\tau)\,\mathrm{d}x\,\mathrm{d}t\\
&=\limsup_{\tau\to 0}\Bigg(\int_0^T\int_U \mathrm{D}H(\nabla^2 y)\vdots\nabla^2\overline{z}_\tau+\partial_F V^J(\nabla\overline{y}_\tau){:}\nabla\overline{z}_\tau-h(\theta{-}\overline{t}_\tau)f(\overline{t}_\tau){\cdot}\overline{z}_\tau\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\int_U h(\theta{-}\overline{t}_\tau)\left(\partial_F W(\nabla\overline{y}_\tau(\overline{A}_\tau)^{-1})(\overline{A}_\tau)^{-\top}{+}\partial_{\dot F}R\left(\nabla\underline{y}_\tau,\nabla\dot{\widehat{y}}_\tau\right)\right){:}\nabla\overline{z}_\tau\,\mathrm{d}x\,\mathrm{d}t\Bigg)=0
\end{aligned}$$

By the coercivity (H8), it follows that $\mathrm{D}H(\nabla^2\overline{y}_\tau)\to\mathrm{D}H(\nabla^2 y)$ strongly in $L^{p'}(Q;\mathbb{R}^{d\times d\times d})$ and thus (4.3.1) follows by passing to the limit in (4.4.15) as $\tau\to 0$. □

Having Propositions 4.4.1 and 4.4.2, we proceed with the proof of Theorem 4.3.1 by the following iterative construction. We first remark that, since $y_0\in\mathcal{A}$ by (H13), ∇y_0 is Hölder continuous and, thus, so is the mapping $x\in U\mapsto\gamma(y_0(x),\nabla y_0(x))$. Denoting by $\widehat{\gamma}$ be any continuous extension of such mapping to $\mathbb{R}^d$ with $c_\gamma\le\widehat{\gamma}(\cdot)\le C_\gamma$, by Proposition 4.4.1 there exists $\theta_0\in C(\overline{U})$ nonnegative viscosity solution to problem

$$\begin{aligned}
&\gamma(y_0(x),\nabla y_0(x))|\nabla\theta_0(x)|=1\quad\text{in } U\setminus\overline{\Omega_0},\\
&\theta_0=0\quad\text{in }\Omega_0,
\end{aligned}$$

satisfying (4.4.3) in $\overline{U}$. Given $\theta=\theta_0$, on the other hand, Proposition 4.4.2 provides the existence of $y^1\in L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d))\cap H^1(0,T;H^1(U;\mathbb{R}^d))$ satisfying (4.3.1).

For $k\ge 1$, given $y^k\in L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d))\cap H^1(0,T;H^1(U;\mathbb{R}^d))$, let the map $x\in U\mapsto\gamma(y^k(\theta^k(x){\wedge}T,x),\nabla y^k(\theta^k(x){\wedge}T,x))$ is Hölder continuous. Extend it continuously to $\mathbb{R}^d$ as $\widehat{\gamma}$, similarly as above, with $c_\gamma\le\widehat{\gamma}(\cdot)\le C_\gamma$. Hence, Proposition 4.4.1 and the locality of the viscosity notion of solution guarantee the existence of a nonnegative $\theta^k\in C(\overline{U})$ solving

$$\begin{aligned}
&\gamma(y^k(\theta^k(x){\wedge}T,x),\nabla y^k(\theta^k(x){\wedge}T,x))|\nabla\theta^k(x)|=1\quad\text{in } U\setminus\overline{\Omega_0},\\
&\theta^k=0\quad\text{in }\Omega_0
\end{aligned}$$

in the viscous sense and such that (4.4.3) holds in $\overline{U}$.

For such θ^k, Proposition 4.4.2 applied for $\theta=\theta^k$ entails the existence of a deformation $y^{k+1}\in L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d))\cap H^1(0,T;H^1(U;\mathbb{R}^d))$ satisfying (4.3.1).

The sequence $(y^k,\theta^k)_{k\in\mathbb{N}}$ generated by this iterative process is, although in general not unique, uniformly bounded in

$$\big(L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d))\cap H^1(0,T;H^1(U;\mathbb{R}^d))\big)\times C^{0,1}(\overline{U})$$

thanks to the bounds (4.4.9), (4.4.11), and (4.4.3). Thus, up to subsequences, by the Banach–Alaoglu and the Ascoli–Arzelà Theorems, there exists a pair (y,θ) such that, for some $\alpha\in(0,1)$,

$$y^k\overset{*}{\rightharpoonup}y\quad\text{weakly-}{*}\text{ in }L^\infty(0,T;W^{2,p}(U;\mathbb{R}^d))\cap H^1(0,T;H^1(U;\mathbb{R}^d)),\tag{4.4.16}$$

$$y^k\to y\quad\text{strongly in }C^{1,\alpha}(\overline{Q};\mathbb{R}^d),\tag{4.4.17}$$

$$\theta^k\to\theta\quad\text{strongly in }C(\overline{U}),\tag{4.4.18}$$

and θ fulfills (4.4.3) in $\overline{U}$. By the uniform Lipschitz continuity of γ, $(y^k)_{k\in\mathbb{N}}$, and $(\nabla y^k)_{k\in\mathbb{N}}$, and by convergences (4.4.17)–(4.4.18), we have that $x \mapsto \gamma(y^k(\theta^k(x)\wedge T, x), \nabla y^k(\theta^k(x)\wedge T, x))$ converges to $x \mapsto \gamma(y(\theta(x)\wedge T, x), \nabla y(\theta(x)\wedge T, x))$ uniformly in $\overline{U}$. Since the eikonal equation is stable with respect to the uniform convergence of the data [44, Prop. 1.2], θ satisfies (4.3.3) with coefficient $x \mapsto \gamma(y(\theta(x)\wedge T, x), \nabla y(\theta(x)\wedge T, x))$. Moreover, since bounds (4.4.9) and (4.4.11) are independent of θ, the same arguments of the proof of Proposition 4.4.2 allow passing to the limit in the Euler–Lagrange equation (4.3.1), thus concluding the proof of Theorem 4.3.1.

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5 Zusammenfassung

Unter Viskoelastizität versteht man die Reaktion von Materialien wie Elastomere, Ton und verschiedenen Polymeren oder Metallen, die sich unter Einwirkung äußerer Kräfte sowohl elastisch als auch viskos verhalten. Das Zusammenspiel zwischen dem festkörperähnlichen Verhalten der Elastizität und dem flüssigkeitsähnlichen der Viskosität ermöglicht die Modellierung verschiedener Phänomene in der Kontinuumsmechanik und hat zu reichhaltigen und interessanten mathematischen Theorien geführt.

Diese Dissertation zielt darauf ab, neuere Entwicklungen in nichtlinearen Variationsrechnungsmodellen für die Entwicklung viskoelastischer Materialien bei endlicher Verzerrung zu untersuchen und konzentriert sich auf zwei Hauptaspekte. Einerseits untersuchen wir das Poynting-Thomson-Modell bei großen Deformationen: Wir zeigen die Existenz von Lösungen in einem geeigneten schwachen Sinn, ohne auf regulierende Terme zweiter Ordnung zurückzugreifen, deren physikalische Interpretation umstritten ist. Darüber hinaus führen wir eine rigorose Linearisierung durch und beweisen, dass das klassische Modell für kleine Deformationen wiederhergestellt wird. Andererseits betrachten wir das Zusammenspiel von viskoelastischen Effekten mit akkretivem Wachstum, wie es bei der Kristallisation, der Quellung von Polymergelen und Erstarrungsprozessen auftritt. Wir zeigen die Existenz von Lösungen für das damit verbundene gekoppelte Problem für verschiedene Modelle: Wir konzentrieren uns auf zweiphasige Materialien mit diffuser und scharfer Grenzfläche sowie auf Festkörper, die während des Wachstums Eigenspannungen akkumulieren.