Lecture notes for Math131A: Real Analysis
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Allen Gehret

Author address:

Department of Mathematics, University of California, Los Angeles, Los Angeles, CA 90095
E-mail address: allen@math.ucla.edu

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Abstract

The goal of this class is to give a complete and rigorous proof of the Fundamental Theorems of Calculus (5.3.3 and 5.3.4) from scratch. We have only ten weeks to do this! These lecture notes are largely based off of the course textbook [**2**], except for the sake of time I exclude/rearrange some material and modify some proofs. I also include some additional material on real powers and the exponential function. I recommend you refer to these notes for learning the mathematical content of the course, and refer to the textbook for examples, pictures, and additional exercises.

Note: these lecture notes are subject to revision, so the numbering of Lemmas, Theorems, etc. may change throughout the course and I do not recommend you print out too many pages beyond the section where we are in lecture. Any and all questions, comments, and corrections are enthusiastically welcome!

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CHAPTER 1

The Real Numbers and the Completeness Axiom

You have probably encountered the following number systems before:

$$\mathbb{N} = \{1, 2, 3, \ldots\} \quad \text{(the natural numbers)}$$

$$\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\} \quad \text{(the integers)}$$

$$\mathbb{Q} = \{k/\ell : k, \ell \in \mathbb{Z}, \ell \neq 0\} = \{\text{all ``fractions''}\} \quad \text{(the rational numbers)}$$

and finally:

$$\mathbb{R} \quad \text{(the set of all real numbers)}$$

Furthermore, we know that these number systems contain one another:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

i.e., every natural number *is an* integer, every integer *is a* rational number, and every rational number *is a* real number.

The development of these number systems somewhat reflects the stages in life you learned about them. For example:

(1) As a young child you probably first learned about the natural numbers (or "counting numbers"), i.e., $1, 2, 3, 4, 5, \ldots$ You also slowly learned how to add and multiply these numbers.
(2) Later on in elementary school you learn about the numbers "0" as well as the negative natural numbers $-1, -2, -3, \ldots$ You also learned how to add and multiply these numbers.
(3) At some point in elementary school you also start to learn about fractions $1/2, 2/3, 3/4, \ldots$ as well as how to add and multiply these numbers as well.
(4) Finally, later in middle school and high school, you start learning about "real numbers" which aren't fractions, like $\sqrt{2}, \pi$, and e. You were probably given a vague (and possibly incorrect) description of "what is the set of real numbers $\mathbb{R}$" suitable enough to learn how to do the computations in calculus. However, you probably didn't spend any time discussing what $\mathbb{R}$ "really is", as a mathematical object.

The first goal of the class is to provide a correct and satisfying answer to the following question:

Question 1.0.1. *What exactly is $\mathbb{R}$, the collection of real numbers?*

We will build towards an answer (Answer 1.4.11). First we study the main properties $\mathbb{N}$. Then we look at $\mathbb{Z}$ and $\mathbb{Q}$. Finally, we look at what distinguishes $\mathbb{Q}$ from $\mathbb{R}$ and provide the defining property of $\mathbb{R}$, the *Completeness Axiom*.

1.1. The natural numbers and induction

In this class[1], the **natural numbers** is the set

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

of positive integers. We will not attempt to construct the natural numbers axiomatically, instead we assume that they are already given and that we are familiar with their basic properties, for instance, how the operations $+, \cdot$ and the ordering $\leq$ work with $\mathbb{N}$. Here is an important basic property about $\mathbb{N}$ which we will take for granted:

Well-Ordering Principle 1.1.1. *Suppose $S \subseteq \mathbb{N}$ is such that $S \neq \emptyset$. Then S has a least element, i.e., there is some $a \in S$ such that for all $b \in S$, $a \leq b$.*

The Well-Ordering Principle of $\mathbb{N}$ gives us the following important *proof principle* about natural numbers:

Principle of Induction 1.1.2. *Suppose $P(n)$ is a property that a natural number n may or may not have. Suppose that*

(1) $P(1)$ holds (this is called the "base case for the induction"), and
(2) for every $n \in \mathbb{N}$, if $P(n)$ holds, then $P(n+1)$ holds (this is called the "inductive step").

Then $P(n)$ holds for every *natural number $n \in \mathbb{N}$.*

Proof. Define the set:

$$S := \big\{n \in \mathbb{N} : P(n) \text{ is false}\big\} \subseteq \mathbb{N}.$$

Assume towards a contradiction that $P(n)$ does not hold for every natural number $n \in \mathbb{N}$. Thus $S \neq \emptyset$. By the Well-Ordering Principle, the set S has a least element a. Since $P(1)$ holds by assumption, we know that $1 < a$ (so $a - 1 \in \mathbb{N}$). Since a is the least element of S, then the natural number $a - 1 \notin S$, so $P(a-1)$ holds. By assumption (2), this implies $P(a)$ holds, a contradiction. □

Warning 1.1.3. In part (2) of the Principle of Induction, it does not say you have to prove $P(n+1)$ is true. It says you have to prove that the following implication holds:

$$\big(P(n) \text{ is true}\big) \implies \big(P(n+1) \text{ is true}\big)$$

Notation 1.1.4. Suppose $M, N \in \mathbb{Z}$ are integers such that $M \leq N$ and suppose we are given numbers $a_M, a_{M+1}, \ldots, a_N$ indexed by all integers between M and N. Then we denote the **finite summation** of the integers $a_M, \ldots, a_N$ by

$$\sum_{k=M}^{N} a_k := a_M + a_{M+1} + \cdots + a_N.$$

The "k" in the expression $\sum_{k=M}^{N} a_k$ is referred to as the **index of summation** and it is a **dummy variable**, i.e., a variable which only exists and takes value inside the summation. This is analogous to the control variable of a `for-loop` in

[1] In other textbooks, sometimes the natural numbers include zero, i.e., $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$. Also sometimes people might write $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. We won't do this here, but it's good to know about it. The important thing is that when you are communicating about the natural numbers with someone else, you are always on the same page about whether you are including 0 or not.

a computer program. Accordingly, the index of summation can be changed to any other variable which is not being used and the meaning will stay the same, i.e.,

$$\sum_{k=M}^{N} a_k \;=\; \sum_{\ell=M}^{N} a_\ell \;=\; \sum_{j=M}^{N} a_j \;=\; \cdots$$

We now arrive at the very first example of a proof by induction. It is the canonical "first proof by induction that everybody should know" and it involves the so-called *triangular numbers*[2]:

Example 1.1.5. The equality

$$\sum_{k=1}^{n} k \;=\; \frac{n(n+1)}{2}$$

holds for all $n \in \mathbb{N}$.

Note: recall that by Notation 1.1.4, $\sum_{k=1}^{n} k = 1 + 2 + \cdots + n$.

Proof. Let $P(n)$ be the assertion:

$$P(n): \quad \text{“}\textstyle\sum_{k=1}^{n} k = n(n+1)/2 \text{ is true.”}$$

We will show that $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

First, we show that $P(1)$ holds outright. This is easy because $P(1)$ says "$1 = \frac{1}{2} \cdot 1 \cdot 2$", which is obviously true.

Next, we will show that $P(n)$ implies $P(n+1)$. Suppose $P(n)$ holds, i.e.,

$$\sum_{k=1}^{n} k \;=\; \frac{1}{2} n(n+1).$$

We must now show that $P(n+1)$ also holds. Note that:

$$\begin{aligned}
\sum_{k=1}^{n+1} k \;&=\; \sum_{k=1}^{n} k + (n+1) \\
&=\; \frac{1}{2} n(n+1) + (n+1) \quad \text{since } P(n) \text{ is assumed to be true} \\
&=\; (n/2+1)(n+1) \\
&=\; \frac{1}{2}(n+2)(n+1) \\
&=\; \frac{1}{2}(n+1)\big((n+1)+1\big).
\end{aligned}$$

Thus $P(n+1)$ holds as well. □

We also have the following variant of the Principle of Induction, which starts at some natural number (or integer!) other than 1:

Corollary 1.1.6 (Principle of Induction starting at N)**.** *Let $N \in \mathbb{Z}$ and suppose $P(n)$ is a property that an integer $n \geq N$ may or may not have. Suppose that*

(1) $P(N)$ holds.

(2) for every $n \geq N$, if $P(n)$ holds, then $P(n+1)$ holds.

Then $P(n)$ holds for every integer $n \geq N$.

[2] https://en.wikipedia.org/wiki/Triangular_number

Proof. We will prove this by reducing it to the original Induction Principle by shifting. Let $Q(n)$ be the statement:

$$Q(n): \quad \text{“}P(n+N-1) \text{ holds.”}$$

Then (1) implies that $Q(1)$ holds. Also, (2) implies that for every $n \geq 1$, $Q(n) \Rightarrow Q(n+1)$. Thus $Q(n)$ is true for all $n \geq 1$ by the Principle of Induction. In other words, $P(n)$ is true for all $n \geq N$. □

Usually, but not always, proofs by induction start at $N = 1$, but starting at $N = 0$ is also common. The next example shows an induction proof starting at $N = 10$:

Example 1.1.7. $2^n > n^3$ for all $n \geq 10$.

Proof. Let $P(n)$ be the assertion:

$$P(n): \quad \text{“}2^n > n^3\text{”}$$

First, we check that $P(10)$ holds outright. Note that $2^{10} = 1024 > 1000 = 10^3$.

Next, we will show that for $n \geq 10$, the implication $P(n) \Rightarrow P(n+1)$ holds. So assume that $P(n)$ is true for some $n \geq 10$, i.e., $2^n > n^3$. We need to use this to show $2^{n+1} > (n+1)^3$. Note that

$$(n+1)^3 \;=\; n^3 + 3n^2 + 3n + 1.$$

Now, $2^n > n^3$ implies $2^n \geq n^3 + 1$ (since 2^n and n^3 are natural numbers), so

$$2^{n+1} \;\geq\; 2(n^3+1) \;=\; 2n^3 + 2.$$

Thus, it suffices to show that

$$2n^3 + 2 \;\geq\; (n+1)^3 + 1 \;=\; n^3 + 3n^2 + 3n + 2.$$

However, this is equivalent to

$$n^3 \;\geq\; 3n^2 + 3n \;=\; 3n(n+1)$$

which is equivalent to

$$n^2 \;\geq\; 3(n+1).$$

However, $n \geq 10$ implies $n \geq 4 \geq 3$, so

$$n^2 \;\geq\; 4n \;\geq\; 3n + 3 \;=\; 3(n+1),$$

as required. □

1.2. The integers and rational numbers

The natural numbers $\mathbb{N}$ have some natural defects as a number system. First and foremost is that we cannot always solve linear equations of the form:

$$x + m \;=\; n \quad \text{for given } m, n \in \mathbb{N}$$

To remedy this, we introduce the **integers**:

$$\mathbb{Z} \;=\; \{0, \pm 1, \pm 2, \pm 3, \ldots\}$$

However, the integers have a similar defect, i.e., you cannot solve arbitrary linear equations:

$$ax + b \;=\; c \quad (a, b, c \in \mathbb{Z}, a \neq 0).$$

To remedy this, we further enlarge $\mathbb{Z}$ to the **rational numbers**:

$$\mathbb{Q} \;=\; \left\{ \frac{k}{\ell} : k, \ell \in \mathbb{Z}, \ell \neq 0 \right\}$$

Here are some nice features of $\mathbb{Q}$:

(1) Linear equations with rational coefficients can be solved over $\mathbb{Q}$. More generally, most of linear algebra can be done using only rational numbers. The main exception to this is that the eigenvalues of a rational matrix might not be themselves rational numbers.
(2) Rational numbers are very *concrete*: you can easily represent a rational number with infinite precision as a fraction k/ℓ of integers. Arithmetic operations with rational numbers also behave quite nicely.

However, $\mathbb{Q}$ still has some defects. In a certain vague sense, the rational numbers are *incomplete*, i.e., there are many theoretical numbers that ought to exist, but don't exist as rational numbers. The best instance of this is the well-known fact that many *polynomial equations*

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \;=\; 0 \quad (a_i \in \mathbb{Q})$$

do not have solutions in $\mathbb{Q}$. In fact, even the equation

$$x^2 - 2 \;=\; 0$$

cannot be solved in $\mathbb{Q}$. We show now that there are no rational solutions to the equation $x^2 - 2 = 0$:

Proposition 1.2.1 ("$\sqrt{2}$" is irrational)**.** *There is no rational number $c \in \mathbb{Q}$ with the property that $c^2 = 2$.*

The proof of Proposition 1.2.1 assumes two basic facts[3] about the integers and rational numbers:

(i) Each rational number c can be expressed as $c = m/n$, where m and n are integers and m or n is odd (an integer m is **odd** if there exists an integer k such that $m = 2k + 1$).
(ii) An integer n is even if its square n^2 is even (an integer n is **even** if there exists an integer k such that $n = 2k$).

PROOF OF PROPOSITION 1.2.1. Suppose towards a contradiction that there is a rational number $c \in \mathbb{Q}$ such that $c^2 = 2$. Then by (i) there are integers $m, n \in \mathbb{Z}$ such that m or n is odd and $c = m/n$. Then $m^2/n^2 = 2$, so $m^2 = 2n^2$. Thus m^2 is even, so by (ii), m is even. Take $k \in \mathbb{Z}$ such that $m = 2k$. Then $m^2 = 2n^2$ implies $4k^2 = 2n^2$. Dividing by 2 shows $2k^2 = n^2$. Thus n^2 is also even and by (ii) again, n is even. Thus both m and n are even, a contradiction. □

You may already have heard about the existence of certain real numbers ($\pm\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$) which are solutions to the equation $x^2 - 2 = 0$. However, at this point you should forget about this and instead adopt a skeptical point of view that real numbers only exist if we can prove they exist, and we haven't proven "$\sqrt{2}$" exists yet so it doesn't make sense to talk about it yet (the existence of such a number "$\sqrt{2}$" will follow from the more general Existence of nth roots 1.6.2 below).

[3]Properties (i) and (ii) should be very believable, although to prove them rigorously would require a small detour into the realm of *elementary number theory* including establishing a fragment of *prime factorization*.

1.3. Inequalities

Recall that the basic operations on the real numbers are addition $+$, multiplication $\cdot$. The real numbers also come equipped with an ordering $\leq$. We assume we are familiar with the basic properties of these operations. Here are some useful facts to keep in mind:

Facts 1.3.1. *Suppose $a, b, c \in \mathbb{R}$. Then*

(1) if $a \leq b$, then $a + c \leq b + c$,
(2) if $a \leq b$ and $0 \leq c$, then $ac \leq bc$,
(3) if $a \leq b$, then $-b \leq -a$,
(4) if $a \leq b$ and $c \leq 0$, then $ac \geq bc$,
(5) if $a \neq 0$, then $a^2 > 0$; in particular, $1 > 0$,
(6) if $a > 0$, then $a^{-1} > 0$,
(7) if $0 < a < b$, then $0 < b^{-1} < a^{-1}$.

Definition 1.3.2. For each $a \in \mathbb{R}$ we define the **absolute value** $|a|$ of a by

$$|a| \ := \ \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

Remark 1.3.3. Observe that if $a, b \in \mathbb{R}$ are such that $b \geq 0$, then

$$|a| \leq b \quad \text{if and only if} \quad -b \leq a \leq b.$$

This will be useful for showing inequalities of the form "$|a| \leq b$".

Here are the basic properties of the absolute value:

Proposition 1.3.4. *Suppose $a, b \in \mathbb{R}$. Then*

(1) $|a| \geq 0$
(2) $|a \cdot b| = |a| \cdot |b|$

PROOF. (1) If $a \geq 0$, then $|a| = a \geq 0$, and if $a < 0$, then $|a| = -a > 0$.

(2) (Case 1: $a \geq 0$ and $b \geq 0$) Then $a \cdot b \geq 0$, so $|a \cdot b| = a \cdot b = |a| \cdot |b|$.

(Case 2: $a < 0$ and $b < 0$) Then $-a, -b > 0$, so

$$a \cdot b \ = \ (-a) \cdot (-b) \ > \ 0$$

hence $|a \cdot b| = a \cdot b = |a| \cdot |b|$.

(Case 3: $a \geq 0$ and $b < 0$) Then $-b \geq 0$, and $a \cdot (-b) \geq 0$, hence

$$|a| \cdot |b| \ = \ a \cdot (-b) \ = \ -(a \cdot b) \ = \ |a \cdot b|.$$

(Case 4: $a < 0$ and $b \geq 0$) This is similar to Case 3. □

The following inequality is perhaps the most important and fundamental inequality in analysis. It will be used frequently, sometimes without explicit mention.

Triangle Inequality 1.3.5. *For every $a, b \in \mathbb{R}$:*

$$|a + b| \leq |a| + |b|$$

PROOF. We have

$$-|a| \leq a \leq |a| \quad \text{and} \quad -|b| \leq b \leq |b|.$$

Applying 1.3.1(1) four times gives

$$\begin{aligned}(-|a|)+(-|b|) &\leq a+(-|b|)\\ &\leq a+b\\ &\leq |a|+b\\ &\leq |a|+|b|,\end{aligned}$$

or in other words:

$$-(|a|+|b|) \ \leq\ a+b \ \leq\ |a|+|b|.$$

By Remark 1.3.3 it follows that

$$|a+b| \ \leq\ |a|+|b|. \qquad \square$$

Related to the triangle inequality is the so-called *reverse triangle inequality*:

Reverse Triangle Inequality 1.3.6. *For every $a, b \in \mathbb{R}$:*

$$|a-b| \ \geq\ \big||a|-|b|\big|$$

PROOF. This is Exercise 1.9.6. $\square$

The following lemma is rather useful:

Power Inequality 1.3.7. *For every $n \in \mathbb{N}$ and for all $x, y \in \mathbb{R}$, if $0 \leq x < y$, then $0 \leq x^n < y^n$.*

PROOF. We begin with proving a variant of Fact 1.3.1(4):

Claim. *Suppose $a, b, c \in \mathbb{R}$. If $a < b$ and $0 < c$, then $ac < bc$.*

PROOF OF CLAIM. Fact 1.3.1(4) implies that $ac \leq bc$. Assume towards a contradiction that $ac = bc$. Then $bc \leq ac$. Since $c > 0$, we have $c^{-1} > 0$ by Fact 1.3.1(6). Applying 1.3.1(4) now to $bc \leq ac$ and $0 \leq c^{-1}$ yields $b = bcc^{-1} \leq acc^{-1} = a$, which contradicts $a < b$. We conclude that $ac < bc$. $\square$

We now proceed to prove the lemma by induction on n. Let $P(n)$ be the assertion:

$$P(n): \quad \text{“for all } x, y \in \mathbb{R}\text{, if } 0 \leq x < y\text{, then } x^n < y^n\text{.”}$$

First, we note that that the base case $P(1)$ is obviously true, since $0 \leq x < y$ implies $0 \leq x^1 < y^1$.

Next, we will show that $P(n)$ implies $P(n+1)$. Suppose $P(n)$ holds, i.e., for every $x, y \in \mathbb{R}$, if $0 \leq x < y$, then $0 \leq x^n < y^n$. We must show that $P(n+1)$ holds. Let $x, y \in \mathbb{R}$ be arbitrary. Assume $0 \leq x < y$. Then

$$\begin{aligned}x^{n+1} \ = \ x^n x &\leq x^n y && \text{by 1.3.1(4), using } x \leq y \text{ and } 0 \leq x^n\\ &< y^n y && \text{by above Claim, using } x^n < y^n \text{ and } 0 < y\\ &= y^{n+1}.\end{aligned}$$

We conclude that $x^{n+1} < y^{n+1}$. Finally, since $0 \leq x^n$ and $0 \leq x$, we conclude that $0 \leq x^{n+1}$ by 1.3.1(4). $\square$

1.4. The real numbers and the completeness axiom

Definition 1.4.1. Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$.

(1) The largest element of S (if there is one) is called the **maximum** of S, denoted by $\max S$.
(2) The least element of S (if there is one) is called the **minimum** of S, denoted by $\min S$.

Example 1.4.2. (1) Every finite nonempty subset of $\mathbb{R}$ has a maximum and minimum.
(2) Let a, b with $a < b$. Then the **closed interval**

$$[a,b] \;:=\; \{x \in \mathbb{R} : a \leq x \leq b\}$$

has $\min[a,b] = a$ and $\max[a,b] = b$. The **open interval**

$$(a,b) \;:=\; \{x \in \mathbb{R} : a < x < b\}$$

has neither a minimum nor a maximum. Note that both closed and open intervals are infinite sets.
(3) The sets $\mathbb{Z}$ and $\mathbb{Q}$ do not have a minimum or a maximum. The set $\mathbb{N}$ has $\min \mathbb{N} = 1$, but no maximum.
(4) (Assume temporarily that we know $\sqrt{2}$ exists). The set

$$\{r \in \mathbb{Q} : r^2 \leq 2\} \;=\; \big[-\sqrt{2}, \sqrt{2}\big] \cap \mathbb{Q}$$

has neither a minimum nor a maximum since $\sqrt{2} \notin \mathbb{Q}$.

Definition 1.4.3. Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$.

(1) We call $M \in \mathbb{R}$ an **upper bound** for S if $M \geq s$ for all $s \in S$. If such an upper bound exists, then we say that S is **bounded from above**.
(2) We call $m \in \mathbb{R}$ a **lower bound** for S if $m \leq s$ for all $s \in S$. If such a lower bound exists, then we say that S is **bounded from below**.
(3) We say that S is **bounded** if it bounded from above and from below. In this case:

$$S \subseteq [-M, M] \quad \text{for some } M \in \mathbb{R} \text{ such that } M > 0.$$

Example 1.4.4. (1) The maximum of a set (if it exists) is an upper bound. The minimum of a set (if it exists) is a lower bound.
(2) Let $a < b \in \mathbb{R}$. Then a is a lower bound for both $[a,b]$ and (a,b), while b is an upper bound for $[a,b]$ and (a,b). In fact, in each case a is the *largest* lower bound and b is the *smallest* upper bound.
(3) Neither $\mathbb{Z}$ nor $\mathbb{Q}$ are bounded from below or above.
(4) The least upper bound of $\{r \in \mathbb{Q} : r^2 \leq 2\}$ is $\sqrt{2}$, and the greatest lower bound is $-\sqrt{2}$.

Definition 1.4.5. Let $S \subseteq \mathbb{R}$ such that $S \neq \emptyset$.

(1) If S is bounded from above and has a least upper bound s_0, then we call s_0 the **supremum**[4] of S, written: $s_0 = \sup S$.
(2) If S is bounded from below and has a great lower bound s_1, then we call s_1 the **infimum**[5] of S, written: $s_1 = \inf S$.

[4] "Sup" is pronounced like the word *soup*.
[5] "Infimum" is pronounced like *in-fee-mum*.

Remark 1.4.6.
(1) Every $S \subseteq \mathbb{R}$ such that $S \neq \emptyset$, can have at most one supremum and one infimum.
(2) If $S \subseteq \mathbb{R}$ has a maximum, then $\max S = \sup S$. If S has a minimum, then $\min S = \inf S$.
(3) The following are equivalent:
   (a) $s_0 = \sup S$,
   (b) (i) $s_0 \geq s$ for all $s \in S$, and
       (ii) if $s_1 \geq s$ for all $s \in S$, then $s_1 \geq s_0$
   (c) (i) $s_0 \geq s$ for all $s \in S$, and
       (ii') if $s_1 < s_0$, then $s_1 < s$ for some $s \in S$

   In practice, to prove something like (a), it is easier to prove (i) and (ii) of (b), or (i) and (ii') of (c) depending on the situation.

Example 1.4.7.
(1) For $a < b$ in $\mathbb{R}$:

$$\sup[a,b] \;=\; \sup(a,b) \;=\; b$$
$$\inf[a,b] \;=\; \inf(a,b) \;=\; a$$

(2) Assume we know already about $\sqrt{2}$, then

$$\sup\{r \in \mathbb{Q} : r^2 \leq 2\} \;=\; \sqrt{2}$$
$$\inf\{r \in \mathbb{Q} : r^2 \leq 2\} \;=\; -\sqrt{2}$$

The following is the characteristic property of $\mathbb{R}$:

Completeness Axiom 1.4.8. *Every non-empty subset of $\mathbb{R}$ which is bounded from above has a least upper bound. In other words, given $S \subseteq \mathbb{R}$, if $S \neq \emptyset$ and S has at least one upper bound, then* $\sup S$ *exists.*

The Completness Axiom 1.4.8 as stated concerns what happens at the top of sets. There is also an equivalent "infimum version" which involves what happens below the set. First, a definition: given a set $S \subseteq \mathbb{R}$, we define $-S$ to be the reflection of S about the origin, i.e.,

$$-S \;:=\; \{-s : s \in S\}.$$

For example, $-(1,2] = [-2,-1)$. Note that $-(-S) = S$. The following allows us to relate results involving supremums with results involving infimums:

Sup-Inf Symmetry 1.4.9. *Suppose $S \subseteq \mathbb{R}$ is nonempty.*

(1) If S is bounded above, then $-S$ is bounded below and $\sup S = -\inf(-S)$
(2) If S is bounded below, then $-S$ is bounded above and $\inf S = -\sup(-S)$

Proof. See Exercise 1.9.11. □

Completeness Axiom 1.4.10 (Infimum version)**.** *Every nonempty subset S of $\mathbb{R}$ which is bounded from below has a greatest lower bound* $\inf S$*.*

Proof. Suppose $S \subseteq \mathbb{R}$ is nonempty and bounded below. Then $-S$ is nonempty and bounded above, by Sup-Inf Symmetry 1.4.9(1). In particular, $\sup(-S) \in \mathbb{R}$ exists by the Completeness Axiom 1.4.8. Then $-\sup(-S) = \inf(S)$ by Sup-Inf Symmetry 1.4.9(2). In particular, $\inf(S) \in \mathbb{R}$ exists. □

We now provide our answer to Question 1.0.1:

Answer 1.4.11. *$\mathbb{R}$ is a number system*[6] *which contains $\mathbb{Q}$ and satisfies the Completeness Axiom.*

Why is 1.4.11 a good answer to 1.0.1? Because it tells you everything you need to know about the real numbers in order to deduce all other properties about the real numbers. For the rest of the course, we will only need to know the basic properties of $+, \cdot, \leq$ for the real numbers, as well as the Completeness Axiom. Everything else will follow.

1.5. Geography of the real numbers

In this section we include a few results on the big picture "landscape" of the real number line, i.e., where can you find what types of numbers. These may seem obvious, but they really require us to use the Completeness Axiom to establish rigorously.

Archimedean Property 1.5.1. *The following properties about real numbers hold:*

1. *For every positive real number $a > 0$, there is a natural number n such that $n > a$.*
2. *For every two positive real numbers $a, b > 0$, there is a natural number n such that $na > b$.*
3. *For every positive real number $\epsilon > 0$, there is a natural number n such that $1/n < \epsilon$.*

Proof. We first prove (1). Assume towards a contradiction that there is a positive real number $a > 0$ such that $a \geq n$ for every natural number n. Thus a is an upper bound for the set $\mathbb{N}$ of natural numbers. By the Completeness Axiom, we can take $b := \sup(\mathbb{N})$. As b is the least upper bound of $\mathbb{N}$, the number $b - 1/2$ is *not* an upper bound of $\mathbb{N}$. In particular, there is a natural number n such that $n > b - 1/2$. Adding 1 yields

$$n + 1 \; > \; (b - 1/2) + 1 \; > \; b.$$

Thus $n + 1$ is a natural number which is larger than b, contradicting that b is an upper bound of $\mathbb{N}$. Thus for every $a > 0$ there is a natural number n such that $a < n$.

Now that we know (1) is true, we will use it to prove (2). Suppose $a, b > 0$ are positive real numbers, in particular, $b/a > 0$. By (1) there is a natural number n such that $n > b/a$. Multiplying through by a we get $na > b$.

Finally, we will prove (3). Suppose $\epsilon > 0$ is a positive real number. Then $1/\epsilon > 0$ is also a positive real number. By (1) there is a natural number n such that $n > 1/\epsilon$. Multiplying both sides by the positive number ϵ/n then yields $\epsilon > 1/n$. □

The following says that if there is a large enough gap between two real numbers, you can always find an integer:

[6] Technically, $\mathbb{R}$ is a so-called *ordered field*. See Appendix B for an explanation of what it means. Essentially it means that $+, \cdot$ and $\leq$ satisfy a certain list of axiom which we are already familiar with. Also, it technically is redundant to say "$\mathbb{R}$ contains $\mathbb{Q}$", since every ordered field automatically contains a copy of $\mathbb{Q}$: you just consider the subset which is generated by $0, 1$ and closed under addition, subtraction, multiplication and division. In abstract algebra terms, we would express this as "all ordered fields are characteristic 0 so their prime subfield is isomorphic to the field $\mathbb{Q}$."

Distribution of Integers 1.5.2. *Suppose $a < b$ are real numbers such that $b - a > 1$. Then there is an integer m such that $a < m < b$.*

Proof. By the Archimedean Property, there is a natural number $k > \max\big(|a|, |b|\big)$ such that

$$-k < a < b < k.$$

Then the sets $K = \{j \in \mathbb{Z} : -k \leq j \leq k\}$ and $\{j \in K : a < j\}$ are finite, and nonempty since they contain k. Define $m = \min\{j \in K : a < j\}$. Then $-k < a < m$. As $m > -k$, we get $m - 1 \in K$, so

$$m - 1 \leq a < a + 1 < b$$

by the definition of m and assumption that $b - a > 1$. Adding 1 yields $m < b$, and thus $a < m < b$. □

Between any two real numbers, no matter how tiny the gap, you can always find a rational number:

Denseness of $\mathbb{Q}$ 1.5.3. *Suppose $a, b \in \mathbb{R}$ are such that $a < b$. Then there is a rational number $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. We need to find integers m, n such that $n > 0$ and

$$a < \frac{m}{n} < b$$

(then $r = m/n$ is the desired rational number we seek). Multiplying through by n, it suffices to find a natural number $n > 0$ and an integer m such that

$$an < m < bn.$$

We need to arrange that the gap between an and bn is large enough to contain an integer. By the Archimedean Property, there is a natural number $n > 0$ such that $n(b - a) > 1$, and thus $bn - an > 1$. By Distribution of integers (applied to the real numbers $an < bn$), we can find an integer m such that $an < m < bn$. □

1.6. Existence of nth roots and real powers

In this section we study an important and useful operation in the real numbers: taking arbitrary real powers of a fixed base number. As motivation, fix a positive real number $b > 0$. Then we can initially define the **power function of base** b to be the function $p_0 \colon \mathbb{Z} \to \mathbb{R}$ defined by:

$$(\dagger) \qquad p_0(k) := \begin{cases} \underbrace{b \times \cdots \times b}_{k \text{ times}} & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ \underbrace{b^{-1} \times \cdots \times b^{-1}}_{-k \text{ times}} & \text{if } k < 0 \end{cases}$$

In other words, p_0 is the power function "b^k". Note that at this point we only have defined this function to have domain $\mathbb{Z}$. As is customary, we will denote $p(k)$ instead as b^k. This function has the following well-known properties:

(P1) (Exponent rule) For every $k, \ell \in \mathbb{Z}$, $b^{k+\ell} = b^k b^\ell$.
(P2) (Monotonicity) For every $k, \ell \in \mathbb{Z}$ such that $k < \ell$:
  (a) if $b > 1$, then $b^k < b^\ell$
  (b) if $b = 1$, then $b^k = b^\ell = 1$

(c) if $b < 1$, then $b^k > b^\ell$

One tempting question to ask at this point is the following:

Question 1.6.1. *Is it possible to extend the function* $b^k\colon \mathbb{Z} \to \mathbb{R}$ *to a function* $b^x\colon \mathbb{R} \to \mathbb{R}$ *(whose domain is all of* $\mathbb{R}$*) in such a way that the new function* b^x *still has properties (P1) and (P2) for all* $k, \ell \in \mathbb{R}$*? If it is possible, how many different ways are there to define such a function?*

Ideally, it ought to be possible to define such a function "b^x" and moreover there should be only one function that has properties (P1) and (P2). This will indeed be the case. However, you should first convince yourself that it is far from obvious how to define such a function. Clearly, we cannot extend definition (†) to non-integers, so we have to do something else. As a warmup, we first show how to define "$b^{1/n}$" for natural numbers n. Our treatment is based on [**3**, 1.21] and will illustrate the power of the Completeness Axiom.

Existence and Uniqueness of nth roots 1.6.2. *Fix a real number* $b > 0$*, a natural number* n*, and define the set*

$$E \;:=\; \{t \in \mathbb{R} : t^n < b\}$$

Then:

(1) *E is nonempty and bounded above.*
(2) *(Existence) The number $y := \sup E$ has the property $y^n = b$.*
(3) *(Uniqueness) If $z \in \mathbb{R}$ is such that $z > 0$ and $z^n = b$, then $y = z$.*
(4) *(Inequality) If $b > 1$, then for every n:*

$$0 \;<\; b^{1/n} - 1 \;\le\; \frac{b-1}{n}$$

We will write the number y above as $b^{1/n}$ or $\sqrt[n]{b}$. The Uniqueness tells us that no other positive real number deserves the name "nth root of b". There is also a similar Inequality (4) for the case $0 < b < 1$, but we won't need it since below we only consider the case $b > 1$ (for simplicity).

Proof. (1) We will prove E is nonempty and bounded above:

- Clearly $0 \in E$ and so $E \neq \emptyset$. However, we will need to know below that E contains a positive element. Set $t := b/(1+b)$. Then $0 < t < \min(1, b)$, so $t^n \le t < b$, which implies that $t \in E$.
- We claim that $1 + b$ is an upper bound for E. Indeed, if $s > 1 + b$, then $s^n \ge s > b$, so $s \notin E$.
- This permits us to define $y := \sup E$ by the Completeness Axiom. Note that since $0 < b/(1+b) \in E$, we know that $y > 0$.

(2) Now we will prove $y^n = b$, by getting two contradictions:

- Assume towards a contradiction that $y^n < b$. We will show that then there is a tiny $h > 0$ such that $(y+h)^n < b$, contradicting that y is an upper bound of E since $y + h \in E$. Choose h small enough such that $0 < h < 1$ and
$$h \;<\; \frac{b - y^n}{n(y+1)^{n-1}}.$$
Then by the Difference of Powers Inequality A.2.3 and choice of h we have
$$(y+h)^n - y^n \;<\; hn(h+h)^{n-1} \;<\; hn(y+1)^{n-1} \;<\; b - y^n,$$

and so $(y+h)^n < b$, a contradiction.

- Now assume towards a contradiction that $y^n > b$. Then we will show there is a tiny $k > 0$ such that $y - k$ is also an upper bound of E, contradicting that y is the least upper bound of E. Set
$$k \ := \ \frac{y^n - b}{ny^{n-1}} \ > \ 0 \quad \text{(using } y > 0 \text{ for the denominator)}$$
Also, since $-b < 0 \le (n-1)y^n$, it follows that $k < y$. Now, suppose $t \ge y - k$ is arbitrary. Then by the Difference of Powers Inequality again we have,
$$y^n - t^n \ \le \ y^n - (y-k)^n \ < \ kny^{n-1} \ = \ y^n - b$$
and so $t^n > b$, so $t \notin E$. Thus $y-k$ is an upper bound of E, a contradiction.

(3) Assume towards a contradiction that $y \neq z$, i.e., that either $z < y$ or $y < z$. Then by the Power Inequality 1.3.7 $z^n < y^n = b$ or $y^n = b < z^n$, contradicting the assumption that $z^n = y^n = b$.
(4) Setting $x := b^{1/n} - 1 > 0$ in Bernoulli's Inequality A.3.1 we get
$$(1 + b^{1/n} - 1)^n \ \ge \ 1 + n(b^{1/n} - 1)$$
which we can rewrite as
$$b^{1/n} - 1 \ \le \ \frac{b-1}{n}.$$
□

The following shows that we can distribute $1/n$th powers the same way as integer powers. This actually requires a subtle argument involving the *Uniqueness* part of 1.6.2.

Corollary 1.6.3. *Given $a, b \in \mathbb{R}$ such that $a, b > 0$ and $n \in \mathbb{N}$, we have*
$$(ab)^{1/n} \ = \ a^{1/n} b^{1/n}.$$

Proof. Define the following numbers:
$$\alpha \ := \ (ab)^{1/n}$$
$$\beta \ := \ a^{1/n}$$
$$\gamma \ := \ b^{1/n}$$
Note that since $n \in \mathbb{N}$, the laws of exponents for *integers* imply
$$(\beta\gamma)^n \ = \ \beta^n \gamma^n \ = \ (a^{1/n})^n (b^{1/n})^n \ = \ ab.$$
Thus the real number $\beta\gamma$ is also a positive nth root of ab. Since α is the *only* positive nth root of ab according to 1.6.2, we must conclude that these numbers are the same, i.e., that $\alpha = \beta\gamma$. In other words:
$$(ab)^{1/n} \ = \ a^{1/n} b^{1/n}$$
□

We now continue with our mission to define and make sense of "b^x". For simplicity, we will assume that $b > 1$. The next order of business is to now extend the power function $b^k \colon \mathbb{Z} \to \mathbb{R}$ to a sensible function $b^q \colon \mathbb{Q} \to \mathbb{R}$. This is the content of the next theorem:

Rational Power Theorem 1.6.4. *Fix $b \in \mathbb{R}$ such that $b > 1$.*

(1) *Define the two-variable function* $p_1 : \mathbb{Z} \times \mathbb{N} \to \mathbb{R}$ *by* $p_1(m, n) := (b^m)^{1/n}$. *Then given* $m, k \in \mathbb{Z}$ *and* $n, \ell \in \mathbb{N}$:

$$\text{if } m/n = k/\ell, \text{ then } p_1(m, n) \;=\; p_1(k, \ell)$$

(2) *(Existence) Define the function* $p_2 : \mathbb{Q} \to \mathbb{R}$ *by*

$$p_2(q) \;:=\; p_1(m, n) \;=\; (b^m)^{1/n} \quad \text{for any } m, n \in \mathbb{Z} \text{ such that } n > 0 \text{ and } q = m/n.$$

Then

(a) *for every* $k \in \mathbb{Z}$, $p_2(k) = p_0(k)$ *(i.e., this candidate power function defined for rational powers agrees with the existing definition of* integer *powers)*

(b) *for every* $q, r \in \mathbb{Q}$, $p_2(q + r) = p_2(q)p_2(r)$

(c) *for every* $q, r \in \mathbb{Q}$, *if* $q < r$, *then* $p_2(q) < p_2(r)$

(3) *(Uniqueness) Given any function* $\tilde{p} : \mathbb{Q} \to \mathbb{R}$ *which satisfies (a) and (b) above with* $\tilde{p}$ *in place of* p_2, *then* $p_2(q) = \tilde{p}(q)$ *for every* $q \in \mathbb{Q}$.

Proof. (1) and (2) follow from Exercise 1.9.12.
(3) is an application of Exercise 1.9.5. □

The Uniqueness in 1.6.4 above permits us to define for rational $q \in \mathbb{Q}$, the power b^q to be the value $p_2(q) = (b^m)^{1/n}$ where $m/n = q$. Thus we now have a well-defined and sensible power function

$$q \mapsto b^q : \mathbb{Q} \to \mathbb{R}$$

which extends the integer power function $k \mapsto b^k : \mathbb{Z} \to \mathbb{R}$. Note that (1) is necessary to establish in order for us to know that the definition in (2) does not depend on the choice of how we represent q has a fraction m/n. Note also that (2)(c) is not needed for the proof of Uniqueness in (3), but it is necessary to establish for Uniqueness in the Real Power Theorem below:

Real Power Theorem 1.6.5. *Fix* $b > 1$ *and for each* $x \in \mathbb{R}$ *define the set*

$$B(x) \;:=\; \{b^q : q \in \mathbb{Q} \text{ and } q \le x\} \;=\; \{p_2(q) : q \in \mathbb{Q} \text{ and } q \le x\}$$

(1) $B(x)$ *is nonempty and bounded above.*

(2) *(Existence) Define the function* $p_3 : \mathbb{R} \to \mathbb{R}$ *by*

$$p_3(x) \;:=\; \sup B(x)$$

Then

(a) *for every* $q \in \mathbb{Q}$, $p_3(q) = p_2(q)$ *(i.e., this candidate power function defined for real powers agrees with the existing definition of* rational *powers)*

(b) *for every* $x, y \in \mathbb{R}$, $p_3(x + y) = p_3(x)p_3(y)$

(c) *for every* $x, y \in \mathbb{R}$, *if* $x < y$, *then* $p_3(x) < p_3(y)$.

(3) *(Uniqueness) For any function* $\tilde{p} : \mathbb{R} \to \mathbb{R}$ *which satisfies (a), (b), and (c) above with* $\tilde{p}$ *in place of* p_3, *then* $p_3(x) = \tilde{p}(x)$ *for every* $x \in \mathbb{R}$.

Proof. Parts (1) and (2) are done in Exercise 1.9.18.
Part (3) will be done later as Proposition 3.4.1. □

By the Uniqueness in 1.6.5, we are permitted to define $b^x : \mathbb{R} \to \mathbb{R}$ to be the function $p_3(x)$ defined in the theorem. This answers our Question 1.6.1:

Answer 1.6.6. *The function* $b^x : \mathbb{R} \to \mathbb{R}$ *defined in Theorem 1.6.5 is the only function that extends* $b^k : \mathbb{Z} \to \mathbb{R}$ *and has properties (P1) and (P2) for all* $k, \ell \in \mathbb{R}$.

1.7. The AGM Inequality

In this section we provide an application of nth roots, the celebrated *Arithmetic and Geometric Mean Inequality* (abbreviated *AGM Inequality*). Given real numbers $a_1, \ldots, a_n$, we define their **arithmetic mean** to be

$$\frac{a_1 + \cdots + a_n}{n} = \frac{1}{n}\sum_{k=1}^{n} a_k$$

and if each $a_i > 0$, then we define their **geometric mean** to be

$$(a_1 \cdots a_n)^{1/n} = \left(\prod_{k=1}^{n} a_k\right)^{1/n}.$$

The AGM Inequality states that the geometric mean is always less than or equal to the arithmetic mean, and moreover, the only way the two means can be equal is if all the a_i's are equal. The AGM Inequality has a very beautiful proof that uses a rare form of induction attributed to Cauchy, our treatment is from [**1**].

AGM Inequality 1.7.1. *For $n \geq 2$, and positive real numbers $a_1, \ldots, a_n \in \mathbb{R}$,*

$$(a_1 \cdots a_n)^{1/n} \leq \frac{a_1 + \cdots + a_n}{n},$$

with equality iff $a_1 = a_2 = \cdots = a_n$.

PROOF. For $n \geq 2$, let $P(n)$ be the statement:

$$P(n): \quad \text{"for all positive } a_1, \ldots, a_n \in \mathbb{R},\ a_1 \cdots a_n \leq \left(\frac{a_1 + \cdots + a_n}{n}\right)^n \text{ with equality iff } a_1 = a_2 = \cdots = a_n\text{"}$$

First for our base case of $P(2)$, we note that

$$a_1 a_2 \leq \left(\frac{a_1 + a_2}{2}\right)^2 \iff (a_1 - a_2)^2 \geq 0,$$

which is always true, and we have equality iff $a_1 - a_2 = 0$.

Next, we will prove two different inductive steps:

(A) for $n \geq 3$, if $P(n)$ holds, then $P(n-1)$ holds
(B) for $n \geq 2$, if $P(n)$ holds, then $P(2n)$ holds.

If we can establish both (A) and (B), then it follows that $P(n)$ is true for all natural numbers $n \geq 2$ (see Exercise 1.9.14).

For (A), let $a_1, \ldots, a_{n-1}$ be $n-1$ positive real numbers. Define an nth positive real number $A := \sum_{k=1}^{n-1} a_k/(n-1)$. Then

$$\begin{aligned}
\left(\prod_{k=1}^{n-1} a_k\right) A &\leq \left(\frac{\sum_{k=1}^{n-1} a_k + A}{n}\right)^n \quad \text{by } P(n) \\
&\qquad \text{with equality iff } a_1 = \cdots = a_{n-1} \\
&= \left(\frac{(n-1)A + A}{n}\right)^n \\
&= A^n.
\end{aligned}$$

For (B), let $a_1, \dots, a_n, a_{n+1}, \dots, a_{2n}$ be $2n$ positive numbers. Then

$$\begin{aligned}
\prod_{k=1}^{2n} a_k &= \left(\prod_{k=1}^{n} a_k\right)\left(\prod_{k=n+1}^{2n} a_k\right)\\
&\le \left(\sum_{k=1}^{n} \frac{a_k}{n}\right)^n \left(\sum_{k=n+1}^{2n} \frac{a_k}{n}\right)^n \quad \text{using } P(n) \text{ twice}\\
&\quad \text{with equality iff } a_1 = \cdots = a_n \text{ and } a_{n+1} = \cdots = a_{2n}\\
&= \left[\left(\sum_{k=1}^{n} \frac{a_k}{n}\right)\left(\sum_{k=n+1}^{2n} \frac{a_k}{n}\right)\right]^n\\
&\le \left[\left(\frac{\sum_{k=1}^{2n} \frac{a_k}{n}}{2}\right)^2\right]^n \quad \text{using } P(2)\\
&\quad \text{with equality iff } \textstyle\sum_{k=1}^{n} \frac{a_k}{n} = \sum_{k=n+1}^{2n} \frac{a_k}{n}, \text{ iff } a_1 = \cdots = a_{2n}\\
&= \left(\frac{\sum_{k=1}^{2n} a_k}{2n}\right)^{2n}.
\end{aligned}$$

□

1.8. The extended real line $\mathbb{R}_{\pm\infty}$

We now adjoin two new symbols $-\infty$ and $+\infty$ to $\mathbb{R}$:

Definition 1.8.1. Define $\mathbb{R}_{\pm\infty} := \mathbb{R} \cup \{-\infty, +\infty\}$. We extend the ordering on $\mathbb{R}$ to all of $\mathbb{R}_{\pm\infty}$ by declaring:

$$-\infty \le a \le +\infty \quad \text{for every } a \in \mathbb{R}_{\pm\infty}.$$

We also define the following **unbounded intervals**. For $a, b \in \mathbb{R}$ we set:

$$\begin{aligned}
[a, +\infty) &:= \{x \in \mathbb{R} : a \le x\}\\
(a, +\infty) &:= \{x \in \mathbb{R} : a < x\}\\
(-\infty, b] &:= \{x \in \mathbb{R} : x \le b\}\\
(-\infty, b) &:= \{x \in \mathbb{R} : x < b\}.
\end{aligned}$$

We also extend the meaning of inf and sup to this context:

Definition 1.8.2. For $S \subseteq \mathbb{R}$, $S \neq \emptyset$, if S is *not* bounded from above, then we declare

$$\sup S := +\infty,$$

and if S is *not* bounded from below, then we declare

$$\inf S := -\infty.$$

Remark 1.8.3. There is not supposed to be anything super deep or special about adjoining $\pm\infty$ to our real line. We primarily introduce it because it makes certain commonly occurring statements and expressions shorter. For example:

(1) Writing "$(-\infty, a)$" is shorter than writing "$\{x \in \mathbb{R} : x < a\}$",
(2) Writing "$\inf S = -\infty$" is shorter than writing "for every $x \in \mathbb{R}$, there is $s \in S$ such that $s < x$",

(3) (Using knowledge of Chapter 2 below) Writing "$\lim_{n\to\infty} a_n = \infty$" is shorter than writing "For every $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n > M$."

In other words, $\pm\infty$ are not numbers, they are just artificial endpoints we attached to the real line to simplify certain expressions. Any expression involving these new points $\pm\infty$ is really just shorthand for a longer statement which does not refer to $\pm\infty$. Unless we state otherwise, we do not extend the arithmetic operations $+, \cdot$ on $\mathbb{R}$ to include $\pm\infty$.

1.9. Exercises

Exercise 1.9.1. Use induction to prove the *Sum of Squares Formula*

$$1^2 + 2^2 + \cdots + n^2 \;=\; \sum_{k=1}^{n} k^2 \;=\; \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n.

Exercise 1.9.2. Use induction to show that

$$\sum_{k=1}^{2^n} \frac{1}{k} \;\geq\; 1 + \frac{n}{2}$$

for every integer $n \geq 0$.

Exercise 1.9.3 (Abel's formula)**.** Suppose $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are real numbers. For each natural number k define the sum $A_k := \sum_{i=1}^{k} a_i$. Prove for every natural number $n \geq 1$ that

$$\sum_{i=1}^{n} a_i b_i \;=\; A_n b_n - \sum_{i=1}^{n-1} A_i (b_{i+1} - b_i).$$

Exercise 1.9.4. This exercise derives the basic number-theoretic facts used in the proof that there is no rational solution to the equation $x^2 - 2 = 0$.

(1) Prove that if n is a natural number, then $2^n > n$.
(2) Prove that if n is a natural number, then

$$n \;=\; 2^{k_0} \ell_0$$

for some odd natural number ℓ_0 and some nonnegative integer k_0. (Hint: consider the set $A = \{k \in \mathbb{N} \cup \{0\} : n = 2^k \ell \text{ for some } \ell \text{ in } \mathbb{N}\}$, what does part (1) tell you about the set A?)
(3) Prove that each rational number x can be expressed as $x = m/n$, where $m, n \in \mathbb{Z}$ and m or n is odd.
(4) Prove that an integer n is even if its square n^2 is even.

Exercise 1.9.5. Suppose $f, g \colon \mathbb{R} \to \mathbb{R}$ are two functions that satisfy:
(a) $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$,
(b) $g(x+y) = g(x)g(y)$ for all $x, y \in \mathbb{R}$, and
(c) $f(1) = g(1)$.

Prove the following:

(1) $f(n) = g(n)$ for every $n \in \mathbb{N}$,
(2) $f(k) = g(k)$ for every $k \in \mathbb{Z}$, and

(3) $f(q) = g(q)$ for every $q \in \mathbb{Q}$.

In other words, properties (a), (b) and (c) imply that the functions f and g are "almost equal" in the sense that they agree on the dense set $\mathbb{Q}$. Must it always be the case that $f(r) = g(r)$ for all $r \in \mathbb{R}$? (This last question is not part of the exercise, just something to think about.)

Exercise 1.9.6. Prove the *Reverse Triangle Inequality*: For every $a, b \in \mathbb{R}$:

$$|a - b| \ \geq \ \big||a| - |b|\big|$$

Exercise 1.9.7. This exercise extends the Triangle Inequality to finite sums.

(1) Prove $|a+b+c| \leq |a| + |b| + |c|$ for all $a, b, c \in \mathbb{R}$ by applying the triangle inequality twice.
(2) Use induction to prove

$$|a_1 + a_2 + \cdots + a_n| \ \leq \ |a_1| + |a_2| + \cdots + |a_n|$$

for n numbers $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

Exercise 1.9.8. Suppose $\alpha, \beta \in \mathbb{R}$ are real numbers with the property:

$$\text{For every } n \in \mathbb{N},\ |\alpha - \beta| < 1/n.$$

Show that then $\alpha = \beta$. [This is basically a fancy way of showing that two real numbers are the same. We will use this fact quite often, sometimes without mention.]

Exercise 1.9.9 (Cauchy's Inequality)**.** Prove that for any numbers $a, b \in \mathbb{R}$:

$$ab \ \leq \ \frac{1}{2}(a^2 + b^2)$$

Exercise 1.9.10. Let $n \in \mathbb{N}$. Suppose $0 \leq x \leq 2/n^2$. Then

$$1 - nx + \frac{n(n-3)}{2}x^2 \ \leq \ (1-x)^n \ \leq \ 1 - nx + \frac{n(n-1)}{2}x^2.$$

[This inequality is like a "second order" version of Bernoulli's Inequality.]

Exercise 1.9.11. Prove *Sup-Inf Symmetry*: Suppose $S \subseteq \mathbb{R}$ is nonempty.

(1) If S is bounded above, then $-S$ is bounded below and $\sup S = -\inf(-S)$
(2) If S is bounded below, then $-S$ is bounded above and $\inf S = -\sup(-S)$

Exercise 1.9.12. In this exercise we prove part of the Rational Power Theorem 1.6.4: Fix $b \in \mathbb{R}$ such that $b > 1$.

(1) Define the two-variable function $p_1\colon \mathbb{Z} \times \mathbb{N} \to \mathbb{R}$ by $p_1(m, n) := (b^m)^{1/n}$. Then given $m, k \in \mathbb{Z}$ and $n, \ell \in \mathbb{N}$ show that if $m/n = k/\ell$, then $p_1(m, n) = p_1(k, \ell)$.

Define the function $p_2\colon \mathbb{Q} \to \mathbb{R}$ by

$$p_2(q) \ := \ p_1(m, n) \ = \ (b^m)^{1/n} \quad \text{for any } m, n \in \mathbb{Z} \text{ such that } n > 0 \text{ and } q = m/n$$

Then

(2) for every $k \in \mathbb{Z}$, $p_2(k) = p_0(k)$ (i.e., this candidate power function defined for rational powers agrees with the existing definition of *integer* powers)
(3) for every $q, r \in \mathbb{Q}$, $p_2(q + r) = p_2(q)p_2(r)$
(4) for every $q, r \in \mathbb{Q}$, if $q < r$, then $p_2(q) < p_2(r)$

Exercise 1.9.13. Prove the *Principle of Strong Induction*: Suppose $P(n)$ is a property that a natural number n may or may not have. Suppose that

(1) $P(1)$ holds, and
(2) For every $n \geq 1$, if $P(k)$ holds for $k = 1, \ldots, n$, then $P(n+1)$ holds.

Then $P(n)$ holds for every natural number $n \geq 2$. [Hint: Use the Principle of Induction with an appropriate inductive hypothesis.]

Exercise 1.9.14. Carefully prove the *Principle of Cauchy Induction*: Suppose $P(n)$ is a property that a natural number n may or may not have. Suppose that

(a) $P(2)$ holds,
(b) For every $n \geq 2$, if $P(n)$ holds, then $P(2n)$ holds, and
(c) For every $n \geq 3$, if $P(n)$ holds, then $P(n-1)$ holds.

Then $P(n)$ holds for every natural number $n \geq 2$.

Exercise 1.9.15. Define the *Fibonacci numbers* recursively as follows:

$$F_1 \;=\; F_2 \;:=\; 1 \quad \text{and} \quad F_{n+1} \;:=\; F_n + F_{n-1} \quad \text{for } n \geq 1.$$

Prove *Binet's formula* for the Fibonacci numbers:

$$F_n \;=\; \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{where} \quad \alpha \;:=\; \frac{1+\sqrt{5}}{2}, \quad \beta \;:=\; \frac{1-\sqrt{5}}{2}.$$

Exercise 1.9.16. Consider the polynomial

$$P(X) \;=\; X^d + a_{d-1}X^{d-1} + \cdots + a_1 X + a_0,$$

where $d \geq 1$ and $a_0, \ldots, a_{d-1} \in \mathbb{R}$. Define the number

$$M \;:=\; 1 + |a_{d-1}| + \cdots + |a_1| + |a_0|.$$

Prove the following:

(1) For every $x \in \mathbb{R}$ such that $|x| \geq M$,
$$\left|\frac{a_{d-1}}{x} + \frac{a_{d-2}}{x^2} + \cdots + \frac{a_1}{x^{d-1}} + \frac{a_0}{x^d}\right| \;<\; 1.$$
(2) For every $x \in \mathbb{R}$ such that $|x| \geq M$, $P(x) = x^d(1+\epsilon)$ for some $\epsilon \in \mathbb{R}$ such that $|\epsilon| < 1$.
(3) For every $x \in \mathbb{R}$ such that $|x| \geq M$, $P(x) \neq 0$.

Thus all real zeros of P must lie in the interval $(-M, M)$.

Exercise 1.9.17. Prove for all natural numbers $n \geq 2$:

$$n! \;<\; \left(\frac{n+1}{2}\right)^n$$

[Hint: the AGM Inequality might be useful.]

Exercise 1.9.18. In this exercise we prove part of the Real Power Theorem 1.6.5: Fix $b > 1$ and for each $x \in \mathbb{R}$ define the set

$$B(x) \;:=\; \{b^q : q \in \mathbb{Q} \text{ and } q \leq x\}$$

(1) $B(x)$ is nonempty and bounded above.

Define the function $p_3 \colon \mathbb{R} \to \mathbb{R}$ by

$$p_3(x) \;:=\; \sup B(x)$$

Then

(2) for every $q \in \mathbb{Q}$, $p_3(q) = p_2(q)$ (i.e., this candidate power function defined for real powers agrees with the existing definition of *rational* powers)

(3) for every $x, y \in \mathbb{R}$, $p_3(x + y) = p_3(x)p_3(y)$
(4) for every $x, y \in \mathbb{R}$, if $x < y$, then $p_3(x) < p_3(y)$.

Exercise 1.9.19. For each of the sets below, do the following things:

(i) Give three upper bounds for the set (no proof needed), or show the set is not bounded above (proof needed).
(ii) Give three lower bounds for the set (no proof needed), or show the set is not bounded below (proof needed).
(iii) Determine the supremum of the set, or show that it does not have one. (Proof needed for either)
(iv) Determine the infimum of the set, or show that it does not have one. (Proof needed for either)

Here are the sets:

(1) $\{1 - 1/3^n : n \in \mathbb{N}\}$
(2) $\bigcap_{n=1}^{\infty}(1 - 1/n, 1 + 1/n)$
(3) $\{x \in \mathbb{R} : x^3 < 8\}$

[Hint: you might find the Archimedean Property to be useful for some of these]

Exercise 1.9.20. Let $S \subseteq \mathbb{R}$ be nonempty and bounded. For a fixed $c > 0$, define $cS := \{cs : s \in S\}$. Show that $\sup(cS) = c \cdot \sup(S)$, and $\inf(cS) = c \cdot \inf(S)$.

Exercise 1.9.21. Suppose $S, T \subseteq \mathbb{R}$ are nonempty subsets of $\mathbb{R}$ such that $S \leq T$, i.e., for every $s \in S$ and $t \in T$, $s \leq t$. Show that $\sup S \leq \inf T$.

Exercise 1.9.22. Suppose $r \in \mathbb{Q} \setminus \{0\}$ and $x \in \mathbb{R} \setminus \mathbb{Q}$ is irrational. Show that $r + x$ and rx are also irrational.

Exercise 1.9.23. Suppose $A, B \subseteq \mathbb{R}$ are nonempty sets bounded below which satisfy the property:

- For every $a^* \in A$ and $\epsilon > 0$, there are $a \in A$ and $b \in B$ such that $a \leq a^*$ and $|a - b| < \epsilon$.

Prove that $\inf A \geq \inf B$.

Exercise 1.9.24. Suppose that $A \subseteq \mathbb{R}$ is a nonempty bounded set. Show that:

$$\sup A - \inf A \;=\; \inf\{b - a : a, b \in \mathbb{R} \text{ and } a \leq A \leq b\}$$

Exercise 1.9.25. Write the following sets in interval notation in the extended real line $\mathbb{R}_{\pm\infty}$:

(1) $\{x \in \mathbb{R} : x < 0\}$
(2) $\{x \in \mathbb{R} : x^5 \leq 32\}$
(3) $\{x^2 : x \in \mathbb{R}\}$
(4) $\{x \in \mathbb{R} : x^4 < 3\}$

In each part, provide justification (i.e., a proof) that the set equals the interval that you claim it equals.

CHAPTER 2

Sequences of Real Numbers

Sequences are a fundamental concept of utmost importance in analysis. Convergent sequences give us a systematic way of talking about particular real numbers which otherwise might be hard to describe.

2.1. Sequences and limits of sequences

Definition 2.1.1. A **sequence** in $\mathbb{R}$ is a function

$$s\colon \{n \in \mathbb{Z} : n \geq m\} \to \mathbb{R} \quad \text{(for some } m \in \mathbb{Z}\text{)}.$$

Usually we denote such a sequence by

$$s \;=\; (s_n)_{n=m}^{\infty} \;=\; (s_n)_{n\geq m} \;=\; (s_m, s_{m+1}, \ldots),$$

where $s_n = s(n)$. If $m = 1$, then we also write

$$s \;=\; (s_n)_{n\in\mathbb{N}} \;=\; (s_1, s_2, \ldots).$$

If the domain of s is clear from the context, then we also just write $s = (s_n)$.

Example 2.1.2. (1) $s_n = 1/n^2$, $n \geq 1$ is the sequence:

$$\left(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\right)$$

(2) $a_n = (-1)^n$, $n \geq 0$ gives the sequence:

$$(1, -1, 1, -1, \ldots)$$

(3) $a_n = \sqrt[n]{n}$, $n \in \mathbb{N}$ gives the sequence:

$$(1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \ldots)$$

A computation on a computer shows that $a_{1,000} \approx 1.0069\ldots$, and in general, a_n seems to be getting closer and closer to 1 as n increases.

(4) $b_n = (1 + 1/n)^n$, $n \in \mathbb{N}$ gives the sequence

$$(2, 2.25, 2.3704, 2.4414, \ldots)$$

which gets closer and closer to $2.71\ldots$

(5) We can also define sequences recursively, for instance:

$$(f_n) = (1, 1, 2, 3, 5, 8, 13, \ldots)$$

is defined by initial conditions $f_1 = f_2 = 1$ and the rule $f_{n+2} = f_{n+1} + f_n$.

Definition 2.1.3. Let $(s_n)_{n\geq m}$ be a sequence, $s \in \mathbb{R}$. We say that $(s_n)_{n\geq m}$ **converges to** s if for each $\epsilon > 0$ there is some number $N \geq m$ such that $|s_n - s| < \epsilon$ for every natural number $n \geq N$.

If s_n converges to s, we write this as $s = \lim_{n\to\infty} s_n$ or $s_n \to s$, and we call s the **limit** of (s_n).

If there is no $s \in \mathbb{R}$ such that $s_n \to s$, then we say that (s_n) **diverges**.

[Note, in the definition of $s_n \to s$, the numbers n, m are natural numbers, but N does not need to be in $\mathbb{N}$.]

Sanity Check 2.1.4. *The following things are true about sequences:*

(1) (Limits are unique) Given a sequence (s_n) and $s, s' \in \mathbb{R}$, if $s_n \to s$ and $s_n \to s'$, then $s = s'$.
(2) (Constant sequences converge to the constant) If $(s_n)_{n \geq m}$ is a constant sequence, i.e., there is $s \in \mathbb{R}$ such that $s_n = s$ for all $n \geq m$, then $s_n \to s$.

PROOF. (1) Let $\epsilon > 0$ be arbitrary. Since $s_n \to s$ and $s_n \to s'$, there is $N \in \mathbb{N}$ such that

$$|s_n - s| < \frac{\epsilon}{2} \quad \text{and} \quad |s_n - s'| < \frac{\epsilon}{2}$$

for every $n \geq N$ (to get such an N, just take the maximum of the two N's which work for $s_n \to s$ and $s_n \to s'$ separately). Then

$$\begin{aligned} |s - s'| &= \big|(s - s_n) - (s' - s_n)\big| \\ &\leq |s_n - s| + |s_n - s'| \quad \text{by Triangle Inequality} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

We have just shown that $|s - s'| < \epsilon$ for every $\epsilon > 0$. Thus $s = s'$ by Exercise 1.9.8.

(2) Let $\epsilon > 0$ be given. We need to find $N \geq m$ such that $|s_n - s| < \epsilon$ for every $n \geq N$. But

$$|s_n - s| = |s - s| = 0 < \epsilon \quad \text{for } \textit{all } n \geq m$$

so we can take $N := m$ (or even let N be any integer $\geq m$). □

Example 2.1.5. $1/\sqrt{n} \to 0$.

PROOF. Let $\epsilon > 0$ be given. We need to find $N \geq 1$ such that

$$\left| \frac{1}{\sqrt{n}} - 0 \right| < \epsilon \quad \text{for every } n \geq N,$$

In other words, we want to know how large n must be so that

$$\frac{1}{\sqrt{n}} < \epsilon,$$

which is equivalent to

$$n > \frac{1}{\epsilon^2}.$$

Thus we can let N be any natural number such that $N > 1/\epsilon^2$. Indeed, if $n \geq N$, then $n > 1/\epsilon^2$, so $|1/\sqrt{n} - 0| < \epsilon$. □

Example 2.1.6. $\big((-1)^n\big)_{n \geq 1}$ diverges.

PROOF. Suppose towards a contradiction that there is $a \in \mathbb{R}$ such that $(-1)^n \to a$. Then either 1 or -1 will have distance ≥ 1 from a (since a will either be positive, negative or 0), so

$$\big|(-1)^n - a\big| < 1$$

will not hold for all large n.

In other words, let $\epsilon = 1$ and take N such that

$$\big|(-1)^n - a\big| < 1 \quad \text{for all } n \geq N,$$

which we can do since we are assuming $(-1)^n \to a$. Then

$$|1-a|, |1+a| \; < \; 1,$$

so

$$2 \; = \; \big|(1-a)+(1+a)\big| \; \leq \; |1-a|+|1+a| \; < \; 1+1 \; = \; 2,$$

a contradiction. □

We want to get more intuition for convergent sequences. The following is a property that a sequence may or may not enjoy:

Definition 2.1.7. A sequence $(s_n)_{n\geq m}$ is said to be **bounded** if the set

$$\{s_n : n \geq m\} \; \subseteq \; \mathbb{R}$$

is bounded.

It should come as no surprise that convergent sequences are bounded:

Proposition 2.1.8. *Every convergent sequence is bounded.*

PROOF. Let $(s_n)_{n\geq m}$ be a convergent sequence, say $s_n \to s \in \mathbb{R}$. We will find a bound for (s_n).

Let $\epsilon := 1$ and take $N \geq m$ such that

$$|s_n - s| \; < \; 1 \quad \text{for } n \geq N.$$

Then for $n \geq N$:

$$\begin{aligned} |s_n| \; &= \; \big|(s_n - s) + s\big| \\ &\leq \; |s_n - s| + |s| \quad \text{(by Triangle Inequality)} \\ &< \; 1 + |s|. \end{aligned}$$

Now put

$$M \; := \; \max\big\{|s_m|, \ldots, |s_{N-1}|, 1+|s|\big\}.$$

Then $|s_n| \leq M$ for *all* $n \geq m$. □

Note that the converse to Proposition 2.1.8 does *not* hold.

Limit Laws for Sequences 2.1.9. *Let $(s_n), (t_n)$ be sequences in $\mathbb{R}$ such that $s_n \to s$ and $t_n \to t$. Then*

(1) $s_n + t_n \to s + t$
(2) $s_n \cdot t_n \to s \cdot t$
(3) if $t_n \neq 0$ for all n and $t \neq 0$, then

$$\frac{s_n}{t_n} \to \frac{s}{t}.$$

PROOF. (1) Let $\epsilon > 0$. [The idea here is that the two terms $|s - s_n|$ and $|t - t_n|$ below will "share the ϵ", i.e., that we will make each one separately smaller than $\epsilon/2$. This "ϵ sharing" is a common trick.] Take N_1, N_2 such that

$$\begin{aligned} n \geq N_1 \; &\Longrightarrow \; |s - s_n| < \frac{\epsilon}{2} \\ n \geq N_2 \; &\Longrightarrow \; |t - t_n| < \frac{\epsilon}{2}. \end{aligned}$$

Then with $N := \max\{N_1, N_2\}$ we get

$$\begin{aligned} \big|(s+t) - (s_n + t_n)\big| &\leq |s - s_n| + |t - t_n| \quad \text{by the Triangle Inequality} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

(2) The idea here is to use the following inequalities:

$$\begin{aligned} |s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\ &\leq \big|s_n(t_n - t)\big| + \big|(s_n - s)t\big| \\ &= |s_n| \cdot |t_n - t| + |s_n - s| \cdot |t|. \end{aligned}$$

Note that the expression $|s_n t_n - st|$ which we want to make arbitrary small is bounded by the last expression. There, the term $|s_n|$ is bounded, the terms $|t_n - t|$ and $|s_n - s|$ can be made arbitrary small, and $|t|$ is some constant.

Now, let $\epsilon > 0$. By Proposition 2.1.8, there is $M > 0$ such that

$$|s_n| \leq M \quad \text{for all } n.$$

Now take N_1 such that

$$|t_n - t| < \frac{\epsilon}{2M} \quad \text{for all } n \geq N_1$$

and take N_2 such that

$$|s_n - s| < \frac{\epsilon}{2\big(|t| + 1\big)} \quad \text{for all } n \geq N_2.$$

[Note: we have to use "$|t| + 1$" in the denominator instead of "$|t|$" to take into account the case where $|t| = 0$. The "2's" in the denominators are there because we are doing an ϵ-sharing trick here.] Let $N := \max\{N_1, N_2\}$ and note that for $n \geq N$:

$$\begin{aligned} |s_n t_n - st| &\leq |s_n| \cdot |t_n - t| + |t| \cdot |s_n - s| \\ &\leq M \cdot \frac{\epsilon}{2M} + |t| \frac{\epsilon}{2\big(|t| + 1\big)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

(3) By (2), it is enough to show that if $t_n \neq 0$ for all n and $t \neq 0$, then $1/t_n \to 1/t$. The idea is the following: note that

$$\left| \frac{1}{t_n} - \frac{1}{t} \right| = \left| \frac{t - t_n}{t_n t} \right|$$

so we need to be able to make the numerator $t - t_n$ small, and the denominator $t_n \cdot t$ large.

Claim. *There is some $r > 0$ such that $|t_n| \geq r$ for all n.*

Proof of Claim. Put $\epsilon := \frac{1}{2}|t| > 0$, and take N such that

$$|t_n - t| \leq \epsilon = \frac{1}{2}|t| \quad \text{for all } n \geq N.$$

Then for $n \geq N$ we have $|t_n| \geq |t|/2$, because otherwise if we assume towards a contradiction that $|t_n| < |t|/2$ for some $n \geq N$ we get

$$\begin{aligned} |t| &= \big|(t - t_n) + t_n\big| \\ &\leq |t - t_n| + |t_n| \quad \text{by Triangle Inequality} \\ &< \frac{1}{2}|t| + \frac{1}{2}|t| \ = \ |t|, \end{aligned}$$

a contradiction. Now we put

$$r \ := \ \min\big\{|t_1|, \dots, |t_{N-1}|, |t|/2\big\} \ > \ 0.$$

Then clearly $|t_n| \geq r > 0$ for all n. □

Now we take $r > 0$ as in the claim. Then

$$\left|\frac{1}{t_n} - \frac{1}{t}\right| \ \leq \ \frac{|t - t_n|}{r|t|} \quad \text{for all } n.$$

Let $\epsilon > 0$. Take N such that

$$|t_n - t| \ < \ \epsilon r|t| \quad \text{for all } n \geq N.$$

Then

$$\left|\frac{1}{t_n} - \frac{1}{t}\right| \ \leq \ \frac{|t - t_n|}{r|t|} \ < \ \frac{\epsilon r|t|}{r|t|} \ = \ \epsilon \quad \text{for } n \geq N.$$ □

The following is an indispensable tool for computing limits of complicated sequences. Usually $(s_n)_{n\geq m}$ below will be some complicated sequence, and $(t_n)_{n\geq m}$ will be some simpler sequence that we already know converges to 0.

Squeeze Lemma for Sequences 2.1.10. *Suppose $(s_n)_{n\geq m}$ and $(t_n)_{n\geq m}$ are sequences such that $t_n \to 0$ and for some $N \geq m$ we have $0 \leq s_n \leq t_n$ for all $n \geq N$. Then $s_n \to 0$.*

Proof. This is Exercise 2.6.1. □

The following example shows how to compute some limits that are a little more complicated:

Example 2.1.11. Here are some basic limits:

(1) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.
(2) $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for all $p \in \mathbb{R}$ such that $p > 0$,
(3) $\lim_{n\to\infty} a^n = 0$ for $|a| < 1$,
(4) $\lim_{n\to\infty} \sqrt[n]{a} = 1$ for $a > 0$.

Proof. (1) Apply the AGM Inequality 1.7.1 to the n numbers

$$\underbrace{1, \dots, 1}_{n-1 \text{ of these}}, \sqrt{n}$$

to get

$$1 \ < \ \underbrace{n^{1/2n} \ < \ \frac{n - 1 + \sqrt{n}}{n}}_{\text{AGM Inequality}} \ < \ 1 + \frac{1}{\sqrt{n}}.$$

Squaring the first, second and fourth parts yield

$$1 \ < \ n^{1/n} \ < \ 1 + \frac{2}{\sqrt{n}} + \frac{1}{n}.$$

Next, by the Limit Laws 2.1.9(1) and (2) and Example 2.1.5 (or (2) below) we have

$$\lim_{n\to\infty} 1 + \frac{2}{\sqrt{n}} + \frac{1}{n} \;=\; 1,$$

so by the Squeeze Theorem 2.1.10 we conclude that $\lim_{n\to\infty} n^{1/n} = 1$.

(2) Let $\epsilon > 0$, and let N be an integer such that

$$N \;>\; \left(\frac{1}{\epsilon}\right)^{1/p}.$$

Then

$$\begin{aligned} n \geq N \;&\Longrightarrow\; n^p \geq N^p > \frac{1}{\epsilon} \\ &\Longrightarrow\; \left|\frac{1}{n^p} - 0\right| = \frac{1}{n^p} < \epsilon. \end{aligned}$$

(3) We may assume that $a \neq 0$. Then

$$|a| \;=\; \frac{1}{1+b} \quad \text{for some } b > 0.$$

Then by Bernoulli's Inequality,

$$(1+b)^n \;\geq\; 1 + nb \;>\; nb$$

which implies

$$|a^n - 0| \;=\; |a|^n \;=\; \left(\frac{1}{1+b}\right)^n \;<\; \frac{1}{nb}.$$

However, since $1/nb \to 0$, the Squeeze Theorem for Sequences tells us that $|a^n| \to 0$, and thus also $a^n \to 0$ (using Squeeze Theorem again and the observation that $-|a^n| \leq a^n \leq |a^n|$).

(4) Suppose first that $a \geq 1$. Then for $n \geq a$ we have

$$1 \leq a^{1/n} \leq n^{1/n} \to 1.$$

Thus $a^{1/n} \to 1$ by the Squeeze Theorem for Limits. Next, suppose that $0 < a < 1$. Then $1/a > 1$, so $(1/a)^{1/n} \to 1$. Thus $a^{1/n} = 1/(1/a)^{1/n} \to 1$ by Limit Law 2.1.9(3). □

Finally, we extend our definition of *limit* to include the case where the sequence "blows up to infinity". This is technically an instance of a divergent sequence (albeit, a very special type of divergence), but the notion can still be useful in computing the convergence of sequences, e.g., see Lemma 2.1.16.

Definition 2.1.12. Let $(s_n)_{n\geq m}$ be a sequence. Then we define $\lim_{n\to\infty} s_n = +\infty$ if for each $M \in \mathbb{R}$, there is $N \geq m$ such that $s_n \geq M$ for all $n \geq N$.

In this case we say that (s_n) **diverges to** $+\infty$. We also define $\lim_{n\to\infty} s_n = -\infty$ to mean $\lim_{n\to\infty}(-s_n) = +\infty$.

We say that (s_n) **has a limit** or that the **limit of** (s_n) **exists** if either

(1) (s_n) converges (to a limit L in $\mathbb{R}$), or
(2) (s_n) diverges to $+\infty$ or $-\infty$.

Example 2.1.13. (1) $\lim_{n\to\infty} n^2 = \lim_{n\to\infty} 2^n = +\infty$.

(2) $\lim_{n\to\infty}(-1)^n n$ does not exist.

Lemma 2.1.14. *Let* $(s_n), (t_n)$ *be sequences, and suppose*

$$\lim_{n\to\infty} s_n = +\infty \quad \textit{and} \quad \lim_{n\to\infty} t_n \in \mathbb{R}^{>0} \cup \{+\infty\}.$$

Then $\lim_{n\to\infty} s_n \cdot t_n = +\infty$.

PROOF. Let $M \in \mathbb{R}$ be given; we need to find N such that

$$s_n t_n \ \geq \ M \quad \text{for } n \geq N.$$

Since $\lim_{n\to\infty} t_n > 0$, we can take a real number m such that $0 < m < \lim_{n\to\infty} t_n$. Then we can take N_1 such that $t_n \geq m$ for all $n \geq N_1$ (using either the $m = r$ from the Claim in the proof of Proposition 2.1.9(3) in the case that $\lim_{n\to\infty} t_n \in \mathbb{R}$, or else we use the definition of $\lim_{n\to\infty} t_n = +\infty$ in the case that $\lim_{n\to\infty} t_n = +\infty$). Since $\lim_{n\to\infty} s_n = +\infty$, we can take N_2 such that

$$s_n \ \geq \ \frac{M}{m} \quad \text{for } n \geq N_2.$$

So for $n \geq \max\{N_1, N_2\}$ we get $s_n t_n \geq \frac{M}{m} \cdot m = M$. □

Example 2.1.15. $(n^2+1)/(n-1) \to +\infty$. To see this, note that for $n \geq 2$

$$\frac{n^2+1}{n-1} \ = \ \frac{n + \frac{1}{n}}{1 - \frac{1}{n}} = \underbrace{\left(n + \frac{1}{n}\right)}_{s_n} \cdot \underbrace{\left(\frac{1}{1-\frac{1}{n}}\right)}_{t_n}$$

and it is easy to show $s_n \to +\infty$ directly from the definition, and $t_n \to 1$ using Proposition 2.1.9 several times. Thus $s_n \cdot t_n \to +\infty$ by Lemma 2.1.14.

Lemma 2.1.16. *Let* (s_n) *be a sequence where* $s_n > 0$ *for each* n. *Then,*

$$\lim_{n\to\infty} s_n = +\infty \quad \Longleftrightarrow \quad \lim_{n\to\infty} \frac{1}{s_n} = 0.$$

PROOF. ($\Rightarrow$) Suppose $\lim_{n\to\infty} s_n = +\infty$. Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} s_n = +\infty$, we can take N such that $s_n > 1/\epsilon$ for all $n \geq N$. Thus for $n \geq N$:

$$\left|\frac{1}{s_n} - 0\right| \ = \ \frac{1}{s_n} \ < \ \epsilon.$$

In other words, $1/s_n \to 0$.

($\Leftarrow$) Suppose $\lim_{n\to\infty} 1/s_n = 0$. Let $M \in \mathbb{R}^{>0}$ be given. Take N such that

$$\left|\frac{1}{s_n} - 0\right| \ < \ \frac{1}{M} \quad \text{for } n \geq N.$$

Then $s_n > M$ for $n \geq N$. □

2.2. Monotone sequences and Cauchy sequences

Recall that in the definition of a sequence (s_n) converging, we need to specify L which it converges to. In this section we will show two theorems that allow us to conclude that certain sequences converge *without* knowing their limit L in advance (or at all).

Definition 2.2.1. We say that a sequence (s_n) is

(1) **increasing** if $s_n \leq s_{n+1}$ for all n,
(2) **decreasing** if $s_n \geq s_{n+1}$ for all n,
(3) **monotone** if it is increasing or decreasing.

Monotone Convergence Theorem 2.2.2. *Every bounded monotone sequence converges.*

PROOF. Suppose $(s_n)_{n\geq 1}$ is bounded and increasing. Let

$$S := \{s_n : n \in \mathbb{N}\}.$$

Then S is bounded from above, so $s := \sup S \in \mathbb{R}$ exists.

Claim. $s = \lim_{n\to\infty} s_n$.

PROOF OF CLAIM. Let $\epsilon > 0$ be given. Then $s - \epsilon$ is *not* an upper bound for S; so there is some N such that

$$s - \epsilon < s_N.$$

Since (s_n) is increasing, for all $n \geq N$ we have

$$s - \epsilon < s_N \leq s_{N+1} \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots \leq s$$

and so

$$s - \epsilon < s_n \leq s \quad \text{for all } n \geq N.$$

This implies $|s - s_n| < \epsilon$ for all $n \geq N$. □

Likewise, if (s_n) is bounded an decreasing, then

$$\lim_{n\to\infty} s_n \;=\; \inf\{s_n : n \in \mathbb{N}\}.$$ □

The following example is the only time we will say anything about decimal expansions for real numbers. You do not need to know it for any homework or exam.

Example 2.2.3 (Decimal expansions)**.** We may view an infinite decimal expansion as a limit of a certain sequence. Consider the infinite decimal expansion:

$$s \;=\; K.d_1d_2d_3\cdots \quad (K \in \mathbb{Z}^{\geq 0}, d_i \in \{0, 1, \ldots, 9\})$$

This may be approximated by

$$s_n \;=\; K + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$$

The sequence (s_n) is increasing and bounded by $K + 1$. By the Monotone Convergence Theorem 2.2.2, it has a limit: $s = \lim_{n\to\infty} s_n$. In some sense, the real number s is *defined* to be this limit.

Proposition 2.2.4. *Suppose (s_n) is unbounded. Then*

(1) if (s_n) is increasing, then $\lim_{n\to\infty} s_n = +\infty$, and
(2) if (s_n) is decreasing, then $\lim_{n\to\infty} s_n = -\infty$.

PROOF. (1) Suppose (s_n) is increasing and unbounded. Then

$$S := \{s_n : n \in \mathbb{N}\}$$

is unbounded from above. Hence for each M there is N such that $s_N \geq M$. Since (s_n) is increasing, $s_n \geq s_N \geq M$ for all $n \geq N$.

(2) is similar and left as an exercise. □

Corollary 2.2.5. *For a monotone sequence (s_n), either*

(1) (s_n) converges, or
(2) (s_n) diverges to $\pm\infty$,

and so in either case $\lim_{n\to\infty} s_n \in \mathbb{R} \cup \{\pm\infty\}$ exists.

We now define the important notion of a *Cauchy sequence*:

Definition 2.2.6. A sequence $(s_n)_{n\geq m}$ in $\mathbb{R}$ is called a **Cauchy**[1] **sequence** if for every $\epsilon > 0$ there is $N \geq m$ such that $|s_n - s_{n'}| < \epsilon$ for all $n, n' \geq N$.

A Cauchy sequence is basically a sequence that behaves like it ought to converge to something. Note that the definition only refers to elements of the sequence (s_n), and *does not* mention any limit s (in contrast with the definition of *convergent sequence*). We will show eventually that for sequences in $\mathbb{R}$, "Cauchy=convergent" (Theorem 2.3.8), although this will take some work. One direction is easy though:

Lemma 2.2.7. *If* (s_n) *is convergent, then* (s_n) *is Cauchy.*

PROOF. Suppose (s_n) is a sequence in $\mathbb{R}$ such that $s_n \to s \in \mathbb{R}$. By the Triangle Inequality we have for all n, n':

$$|s_n - s_n'| \;=\; \big|(s_n - s) + (s - s_{n'})\big| \;\leq\; |s_n - s| + |s - s_{n'}|.$$

Let $\epsilon > 0$ be given and take N such that

$$|s_n - s| \;<\; \frac{\epsilon}{2} \quad \text{for all } n \geq N.$$

Thus if $n, n' \geq N$, then

$$|s_n - s|, |s_{n'} - s| \;<\; \frac{\epsilon}{2}.$$

Hence $|s_n - s_{n'}| < \epsilon/2 + \epsilon/2 = \epsilon$. □

We can't yet directly show that Cauchy sequences are convergent, but at least we can show that Cauchy sequences are *bounded*, which is a necessary property if they are going to be convergent (by Proposition 2.1.8).

Lemma 2.2.8. *If* (s_n) *is Cauchy, then* (s_n) *is bounded.*

PROOF. Let $\epsilon := 1$ and take N such that

$$|s_n - s_{n'}| \;<\; \epsilon \quad \text{for all } n, n' \geq N.$$

Then $|s_n - s_N| < 1$ for all $n \geq N$, so for all $n \geq N$ we have

$$\begin{aligned} |s_n| \;&=\; \big|(s_n - s_N) + s_N\big| \\ &\leq\; |s_n - s_N| + |s_N| \\ &<\; 1 + |s_N|. \end{aligned}$$

Now, put

$$M \;:=\; \max\big\{|s_m|, |s_{m+1}|, \ldots, |s_{N-1}|, 1 + |s_N|\big\}.$$

Then $|s_n| \leq M$ for all $n \geq m$. □

To summarize so far, we have

$$\text{convergent} \Longrightarrow \text{Cauchy} \Longrightarrow \text{bounded}.$$

Our goal is to show that

$$\text{Cauchy} \Longrightarrow \text{convergent}.$$

This will take a little work.

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[1] "Cauchy" is pronounced like *coh-shee*, *coh* as in the word <u>*co*</u>*ach*, and *shee* as in the word *she*. See also `https://www.youtube.com/watch?v=eHsmDFKLpZU`.

2.3. Subsequences

Sometimes if you have a poorly behaved sequence (s_n), it might have a better behaved *subsequence*, and dealing with the subsequence may be useful in dealing with the original sequence.

Definition 2.3.1. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. A **subsequence** of (s_n) is a sequence of the form $(s_{n_k})_{k\in\mathbb{N}}$ where $(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$:

$$1 \leq n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots .$$

Of course, every sequence is a subsequence of itself (just take $n_k = k$).

Example 2.3.2.
(1) The constant sequences (-1) and (1) are subsequences of $((-1)^n)_{n\in\mathbb{N}}$. Just take $n_k = 2k+1$ and $n_k = 2k$, respectively, to get each of the two subsequences
(2) Given the sequence $(s_n) = (1/n)$, then $(1/2^k)_{k\in\mathbb{N}}$ is a subsequence (use $n_k = 2^k$).

Remark 2.3.3. Since $(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$, a proof by induction shows that

$$n_k \ \geq \ k \quad \text{for all } k \in \mathbb{N}.$$

Lemma 2.3.4. *Suppose $s_n \to s \in \mathbb{R}$. Then $s_{n_k} \to s$ for every subsequence (s_{n_k}) of (s_n).*

Proof. Let $\epsilon > 0$. Take N such that

$$|s_n - s| \ < \ \epsilon \quad \text{for every } n \geq N.$$

Suppose $k \geq N$. Then $n_k \geq k \geq N$, so

$$|s_{n_k} - s| \ < \ \epsilon \quad \text{for every } k > N.$$

Thus $s_{n_k} \to s$. □

In general we think of a non-convergent sequence as possibly being rather *chaotic*. The next lemma says that a sequence has a subsequence which converges to s iff there is always enough "chaos" from the original sequence happening sufficiently close to s.

Lemma 2.3.5 (Extracting a convergent subsequence)**.** *Let (s_n) be a sequence and $s \in \mathbb{R}$. Then there is a subsequence of (s_n) converging to s iff for all $\epsilon > 0$, $\{n : |s_n - s| < \epsilon\}$ is infinite.*

Proof. ($\Rightarrow$) This direction is clear, since if (s_{n_k}) is a subsequence of (s_n) which converges to s, then some infinite "tail" of (s_{n_k}) will be within distance ϵ of s. Formally: Suppose $s_{n_k} \to s$ as $k \to \infty$. Let $\epsilon > 0$. Then there is N such that for every $k \geq N$, $|s_{n_k} - s| < \epsilon$. Thus

$$\{n : |s_n - s| < \epsilon\} \ \supseteq \ \{n_k : k \geq N\}$$

is infinite.

($\Leftarrow$) By induction on k we construct a sequence of natural numbers

$$1 \leq n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$$

such that $|s_{n_k} - s| < 1/2^k$ for all $k \in \mathbb{N}$. If we can accomplish this, then (s_{n_k}) will be a subsequence which converges to s.

For the base case $k = 1$, take any n_1 such that $|s_{n_1} - s| < 1/2$.

For the inductive step, assume we have constructed already $n_1, \ldots, n_k$ as desired. By assumption the set

$$\{n : |s_n - s| < 1/2^{k+1}\}$$

is infinite, so

$$\{n : |s_n - s| < 1/2^{k+1} \text{ and } n > n_k\}$$

is also infinite. Let n_{k+1} be any element from this last set.

It is easily checked that $s_{n_k} \to s$ as $k \to \infty$. □

Nested Intervals Lemma 2.3.6. *Let $I_n = [a_n, b_n]$, where $a_n, b_n \in \mathbb{R}$, $a_n < b_n$ are such that*

$$I_n \supseteq I_{n+1} \quad \text{for all } n.$$

Then there is some $s \in \mathbb{R}$ such that $s \in I_n$ for all n. Moreover, if $b_n - a_n \to 0$, then there is a unique $s \in \mathbb{R}$ such that $s \in I_n$ for all n.

Proof. Consider the set of left endpoints:

$$S := \{a_n : n \in \mathbb{N}\}$$

The set S is bounded from above: every b_n is an upper bound of S since

$$I_n \supseteq I_{n+1} \quad \text{means} \quad a_n \leq a_{n+1} \leq b_{n+1} \leq b_n.$$

Put $s := \sup S \in \mathbb{R}$. Then $a_n \leq s \leq b_n$ for all n.

In the case that $b_n - a_n \to 0$, suppose $t \in \mathbb{R}$ is also such that $t \in I_n$ for every n. Then $a_n \leq s, t \leq b_n$, so $0 \leq |s - t| \leq b_n - a_n \to 0$ so $s = t$. □

Bolzano-Weierstrass Theorem 2.3.7. *Every bounded sequence has a convergent subsequence.*

Proof. The proof will proceed by a standard "bisection" argument.

Let (s_n) be a bounded sequence. Take M such that $|s_n| \leq M$ for all n. We will construct a sequence of nested intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

such that for each k:

(1) $I_k = [a_k, b_k]$ and $b_k - a_k = M \cdot 2^{2-k}$, and
(2) $\{n : s_n \in I_k\}$ is infinite.

Let $I_1 := [-M, M]$. Suppose we have already constructed I_k. Consider the midpoint $c_k = (a_k + b_k)/2$. Then each of the intervals $[a_k, c_k], [c_k, b_k]$ has length $M \cdot 2^{2-(k+1)}$. Furthermore, at least one of the sets

$$\{n : s_n \in [a_k, c_k]\} \quad \text{or} \quad \{n : s_n \in [c_k, b_k]\}$$

is infinite (since their union $\{n : s_n \in I_k\}$ is infinite). If the first set is infinite, then we set $I_{k+1} := [a_k, c_k]$. Otherwise, we set $I_{k+1} := [c_k, b_k]$. By the Nested Intervals Lemma 2.3.6, there is some $s \in \mathbb{R}$ such that

$$s \in I_k \quad \text{for every } k.$$

Claim. *Some subsequence of (s_n) converges to s.*

Proof of claim. By 2.3.5, it is enough to show that for all $\epsilon > 0$,

$$\{n : |s_n - s| < \epsilon\} \quad \text{is infinite.}$$

To see this, let $\epsilon > 0$ be given, and choose k_0 such that $M \cdot 2^{2-k_0} < \epsilon$. Then for infinitely many many n, we have $s_n \in I_{k_0}$, as well as $s \in I_{k_0}$. So

$$|s - s_n| \;\leq\; \text{length of } I_{k_0} \;=\; M \cdot 2^{2-k_0} \;<\; \epsilon,$$

for infinitely many n. □

This finishes the proof of the Bolzano-Weierstrass Theorem. □

The next theorem shows how we can check whether a sequence converges *without* having to know its limit. We call the theorem the "Cauchy Completeness Theorem" because it asserts that no Cauchy sequence is missing a limit (just like the Completeness Axiom asserts that no nonempty bounded-above set is missing a supremum).

Cauchy Completeness Theorem 2.3.8. *Let (s_n) be a sequence. Then (s_n) is Cauchy iff (s_n) is convergent.*

Proof. ($\Leftarrow$) This is Lemma 2.2.7.

($\Rightarrow$) Suppose (s_n) is Cauchy. Then (s_n) is bounded by 2.2.8, thus (s_n) has a convergent subsequence by the Bolzano-Weierstrass Theorem 2.3.7. Let (s_{n_k}) be this convergent subsequence and $s \in \mathbb{R}$ such that $s_{n_k} \to s$. Then it suffices to show

Claim. *(s_n) also converges to s.*

Proof of claim. Let $\epsilon > 0$ be given. Take N_0 such that $|s_n - s_{n'}| < \epsilon/2$ for all $n, n' \geq N_0$ (since (s_n) is Cauchy). Next, take k_0 large enough such that

(i) $|s - s_{n_k}| < \epsilon/2$ for $k \geq k_0$, and
(ii) $n_{k_0} \geq N_0$.

Then for $n \geq n_{k_0}$ we have:

$$\begin{aligned} |s - s_n| &= \big|(s - s_{n_{k_0}}) + (s_{n_{k_0}} - s_n)\big| \\ &= |s - s_{n_{k_0}}| + |s_{n_{k_0}} - s_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon. \end{aligned}$$ □

This concludes the proof of 2.3.8. □

2.4. Series

Series (aka "infinite sums") are really just a special case of sequences. Suppose we are given a sequence $(a_n)_{n \geq m}$ in $\mathbb{R}$. Our goal is to try to assign, *if possible*, some mathematical meaning to the infinite sum:

$$a_m + a_{m+1} + a_{m+2} + \cdots$$

For $N \geq m$ we define

$$s_N \;:=\; \sum_{n=m}^{N} a_n \;=\; a_m + a_{m+1} + \cdots + a_N.$$

We call $(s_N)_{N\geq m}$ the **sequence of partial sums** associated to (a_n). We say that the **infinite series**

$$\sum_{n=m}^{\infty} a_n$$

converges if the sequence (s_N) of partial sums converges to a real number S, i.e., if

$$S = \lim_{N\to\infty} \sum_{n=m}^{N} a_n = \lim_{N\to\infty} s_N.$$

In this case, we write

$$\sum_{n=m}^{\infty} a_n := S.$$

An infinite sum which does not converge is said to **diverge**.

If (s_N) diverges to $+\infty$ (resp., $-\infty$), then we say that $\sum_{n=m}^{\infty} a_n$ **diverges to** $+\infty$ (resp., to $-\infty$), and we write

$$\sum_{n=m}^{\infty} a_n = +\infty \quad (\text{resp.}, -\infty)$$

Remark 2.4.1. If $a_n \geq 0$ for all $n \geq m$, then the sequence $(s_N)_{N\geq m}$ of partial sums is increasing, hence $\sum_{n=m}^{\infty} a_n$ either converges, or diverges to $+\infty$.

In particular, $\sum_{n=m}^{\infty} |a_n|$ either converges or diverges to $+\infty$ for *any* sequence $(a_n)_{n\geq m}$.

Definition 2.4.2. An infinite series $\sum_{n=m}^{\infty} a_n$ is said to be **absolutely convergent** if

$$\sum_{n=m}^{\infty} |a_n|$$

converges.

Example 2.4.3 (Geometric series)**.** Suppose $m = 0$, $a_n = r^n$ ($r \in \mathbb{R}$, $r \neq 1$). Then for $N \geq 0$,

$$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r}$$

For $|r| < 1$, we have $\lim_{N\to\infty} r^{N+1} = 0$ by 2.1.11(3), and so

$$\sum_{n=0}^{\infty} r^n = \lim_{N\to\infty} \left(\frac{1 - r^{N+1}}{1 - r}\right) = \frac{1}{1 - r}.$$

In fact, when $|r| < 1$, we have shown that $\sum_{n=0}^{\infty} r^n$ converges absolutely.

Definition 2.4.4. We say that an infinite series

$$\sum_{n=m}^{\infty} a_n$$

satisfies the **Cauchy criterion** if the sequence $(s_N)_{N\geq m}$ of partial sums is a Cauchy sequence:

$$\begin{cases} \text{for each } \epsilon > 0 \text{ there is } N_0 \geq m \text{ such that} \\ |s_N - s_{N'}| < \epsilon \text{ for all } N, N' \geq N_0 \end{cases}$$

In this definition, we may also assume $N > N'$. Then

$$s_N - s_{N'} = a_{N'+1} + a_{N'+2} + \cdots + a_N,$$

so this definition now reads:

$$\begin{cases} \text{for each } \epsilon > 0 \text{ there is } N_0 \geq m \text{ such that} \\ |a_{N'+1} + a_{N'+2} + \cdots + a_N| < \epsilon \text{ for all } N > N' \geq N_0. \end{cases}$$

Taking $N' + 1$ instead of N', we get:

$$(*) \qquad \begin{cases} \text{for each } \epsilon > 0 \text{ there is } N_0 \geq m \text{ such that} \\ |a_{N'} + a_{N'+1} + \cdots + a_N| < \epsilon \text{ for all } N \geq N' > N_0. \end{cases}$$

Corollary 2.4.5. *$\sum_{n=m}^{\infty} a_n$ converges iff it satisfies the Cauchy criterion.*

Proof. This is an application of Theorem 2.3.8 to series. □

When dealing with infinite series, the first question you want to ask is "does this series converge or diverge?". To handle this question, we have various "tests" for convergence/divergence.

Divergence Test 2.4.6. *If $\sum_{n=m}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.*

Proof. Take $N = N'$ in $(*)$. □

The *Comparison Test* is basically like the Squeeze Lemma but for series, and you use it the same way: given a complicated series, you try to find a simpler series to compare it to.

Comparison Test 2.4.7. *Let $\sum_{n=m}^{\infty} a_n$ be an infinite series where $a_n \geq 0$ for all $n \geq m$.*

(1) *if $\sum a_n$ converges, and $|b_n| \leq a_n$ for all n, then $\sum b_n$ and $\sum |b_n|$ converges.*
(2) *if $\sum a_n = +\infty$ and $b_n \geq a_n$ for all n, then $\sum b_n = +\infty$.*

Proof. (1) For $N' \geq N > m$ we have by the Triangle Inequality

$$\left| \sum_{n=N}^{N'} b_n \right| \leq \sum_{n=N}^{N'} |b_n| \leq \sum_{n=N}^{N'} a_n.$$

Since $\sum a_n$ satisfies the Cauchy criterion, so does $\sum b_n$ and $\sum |b_n|$.

(2) Note that $\sum_{n=m}^{N} b_n \geq \sum_{n=m}^{N} a_n$ and that $\sum_{n=m}^{N} a_n \to +\infty$. □

Corollary 2.4.8. *If $\sum b_n$ is absolutely convergent, then it is convergent.*

Proof. Suppose $\sum b_n$ is absolutely convergent. In other words, $\sum a_n$ is convergent, for $a_n := |b_n|$. Since we have $|b_n| \leq a_n$ for every n, by the Comparison Test 2.4.7(1), $\sum b_n$ converges. □

Comparing a given series with a convergent geometric series with the Comparison Test is very useful. It is so useful, in fact, that it has been codified into a test of its own: *The Root Test.* The formulation of the root test we give here is more general than the one given in calculus because we do not require $\lim |a_n|^{1/n}$ to exist.

Root Test 2.4.9. *Suppose (a_n) is a sequence in $\mathbb{R}$.*

(1) If there is $\alpha < 1$ and N such that for every $n \geq N$,

$$|a_n|^{1/n} \;\leq\; \alpha,$$

then $\sum a_n$ converges absolutely.

(2) If $|a_n| \geq 1$ for infinitely many n, then $\sum a_n$ diverges.

(3) If neither (1) nor (2) hold, then the test is inconclusive.

In particular, if $\ell = \lim_{n\to\infty} |a_n|^{1/n}$ exists and $\ell < 1$, then (1) holds, and if $\ell > 1$, then (2) holds.

Proof. (1) Let N be such that for every $n \geq N$, $|a_n|^{1/n} \leq \alpha$. Then $|a_n| < \alpha^n$ for every $n \geq N$. By Example 2.4.3, we know that $\sum_{k=N}^{\infty} \alpha^k$ converges, since $|\alpha| < 1$. By the Comparison Test 2.4.7(1), $\sum_{k=N}^{\infty} |a_k|$ also converges. Thus $\sum |a_k|$ converges, so $\sum a_k$ converges absolutely.

(2) Suppose $|a_n| \geq 1$ infinitely often. Then $a_n \not\to 0$, and so $\sum a_n$ is divergent by the Divergence Test 2.4.6.

(3) To show that the root test is inconclusive if we are not in case (1) or (2), we just need to demonstrate examples of convergent series and divergent series which fall under this case. This will be the next example. □

Example 2.4.10. The Root Test is inconclusive for the following two series:

(1) The **harmonic series** $\sum_{k=1}^{\infty} 1/k$ diverges. To see this, recall that in Exercise 1.9.2 we shows that

$$\sum_{k=1}^{2^n} \frac{1}{k} \;\geq\; 1 + \frac{n}{2},$$

in particular, since $1 + n/2 \to +\infty$ as $n \to \infty$, it follows that $\sum_{k=1}^{\infty} 1/k = +\infty$. Note also that since $n^{1/n} \to 1$, we also have $1/n^{1/n} = (1/n)^{1/n} \to 1$, and $(1/n)^{1/n} < 1$ for all n. Thus this series does not fall into either case (1) of case (2) of the Root Test.

(2) Next consider the series $\sum_{k=1}^{\infty} \frac{1}{n^2}$. Exercise 2.6.6 we have

$$\sum_{k=2}^{\infty} \frac{1}{k(k-1)} \;=\; 1.$$

So by the Comparison Test we conclude that $\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges since $1/k^2 \leq 1/k(k-1)$ for all $k \geq 2$. Thus $\sum_{k=1}^{\infty} 1/k^2$ also converges. However, since $(1/n^2)^{1/n} = (1/n)^{1/n}(1/n)^{1/n} \to 1$ and $(1/n^2)^{1/n} < 1$ for all $n \geq 2$, this series does not fall into either case (1) or case (2) of the Root Test.

The Root Test has a little sibling: *The Ratio Test.* In practice it is easier to use than the Root Test, but it works for fewer series. In fact, the proof of the Ratio Test will show that any time the Ratio Test shows the convergence of a series, then the Root Test also works for the same series. However, the converse is not true: there are series for which the Ratio Test is inconclusive but the Root Test is conclusive.

Ratio Test 2.4.11. *Suppose (a_n) is a sequence in $\mathbb{R}$.*

(1) If there is $\alpha < 1$ and N such that for every $n \geq N$,

$$\left|\frac{a_{n+1}}{a_n}\right| \;\leq\; \alpha,$$

then $\sum a_n$ converges absolutely.

(2) *If there is N such that for every $n \geq N$,*

$$\left|\frac{a_{n+1}}{a_n}\right| \;\geq\; 1,$$

then $\sum a_n$ diverges.

(3) *If neither (1) nor (2) hold, then the test is inconclusive.*

In particular, if $\lim_{n\to\infty} |a_{n+1}/a_n| = \ell$ exists and $\ell < 1$, then (1) holds, and if $\ell > 1$, then (2) holds.

[Note: it is implicit in the statement of (1) and (2) of the Ratio Test that for every $n \geq N$, $a_n \neq 0$, otherwise the ratio would not make sense.]

Proof. (1) Let $\alpha < 1$ and N be such that for every $n \geq N$, $|a_{n+1}/a_n| \leq \alpha$. Then for $n \geq N$ we have

$$\begin{aligned} |a_n| \;&=\; \left|\frac{a_n}{a_{n-1}}\right| \cdot \left|\frac{a_{n-1}}{a_{n-2}}\right| \cdots \left|\frac{a_{N+1}}{a_N}\right| \cdot |a_N| \\ &\leq\; \alpha^{n-N}|a_N| \quad \text{(formally this follows by induction on } n \geq N) \\ &=\; \alpha^n |a_N \alpha^{-N}|. \end{aligned}$$

Since $|a_N\alpha^{-N}| > 0$ and $(\alpha+1)/2\alpha > 1$, there is $N_1 \geq N$ such that for all $n \geq N_1$ we have

$$|a_N\alpha^{-N}|^{1/n} \;\leq\; \frac{\alpha+1}{2\alpha},$$

by Example 2.1.11(4) which says that $|a_N\alpha^{-N}|^{1/n} \to 1$. Now suppose $n \geq N_1$. Then

$$|a_n|^{1/n} \;\leq\; \alpha|a_N\alpha^{-N}|^{1/n} \;\leq\; \frac{\alpha+1}{2} \;<\; 1.$$

Thus we are in case (1) of the Root Test and so $\sum a_n$ converges absolutely.

(2) Let N be such that for every $n \geq N$, $|a_{n+1}| \geq |a_n|$ and $a_n \neq 0$. Then in particular, $|a_n| \not\to 0$, so $\sum a_n$ diverges by the Divergence Test 2.4.6.

(3) To show the Ratio Test is inconclusive if we are not in case (1) or (2), simply observe that the series in Example 2.4.10 also fall under case (3) here. □

Example 2.4.12. (1) Consider the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$. Here $a_n = n/2^n$. To apply the Ratio Test, the relevant ratios are

$$\left|\frac{a_{n+1}}{a_n}\right| \;=\; \frac{(n+1)2^n}{2^{n+1}n} = \frac{n+1}{2n} \to \frac{1}{2} < 1.$$

By the Ratio Test, $\sum_{n=1}^{\infty} n/2^n$ converges.

(2) Consider the series

$$\sum_{n=0}^{\infty} 2^{(-1)^n - n} \;=\; 2 + \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \frac{1}{8} + \frac{1}{16} + \cdots$$

The relevant ratios are

$$\frac{a_{n+1}}{a_n} \;=\; \frac{2^{(-1)^{n+1}-(n+1)}}{2^{(-1)^n - n}}$$

$$=\; 2^{(-1)^{n+1}-(n+1)-(-1)^n+n} \;=\; 2^{2(-1)^{n+1}-1} \;=\; \begin{cases} \frac{1}{8} & \text{if } n \text{ even} \\ 2 & \text{if } n \text{ odd.} \end{cases}$$

This means the Ratio Test is *not* applicable here. We can use the Root Test:

$$|a_n|^{1/n} \;=\; |2^{(-1)^n - n}|^{1/n} \;=\; \begin{cases} 2^{1/n-1} & \text{if } n \text{ even} \\ 2^{-1/n-1} & \text{if } n \text{ odd.} \end{cases}$$

Since $a_n^{1/n} \to 1/2 < 1$, we can conclude that the series converges by the Root Test.

(3) Now consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \;=\; 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

Note that:

$$\frac{a_{n+1}}{a_n} \;=\; \sqrt{\frac{n}{n+1}} \to 1 \quad \text{and} \quad \sqrt[n]{a_n} \;=\; \sqrt{n^{-1/n}} \to 1,$$

so neither the Ratio Test nor the Root Test are applicable here.

Example 2.4.13. Fix $x \in \mathbb{R}$ and consider the sequence $a_n := x^n/n!$ for $n \geq 0$. Then we have

$$\left|\frac{a_{n+1}}{a_n}\right| \;=\; \left|\frac{x^{n+1}n!}{(n+1)!x^n}\right| \;=\; \frac{|x|}{n+1} \;\to\; 0.$$

Thus by the Ratio Test we conclude that $\sum_{n=0}^{\infty} x^n/n!$ converges. We are now in a position to define "the most important[2] function in mathematics".

Definition 2.4.14. Define the **exponential function** $\exp\colon \mathbb{R} \to \mathbb{R}$ by

$$\exp(x) \;:=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for $x \in \mathbb{R}$.

We will return the exponential function again in these notes (how could we not?).

The *Alternating Series Test* is usually the best bet for alternating series where the *Root/Ratio* Tests fail. It is the only test we are giving which also provides an estimate for the sum.

Alternating Series Test 2.4.15 (Leibniz)**.** *Let $(a_n)_{n\geq 1}$ be a decreasing sequence with $a_n \geq 0$ for all n. If $\lim_{n\to\infty} a_n = 0$, then $s = \sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges. Furthermore, for $s_N = \sum_{n=1}^{N}(-1)^{n+1}a_n$ we then have*

$$|s - s_N| \leq a_n \quad \textit{for all } N.$$

Proof. First note that the subsequence $(s_{2n})_{n\geq 1}$ of $(s_N)_{N\geq 1}$ is increasing:

$$s_{2n+2} - s_{2n} = -a_{2n+2} + a_{2n+1} \geq 0. \tag{1}$$

Similarly, $(s_{2n-1})_{n\geq 1}$ is decreasing:

$$s_{2n+1} - s_{2n-1} = a_{2n+1} - a_{2n-1} \leq 0. \tag{2}$$

Claim. *$s_{2m} \leq s_{2n+1}$ for all $m, n \in \mathbb{N}$ (even partial sums are $\leq$ odd partial sums).*

[2] According to Walter Rudin [**4**, pg. 1].

Proof of claim. First note that

$$s_{2n+1} - s_{2n} = a_{2n+1} \geq 0,$$

so

(3) $$s_{2n+1} \geq s_{2n}.$$

Thus if $m \leq n$ we have

$$s_{2m} \leq s_{2n} \leq s_{2n+1}$$

using (1) for the first inequality and (3) for the second inequality. Otherwise, if $m > n$, then

$$s_{2m} \leq s_{2m+1} \leq s_{2n+1}$$

using (3) for the first inequality and (2) for the second inequality. □

By the Claim and the Monotone Convergence Theorem 2.2.2 we have that (s_{2n}) and (s_{2n-1}) both converge, say $s_{2n} \to s \in \mathbb{R}$ and $s_{2n+1} \to t \in \mathbb{R}$.

Claim. *$s = t$.*

Proof of claim. Note that

$$\begin{aligned} t - s &= \lim_{n\to\infty} s_{2n+1} - \lim_{n\to\infty} s_{2n} \\ &= \lim_{n\to\infty} (s_{2n+1} - s_{2n}) \quad \text{(by limit laws)} \\ &= \lim_{n\to\infty} a_{2n+1} \\ &= 0. \end{aligned}$$

By the "sequence splicing" homework exercise, we conclude that $\lim_{N\to\infty} s_N = s$. □

Finally, note that for each n we have $s_{2n} \leq s = t \leq s_{2n+1}$. Thus

$$0 \leq s_{2n+1} - s \leq s_{2n+1} - s_{2n} = a_{2n+1},$$

and

$$0 \leq s - s_{2n} \leq s_{2n+1} - s_{2n} = a_{2n+1} \leq a_{2n}.$$

Thus $|s - s_N| \leq a_N$ for all N (regardless of whether N is odd or even). □

Example 2.4.16. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ both converge. The second series converges to $\log 2$ (where $\log 2$ is the unique number $y \in \mathbb{R}$ such that $\exp(y) = 2$, we won't ever talk about logarithms in this class unfortunately).

The following test says that given a nonnegative monotonically decreasing sequence (a_n), the convergence of $\sum a_n$ is determined by a rather "thin" subsequence of (a_n):

Cauchy Condensation Test 2.4.17. *Let $(a_n)_{n\geq 1}$ be a decreasing sequence with $a_n \geq 0$. Then*

$$\sum_{n=1}^{\infty} a_n \quad \textit{converges} \quad \iff \quad \sum_{m=0}^{\infty} 2^m a_{2^m} \quad \textit{converges.}$$

PROOF. For each $N \geq 1$ consider the partial sums $S_N := \sum_{n=1}^{N} a_n$ and $T_N := \sum_{n=0}^{N} 2^n a_{2^n}$. Recall from Exercise 2.6.27 that for every $N \geq 1$:

$$S_{2^{N+1}-1} \ \leq \ T_N \ \leq \ 2S_{2^N}$$

($\Rightarrow$) Suppose $\sum_{n=1}^{\infty} a_n$ converges. Then $(S_N)_{N\geq 1}$ converges (by definition), so $(S_{2^N})_{N\geq 1}$ converges as it is a subsequence of $(S_N)_{N\geq 1}$, so $(2S_{2^N})_{N\geq 1}$ converges, which implies $(T_N)_{N\geq 1}$ converges by the Comparison Test. In other words, $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

($\Leftarrow$) Suppose $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges, i.e., that $(T_N)_{N\geq 1}$ converges. Then $(S_{2^{N+1}-1})_{N\geq 1}$ converges by the Comparison Test. Since $(S_N)_{N\geq 1}$ is a monotonically increasing sequence (because $a_n \geq 0$ for all n), and it has a convergent subsequence, then by Exercise 2.6.11 we conclude that $(S_N)_{N\geq 1}$ converges. □

The Cauchy Condensation Test works for certain series which would otherwise require the Integral Test. This is advantageous to us because we don't want to wait until Chapter 5 (when we do integrals) to be able to analyze the following series:

Example 2.4.18 (p-series)**.** Suppose $p \in \mathbb{R}$. Then

(1) If $p > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. By the Cauchy Condensation Test, it suffices to check the convergence of

$$\sum_{m=0}^{\infty} 2^m \frac{1}{(2^m)^p} \ = \ \sum_{m=0}^{\infty} \frac{1}{(2^m)^{p-1}} \ = \ \sum_{m=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^m \ = \ \frac{1}{1-\left(\frac{1}{2}\right)^{p-1}}$$

which we recognize as a convergent geometric series.

(2) if $0 < p \leq 1$: Then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges. By the Cauchy Condensation Test, it suffices to check the divergence of

$$\sum_{m=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^m$$

which we recognize as a divergence geometric series since $1/2^{p-1} \geq 1$.

The following gives some interesting background concerning some important series in mathematics. You do not need to know it for any homework or exam:

Application 2.4.19. For $p > 1$, define the $\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p}$. The function ζ is called the *Riemann ζ-function*. Some known values of the ζ-function are

$$\begin{aligned}\zeta(2) &= \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \\ \zeta(4) &= \frac{\pi^4}{90} \\ \zeta(6) &= \frac{\pi^6}{945}\end{aligned}$$

Computing the value of $\zeta(2)$ is known as the *Basel problem*[3], first asked in 1644, solved by Euler in 1734, announced in 1735, and rigorously proved in 1741. In general there is a formula for $\zeta(p)$ when p is even, however computing the value of $\zeta(p)$ when p is odd is *hard*. Here are some things which are known[4] (note the year!):

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[3] https://en.wikipedia.org/wiki/Basel_problem

[4] https://en.wikipedia.org/wiki/Particular_values_of_the_Riemann_zeta_function

(1) It was proved in 1978 by R. Apéry[5] that $\zeta(3) \notin \mathbb{Q}$, although it is unknown whether $\zeta(3)/\pi^3 \in \mathbb{Q}$.
(2) It is known that infinitely many $\zeta(2n+1)$ are irrational (T. Rivoal, 2000).
(3) It is also known that at least one of $\zeta(5), \zeta(7), \zeta(9)$, or $\zeta(11)$ are irrational (W. Zudilin, 2001).

2.5. The Exponential Function

We have already met one definition of the exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ in Definition 2.4.14 above. In these notes we will try to prove as much as we can about the exponential function subject to the rule that we only allow ourselves to use theorems taught in Math131a (so nothing about uniform convergence allowed). Given this restriction, Definition 2.4.14 doesn't seem to be a very useful starting point. Instead, we will pursue the following roundabout strategy:

(1) Define a second function $\Phi\colon \mathbb{R} \to \mathbb{R}$ below.
(2) Prove a bunch of cool things about Φ.
(3) Eventually develop enough machinery to show that $\exp = \Phi$ are the same function (so the cool things we proved about Φ in (2) are automatically true for exp).
(4) In fact, after we define $\mathrm{e} := \Phi(1) > 1$, we will show that the power function $\mathrm{e}^x\colon \mathbb{R} \to \mathbb{R}$ (from the Real Power Theorem 1.6.5 for $b := \mathrm{e}$) is also the same function as Φ and exp. (So we have only one true exponential function!)

First we need the following proposition:

Proposition 2.5.1. *Fix $x \in \mathbb{R}$. Then:*

(1) For every natural number $n > |x|$,

$$\left(1+\frac{x}{n}\right)^n \;<\; \left(1-\frac{x}{n}\right)^{-n},$$

(2) The sequence $\big((1+x/n)^n\big)_{n>|x|}$ is increasing,
(3) The sequence $\big((1-x/n)^{-n}\big)_{n>|x|}$ is decreasing,
(4) $(1-x/n)^{-n} - (1+x/n)^n \to 0$ as $n \to \infty$.

Proof. (1) Suppose $n > |x|$. Then $0 \le x^2/n^2 < 1$ and so

$$\left(1+\frac{x}{n}\right)^n \left(1-\frac{x}{n}\right)^n \;=\; \left(1-\frac{x^2}{n^2}\right)^n \;<\; 1,$$

which implies

$$\left(1+\frac{x}{n}\right)^n \;<\; \left(1-\frac{x}{n}\right)^{-n}.$$

(2) Suppose $n > |x|$ and apply the AGM Inequality 1.7.1 to the following $n+1$ numbers:

$$1, \underbrace{\left(1+\frac{x}{n}\right), \ldots, \left(1+\frac{x}{n}\right)}_{n \text{ of these}}$$

[5]https://en.wikipedia.org/wiki/Ap%C3%A9ry%27s_constant

This yields

$$\begin{aligned}\left(1\cdot\left(1+\frac{x}{n}\right)^n\right)^{\frac{1}{n+1}} &\le \frac{1}{n+1}\left[1+n\left(1+\frac{x}{n}\right)\right]\\ &= \frac{1}{n+1}\left(1+n+n\frac{x}{n}\right)\\ &= 1+\frac{x}{n+1}.\end{aligned}$$

Raising each side to the $(n+1)$st power yields

$$\left(1+\frac{x}{n}\right)^n \le \left(1+\frac{x}{n+1}\right)^{n+1}.$$

(3) Repeat the same argument as for (2) except with the numbers

$$1,\underbrace{\left(1-\frac{x}{n}\right),\ldots,\left(1-\frac{x}{n}\right)}_{n\text{ of these}}$$

(4) Note that

$$0 < \left(1-\frac{x}{n}\right)^{-n} - \left(1+\frac{x}{n}\right)^n = \left(1-\frac{x}{n}\right)^{-n}\left(1-\left(1-\frac{x^2}{n^2}\right)^n\right).$$

Then for $n>|x|$, Bernoulli's Inequality A.3.1 applied to $x^2/n^2 > -1$ yields:

$$\left(1-\frac{x^2}{n^2}\right)^n \ge 1-\frac{x^2}{n}.$$

Multiplying by -1 and then adding 1 to both sides yields:

$$1-\left(1-\frac{x^2}{n^2}\right)^n \le \frac{x^2}{n}$$

In particular,

$$0 < \left(1-\frac{x}{n}\right)^{-n} - \left(1+\frac{x}{n}\right)^n \le \underbrace{\left(1-\frac{x}{n}\right)^{-n}}_{\text{decreasing, }\ge 0}\underbrace{\frac{x^2}{n}}_{\to 0} \to 0,$$

which gives the desired convergence to 0 by the Squeeze Lemma. □

Note that by (2), (3) and (4) of Proposition 2.5.1, for $n>|x|$, the sequence of intervals

$$\left[\left(1+\frac{x}{n}\right)^n, \left(1-\frac{x}{n}\right)^{-n}\right]$$

is nested, and the endpoints converge to each other, so by the Nested Intervals Lemma 2.3.6 there is a unique point in each of these intervals, i.e., the common limit of the sequence of left endpoints and the sequence of right endpoints.. This enables us to define the following:

Definition 2.5.2. Define the function[6] $\Phi\colon \mathbb{R}\to\mathbb{R}$ by setting for each $x\in\mathbb{R}$,

$$\Phi(x) := \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty}\left(1-\frac{x}{n}\right)^{-n}.$$

[6]We will show later that $\Phi=\exp$, however for the time being we will pretend that Φ is a function completely unrelated to exp.

Note that $\Phi(0) = 1$. We also define the number $\mathrm{e} := \Phi(1)$. Note that by Proposition 2.5.1(1), (2) and (3) we have:

(a) for all natural numbers $n > |x|$,

$$\left(1 + \frac{x}{n}\right)^n \;\leq\; \Phi(x) \;\leq\; \left(1 - \frac{x}{n}\right)^{-n}$$

(b) if $|x| < 1$, then for all natural numbers n,

$$1 + x \;\leq\; \left(1 + \frac{x}{n}\right)^n \;\leq\; \Phi(x) \;\leq\; \left(1 - \frac{x}{n}\right)^{-n} \;\leq\; \frac{1}{1 - x}.$$

In particular, $\mathrm{e} \geq (1 + 1/2)^2 = 9/4 > 2$.

We will get a lot of mileage out of the following:

Functional Equation 2.5.3. *For all $x, y \in \mathbb{R}$,*

$$\Phi(x)\Phi(y) \;=\; \Phi(x + y).$$

Proof. For each natural number $n \geq 1$ define

$$h_n \;:=\; \frac{xy}{n + x + y}.$$

Then $h_n \to 0$, so for n large enough (specifically, take N such that $|h_n| < 1$ for all $n \geq N$ and assume $n \geq N$), we have

$$1 + h_n \;\leq\; \left(1 + \frac{h_n}{n}\right)^n \;\leq\; \frac{1}{1 - h_n}$$

by inequality (b) in Definition 2.5.2. Since $h_n \to 0$, by the Squeeze Lemma, we get $(1 + h_n/n)^n \to 1$, i.e.,

$$\left(1 + \frac{xy}{n(n + x + y)}\right)^n \;=\; \frac{\left(1 + \frac{x}{n}\right)^n \left(1 + \frac{y}{n}\right)^n}{\left(1 + \frac{x+y}{n}\right)^n} \;\to\; 1 \quad \text{as } n \to \infty.$$

Thus $\Phi(x)\Phi(y)/\Phi(x + y) = 1$, or rather, $\Phi(x)\Phi(y) = \Phi(x + y)$. □

Here are some immediate consequences of Functional Equation 2.5.3:

(1) By Exercise 1.9.5, the functions e^x and $\Phi(x)$ agree on all rational numbers, i.e., for every $q \in \mathbb{Q}$, $\mathrm{e}^q = \Phi(q)$. We will show later that in fact $\mathrm{e}^x = \Phi(x)$ for every $x \in \mathbb{R}$.
(2) For all $x \in \mathbb{R}$, $\Phi(-x)\Phi(x) = \Phi(0) = 1$, so in particular, $\Phi(x) \neq 0$ for every $x \in \mathbb{R}$.
(3) For all $x \in \mathbb{R}$, $\big(\Phi(x/2)\big)^2 = \Phi(x)$, so $\Phi(x) > 0$.

2.6. Exercises

Exercise 2.6.1. Prove the Squeeze Lemma for Sequences 2.1.10: Suppose $(s_n)_{n\geq m}$ and $(t_n)_{n\geq m}$ are sequences such that $t_n \to 0$ and for some $N \geq m$ we have $0 \leq s_n \leq t_n$ for all $n \geq N$. Then $s_n \to 0$.

Exercise 2.6.2. Calculate $\lim_{n\to\infty} \sqrt{n^2 + n} - n$. (For us, "calculate" means "somehow figure out what number the sequence converges to, and give a proof that your claimed convergence is correct.")

Exercise 2.6.3. Fix $\alpha > 0$ and $x_1 > \sqrt{\alpha}$. Recursively define a formula $(x_n)_{n\geq 1}$ by the formula

$$x_{n+1} := \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

(1) Show that (x_n) is monotonically decreasing and that $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.
(2) For each $n \geq 1$ define $\epsilon_n := x_n - \sqrt{\alpha}$. Show for each $n \geq 1$ that

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$$

and that if we set $\beta := 2\sqrt{\alpha}$, then

$$\epsilon_{n+1} < \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n}.$$

(3) Set $\alpha := 3$ and $x_1 := 2$. Show that:
   (a) $\epsilon_5 < 4 \cdot 10^{-16}$,
   (b) $\epsilon_6 < 4 \cdot 10^{-32}$.

The virtue of this exercise is that it gives a good algorithm for approximating $\sqrt{\alpha}$. Parts (2) and (3) show that this approximation converges very rapidly.

Exercise 2.6.4. Let (t_n) be a bounded sequence, i.e., there exists $M > 0$ such that $|t_n| \leq M$ for all n, and let (s_n) be a sequence such that $s_n \to 0$. Prove that $s_n t_n \to 0$. [Note: we are *not* assuming that (t_n) is convergent.]

Exercise 2.6.5. Let (s_n) be a sequence that converges.

(1) Show that if $s_n \geq a$ for all but finitely many n, then $\lim_{n\to\infty} s_n \geq a$.
(2) Show that if $s_n \leq b$ for all but finitely many n, then $\lim_{n\to\infty} s_n \leq b$.
(3) Conclude that if all but finitely many s_n belong to $[a, b]$, then $\lim_{n\to\infty} s_n$ belongs to $[a, b]$.

Exercise 2.6.6. Define the sequence (s_n) by

$$s_n := \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \cdots + \frac{1}{(n+1)\cdot n} \quad \text{for every } n \geq 1.$$

Prove that $\lim_{n\to\infty} s_n = 1$.

Exercise 2.6.7. For each of the following statements, determine whether it is true or false and justify your answer (give proof or counterexample).

(1) If the sequence (a_n^2) converges, then sequence (a_n) also converges.
(2) If the sequence $(a_n + b_n)$ converges, then the sequences (a_n) and (b_n) also converge.
(3) If the sequence $(|a_n|)$ converges, then the sequence (a_n) also converges.

Exercise 2.6.8 (Sequence splicing)**.** This exercise shows that the limit of a sequence can be computed from the limits of two subsequences which "sufficiently cover" the original sequence. (1) is a warmup, (2) is the general case.

(1) Suppose $(a_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ and $L \in \mathbb{R}_\pm$ is such that

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = L$$

(i.e., the even subsequence and the odd subsequence both have the same limit L). Show that $a_n \to L$.

(2) Suppose $(a_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ and $L \in \mathbb{R}_{\pm\infty}$. Let $(a_{m_j})_{j\in\mathbb{N}}$ and $(a_{n_k})_{k\in\mathbb{N}}$ be two subsequences of (a_n) such that
   (a) $\lim_{j\to\infty} a_{m_j} = L$,
   (b) $\lim_{k\to\infty} a_{n_k} = L$, and
   (c) there is $N \in \mathbb{N}$ such that
$$\{n \in \mathbb{N} : n \geq N\} \;\subseteq\; \{m_j : j \in \mathbb{N}\} \cup \{n_k : k \in \mathbb{N}\}.$$
   Show that $\lim_{n\to\infty} a_n = L$.

Exercise 2.6.9. Suppose (x_n) is a sequence in $\mathbb{R}$ and $x \in \mathbb{R}$ have the property that given any subsequence (x_{n_k}) of (x_n), there exists a subsequence $(x_{n_{k_\ell}})$ of (x_{n_k}) such that $x_{n_{k_\ell}} \to x$ as $\ell \to \infty$. Show that $x_n \to x$ as $n \to \infty$.

Exercise 2.6.10. For each of the following statements, determine whether it is true or false and justify your answer:
(1) A subsequence of a bounded sequence is bounded.
(2) A subsequence of a monotone sequence is monotone.
(3) A subsequence of a convergent sequence is convergent.
(4) A sequence converges if it has a convergent subsequence.

Exercise 2.6.11. Suppose that the sequence (a_n) is monotone and that it has a convergent subsequence. Show that (a_n) converges.

Exercise 2.6.12. Let (s_n) be a sequence such that
$$|s_{n+1} - s_n| \;<\; 2^{-n} \quad \text{for every } n \in \mathbb{N}.$$
Prove that (s_n) is a Cauchy sequence and hence a convergent sequence.

Exercise 2.6.13. For $c > 0$, consider the quadratic equation
$$x^2 - x - c = 0, \quad x > 0.$$
Define the sequence (x_n) recursively by fixing first some $x_1 > 0$ and then, if n is an index for which x_n has already been defined, defining
$$x_{n+1} \;:=\; \sqrt{c + x_n}.$$
Prove that the sequence (x_n) converges monotonically to a solution of the above equation.

Exercise 2.6.14 (Convergence of Cesaro Averages)**.** Suppose that the sequence (a_n) converges to a. Define the sequence (σ_n) by
$$\sigma_n \;:=\; \frac{a_1 + a_2 + \cdots + a_n}{n} \quad \text{for each } n \in \mathbb{N}.$$
Prove that the sequence (σ_n) also converges to a.

Exercise 2.6.15. Suppose $a_n \to a$.
(1) Show that $|a_n| \to |a|$.
(2) Suppose that $a_n \geq 0$ for every $n \in \mathbb{N}$. Show that $a \geq 0$ and $\sqrt{a_n} \to \sqrt{a}$.

Exercise 2.6.16. Let (b_n) be a bounded sequence of nonnegative numbers and r be any number such that $0 \leq r < 1$. Define
$$s_n \;:=\; b_1 r + b_2 r^2 + \cdots + b_n r^n \quad \text{for every } n \in \mathbb{N}.$$
Prove that (s_n) converges.

Exercise 2.6.17. Suppose $\sum_{n=m}^{\infty} a_n = a$ and $\sum_{n=m}^{\infty} b_n = b$ are two convergent series. Prove:

(1) $\sum_{n=m}^{\infty}(a_n + b_n) = a + b$, and
(2) for $r \in \mathbb{R}$, $\sum_{n=m}^{\infty} r \cdot a_n = r \cdot a$.

[Note: part of the proof should be justification that the series converges.]

Exercise 2.6.18. For each of the following, completely determine (with proof) for which real numbers $x \in \mathbb{R}$ the series converges and for which real numbers $x \in \mathbb{R}$ the series diverges:

(1) $\sum_{n=0}^{\infty} a^n x^n$ where $a \in \mathbb{R}$ (your answer may depend on what a is)
(2) $\sum_{n=0}^{\infty} a^{n^2} x^n$ where $a \in \mathbb{R}$ (your answer may depend on what a is)
(3) $\sum_{n=0}^{\infty} x^{n!}$
(4) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^{n(n+1)}$

Exercise 2.6.19. For (1) and (3) below, you need to provide justification as to why your example has the indicated property.

(1) Give an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.
(2) Show that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ also converges.
(3) Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

Exercise 2.6.20. Suppose we have a sequence $(a_n)_{n \geq 1}$ such that $a_n > 0$ for all n, define $s_n := a_1 + \cdots + a_n$, and suppose $\sum a_n$ diverges.

(1) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.
(2) Prove that for each $N, k \in \mathbb{N}$,

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \; \geq \; 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

(3) Prove for each $n \geq 1$ that

$$\frac{a_n}{s_n^2} \; \leq \; \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

(4) Determine the convergence/divergence of

$$\sum \frac{a_n}{1 + na_n} \quad \text{and} \quad \sum \frac{a_n}{1 + n^2 a_n}.$$

The point of this exercise is to show that given any divergent series, you can always find a series which diverges asymptotically slower.

Exercise 2.6.21. Evaluate

$$\lim_{n \to \infty} \left(n^2 \left[\left(1 + \frac{1}{n+1}\right)^{n+1} - \left(1 + \frac{1}{n}\right)^{n} \right] \right).$$

Hint: Exercise 1.9.10 might be useful.

Exercise 2.6.22. Determine the convergence/divergence of the following:

$$\sum_{n=1}^{\infty} (n^{1/n} - 1)$$

Exercise 2.6.23. In this exercise we will compare two different methods of approximating the number $\mathrm{e} := \Phi(1)$ (technically we don't know yet that the numbers $\exp(1)$ and e are the same, but it doesn't matter for this problem). For each $n \geq 0$ define the two "error terms":

$$a_n \;:=\; \mathrm{e} - \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad b_n \;:=\; \exp(1) - \sum_{k=0}^{n} \frac{1}{k!}$$

Determine the convergence/divergence of each of the following series:

(1) $\sum_{n=0}^{\infty} a_n$
(2) $\sum_{n=0}^{\infty} b_n$

When answered correctly, this exercise shows that $(a_n)_{n\geq 0}$ converges to 0 much "slower" than $(b_n)_{n\geq 1}$, and thus the sequence of partial sums $\sum_{k=0}^{n} \frac{1}{k!}$ converges to e much "faster" than the sequence $(1+1/n)^n$.

Exercise 2.6.24. Suppose $(x_n)_{n\geq 1}$ is a sequence in $\mathbb{R}$ such that $x_n \to x \in \mathbb{R}$ as $n \to \infty$. Show that

$$\lim_{n\to\infty} \left(1 + \frac{x_n}{n}\right)^n \;=\; \Phi(x).$$

[This limit is used in the proof of the *Central Limit Theorem* in probability]

Exercise 2.6.25. Suppose $L\colon (0,+\infty) \to \mathbb{R}$ is a function which enjoys the following properties:

(a) for every $a, b \in (0,+\infty)$, $L(ab) = L(a) + L(b)$;
(b) for every $a, b \in (0,+\infty)$, if $a < b$, then $L(a) < L(b)$.

Do the following:

(1) Show that $L(n) > 0$ for every $n \in \mathbb{N}$ such that $n \geq 2$.
(2) Suppose $b > 1$. Show that $L(b^n) = nL(b)$ for every $n \in \mathbb{N}$.
(3) Determine for which positive real numbers $p > 0$ the series

$$\sum_{n=2}^{\infty} \frac{1}{n\big(L(n)\big)^p}$$

converges.

Exercise 2.6.26. (Decimal expansion!) For any $x \in \mathbb{R}$ there is a unique integer n, denoted $[x]$, such that $n \leq x < n+1$. We also define $(x) := x - [x]$, and have $0 \leq (x) < 1$. Let $x \in \mathbb{R}$. Define two sequences $(a_n)_{n\geq 0}$ and $(\alpha_n)_{n\geq 0}$ recursively by:

$$a_0 \;:=\; [x], \quad \alpha_0 \;:=\; x - a_0 \;=\; (x), \quad a_n \;:=\; [10\alpha_{n-1}], \quad \alpha_n \;:=\; (10\alpha_{n-1}),$$

for $n \geq 1$. Prove:

(1) For every $n \geq 1$, $0 \leq a_n < 10$.
(2) The sequence $(b_n)_{n\geq 0}$ where $b_n \;:=\; \sum_{k=0}^{n} a_k 10^{-k}$, converges to x as $n \to \infty$.
(3) Show that given any sequence $(c_n)_{n\geq 1}$ such that $c_n \in \{0,1,2,\dots,8,9\}$ for every n, that the sequence

$$\left(\sum_{k=1}^{n} c_k 10^{-k}\right)_{n\geq 1}$$

converges as $n \to \infty$.

Exercise 2.6.27. Let $(a_n)_{n\geq 1}$ be a decreasing sequence with $a_n \geq 0$. For each $N \in \mathbb{N}$ define the partial sums $S_N := \sum_{n=1}^{N} a_n$ and $T_N := \sum_{n=0}^{N} 2^n a_{2^n}$. Prove for each $N \in \mathbb{N}$ that

$$S_{2^{N+1}-1} \;\leq\; T_N \;\leq\; 2S_{2^N}.$$

CHAPTER 3

Continuity and Continuous Functions

We still owe ourselves a proof of Uniqueness in the Real Power Theorem 1.6.5. Picking up where we left off, we have two functions $p, \tilde{p}\colon \mathbb{R} \to \mathbb{R}$ such that:

(1) $p(q) = \tilde{p}(q)$ for all $q \in \mathbb{Q}$ (by the Rational Power Theorem 1.6.4)
(2) Both $p(x)$ and $\tilde{p}(x)$ seem to *vary continuously* as you vary x; specifically, if you increase x a little bit from x to $x+\epsilon$ for some tiny ϵ, then both $p(x)$ and $\tilde{p}(x)$ increase a little bit to $p(x+\epsilon) = p(x)+\epsilon_1$ and $\tilde{p}(x+\epsilon) = \tilde{p}(x)+\epsilon_2$ for some other tiny $\epsilon_1, \epsilon_2 > 0$ (i.e., "no big jumps").

We hope that properties (1) and (2) above should be enough to conclude:

(3) $p(x) = \tilde{p}(x)$ for *every* $x \in \mathbb{R}$.

To show this, we will essentially do the following:

(4) Pinpoint and define a magical property which both functions p and $\tilde{p}$ enjoy which gets at the idea in (2) above.
(5) Show that the magical property from (4) plus item (1) above are sufficient to prove (3). (See Proposition 3.4.1)

This magical property is *continuity*, the subject of this chapter.

3.1. Limits of functions

Given a set $S \subseteq \mathbb{R}$, a point $x_0 \in S$ and a function $f\colon S \to \mathbb{R}$, we want to talk about the function values of $f(x)$ as x gets really close to x_0. However, it also makes sense to do this even if x_0 is technically not a point in S, for instance, if $S = (0,1)$ is an open interval and $x_0 = 0$. To allow for this possibility, we make the following flexible definition which relates a point $x_0 \in \mathbb{R}$ and a set $S \subseteq \mathbb{R}$ (where S can be thought of as possibly the domain of some function):

Definition 3.1.1. Suppose $S \subseteq \mathbb{R}$ and $x_0 \in \mathbb{R}_{\pm\infty}$. We say that x_0 is **in the closure of** S if there is at least one sequence $(x_n)_{n\geq 1}$ in S such that $\lim_{n\to\infty} x_n = x_0$.

In other words, x_0 is in the closure of S if it is possible to approximate x_0 by some sequence from S.

The above definition is often very easily satisfied in practice, for instance in each of the following, x_0 is in the closure of S:

(1) if $S \subseteq \mathbb{R}$ is any set and $x_0 \in S$,
(2) if $S = (a,b)$ and $x_0 = a$ or $x_0 = b$,
(3) if $S = \mathbb{R}$ or $S = (0,+\infty)$, and $x_0 = +\infty$,
(4) if $S = \mathbb{R} \setminus \{a\}$, and $x_0 = a$, etc.

The reason for defining "x_0 is in the closure of S" is because it encapsulates all possible ways we would wish to take the limit of a function $f\colon S \to \mathbb{R}$ as $x \to x_0$ in the following definition:

Definition 3.1.2. Let $f\colon S \to \mathbb{R}$ be a function with $S \subseteq \mathbb{R}$, let $x_0 \in \mathbb{R}_{\pm\infty}$ be in the closure of S, and let $L \in \mathbb{R}_{\pm\infty}$. We say

$$\lim_{x \in S, x \to x_0} f(x) = L$$

if for every sequence $(x_n)_{n\geq 1}$ in S such that $x_n \to x_0$, we have $f(x_n) \to L$.

If instead x_0 is not in the closure of S, or if x_0 is in the closure of S but there is no such $L \in \mathbb{R}_{\pm\infty}$ such that $\lim_{x\in S, x\to x_0} f(x) = L$, then we say the limit $\lim_{x\in S, x\to x_0} f(x)$ **does not exist**.

Notation 3.1.3. Let I be an open interval, suppose $a \in I$ and that g is a function whose domain contains $I \setminus \{a\}$. Then we write

$$\lim_{x\to a} g(x) \quad \text{instead of} \quad \lim_{x \in I\setminus\{a\}, x\to a} g(x).$$

This is often referred to the **two-sided limit** of f at a.

Remark 3.1.4. (1) (Limits are unique) If L and L' are from $\mathbb{R}_{\pm\infty}$ such that

$$\lim_{x\in S, x\to x_0} f(x) = L \quad \text{and} \quad \lim_{x\in S, x\to x_0} f(x) = L',$$

then $L = L'$.

(2) (Limits and restriction) Suppose $S' \subseteq S$ is such that x_0 is still in the closure of S'. Then

$$\lim_{x\in S, x\to x_0} f(x) = L \quad \Longrightarrow \quad \lim_{x\in S', x\to x_0} f(x) = L.$$

Proof. (1) Suppose $(x_n)_{n\geq 1}$ is a sequence in S such that $x_n \to x_0$. Then by Definition 3.1.2, we have $f(x_n) \to L$ and $f(x_n) \to L'$. Thus $L = L'$ by uniqueness for limits of *sequences*.

(2) Exercise. □

Example 3.1.5. We have

$$\lim_{x\to 1} \frac{x^2-1}{x-1} \;=\; 2.$$

Proof. Here, the function is $f(x) = (x^2-1)/(x-1)\colon \mathbb{R}\setminus\{1\} \to \mathbb{R}$. Let $(x_n)_{n\geq 1}$ be an arbitrary sequence from $\mathbb{R}\setminus\{1\}$ such that $x_n \to 1$. Then for all n, $(x_n^2 - 1)/(x_n-1) = x_n + 1$. Thus

$$\lim_{n\to\infty} \frac{x_n^2-1}{x_n-1} \;=\; \lim_{n\to\infty} x_n + 1 \;=\; 2.$$

As (x_n) was arbitrary, this shows that $\lim_{x\to 1}(x^2-1)/(x-1) = 2$. □

Example 3.1.6. Define the function $f\colon \mathbb{R}\to\mathbb{R}$ by

$$f(x) \;=\; \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

for $x \in \mathbb{R}$. Then $\lim_{x\to 0} f(x) = 0$.

Proof. Let $(x_n)_{n\geq 1}$ be an arbitrary sequence from $\mathbb{R}\setminus\{0\}$ such that $x_n \to 0$. Then by definition of f, $\big|f(x_n) - 0\big| \leq |x_n|$. Let $\epsilon > 0$. Take $N \geq 1$ such that $|x_n - 0| < \epsilon$ for all $n \geq N$. Then $\big|f(x_n) - 0\big| < \epsilon$ for all $n \geq N$. Thus $f(x_n) \to 0$. As (x_n) was arbitrary, this implies that $\lim_{x\to 0} f(x) = 0$. □

Lemma 3.1.7 (Limit laws for functions)**.** *Let $f_1, f_2 \colon S \to \mathbb{R}$ be functions, x_0 is in the closure of S, and suppose*

$$L_i \;=\; \lim_{x\in S, x\to x_0} f_i(x) \in \mathbb{R} \quad \textit{(for } i = 1,2\textit{)}.$$

Then

(1) $\lim_{x\in S,x\to x_0}(f_1+f_2)(x) = L_1 + L_2$,
(2) $\lim_{x\in S,x\to x_0}(f_1\cdot f_2)(x) = L_1\cdot L_2$,
(3) *if $f_2(x)\neq 0$ for $x\in S$, and $L_2\neq 0$, then*

$$\lim_{x\in S,x\to x_0}\left(\frac{f_1}{f_2}\right)(x) \;=\; \frac{L_1}{L_2}.$$

Proof. These follow from the corresponding limit laws for sequences (Proposition 2.1.9).

(1) Let (x_n) be an arbitrary sequence from S such that $x_n \to x_0$. Then by Proposition 2.1.9(1) we have

$$\lim_{n\to\infty}(f_1+f_2)(x_n) \;=\; \lim_{n\to\infty}\big[f_1(x_n)+f_2(x_n)\big] \;=\; \lim_{n\to\infty} f_1(x_n)+\lim_{n\to\infty} f_2(x_n) \;=\; L_1+L_2.$$

(2) and (3) are similar, use instead 2.1.9(2) and 2.1.9(3). □

Proposition 3.1.8 (ϵ-δ definition of limit)**.** *Let $f\colon S\to\mathbb{R}$, $x_0, L\in\mathbb{R}$ (**not** $\pm\infty$) such that x_0 is in the closure of S. Then the following are equivalent:*

(1) $\lim_{x\in S,x\to x_0} f(x) = L$
(2) *for every $\epsilon>0$ there is a $\delta>0$ such that: if $x\in S$ and $|x-x_0|<\delta$, then $|f(x)-L|<\epsilon$.*

Proof. (1)⇒(2) Suppose (2) does not hold. Then we can find some $\epsilon>0$ such that for *every* $\delta>0$, the implication

$$\text{“}x\in S \text{ and } |x-x_0|<\delta \quad\Longrightarrow\quad \big|f(x)-L\big|<\epsilon\text{”}$$

fails. So for each $n\geq 1$, there is some $x_n\in S$ with

$$|x_n-x_0|<\frac{1}{n} \quad\text{and}\quad \big|f(x_n)-L\big|\geq\epsilon.$$

(using the failure of the implication with $\delta := 1/n$) In this case, we have $x_n\to x_0$, but $f(x_n)\not\to L$. This shows (1) fails.

(2)⇒(1) Suppose (2) holds, and let $(x_n)_{n\geq 1}$ be a sequence in S with $x_n\to x_0$. It suffices to show that $f(x_n)\to L$. Let $\epsilon>0$, and take $\delta>0$ as in (2). Since $x_n\to x_0$, we can take $N\geq 1$ such that $|x_n-x_0|<\delta$ for every $n\geq N$. By (2), $\big|f(x_n)-L\big|<\epsilon$ for every $n\geq N$. □

Corollary 3.1.9 (Special case of ϵ-δ definition of limit)**.** *Let $L\in\mathbb{R}$, I an open interval, $a\in I$. If $f\colon I\setminus\{a\}\to\mathbb{R}$, then the following are equivalent:*

(1) $\lim_{x\to a} f(x) = L$,
(2) *for each $\epsilon>0$, there is a $\delta>0$ such that $x\in I$ and $0<|x-a|<\delta$ implies $|f(x)-L|<\epsilon$.*

3.2. Continuity and continuous functions

Definition 3.2.1. Let $f\colon S \to \mathbb{R}$ ($S \subseteq \mathbb{R}$) be a function and $x_0 \in S$. We say that f is **continuous at** x_0 if for every sequence (x_n) in S, if $x_n \to x_0$, then $f(x_n) \to f(x_0)$. Equivalently, f is continuous at x_0 if $\lim_{x \in S, x \to x_0} f(x) = f(x_0)$.

If f is continuous at every $x_0 \in S$, then we say that f is **continuous**.

Example 3.2.2. Here are some continuous functions:

(1) Every constant function $x \mapsto c\colon \mathbb{R} \to \mathbb{R}$ (where $c \in \mathbb{R}$) is continuous.
(2) The identity function $x \mapsto x\colon \mathbb{R} \to \mathbb{R}$ is continuous.
(3) The absolute value function $x \mapsto |x|\colon \mathbb{R} \to \mathbb{R}$ is continuous. This was verified in Exercise 2.6.15(1).
(4) The square root function $x \mapsto \sqrt{x}\colon [0, +\infty) \to \mathbb{R}$ is also continuous. This was verified in Exercise 2.6.15(2).

Proposition 3.2.3. *Let $f, g\colon S \to \mathbb{R}$ be continuous at $x_0 \in S$. Then the following functions are continuous at x_0:*

(1) $f + g\colon S \to \mathbb{R}$,
(2) $f \cdot g\colon S \to \mathbb{R}$
(3) $f/g\colon \{x \in S : g(x) \neq 0\} \to \mathbb{R}$, *provided that* $g(x_0) \neq 0$.

PROOF. (1) By definition of continuous at x_0, we know that

$$\lim_{x \in S, x \to x_0} f(x) = f(x_0) \quad \text{and} \quad \lim_{x \in S, x \to x_0} g(x) = g(x_0).$$

By Lemma 3.1.7(1), it follows that

$$\lim_{x \in S, x \to x_0} (f+g)(x) = f(x_0) + g(x_0) = (f+g)(x_0),$$

i.e., $f + g$ is continuous at x_0.

(2) This is similar and uses Lemma 3.1.7(2).
(3) This is similar and uses Lemma 3.1.7(3). □

Proposition 3.2.4 (Composition and continuity)**.** *Let $f\colon S \to \mathbb{R}$ be continuous at $x_0 \in S$ and $g\colon T \to \mathbb{R}$ with $T \supseteq f(S)$ be continuous at $f(x_0)$. Then $g \circ f\colon S \to \mathbb{R}$ is continuous at x_0.*

PROOF. Let (x_n) be a sequence in S such that $x_n \to x_0$. Then $\big(f(x_n)\big)$ is a sequence in T with $f(x_n) \to f(x_0)$, since f is continuous at x_0. Thus $g\big(f(x_n)\big) \to g\big(f(x_0)\big)$ since g is continuous at $f(x_0)$. Thus $g \circ f$ is continuous at x_0. □

Corollary 3.2.5. *If $f\colon S \to \mathbb{R}$ is continuous at $x_0 \in S$, then so is $|f|\colon S \to \mathbb{R}$, given by*

$$|f|(x) := |f(x)|, \quad \textit{for } x \in S.$$

Corollary 3.2.6 (Rational functions)**.** *Let P, Q be polynomials:*

$$\begin{aligned} P &= a_0 + a_1 x + \cdots + a_m x^m \\ Q &= b_0 + b_1 x + \cdots + b_n x^n, \end{aligned}$$

where $a_i, b_j \in \mathbb{R}$ and $a_m, b_n \neq 0$. Then the function

$$x \mapsto \frac{P(x)}{Q(x)}\colon S \to \mathbb{R}$$

is continuous, where

$$S \;:=\; \big\{x \in \mathbb{R} : Q(x) \neq 0\big\}.$$

The following is needed in the proof of Proposition 4.1.1 below:

Corollary 3.2.7 (Continuity at a point in an interval)**.** *Suppose I is an open interval, $f\colon I \to \mathbb{R}$ is a function and $x_0 \in I$. Then the following are equivalent:*

(1) f is continuous at x_0,
(2) $\lim_{x\to x_0} f(x) = f(x_0)$.

[Note: these might appear to be literally the same statements, but the difference is that the first one involves the limit $\lim_{x\in I, x\to x_0} f(x_0)$ and the second involves the limit $\lim_{x\in I\setminus\{x_0\}, x\to x_0} f(x)$. This distinction will be used later.]

Proof. (1)⇒(2) This is clear since

$$\lim_{x\in I, x\to x_0} f(x) = f(x_0) \quad \Longrightarrow \quad \lim_{x\to x_0} f(x) = f(x_0)$$

by Remark 3.1.4(2).

(2)⇒(1) Suppose $\lim_{x\to x_0} f(x_0) = f(x)$. Then by Corollary 3.1.9, for every $\epsilon > 0$ there is a $\delta > 0$ such that for $x \in I$ if $0 < |x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$. However, $|f(x_0) - f(x_0)| < \epsilon$. Thus, for every $\epsilon > 0$, there is $\delta > 0$ such that for $x \in I$, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$. In other words, $\lim_{x\in S, x\to x_0} f(x) = f(x_0)$ by Proposition 3.1.8. ☐

Definition 3.2.8. We say a function $f\colon S \to \mathbb{R}$ is **bounded** if $f(S) = \big\{f(x) : x \in S\big\} \subseteq \mathbb{R}$ is bounded, i.e., there is $M \in \mathbb{R}$ such that $|f(x)| \le M$ for every $x \in S$.

Extreme Value Theorem 3.2.9. *Let $a < b \in \mathbb{R}$ and suppose $f\colon [a,b] \to \mathbb{R}$ is continuous. Then f is bounded function. Moreover, f attains its maximum and minimum values on $[a,b]$, i.e., there are $x_m, x_M \in [a,b]$ such that $f(x_m) \le f(x) \le f(x_M)$ for all $x \in [a,b]$.*

Proof. We first show that f is bounded. Suppose otherwise, then for each $n \in \mathbb{N}$ there is some $x_n \in [a,b]$ such that $|f(x_n)| > n$ (in particular, the sequence $\big(f(x_n)\big)$ does not converge *and* no subsequence of it converges either). By Bolzano-Weierstrass 2.3.7 there is a subsequence (x_{n_k}) of (x_n) converging to some $x_0 \in \mathbb{R}$. Since $[a,b]$ is closed and bounded, we have $x_0 \in [a,b]$. Also, since f is continuous at x_0, we have

$$f(x_n) \to f(x_0).$$

This is a contradiction. Thus f is bounded.

Next, define

$$M \ := \ \sup\big\{f(x) : x \in [a,b]\big\} \in \mathbb{R}.$$

For each $n \in \mathbb{N}$ take $x_n \in [a,b]$ such that

$$M - \frac{1}{n} \ \le \ f(x_n) \ \le \ M.$$

Then $f(x_n) \to M$. By Bolzano-Weierstrass 2.3.7 we take a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to x_M \in [a,b]$. Then $f(x_{n_k}) \to f(x_M)$ since f is continuous at x_M. Also, $f(x_{n_k}) \to M$ since $f(x_n) \to M$. Thus $f(x_M) = M \in f\big([a,b]\big)$. In particular, $M = \max f\big([a,b]\big)$. The proof that $f\big([a,b]\big)$ has a minimum is similar and left as an exercise. ☐

Intermediate Value Theorem 3.2.10. *Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous, with $a < b \in \mathbb{R}$. Let y be a number strictly between $f(a)$ and $f(b)$, i.e.,*

$$f(a) < y < f(b) \quad \text{or} \quad f(b) < y < f(a).$$

Then there is $x_0 \in (a, b)$ *such that* $f(x_0) = y$.

Proof. We will prove the case where $f(a) < y < f(b)$ (the case $f(b) < y < f(a)$ is similar). Define the set

$$S \ := \ \big\{x \in [a, b] : f(x) \leq y\big\}.$$

Then $a \in S$, so $x_0 := \sup S \in [a, b]$ exists. For each $n \in \mathbb{N}$, $x_0 - 1/n$ is not an upper bound for S, so there is $x_n \in S$ with

$$x_0 - \frac{1}{n} \ \leq \ x_n \ \leq \ x_0.$$

Thus $x_n \to x_0$, so $f(x_n) \to f(x_0)$, and $f(x_n) \leq y$ for all n, hence $f(x_0) \leq y$. Now, let $x_n^* := \min\big\{b, x_0 + 1/n\big\}$. Then $x_n^* \to x_0$ and $x_n^* \in [a, b] \setminus S$, so $f(x_n^*) > y$ for all n. Since $f(x_n^*) \to f(x_0)$, we must have $f(x_0) \geq y$. We conclude that $f(x_0) = y$. □

Example 3.2.11. Let $n \in \mathbb{N}$ and suppose $f\colon \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^n$. Then f is continuous, $f(0) = 0$, and $\lim_{x\to\infty} f(x) = +\infty$. Suppose $b > 0 = f(0)$, then there is $c > 0$ such that $f(c) = c^n > b$. By the Intermediate Value Theorem, there is $y \in (0, c)$ such that $f(y) = y^n = b$. This gives an alternative way of showing the existence of $b^{1/n}$.

You may be asking yourself "why did we bother doing the technical proof of Existence and Uniqueness of nth roots 1.6.2 when we could just obtain $b^{1/n}$ easily using the Intermediate Value Theorem?" The reason is to illustrate the following point: one of the virtues of developing enough abstract theory is that it often enables you avoid tedious calculations.

The Intermediate Value Theorem can be useful for showing the existence of all sorts of points (not just intermediate values). This involves employing a standard trick where you cook up some "auxiliary function" which you apply the Intermediate Value Theorem to, in order to say something about the original function of interest. For example, in the next lemma, in order to find a fixed point of f, we apply the Intermediate Value Theorem to the auxiliary function g:

Fixed Point Lemma 3.2.12. *Let* $f\colon [0, 1] \to [0, 1]$ *be continuous. Then* f *has a fixed point, i.e., there is* $x \in [0, 1]$ *such that* $f(x) = x$

Proof. Consider the function $g(x) := f(x) - x$. Then

$$g(0) = f(0) - 0 \geq 0, \quad \text{and} \quad g(1) = f(1) - 1 \leq 0.$$

Thus either $g(0) = 0$, $g(1) = 0$, or else we can apply the Intermediate Value Theorem 3.2.10 on g with $y := 0$. Then there is $x \in [0, 1]$ such that $g(x) = 0$, i.e., $f(x) = x$. □

3.3. Uniform continuity

In this section, we introduce a stronger version of continuity: *uniform continuity.* To motivate the definition, first recall the definition of continuity:

Given a function $f\colon S \to \mathbb{R}$, f is **continuous** if

$$\underbrace{\text{for every } \epsilon > 0}_{A} \quad \text{and} \quad \underbrace{\text{for every } x_0 \in S,}_{B} \quad \underbrace{\text{there exists } \delta > 0}_{C} \quad \text{such that}$$

$$x \in S \text{ and } |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - f(x_0)| < \epsilon.$$

Note that in this definition, the δ in part C depends on whatever ϵ and x_0 are in parts A and B.

Example 3.3.1. Consider $f\colon \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ given by $f(x) = \frac{1}{x}$. We already know that f is continuous. We will reprove this to see what δ will look like. Suppose we are given $x_0 \in \mathbb{R}^{>0}$ and $\epsilon > 0$. We want

$$\epsilon \ > \ \left|\frac{1}{x} - \frac{1}{x_0}\right| \ = \ \frac{|x - x_0|}{x x_0}.$$

If $|x - x_0| < \frac{x_0}{2}$, then $x > \frac{x_0}{2}$, so

$$\frac{1}{x x_0} \ < \ \frac{2}{x_0^2} \quad \implies \quad \frac{|x - x_0|}{x x_0} \ < \ \frac{2}{x_0^2}|x - x_0|,$$

so $\delta := \min\left\{\frac{x_0}{2}, \epsilon\frac{x_0^2}{2}\right\}$ works.

Note that in the above example, δ depends on *both* ϵ *and* x_0. It would be better if δ only depended on ϵ and the same δ worked for any x_0.

Definition 3.3.2. Given a function $f\colon S \to \mathbb{R}$ with $S \subseteq \mathbb{R}$. We say that f is **uniformly continuous on** S if

$$\underbrace{\text{for every } \epsilon > 0}_{A}, \quad \underbrace{\text{there exists } \delta > 0}_{C} \quad \text{such that} \quad \underbrace{\text{for every } x_0 \in S}_{B},$$

$$x \in S \text{ and } |x - x_0| < \delta \quad \implies \quad |f(x) - f(x_0)| < \epsilon.$$

In other words, the definition of "uniformly continuous" is the same as "continuous" except that we switched the quantifiers (B) and (C).

Remark 3.3.3. If $f\colon S \to \mathbb{R}$ is uniformly continuous, then f is continuous.

A convenient feature of uniformly continuous functions is that it "preserves Cauchy sequences":

Lemma 3.3.4. *Let $f\colon S \to \mathbb{R}$ be uniformly continuous, and let (x_n) be a Cauchy sequence in S. Then $\big(f(x_n)\big)$ is also a Cauchy sequence.*

Proof. Let $\epsilon > 0$, and take $\delta > 0$ such that

$$x, y \in S \text{ and } |x - y| < \delta \quad \implies \quad \big|f(x) - f(y)\big| < \epsilon.$$

Since (x_n) is Cauchy, there is N such that

$$|x_n - x_{n'}| \ < \ \delta \quad \text{for every } n, n' \geq N.$$

Then $\big|f(x_n) - f(x_{n'})\big| < \epsilon$ for every $n, n' \geq N$. Thus $\big(f(x_n)\big)$ is Cauchy. □

Remark 3.3.5. Suppose $f\colon S \to \mathbb{R}$ is a continuous function but *not* uniformly continuous. Then f still preserves many Cauchy sequences, just maybe not all of them. Indeed, the definition of continuity implies that if $x_n \to x_0$ where both the sequence (x_n) and the limit x_0 are in S, then $f(x_n) \to f(x_0)$. In particular, f preserves the Cauchy sequence (x_n). The Cauchy sequences which f might not preserve are the ones whose limits are not in S (see next example).

Lemma 3.3.4 gives us an easy method for showing that a function is *not* uniformly continuous:

Example 3.3.6. Returning to our Example 3.3.1, we have $(1/n)$ is a Cauchy sequence in $\mathbb{R}^{>0}$, but $\big(f(1/n)\big) = (n)$ is *not* a Cauchy sequence. Thus $f(x) = 1/x$ is not uniformly continuous as a function $\mathbb{R}^{>0} \to \mathbb{R}^{>0}$.

The following shows that a large class of continuous functions are also automatically uniformly continuous:

Proposition 3.3.7. *A continuous function on a closed bounded interval*

$$f\colon [a,b] \to \mathbb{R}$$

is uniformly continuous.

Proof. Suppose towards a contradiction that f is not uniformly continuous. Then (by negating the definition of "uniformly continuous"), there is an $\epsilon > 0$ such that for each $\delta > 0$, the implication

$$\text{“}|x-y| < \delta \Longrightarrow \big|f(x) - f(y)\big| < \epsilon\text{”}$$

fails for some $x, y \in [a,b]$. Fix such an ϵ. Then for each $n \in \mathbb{N}$, there are $x_n, y_n \in [a,b]$ such that

$$|x_n - y_n| < \frac{1}{n} \quad \text{but} \quad \big|f(x_n) - f(y_n)\big| \geq \epsilon.$$

By Bolzano-Weierstrass 2.3.7, some subsequence (x_{n_k}) of (x_n) converges to some $x_0 \in [a,b]$. Since $|x_{n_k} - y_{n_k}| < 1/n_k$, we also have $y_{n_k} \to x_0$ (by a homework problem). Since f is continuous at x_0,

$$\lim_{k\to\infty} f(x_{n_k}) \;=\; f(x_0) \;=\; \lim_{k\to\infty} f(y_{n_k}),$$

and so

$$\lim_{k\to\infty} \big(f(x_{n_k}) - f(y_{n_k})\big) \;=\; 0.$$

This is a contradiction since $\big|f(x_{n_k}) - f(y_{n_k})\big| \geq \epsilon$ for every k. □

3.4. Power functions

We now return to our investigation of the power functions $b^x\colon \mathbb{R} \to \mathbb{R}$. We are now able to finish the Uniqueness part of the proof of the Real Power Theorem 1.6.5. Along the way[1] we also show that $b^x\colon \mathbb{R} \to \mathbb{R}$ is continuous.

Proposition 3.4.1. *Fix $b > 1$ and suppose $\tilde{p}\colon \mathbb{R} \to \mathbb{R}$ is a function such that*

1. *$\tilde{p}(1) = b$,*
2. *for every $x, y \in \mathbb{R}$, $\tilde{p}(x+y) = \tilde{p}(x)\tilde{p}(y)$,*
3. *for every $x, y \in \mathbb{R}$, if $x < y$, then $\tilde{p}(x) < \tilde{p}(y)$.*

Then $\tilde{p}$ is continuous and $\tilde{p}(x) = b^x$ for every $x \in \mathbb{R}$. In particular, $b^x\colon \mathbb{R} \to \mathbb{R}$ is continuous.

Proof. By assumptions (1) and (2) and Exercise 1.9.5, we know that $b^q = \tilde{p}(q)$ for every $q \in \mathbb{Q}$. By Exercise 3.5.6, it suffices to show that $\tilde{p}(x)$ is continuous at 0. Let $\epsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $(b-1)/n_0 < \epsilon$. Set $\delta := 1/n_0$. Let $x \in \mathbb{R}$ be such that $|x| < \delta$. Suppose first that $x \geq 0$. Then $x < 1/n_0$, so

$$|\tilde{p}(x) - 1| \;\leq\; |\tilde{p}(1/n_0) - 1| \;=\; |b^{1/n_0} - 1| \;\leq\; \frac{b-1}{n_0} \;<\; \epsilon,$$

[1] Side question: Is there a proof of Uniqueness that avoids the concept of continuity? It seems Question 1.6.1 is a good motivation for why we need to develop continuity.

since $\tilde{p}$ is strictly increasing and agrees with b^x on the rationals and using the Inequality from 1.6.2. Next, suppose that $x < 0$. Then

$$|\tilde{p}(x) - 1| \;=\; \underbrace{|-\tilde{p}(x)|}_{\leq 1} \cdot |\tilde{p}(-x) - 1| \;\leq\; |\tilde{p}(-x) - 1| \;<\; \epsilon,$$

using the above argument for $0 \leq -x < \delta$. We conclude that $\tilde{p}(x)$ is continuous at 0, and thus continuous everywhere. This argument also shows that $b^x \colon \mathbb{R} \to \mathbb{R}$ is continuous as well. □

In particular, the function $\mathrm{e}^x \colon \mathbb{R} \to \mathbb{R}$ is continuous. We also observe the following:

Proposition 3.4.2. *Fix $b > 1$. Then the function $b^x \colon \mathbb{R} \to \mathbb{R}$ has the following properties:*

(1) $\lim_{x\to\infty} b^x = \infty$,
(2) $\lim_{x\to-\infty} b^x = 0$,
(3) $\mathrm{Range}(b^x) = \{b^x : x \in \mathbb{R}\} = (0, +\infty)$.

Proof. (1) Let $M > 0$ be arbitrary. Since $b - 1 > 0$, there is $n \in \mathbb{N}$ such that $b - 1 > (M-1)/n$. Applying Bernoulli's Inequality A.3.1 with $x := b - 1$ yields

$$b^n \;=\; \big(1 + (b-1)\big)^n \;\geq\; 1 + n(b-1) \;>\; M.$$

Now suppose $x > n$ is arbitrary. Then since $b^x \colon \mathbb{R} \to \mathbb{R}$ is strictly increasing, $b^x > b^n > M$. Thus $\lim_{x\to\infty} b^x = \infty$.

(2) Let $\epsilon < 0$. By (1), there is $N \in \mathbb{N}$ such that if $x > N$, then $b^x > 1/\epsilon$. Thus for $x < -N$, $b^{-x} > 1/\epsilon$, so taking reciprocals yields $b^x < \epsilon$. Thus $\lim_{x\to-\infty} b^x = 0$.

(3) follows from the Intermediate Value Theorem 3.2.10 and (1) and (2). □

3.5. Exercises

Exercise 3.5.1. Find the following limits or determine that they do not exist:

(1) $\lim_{x\to 0} |x|$
(2) $\lim_{x>0, x\to 0} \frac{x+\sqrt{x}}{2+\sqrt{x}}$
(3) $\lim_{x\to 0} \frac{|x|^2}{x}$
(4) $\lim_{x\to 0} \frac{1}{x}$

Exercise 3.5.2. Suppose the function $f \colon \mathbb{R} \to \mathbb{R}$ has the property that there is some $M > 0$ such that

$$\big|f(x)\big| \;\leq\; M|x|^2 \quad \text{for all } x.$$

Prove that

$$\lim_{x\to 0} f(x) \;=\; 0 \quad \text{and} \quad \lim_{x\to 0} \frac{f(x)}{x} \;=\; 0.$$

Exercise 3.5.3. Let $k \in \mathbb{N}$. Prove that

$$\lim_{x\to 1} \frac{x^k - 1}{x - 1} \;=\; k.$$

Exercise 3.5.4. For each of the following statements, determine whether it is true or false and justify your answer:

(1) If the function $f + g \colon \mathbb{R} \to \mathbb{R}$ is continuous, then the functions $f \colon \mathbb{R} \to \mathbb{R}$ and $g \colon \mathbb{R} \to \mathbb{R}$ are also continuous.
(2) If the function $f^2 \colon \mathbb{R} \to \mathbb{R}$ is continuous, then so is the function $f \colon \mathbb{R} \to \mathbb{R}$.

(3) If the functions $f+g\colon \mathbb{R} \to \mathbb{R}$ is continuous and $g\colon \mathbb{R} \to \mathbb{R}$ is continuous, then so is the function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous.
(4) Every function $f\colon \mathbb{N} \to \mathbb{R}$ is continuous.

Exercise 3.5.5 (Bump Lemma)**.** Let $g\colon S \to \mathbb{R}$ be continuous at $x_0 \in S$, and suppose $g(x_0) > 0$. Given a value $\alpha \in (0, g(x_0))$, show that there is an open interval I such that $x_0 \in I$ and $g(x_1) \geq \alpha$ for every $x_1 \in I \cap S$.

Exercise 3.5.6. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is such that for every $x, y \in \mathbb{R}$, $f(x+y) = f(x)f(y)$. Furthermore, suppose that f is continuous at 0. Show that f is continuous (i.e., f is continuous at every $x \in \mathbb{R}$).

Exercise 3.5.7. We say a subset $D \subseteq \mathbb{R}$ is **dense** if for every $a, b \in \mathbb{R}$ with $a < b$, there is $d \in D$ such that $a < d < b$ (for example, $\mathbb{Q} \subseteq \mathbb{R}$ is a dense subset of $\mathbb{R}$). Let D be a dense subset of $\mathbb{R}$. Suppose $f, g\colon \mathbb{R} \to \mathbb{R}$ are functions such that $f(q) = g(q)$ for every $q \in D$. Show the following:
(1) If f is continuous and g is monotone, then $f(x) = g(x)$ for every $x \in \mathbb{R}$.
(2) If f and g are both continuous, then $f(x) = g(x)$ for every $x \in \mathbb{R}$.

Exercise 3.5.8. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) \;:=\; \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Determine (with proof) all points $x_0 \in \mathbb{R}$ such that f is continuous at x_0.

Exercise 3.5.9. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a function that for every $x, y \in \mathbb{R}$, $f(x+y) = f(x) + f(y)$. Furthermore, suppose f is continuous at 0.
(1) Show that f is uniformly continuous (on all of $\mathbb{R}$).
(2) Give a more explicit description of the function f.

Exercise 3.5.10. A function $f\colon (a,b) \to \mathbb{R}$ is said to be **convex** if

$$f\bigl(\lambda x + (1-\lambda)y\bigr) \;\leq\; \lambda f(x) + (1-\lambda) f(y)$$

whenever $x, y \in (a,b)$ and $\lambda \in (0,1)$. Suppose $f\colon (a,b) \to \mathbb{R}$ is convex.
(1) Prove that f is continuous.
(2) Suppose $g\colon \mathbb{R} \to \mathbb{R}$ is an increasing convex function. Prove that $g \circ f\colon (a,b) \to \mathbb{R}$ is convex.
(3) Suppose $a < s < t < u < b$. Show that

$$\frac{f(t)-f(s)}{t-s} \;\leq\; \frac{f(u)-f(s)}{u-s} \;\leq\; \frac{f(u)-f(t)}{u-t}.$$

Exercise 3.5.11. Assume $f\colon (a,b) \to \mathbb{R}$ is continuous and has the property that

$$f\left(\frac{x+y}{2}\right) \;\leq\; \frac{f(x)+f(y)}{2}$$

for every $x, y \in (a,b)$. Prove that f is convex.

Exercise 3.5.12. For each of the following statements, determine whether it is true or false, and justify your answer.
(1) Every function $f\colon [0,1] \to \mathbb{R}$ has a maximum.
(2) Every continuous function $f\colon [a,b] \to \mathbb{R}$ has a minimum.
(3) Every continuous function $f\colon (a,b) \to \mathbb{R}$ has a maximum.

(4) Every continuous function $f\colon (0,1) \to \mathbb{R}$ has a bounded image (i.e., that the set $f\big((0,1)\big)$ is bounded).
(5) If a continuous function $f\colon (0,1) \to \mathbb{R}$ is such that the image is bounded below (i.e., the set $f\big((0,1)\big)$ is bounded below), then the function attains a minimum.

Exercise 3.5.13. Let $f, g\colon [a,b] \to \mathbb{R}$ be continuous functions such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove there is $x_0 \in [a,b]$ such that $f(x_0) = g(x_0)$.

Exercise 3.5.14. Consider the polynomial $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ (with $a_0, \ldots, a_n \in \mathbb{R}$). Prove that if $a_0a_n < 0$, then p has a positive root.

Exercise 3.5.15. Suppose that the function $f\colon [a,b] \to \mathbb{R}$ is continuous. For a natural number k, let $x_1, \ldots, x_k \in [a,b]$. Prove there is $z \in [a,b]$ at which:

$$f(z) \ = \ \frac{1}{k}\sum_{\ell=1}^{k} f(x_\ell).$$

Exercise 3.5.16. Let $p\colon \mathbb{R} \to \mathbb{R}$ be a polynomial of odd degree. Prove there is a solution of the equation:

$$p(x) \ = \ 0, \quad \text{where } x \in \mathbb{R}.$$

Exercise 3.5.17. Suppose $f, g\colon D \to \mathbb{R}$ are both uniformly continuous functions. Is the product $fg\colon D \to \mathbb{R}$ also uniformly continuous? Prove or give a counterexample.

Exercise 3.5.18. Suppose $f\colon (a,b) \to \mathbb{R}$ is uniformly continuous. Prove that $f\colon (a,b) \to \mathbb{R}$ is bounded.

Exercise 3.5.19. Suppose $f\colon [0,1] \to \mathbb{R}$ is continuous, with $f(0) = f(1)$. Show there is $c \in [0, 1/2]$ such that $f(c + 1/2) = f(c)$.

Exercise 3.5.20. Let $f\colon S \to \mathbb{R}$ be a uniformly continuous and bounded function. The function $\omega\colon (0, +\infty) \to \mathbb{R}$ given by

$$\omega(\delta) \ := \ \sup\big\{|f(x) - f(y)| : x, y \in S, |x - y| < \delta\big\}$$

is called the **modulus of continuity** of f. Show that ω is increasing and determine $\lim_{\delta>0,\delta\to 0} \omega(\delta)$.

CHAPTER 4

Differentiation

In Chapter 3 we encountered the magical property of *continuity*. Continuity is a *local property*[1], i.e., it is a property which is determined in a tiny neighborhood of a point. A function is globally continuous iff it is locally continuous at each point.

In this chapter we will encounter an even stronger local property: *differentiability*. This property says that in a tiny neighborhood around a point, the function can be approximated suspiciously well by a straight line. This is a much more specific type of local property than continuity. Naturally, we will be able to produce stronger results about functions which have this stronger property.

For example, if a function $f\colon [a,b] \to \mathbb{R}$ is continuous, then we know by the Extreme Value Theorem 3.2.9 that f attains a minimum and maximum *somewhere*. However, knowing this fact alone does us little good if we want to find *where* f attains this min and max. If we know in addition that f is *differentiable*, then Fermat's Theorem 4.3.1 is able to help us find the min and max by solving $f'(x) = 0$.

Throughout this chapter $I \subseteq \mathbb{R}$ is an open interval.

4.1. Differentiability and derivatives

We will actually give three equivalent definitions of differentiability at a point. As is typical when giving multiple equivalent definitions, we will first prove that three properties are equivalent, then afterwards define *differentiability* to mean any[2] one of the three equivalent properties.

Proposition 4.1.1. *Suppose $f\colon I \to \mathbb{R}$ and $a \in I$. The following are equivalent:*

1. *(Standard definition) The limit*
$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \;=\; \ell$$
exists and is finite (i.e., $\ell \in \mathbb{R}$).
2. *(Taylor definition) There exists a number $d \in \mathbb{R}$ and a function $R\colon I \to \mathbb{R}$ such that*
$$f(x) \;=\; f(a) + d(x-a) + R(x) \quad \text{and} \quad \lim_{x\to a} \frac{R(x)}{x-a} \;=\; 0.$$
3. *(Carathéodory definition) There exists a function $q\colon I \to \mathbb{R}$ which is continuous at a such that*
$$f(x) \;=\; f(a) + q(x)(x-a).$$

[1] This is actually an important mathematical theme you should take seriously.

[2] We do it in this way as to not "play favorites" with any one particular version of the definition. This is just a matter of style and taste.

Furthermore, if any (equivalently all) of (1), (2), and (3) holds, then

(4) $\ell = d = q(a)$, and

(5) f is continuous at a.

Proof. (1)⇒(2) Suppose the limit in (1) exists with limit $\ell \in \mathbb{R}$. Define the function $R\colon I \to \mathbb{R}$ by

$$R(x) \;:=\; f(x) - f(a) - \ell(x-a)$$

for $x \in I$. Then since

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \;=\; \ell,$$

it follows from subtracting ℓ from both sides that

$$0 \;=\; \lim_{x\to a} \frac{f(x)-f(a)-\ell(x-a)}{x-a} \;=\; \lim_{x\to a} \frac{R(x)}{x-a}.$$

Thus (2) holds with $d := \ell$.

(2)⇒(3) Suppose (2) holds for some $d \in \mathbb{R}$ and $R\colon I \to \mathbb{R}$. Define the function $q\colon I \to \mathbb{R}$ by

$$q(x) \;:=\; \begin{cases} d + \frac{R(x)}{x-a} & \text{if } x \neq a \\ d & \text{if } x = a, \end{cases}$$

for $x \in I$. Then by (2),

$$\lim_{x\to a} q(x) \;=\; d \;=\; q(a),$$

so q is continuous at a by Corollary 3.2.7.

(3)⇒(1) Suppose we have $q\colon I \to \mathbb{R}$ as in (3). Note that for $x \in I\setminus\{a\}$ we have

$$\frac{f(x)-f(a)}{x-a} \;=\; q(x),$$

so in particular since q is continuous at a the limit

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \;=\; \lim_{x\to a} q(x) \;=\; q(a)$$

exists. Thus (1) holds with $\ell = q(a)$.

(4) The above arguments show that if any of (1), (2) or (3) hold, they all hold and that necessarily $\ell = d = q(a)$.

(5) The representation of f as in (3) shows that f is continuous at a since each of the functions $f(a)$, $q(x)$ and $x - a$ are continuous at a. □

Note that each of (1), (2) and (3) in the above proposition is expressing that in some sense f can be well-approximated by a linear function when you are very close to a. We now define differentiability to mean any one of the three equivalent conditions in the above proposition:

Definition 4.1.2. Suppose $f\colon I \to \mathbb{R}$ and $a \in I$. We say that f is **differentiable at** a, if any of the three equivalent conditions (1), (2), or (3) in Proposition 4.1.1 hold. In the case f is differentiable at a, we write:

$$f'(a) \;:=\; \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \quad \text{(also equals } \ell = d = q(a) \text{ from Proposition 4.1.1)}$$

and we call $f'(a)$ the **derivative** of f at a. If $f\colon I \to \mathbb{R}$ is differentiable at every $a \in I$, then we say f is **differentiable** (on I).

Example 4.1.3. (1) Constant functions are differentiable with derivative 0.

(2) Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that $f(x) = x^n$. Then f is differentiable at all $a \in \mathbb{R}$, and $f'(a) = na^{n-1}$. To see this, note by The Difference of Powers Formula,

$$f(x) - f(a) \;=\; x^n - a^n \;=\; (x-a)\cdot(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}),$$

thus for $x \neq a$, we have

$$\frac{f(x)-f(a)}{x-a} \;=\; x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1},$$

and so

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \;=\; n \cdot a^{n-1}.$$

4.2. Differentiation rules

In this section we prove the usual differentiation rules.

Proposition 4.2.1. *Suppose $f, g\colon I \to \mathbb{R}$ are differentiable at $a \in I$. Then $f + g, f \cdot g\colon I \to \mathbb{R}$ are differentiable at a, with*

(1) $(f+g)'(a) = f'(a) + g'(a)$,

(2) (product rule) $(f \cdot g)'(a) = f(a)g'(a) + f'(a)g(a)$,

and if $g(a) \neq 0$, then $f/g\colon I \to \mathbb{R}$ is differentiable at a with

(3) (quotient rule)

$$\left(\frac{f}{g}\right)'(a) \;=\; \frac{g(a)f'(a) - f(a)g'(a)}{g^2(a)}$$

Proof. For (1) and (2) we will use the Carathéodory definition, so we write

$$f(x) \;=\; f(a) + q(x)(x-a)$$
$$g(x) \;=\; g(a) + r(x)(x-a),$$

where $q, r\colon I \to \mathbb{R}$ are continuous at a and $f'(a) = q(a)$ and $g'(a) = r(a)$.

For (1) we have

$$(f+g)(x) \;=\; (f+g)(a) + (q+r)(x)(x-a).$$

By Lemma 3.1.7 the function $q + r\colon I \to \mathbb{R}$ is also continuous at a and thus $f + g$ is differentiable at a and

$$(f+g)'(a) \;=\; (q+r)(a) \;=\; q(a) + r(a) \;=\; f'(a) + g'(a).$$

(2) Note that for $x \in I$,

$$\begin{aligned}(fg)(x) \;=\; f(x)\cdot g(x) \;&=\; \big(f(a) + q(x)(x-a)\big)\big(g(a) + r(x)(x-a)\big)\\ &=\; f(a)g(a) + \underbrace{\big(f(a)r(x) + q(x)g(a) + q(x)r(x)(x-a)\big)}_{p(x)}(x-a)\\ &=\; (fg)(a) + p(x)(x-a),\end{aligned}$$

and by Lemma 3.1.7 it follows that $p(x)$ is continuous at a. Thus $f \cdot g$ is differentiable at a and

$$(fg)'(a) \;=\; p(a) \;=\; f(a)r(a) + q(a)g(a) \;=\; f(a)g'(a) + f'(a)g(a).$$

(3) We will use the Standard definition of differentiable and prove the quotient rule first in the special case that $f \equiv 1$ (i.e., f is the constant function 1). Now, suppose $g(a) \neq 0$. Then by the Bump Lemma (Exercise 3.5.5), there is an open

interval $J \subseteq I$ such that $a \in J$ and $g(x) \neq 0$ for all $x \in J$. Thus, for $x \in J \setminus \{a\}$, we have

$$\frac{(1/g)(x) - (1/g)(a)}{x - a} \;=\; \frac{g(a) - g(x)}{x - a} \cdot \frac{1}{g(x)g(a)},$$

and taking limits as $x \to a$ (specifically, the limit $\lim_{x \to a}$ which is shorthand here for $\lim_{x \in J \setminus \{a\}, x \to a}$) gives

$$(1/g)'(a) \;=\; -\frac{g'(a)}{g(a)^2}.$$

The general quotient rule now follows from combining what we just proved with product rule, i.e., by viewing f/g as the product $f \cdot (1/g)$. □

Remark 4.2.2. An immediate consequence of Proposition 4.2.1(1) and (2) is that if we have constants $c, d \in \mathbb{R}$ and differentiable functions $f, g\colon I \to \mathbb{R}$, then

$$(cf + dg)' \;=\; cf' + dg'.$$

In linear algebra terms, differentiation is $\mathbb{R}$-*linear* (i.e., it is a linear transformation on the $\mathbb{R}$-vector space of differentiable functions $I \to \mathbb{R}$).

One of the advantages of the Carathéodory definition of differentiability is that it allows for an elegant proof of the chain rule:

The Chain Rule 4.2.3. *Let $f\colon I \to \mathbb{R}$, $g\colon J \to \mathbb{R}$ such that $J \subseteq \mathbb{R}$ is an open interval, $f(I) \subseteq J$, and $a \in I$. Suppose f is differentiable at a and g is differentiable at $f(a)$. Then $g \circ f$ is differentiable at a, with*

$$(g \circ f)'(a) \;=\; g'\big(f(a)\big) \cdot f'(a).$$

Proof. By assumption, there are functions $q\colon I \to \mathbb{R}$ and $r\colon J \to \mathbb{R}$ continuous at a and $f(a)$ respectively, such that

$$\begin{aligned} f(x) \;&=\; f(a) + q(x)(x - a) \quad \text{for } x \in I \\ g(x) \;&=\; g\big(f(a)\big) + r(x)\big(x - f(a)\big) \quad \text{for } x \in J. \end{aligned}$$

Then since $f(I) \subseteq J$, we have for all $x \in I$:

$$\begin{aligned} (g \circ f)(x) \;=\; g\big(f(x)\big) \;&=\; g\big(f(a)\big) + r\big(f(x)\big)\big(f(x) - f(a)\big) \\ &=\; (g \circ f)(x) + \underbrace{r\big(f(x)\big)q(x)}_{s(x)}(x - a). \end{aligned}$$

The function $s\colon I \to \mathbb{R}$ is continuous at a because q is continuous at a and by Proposition 3.2.4 the composition $r\big(f(x)\big)$ is continuous at a (using that f is continuous at a since it is differentiable at a). Thus $g \circ f$ is differentiable at a with

$$(g \circ f)'(a) \;=\; r\big(f(a)\big)q(a) \;=\; g'\big(f(a)\big)f'(a).$$ □

4.3. Differentiation theorems

Fermat's Theorem 4.3.1. *Suppose a function $f\colon (a, b) \to \mathbb{R}$ assumes a maximum or minimum at a point $x_0 \in (a, b)$. Then either $f'(x_0) = 0$ or f is not differentiable at x_0.*

PROOF. We may assume that f assumes a maximum at x_0 (if f assumes a minimum at x_0, then apply this argument to the function $-f$). Thus $f(x) \leq f(x_0)$ for all $x \in (a,b)$. Now let (x_n) be a sequence in (a,b) such that $x_n \to x_0$ and $x_n \neq x_0$ for all n. Thus $f(x_0) \geq f(x_n)$ for all n. It follows that whenever $x_0 > x_n$, then

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} \;\geq\; 0$$

and whenever $x_0 < x_n$, then

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} \;\leq\; 0.$$

In particular, if $f'(x_0)$ exists, then necessarily $f'(x_0) = 0$. □

Rolle's Theorem 4.3.2. *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function which is differentiable on (a,b) such that $f(a) = f(b)$. Then there is some $c \in (a,b)$ such that $f'(c) = 0$.*

PROOF. By the Extreme Value Theorem 3.2.9, there are $x_m, x_M \in [a,b]$ such that $f(x_m) \leq f(x) \leq f(x_M)$ for every $x \in [a,b]$.

If x_m, x_M are both endpoints of $[a,b]$, then f is a constant function since $f(a) = f(b)$. Thus $f'(c) = 0$ for *every* $c \in [a,b]$.

Otherwise, f assumes a maximum or minimum at some point $c \in (a,b)$, and so $f'(c) = 0$ by Fermat's Theorem 4.3.1. □

Mean Value Theorem 4.3.3. *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function which is differentiable on (a,b). Then there exists $c \in (a,b)$ such that*

$$f'(c) \;=\; \frac{f(b) - f(a)}{b - a}.$$

PROOF. Define $m := (f(b) - f(a))/(b - a)$. Define

$$L(x) \;:=\; f(a) + m(x - a),$$

the line that connects the endpoints $(a, f(a))$ and $(b, f(b))$ of the graph of f. Then

$$L(a) = f(a), \quad L(b) = f(b), \quad L'(x) = m.$$

Now define the function $g := f - L$. Then g is continuous on $[a,b]$ and differentiable on (a,b) with

$$g(a) \;=\; 0 \;=\; g(b).$$

Thus by Rolle's Theorem 4.3.2, there is $c \in (a,b)$ such that $g'(c) = 0$. Thus

$$f'(c) \;=\; L'(c) \;=\; m.$$ □

Corollary 4.3.4. *Suppose $f\colon (a,b) \to \mathbb{R}$ is differentiable. Then f is a constant function iff $f'(x) = 0$ for all $x \in (a,b)$.*

PROOF. ($\Rightarrow$) This follows from Example 4.1.3.

($\Leftarrow$) Suppose towards a contradiction that f is not constant on (a,b). Then there are x_1, x_2 such that $a < x_1 < x_2 < b$ and $f(x_1) \neq f(x_2)$. By the Mean Value Theorem 4.3.3, there is $x_3 \in (x_1, x_2)$ such that

$$f'(x_3) \;=\; \frac{f(x_2) - f(x_1)}{x_2 - x_1} \;\neq\; 0,$$

which contradicts the assumption that $f'(x) = 0$ for all $x \in (a,b)$. □

A common question in analysis is: when are two functions $f, g\colon I \to \mathbb{R}$ the same? If f and g are differentiable, the following makes this question easier to answer:

The Identity Criterion 4.3.5. *Let $f, g\colon (a,b) \to \mathbb{R}$ be differentiable such that $f' = g'$ on (a,b). Then there exists a constant $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for all $x \in (a,b)$. Furthermore, if there is a point $x_0 \in (a,b)$ such that $f(x_0) = g(x_0)$, then $f(x) = g(x)$ for all $x \in (a,b)$.*

Proof. The function $f - g\colon (a,b) \to \mathbb{R}$ is differentiable by Proposition 4.2.1, and $(f-g)'(x) = f'(x) - g'(x) = 0$ for all $x \in (a,b)$. By Corollary 4.3.4, there is a constant $c \in \mathbb{R}$ such that $(f-g)(x) = c$ for all $x \in (a,b)$, i.e., $f(x) = g(x) + c$ for all $x \in (a,b)$.

Now, suppose there is $x_0 \in (a,b)$ such that $f(x_0) = g(x_0)$. Then also $f(x_0) = g(x_0) + c$, so we can conclude that $c = 0$. Thus $f(x) = g(x)$ for all $x \in (a,b)$. □

The Identity Criterion 4.3.5 is the impetus for the *uniqueness* part of many *existence and uniqueness* theorems for ordinary differential equations (ODEs).

Up until this point if we want to show that a function $f\colon I \to \mathbb{R}$ is increasing, we usually have to give some tedious argument involving inequalities and identities. If we know that f is differentiable, then the following provides an easier method of proving that f is increasing (or decreasing, etc.).

Corollary 4.3.6. *Let $f\colon (a,b) \to \mathbb{R}$ be a differentiable function. Then*

(1) f is strictly increasing if $f'(x) > 0$ for all $x \in (a,b)$,
(2) f is strictly decreasing if $f'(x) < 0$ for all $x \in (a,b)$,
(3) f is increasing if $f'(x) \geq 0$ for all $x \in (a,b)$, and
(4) f is decreasing if $f'(x) \leq 0$ for all $x \in (a,b)$.

Proof. (1) Let x_1, x_2 be such that $a < x_1 < x_2 < b$. By the Mean Value Theorem 4.3.3, for some $x \in (x_1, x_2)$ we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \;=\; f'(x) \;>\; 0.$$

Since $x_2 - x_1 > 0$, we get $f(x_2) - f(x_1) > 0$ which implies $f(x_2) > f(x_1)$.

(2), (3), and (4) are similar and left as an exercise to the reader. □

4.4. The exponential function

We now return to our study of the various exponential functions $\Phi(x)$, $\exp(x)$, and e^x. Ultimately we will show these three functions are the same. As a warmup, we will show that Φ is differentiable:

Proposition 4.4.1. *The function $\Phi(x)$ has the following properties:*

(1) $\Phi\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function and $\Phi'(x) = \Phi(x)$ for all $x \in \mathbb{R}$.
(2) In particular, by Proposition4.1.1 (5), $\Phi\colon \mathbb{R} \to \mathbb{R}$ is continuous.

Proof. We only need to prove (1). By Exercise 4.6.5, it suffices to show that Φ is differentiable at 0 and that $\Phi'(0) = 1$. Suppose $h \in \mathbb{R}$ is such that $h \neq 0$ and $|h| < 1$. Then by inequality (b) from Definition 2.5.2, we have

$$1 + h \;\leq\; \Phi(h) \;\leq\; \frac{1}{1-h} \;=\; 1 + \frac{h}{1-h}.$$

Subtracting 1 and dividing by h yields

$$1 \le \frac{\Phi(h) - 1}{h} \le \frac{1}{1-h}.$$

Taking a limit as $h \to 0$, the Squeeze Lemma implies

$$\lim_{h\to 0} \frac{\Phi(h) - 1}{h} = 1.$$

Thus Φ is differentiable at 0 (since $\Phi(0) = 1$), and $\Phi'(0) = 1 = \Phi(0)$. □

We can now conclude from Propositions 4.4.1, 3.4.1, and Exercise 3.5.7(2) the following:

Corollary 4.4.2. *For every* $x \in \mathbb{R}$, $\Phi(x) = \mathrm{e}^x$.

4.5. Derivatives of higher order

So far, we have only studied the *first derivative* of a function f. We will occasionally need *higher derivatives* of a function (which might not exist).

Definition 4.5.1. Set $f^{(0)} := f$ (the zeroeth derivative is the function itself). Then for $n \ge 1$, and $x \in I$, suppose that $f^{(n-1)} \colon I \to \mathbb{R}$ exists. If $f^{(n-1)}$ is differentiable at x, then we define

$$f^{(n)}(x) := (f^{(n-1)})'(x) \quad \text{the } n\textbf{th derivative of } f \textbf{ at } x$$

If $f^{(n-1)} \colon I \to \mathbb{R}$ is differentiable at every $x \in I$, then we set

$$f^{(n)} := (f^{(n-1)})' \colon I \to \mathbb{R},$$

and we call $f^{(n)}$ the n**th derivative of** f. In this case we also say that $f^{(n)}$ **exists**.

Example 4.5.2. By Proposition 4.4.1, for every $n \ge 0$, $\Phi^{(n)} \colon \mathbb{R} \to \mathbb{R}$ exists and $\Phi^{(n)}(x) = \Phi(x)$ for every $x \in \mathbb{R}$ (this follows from an easy induction argument).

4.6. Exercises

Exercise 4.6.1. Suppose that the function $f \colon \mathbb{R} \to \mathbb{R}$ is differentiable at 0. For real numbers $a, b, c \in \mathbb{R}$ such that $c \neq 0$, determine (with proof!) the limit:

$$\lim_{x\to 0} \frac{f(ax) - f(bx)}{cx}$$

Exercise 4.6.2. A function $f \colon \mathbb{R} \to \mathbb{R}$ is called **even** if

$$f(x) = f(-x) \quad \text{for all } x,$$

and $f \colon \mathbb{R} \to \mathbb{R}$ is called **odd** if

$$f(x) = -f(-x) \quad \text{for all } x.$$

Prove that if $f \colon \mathbb{R} \to \mathbb{R}$ is differentiable and odd, then $f' \colon \mathbb{R} \to \mathbb{R}$ is even.

Exercise 4.6.3. Let $n \in \mathbb{N}$.

(1) Suppose $f \colon \mathbb{R} \to \mathbb{R}$ is differentiable and the equation $f'(x) = 0$ has at most $n - 1$ solutions. Prove that the equation $f(x) = 0$ has at most n solutions.
(2) Consider the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, \ldots, a_n \in \mathbb{R}$ and $a_n \neq 0$. Prove that the equation $p(x) = 0$ has at most n solutions.

Exercise 4.6.4. Suppose $S, C\colon \mathbb{R} \to \mathbb{R}$ are differentiable and that

(1) $S'(x) = C(x)$ and $C'(x) = -S(x)$ for all $x \in \mathbb{R}$, and
(2) $S(0) = 0$ and $C(0) = 1$.

Prove that $S^2(x) + C^2(x) = 1$ for all $x \in \mathbb{R}$.

Exercise 4.6.5. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ has the property that for every $x, y \in \mathbb{R}$, $f(x+y) = f(x)f(y)$. Furthermore, assume that f is differentiable at 0. Prove that f is differentiable and that for every $x \in \mathbb{R}$, $f'(x) = f'(0) \cdot f(x)$.

Exercise 4.6.6. Suppose $f, g\colon I \to \mathbb{R}$ are functions such that for some $n \geq 0$, $f^{(k)}, g^{(k)}$ exists for $k = 0, 1, \ldots, n$. Let $x \in I$ and show that

$$(fg)^{(n)}(x) \;=\; \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x).$$

Exercise 4.6.7. Let $b > 1$. Show that the power function $b^x\colon \mathbb{R} \to \mathbb{R}$ is differentiable and describe its derivative as explicitly as possible. [Hint: this is easy if you appropriately use everything we proved so far about Φ and real power functions.]

Exercise 4.6.8. Suppose $f, g\colon \mathbb{R} \to \mathbb{R}$ are functions which are differentiable at x_0, $g'(x_0) \neq 0$, and $f(x_0) = g(x_0) = 0$. Prove that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} \;=\; \frac{f'(x_0)}{g'(x_0)}.$$

Exercise 4.6.9. Suppose $f\colon [a, b] \to \mathbb{R}$ is a continuous function, which is differentiable on (a, b). Furthermore, suppose $f(a) = f(b) = 0$, but f is not the constant zero function. Show there is $c \in (a, b)$ such that $f'(c) = f(c)$. [Hint: Apply Rolle's Theorem to an appropriate auxiliary function, consider using the most important function in mathematics.]

CHAPTER 5

Integration

In this chapter we will construct the *Darboux integral*. Among other things, this integral gives us a suitable inverse operation to differentiation. The development of the Darboux integral is different from the *Riemann integral* (the style of integral usually encountered in freshman calculus). There is actually no harm in taking this route in light of the theorem[1] which says that the Darboux integral and the Riemann integral are actually the same thing [**2**, 32.9].

One of the reasons that the Darboux integral is not taught in freshman calculus is that it is a bit more abstract (it's definition involves taking a supremum!). For us however, this is a feature and not a bug. The effort we put in in Chapter 1 will now bear fruit in the form of an efficient development of a transparent and robust theory of integration.

We previously remarked that both continuity and differentiability are *local properties*. The property of *integrability* which we will study in this chapter is not a local property (i.e., there is no such property "integrable at a point" such that a function is globally integrable iff it is integrable at every point). This sets integrability apart from continuity and differentiability on both logical and philosophical grounds and it requires us to take a "global" approach from the very beginning.

Throughout this chapter, $a, b \in \mathbb{R}$ and $a < b$. We will focus our attention on functions defined on the closed bounded interval $[a, b]$.

5.1. Partitions, Darboux sums, and the Darboux integral

In this section, we fix a bounded function

$$f : [a, b] \to \mathbb{R}.$$

Notation 5.1.1. For $S \subseteq [a, b]$, we set

$$\begin{aligned} M(f, S) &:= \sup f(S), \\ m(f, S) &:= \inf f(S). \end{aligned}$$

Definition 5.1.2. A **partition** of $[a, b]$ is a finite set

$$P = \{t_0, \ldots, t_n\} \quad (n \geq 1)$$

where $a = t_0 < t_1 < \cdots < t_n = b$. We also define the **size** of the partition to be $|P| = n$ (number of subintervals).

[1]We will not prove this theorem in this course.

Given such a partition $P = \{t_0, \dots, t_n\}$ of $[a, b]$, we define

$$U(f, P) := \sum_{k=1}^{n} M\big(f, [t_{k-1}, t_k]\big) \cdot (t_k - t_{k-1})$$
$$L(f, P) := \sum_{k=1}^{n} m\big(f, [t_{k-1}, t_k]\big) \cdot (t_k - t_{k-1}).$$

We call $U(f, P)$ and $L(f, P)$ the **upper** (respectively, **lower**) **Darboux sum of** f **with respect to** P.

Remark 5.1.3. Given a partition P of $[a, b]$, we have

$$m\big(f, [a, b]\big) \cdot (b - a) \le L(f, P) \le U(f, P) \le M\big(f, [a, b]\big) \cdot (b - a).$$

Partition Refinement Lemma 5.1.4. *Suppose $f : [a, b] \to \mathbb{R}$ is bounded, and P, Q are partitions of $[a, b]$ such that $P \subseteq Q$ (i.e., "Q is a refinement of P"). Then*

$$L(f, P) \le L(f, Q) \le U(f, Q) \le U(f, P).$$

Proof. The middle inequality is obvious. We will prove only the first inequality $L(f, P) \le L(f, Q)$. The third inequality $U(f, Q) \le U(f, P)$ is similar and left as an exercise.

We will prove this by induction on $d := |Q| - |P|$.

If $d = 0$, then $P = Q$, so $L(f, P) = L(f, Q)$.

For the inductive step, suppose $d \ge 1$ and that we know the claim is true when the difference in partition sizes is $d - 1 \ge 0$. Suppose $d = |Q| - |P|$. Take $t \in Q \setminus P$, and define $Q' := Q \setminus \{t\}$. Then Q' is also a partition of $[a, b]$, $P \subseteq Q' \subseteq Q$ and $|Q'| - |P| = d - 1$, and $|Q| - |Q'| = 1$. By the inductive hypothesis, it is enough to show that $L(f, Q') \le L(f, Q)$ (this is basically proving the $d = 1$ case). Thus we can assume:

$$Q' = \{t_0, \dots, t_n\} \quad (a = t_0 < t_1 < \cdots < t_n = b)$$
$$Q = \{t_0, \dots, t_{k-1}, s, t_k, \dots, t_n\} \quad (t_{k-1} < s < t_k, k \in \{1, \dots, n\})$$

Now we have

$$L(f, Q) - L(f, Q') = m\big(f, [t_{k-1}, s]\big) \cdot (s - t_{k-1}) + m\big(f, [s, t_k]\big) \cdot (t_k - s) - m\big(f, [t_{k-1}, t_k]\big) \cdot (t_k - t_{k-1})$$

(by cancelling everything that $L(f, Q)$ and $L(f, Q')$ have in common). However,

$$m\big(f, [t_{k-1}, t_k]\big) \cdot (t_k - t_{k-1}) = m\big(f, [t_{k-1}, t_k]\big) \cdot \big((t_k - s) + (s - t_{k-1})\big)$$
$$\le m\big(f, [s, t_k]\big) \cdot (t_k - s) + m\big(f, [t_{k-1}, s]\big) \cdot (s - t_{k-1}),$$

and so $L(f, Q) - L(f, Q') \ge 0$. □

Definition 5.1.5. We define

$$U(f) := \inf\big\{U(f, P) : P \text{ is a partition of } [a, b]\big\}$$
(**upper Darboux integral of** f)

$$L(f) := \sup\big\{L(f, P) : P \text{ is a partition of } [a, b]\big\}$$
(**lower Darboux integral of** f)

Corollary 5.1.6. *If P, Q are partitions of $[a, b]$, then*

$$L(f, P) \ \le\ U(f, Q).$$

Thus, $L(f) \le U(f)$.

PROOF. First, suppose P and Q are arbitrary partitions of $[a, b]$. Then $P \cup Q$ is a common refinement of both P and Q. Using the Partition Refinement Lemma 5.1.4 in two different ways we get

$$L(f, P) \ \le\ L(f, P \cup Q) \ \le\ U(f, P \cup Q) \ \le\ U(f, Q).$$

Thus:

$$\big\{L(f, P) : P \text{ is a partition of } [a, b]\big\} \ \le\ \big\{U(f, Q) : Q \text{ is a partition of } [a, b]\big\}$$

(i.e., every element of the set on the left is less than or equal to every element of the set on the right). By Exercise 1.9.21, this implies that

$$\begin{aligned} L(f) \ &=\ \sup\big\{L(f, P) : P \text{ is a partition of } [a, b]\big\} \\ &\le\ \inf\big\{U(f, Q) : Q \text{ is a partition of } [a, b]\big\} \ =\ U(f). \end{aligned}$$

□

Definition 5.1.7. We say that $f\colon [a, b] \to \mathbb{R}$ is **integrable** on $[a, b]$ if $L(f) = U(f)$. In this case we write either $\int_a^b f$ or $\int_a^b f(x)\, dx$ for this common value, i.e.,

$$\int_a^b f \ =\ \int_a^b f(x)\, dx \ =\ L(f) \quad \text{(which also equals } U(f))$$

This value is known as the **Darboux integral** of f. If $f\colon [a, b] \to \mathbb{R}$ is integrable, then it will also be convenient to define

$$\int_a^a f := 0 \quad \text{and} \quad \int_b^a f := -\int_a^b f.$$

Example 5.1.8.

(1) The constant function $c\colon [a, b] \to \mathbb{R}$ defined by $c(x) = c$ for some $c \in \mathbb{R}$ is integrable, and $\int_a^b c = c(b - a)$. To see this, note that for any nonempty $S \subseteq [a, b]$ we have $M(c, S) = m(c, S) = c$. Thus for any partition P of $[a, b]$ we have $L(c, P) = U(c, P) = c(b - a)$ (by writing out the relevant summation, pulling out c, and telescoping). Thus $L(c) = U(c) = c(b - a)$.

(2) The function $f\colon [0, b] \to \mathbb{R}$ given by $f(x) = x^2$ is integrable and $\int_0^b f = b^3/3$. To prove this directly (without the Fundamental Theorem of Calculus) take a little bit of work, but is not too hard. Given a partition $P = \{0 = t_0 < t_1 < \cdots < t_n = b\}$, we have

$$U(f, P) \ =\ \sum_{k=1}^{n} \sup\{x^2 : x \in [t_{k-1}, t_k]\} \cdot (t_k - t_{k-1}) \ =\ \sum_{k=1}^{n} t_k^2 (t_k - t_{k-1}).$$

If we choose the "regular" partition where $t_k = kb/n$, then the Sum Of Squares Formula A.2.2 shows that

$$U(f, P) \ =\ \sum_{k=1}^{n} \frac{k^2 b^2}{n} \left(\frac{b}{n}\right) \ =\ \frac{b^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}.$$

For very large n, this quantity can be made arbitrarily close to $\frac{b^3}{3}$, so $U(f) \leq \frac{b^3}{3}$. Using the same regular partition, we also get

$$L(f,P) \;=\; \sum_{k=1}^{n} \frac{(k-1)^2 b^2}{n^2}\left(\frac{b}{n}\right) \;=\; \frac{b^3}{n^3}\cdot\frac{(n-1)\cdot n\cdot(2n-1)}{6}$$

which shows that $L(f) \geq \frac{b^3}{3}$. Thus $L(f) = U(f)$, so f is integrable and $\int_0^b f = \frac{b^3}{3}$.

(3) Consider the function $f\colon [a,b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \text{ and} \\ 1 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$$

Then f is *not* integrable. Note that for any nonempty subinterval $S \subseteq [a,b]$, we have $m(f,S) = 0$ and $M(f,S)$. Thus for any partition P of $[a,b]$ we have $L(f,P) = 0$ and $U(f,P) = (b-a)$, and so

$$L(f) \;=\; 0 \;<\; (b-a) \;=\; U(f).$$

The following is very useful in practice for showing that a function is integrable:

Cauchy Criterion for Integrability 5.1.9. *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. The following are equivalent:*

(1) f is integrable,

(2) for every $\epsilon > 0$ there is a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) \;<\; \epsilon.$$

Proof. (1) ⇒ (2) Suppose f is integrable and let $\epsilon > 0$. Take partitions P_1, P_2 of $[a,b]$ such that

$$\begin{aligned} L(f,P_1) \;&>\; L(f) - \frac{\epsilon}{2} \\ U(f,P_2) \;&<\; U(f) + \frac{\epsilon}{2}. \end{aligned}$$

Now, set $P := P_1 \cup P_2$. By the Partition Refinement Lemma 5.1.4, we have

$$\begin{aligned} U(f,P) - L(f,P) \;&\leq\; U(f,P_2) - L(f,P_1) \\ &<\; \big(U(f) + \frac{\epsilon}{2}\big) - \big(L(f) - \frac{\epsilon}{2}\big) \\ &=\; U(f) - L(f) + \epsilon \;=\; \epsilon. \end{aligned}$$

(2) ⇒ (1) Suppose (2) holds, and suppose towards a contradiction that $U(f) > L(f)$. Set $\epsilon := U(f) - L(f) > 0$. Take a partition P of $[a,b]$ as in (2), then

$$U(f,P) - L(f,P) \;<\; \epsilon \;=\; U(f) - L(f) \;\leq\; U(f,P) - L(f,P),$$

a contradiction. □

5.2. Properties of the Darboux integral

Definition 5.2.1. The **mesh** of a partition P is the maximum length of the subintervals in P, i.e., if $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$, then

$$\operatorname{mesh}(P) \;:=\; \max\{t_k - t_{k-1} : k = 1,2,\ldots,n\}.$$

Proposition 5.2.2 (Monotonic functions are integrable)**.** *If $f\colon [a,b] \to \mathbb{R}$ is monotonic, then f is integrable.*

PROOF. We will only do the case where f is increasing (the case where f is decreasing is similar and left as an exercise). Furthermore, we may assume $f(a) < f(b)$ (otherwise, since f is increasing, if $f(a) = f(b)$ then f is a constant function, hence integrable).

Furthermore, it is clear that f is bounded by $\max\{|f(a)|, |f(b)|\}$. We will use the Cauchy Criterion 5.1.9 to show that f is integrable. Let $\epsilon > 0$. Pick a partition P of $[a, b]$ such that $\text{mesh}(P) < \epsilon/\big(f(b) - f(a)\big)$. Then

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{k=1}^{n} \big[M\big(f,[t_{k-1},t_k]\big) - m\big(f,[t_{k-1},t_k]\big)\big] \cdot (t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} \big[f(t_k) - f(t_{k-1})\big] \cdot (t_k - t_{k-1}) \quad \text{(since } f \text{ increasing)} \\ &< \sum_{k=1}^{n} \big[f(t_k) - f(t_{k-1})\big] \cdot \frac{\epsilon}{f(b) - f(a)} \\ &\qquad \Big(\text{since } \text{mesh}(P) < \frac{\epsilon}{f(b)-f(a)}\Big) \\ &= \big[f(b) - f(a)\big] \cdot \frac{\epsilon}{f(b) - f(a)} \\ &= \epsilon. \end{aligned}$$

□

Proposition 5.2.3 (Continuous functions are integrable)**.** *If $f \colon [a, b] \to \mathbb{R}$ is continuous, then f is integrable.*

PROOF. We will use the Cauchy Criterion 5.1.9 to show that f is integrable. Let $\epsilon > 0$. By Proposition 3.3.7, f is uniformly continuous, so there is $\delta > 0$ such that for every $x, y \in [a, b]$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon/(b - a)$. Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition such that $\text{mesh}(P) < \delta$. By the Extreme Value Theorem 3.2.9, on each closed interval $[t_{k-1}, t_k]$, the function f attains its maximum and minimum. Thus

$$M\big(f,[t_{k-1},t_k]\big) - m\big(f,[t_{k-1},t_k]\big) < \frac{\epsilon}{b-a}$$

for each k. Thus

$$U(f,P) - L(f,P) < \sum_{k-1}^{n} \frac{\epsilon}{b-a}(t_k - t_{k-1}) = \frac{\epsilon}{b-a} \cdot (b-a) = \epsilon.$$

□

Lemma 5.2.4 (Linearity of Integration)**.** *Let $f, g \colon [a, b] \to \mathbb{R}$ be integrable functions, and let $\alpha \in \mathbb{R}$. Then*

(1) $\alpha f \colon [a, b] \to \mathbb{R}$ is integrable, and $\int_a^b \alpha f = \alpha \int_a^b f$,
(2) $f + g \colon [a, b] \to \mathbb{R}$ is integrable, and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

PROOF. (1) We prove this by considering three cases.

(Case 1: $\alpha > 0$) Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be an arbitrary partition of $[a, b]$. Exercise 1.9.20 shows that

$$M\big(\alpha f,[t_{k-1},t_k]\big) = \alpha \cdot M\big(f,[t_{k-1},t_k]\big)$$

for all $k = 1, \ldots, n$, so

$$U(\alpha f, P) = \alpha \cdot U(f, P).$$

Now, the same exercise again shows that

$$\begin{aligned} U(\alpha f) &= \inf\big\{U(\alpha f, P) : P \text{ a partition of } [a,b]\big\} \\ &= \inf\big\{\alpha \cdot U(f, P) : P \text{ a partition of } [a,b]\big\} \\ &= \alpha \cdot \inf\big\{U(f, P) : P \text{ a partition of } [a,b]\big\} \\ &= \alpha \cdot U(f). \end{aligned}$$

A similar argument using inf's also shows that $L(\alpha f) = \alpha \cdot L(f)$. Since f itself is integrable, we have

$$L(\alpha f) \;=\; \alpha \cdot L(f) \;=\; \alpha \cdot U(f) \;=\; U(\alpha f),$$

thus αf is integrable, and

$$\int_a^b \alpha f \;=\; U(\alpha f) \;=\; \alpha \cdot U(f) \;=\; \alpha \int_a^b f.$$

(Case 2: $\alpha = -1$) This is Exercise 5.5.1.

(Case 3: $\alpha < 0$) This follows from recognizing $\alpha = -(-\alpha)$, and then applying Case 2, and then Case 1 with $-\alpha$.

(2) We will use the Cauchy Criterion 5.1.9 to show that $f+g$ is integrable. Let $\epsilon > 0$. First, since f and g are separately integrable, the Cauchy Criterion 5.1.9 (using $\epsilon/2$ instead of ϵ) gives partitions P_1 and P_2 of $[a,b]$ such that

$$U(f, P_1) - L(f, P_1) \;<\; \frac{\epsilon}{2} \quad \text{and} \quad U(g, P_2) - L(g, P_2) \;<\; \frac{\epsilon}{2}.$$

By the Partition Refinement Lemma 5.1.4 applied to the common refinement $P = P_1 \cup P_2$ of P_1 and P_2 yields

$$U(f, P) - L(f, P) \;<\; \frac{\epsilon}{2} \quad \text{and} \quad U(g, P) - L(g, P) \;<\; \frac{\epsilon}{2}.$$

Furthermore, Exercise 5.5.4 yields

$$L(f+g, P) \;\geq\; L(f, P) + L(g, P) \quad \text{and} \quad U(f+g, P) \;\leq\; U(f, P) + U(g, P).$$

Combining these four inequalities gives

$$U(f+g, P) - L(f+g, P) \;<\; \epsilon.$$

This shows, by the Cauchy Criterion 5.1.9, that $f+g$ is integrable.

Using the same ϵ and partition P, we also have

$$\begin{aligned} \int_a^b (f+g) &= U(f+g) \;\leq\; U(f+g, P) \;\leq\; U(f, P) + U(g, P) \\ &< L(f, P) + L(g, P) + \epsilon \leq L(f) + L(g) + \epsilon \\ &= \int_a^b f + \int_a^b g + \epsilon, \end{aligned}$$

and

$$\begin{aligned} \int_a^b f &= L(f+g) \;\geq\; L(f+g, P) \;\geq\; L(f, P) + L(g, P) \\ &> U(f, P) + U(g, P) - \epsilon \;\geq\; U(f) + U(g) - \epsilon \\ &= \int_a^b f + \int_a^b g - \epsilon. \end{aligned}$$

Combining these (see Remark 1.3.3) gives

$$\left|\int_a^b (f+g) - \left[\int_a^b f + \int_a^b g\right]\right| \;<\; \epsilon,$$

and since we can show this for any ϵ (perhaps using a different partition P), we conclude that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$. □

Lemma 5.2.5. *Suppose $f, g\colon [a,b] \to \mathbb{R}$. Then:*

(1) (Monotonicity) If f, g are integrable, and $f(x) \le g(x)$ for every $x \in [a,b]$, then $\int_a^b f \le \int_a^b g$.

(2) If g is a continuous nonnegative function and $\int_a^b g = 0$, then $g(x) = 0$ for every $x \in [a,b]$ (i.e., "g is identically zero").

PROOF. (1) By Lemma 5.2.4, the function $h := g - f$ is integrable on $[a,b]$. Since $g(x) \ge f(x)$ for every $x \in [a,b]$, it follows that $h(x) \ge 0$ for all $x \in [a,b]$. In particular, $L(h,P) \ge 0$ for every partition P of $[a,b]$. Thus $\int_a^b h = L(h) \ge 0$. By Lemma 5.2.4 again, we get

$$\int_a^b g \;=\; \int_a^b f + \int_a^b h \;\ge\; \int_a^b f.$$

(2) Assume towards a contradiction that there is $x_0 \in [a,b]$ such that $g(x_0) > 0$. Define $\alpha := g(x_0)/2 > 0$. By the Bump Lemma (Exercise 3.5.5), there is an open interval I which contains x_0 such that $g(x) > g(x_0)/2$ for every $x \in I$. By making I smaller, we obtain a closed interval $[c,d] \subseteq [a,b]$ which contains x_0 such that $g(x) > g(x_0)/2$ for every $x \in [c,d]$. Now

$$\begin{aligned}
\int_a^b g \;&\ge\; \int_c^d g \quad \text{(because } g \text{ is nonnegative)}\\
&\ge\; \int_c^d \frac{g(x_0)}{2} \quad \text{(by Monotonicity)}\\
&=\; \frac{g(x_0)}{2}(d-c) \quad \text{(by Example 5.1.8)}\\
&>\; 0.
\end{aligned}$$

This contradicts the assumption $\int_a^b g = 0$. □

Lemma 5.2.6. *If $f\colon [a,b] \to \mathbb{R}$ is integrable, then $|f|\colon [a,b] \to \mathbb{R}$ is integrable and*

$$\left|\int_a^b f\right| \;\le\; \int_a^b |f|.$$

PROOF. For any $S \subseteq [a,b]$ we have

$$M(|f|,S) - m(|f|,S) \;\le\; M(f,S) - m(f,S),$$

by Exercise 5.5.5. It follows from this inequality that

$$U(|f|,P) - L(|f|,P) \;\le\; U(f,P) - L(f,P)$$

for each partition P of $[a,b]$. Now, let $\epsilon > 0$. By the Cauchy Criterion 5.1.9, there is a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) \;<\; \epsilon,$$

but then it follows that

$$U(|f|,P) - L(|f|,P) \;<\; \epsilon.$$

Thus $|f|$ is integrable also by the Cauchy Criterion 5.1.9.

Next, note that for every $x \in [a,b]$ we have

$$-|f(x)| \;\leq\; f(x) \;\leq\; |f(x)|.$$

Thus by Monotonicity (Lemma 5.2.5(1)) and Linearity (Lemma 5.2.4(1)), we get

$$-\int_a^b |f(x)|\,dx \;\leq\; \int_a^b f(x)\,dx \;\leq\; \int_a^b |f(x)|\,dx,$$

which is equivalent to what we want to show; see Lemma 1.3.3. □

Lemma 5.2.7 (Additivity over intervals)**.** *Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded and $c \in (a,b)$. Then f is integrable on $[a,b]$ iff f is integrable on $[a,c]$ and $[c,b]$. In this case, we have*

$$\int_a^b f \;=\; \int_a^c f + \int_c^b f.$$

In fact, if $f\colon [a,b] \to \mathbb{R}$ is integrable, then the above equation holds for all $c \in [a,b]$ by definition of $\int_a^a f = 0$.

Proof. ($\Rightarrow$) This is Exercise 5.5.6.

($\Leftarrow$) Assume f is integrable on both $[a,c]$ and $[c,b]$. Let $\epsilon > 0$ be arbitrary. By the Cauchy Criterion 5.1.9 applied twice, we get partitions P_1 of $[a,c]$ and P_2 of $[c,b]$ such that

$$U(f,P_1) - L(f,P_1) \;<\; \frac{\epsilon}{2} \quad\text{and}\quad U(f,P_2) - L(f,P_2) \;<\; \frac{\epsilon}{2}.$$

Combining these partitions gives a partition $P = P_1 \cup P_2$ of the full interval $[a,b]$. Using the equalities

$$\begin{aligned} U(f,P) &= U(f,P_1) + U(f,P_2) \quad\text{and} \\ L(f,P) &= U(f,P_1) + U(f,P_2) \end{aligned}$$

gives us

$$U(f,P) - L(f,P) \;<\; \epsilon.$$

As ϵ was arbitrary, by the Cauchy Criterion 5.1.9 we conclude that $f\colon [a,b] \to \mathbb{R}$ is integrable.

We now assume that $f\colon [a,b] \to \mathbb{R}$ is integrable. Using the $\epsilon > 0$ and partitions P_1, P_2, and P as above, we have

$$\begin{aligned} \int_a^b f \;\leq\; U(f,P) \;&=\; U(f,P_1) + U(f,P_2) \\ <\; L(f,P_1) + L(f,P_2) + \epsilon \;&\leq\; \int_a^c f + \int_c^b f + \epsilon. \end{aligned}$$

Also

$$\begin{aligned} \int_a^b f \;\geq\; L(f,P) \;&=\; L(f,P_1) + L(f,P_2) \\ >\; U(f,P_1) + U(f,P_2) - \epsilon \;&\geq\; \int_a^c f + \int_c^b f - \epsilon. \end{aligned}$$

Combining these yields

$$\left|\int_a^b f - \left[\int_a^c f + \int_c^a f\right]\right| \;<\; \epsilon.$$

As $\epsilon > 0$ was arbitrary, we conclude that

$$\int_a^b f \;=\; \int_a^c f + \int_c^b f. \qquad \square$$

5.3. Integration theorems

In this section we prove some main theorems about the Darboux integral.

The first theorem says that the "average (or mean[2]) function value" of f on $[a, b]$ is equal to the function value of f at some point $x_0 \in (a, b)$, provided f is continuous.

Mean Value Theorem for Integrals 5.3.1. *Suppose $f, g\colon [a, b] \to \mathbb{R}$ are continuous, $g \geq 0$ or $g \leq 0$ (so g does not change sign on $[a, b]$) and $g \not\equiv 0$ (g is not the constant zero function). Then there is $c \in [a, b]$ such that*

$$f(c) \;=\; \frac{\int_a^b fg}{\int_a^b g}.$$

In particular, there is $c \in [a, b]$ (possibly different c) such that

$$f(c) \;=\; \frac{1}{b-a}\int_a^b f.$$

Proof. By possibly multiplying by -1, we arrange that $g \geq 0$. By the Extreme Value Theorem 3.2.9 there are $x_m, x_M \in [a, b]$ such that

$$f(x_m) \;\leq\; f(x) \;\leq\; f(x_M) \quad \text{for every } x \in [a, b].$$

Since $g \geq 0$, this gives

$$f(x_m)g(x) \;\leq\; f(x)g(x) \;\leq\; f(x_M)g(x) \quad \text{for every } x \in [a, b].$$

Then by Lemmas 5.2.5(1) and 5.2.4(1) we get

$$f(x_m)\int_a^b g \;\leq\; \int_a^b fg \;\leq\; f(x_M)\int_a^b g.$$

Since g is continuous, $g \geq 0$ and $g \not\equiv 0$, Lemma 5.2.5(2) implies that $\int_a^b g > 0$, so we can divide by this number:

$$f(x_m) \;\leq\; \frac{\int_a^b fg}{\int_a^b g} \;\leq\; f(x_M).$$

Finally, by the Intermediate Value Theorem 3.2.10, there is $c \in [a, b]$ such

$$f(c) \;=\; \frac{\int_a^b fg}{\int_a^b g}.$$

[2]The textbook refers to this theorem as the "Intermediate Value Theorem for Integrals", see [**2**, 33.9]. This is probably because the proof uses the Intermediate Value Theorem (for continuous functions), Theorem 3.2.10. I believe it's more common to call this the "Mean Value Theorem (for integrals)" since the quantity $1/(b-a)\int_a^b f$ is like an average (i.e., mean) function value of f on $[a, b]$.

The second statement follows by considering the special case of $g(x) = 1$ for every $x \in [a, b]$. $\square$

We now extend the definition of *integrable* slightly to include bounded functions defined on open intervals $f\colon (a, b) \to \mathbb{R}$.

Definition 5.3.2. We say a bounded function $h\colon (a, b) \to \mathbb{R}$ is **integrable on** $[a, b]$ if every extension of h to some function $\tilde{h}\colon [a, b] \to \mathbb{R}$ is integrable. In this case, we define $\int_a^b h := \int_a^b \tilde{h}$. By Exercise 5.5.2, $\int_a^b \tilde{h}$ does not depend on the choice of extension (since any two extensions of h to functions with domain $[a, b]$ differ only at the two endpoints).

First Fundamental Theorem of Calculus 5.3.3. *Suppose $F\colon [a, b] \to \mathbb{R}$ is a function such that:*

(1) F is continuous on $[a, b]$,
(2) F is differentiable on (a, b), and
(3) F' is bounded on (a, b) and integrable on $[a, b]$ (in the sense of Definition 5.3.2).

Then

$$\int_a^b F' \;=\; F(b) - F(a).$$

Proof. Since $F'\colon (a, b) \to \mathbb{R}$ is "integrable on $[a, b]$", we extend $F'\colon (a, b) \to \mathbb{R}$ to a function $F'\colon [a, b] \to \mathbb{R}$ (which we also denote by F'). The value of the extension of F' on the endpoints does not matter at all.

Let $\epsilon > 0$. By the Cauchy Criterion 5.1.9, there is a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$ such that

$$U(F', P) - L(F', P) \;<\; \epsilon.$$

For each $k = 1, \ldots, n$, we know that F is continuous on $[t_{k-1}, t_k]$ and differentiable on (t_{k-1}, t_k), so by the Mean Value Theorem (for derivatives) 4.3.3, there is $c_k \in (t_{k-1}, t_k)$ such that

$$(t_k - t_{k-1})F'(c_k) \;=\; F(t_k) - F(t_{k-1}).$$

By telescoping, this gives us

$$F(b) - F(a) \;=\; \sum_{k=1}^{n} \big[F(t_k) - F(t_{k-1})\big] \;=\; \sum_{k=1}^{n} F'(c_k)(t_k - t_{k-1}).$$

By definition of upper and lower Darboux sums, this gives

$$L(F', P) \;\le\; F(b) - F(a) \;\le\; U(F', P).$$

Also, we have

$$L(F', P) \;\le\; \int_a^b F' \;\le\; U(F', P).$$

Putting these two inequalities together yields

$$\left| \int_a^b F' - \big[F(b) - F(a)\big] \right| \;<\; \epsilon.$$

As $\epsilon > 0$ was arbitrary, this implies $\int_a^b F' = F(b) - F(a)$. $\square$

Second Fundamental Theorem of Calculus 5.3.4. *Let $f\colon [a,b] \to \mathbb{R}$ be integrable and define $F\colon [a,b] \to \mathbb{R}$ by*

$$F(x) \ := \ \int_a^x f(t)\ dt, \quad \text{for } x \in [a,b].$$

Then F is (uniformly) continuous on $[a,b]$. Moreover, if f is continuous at $x_0 \in (a,b)$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Proof. Let $M > 0$ be such that $|f(x)| \le M$ for all $x \in [a,b]$. Let $\epsilon > 0$. Suppose $x, y \in [a,b]$ are such that $x < y$ and $|x - y| < \epsilon/M$. Then

$$\begin{aligned} |F(y) - F(x)| &= \left| \int_a^y f(t)\ dt - \int_a^x f(t)\ dt \right| = \left| \int_x^y f\ dt \right| \quad \text{(by Lemma 5.2.7)} \\ &\le \int_x^y |f(t)|\ dt \quad \text{(by Lemma 5.2.6)} \\ &\le \int_x^y M\ dt \quad \text{(by Lemma 5.2.5)} \\ &= M(y-x) \quad \text{(by Example 5.1.8)} \\ &< \epsilon. \end{aligned}$$

Thus F is uniformly continuous on $[a,b]$ (using $\delta := \epsilon/M$).

Now suppose f is continuous at $x_0 \in (a,b)$. Let $\epsilon > 0$. Choose $\delta > 0$ such that for all $t \in [a,b]$, if $|t - x_0| < \delta$, then $|f(t) - f(x_0)| < \epsilon$. Then for s, t such that $x_0 - \delta < s \le x_0 \le t < x_0 + \delta$ and $a \le s < t \le b$, we have

$$\begin{aligned} \left| \frac{F(t) - F(s)}{t - s} - f(x_0) \right| &= \left| \frac{1}{t-s} \int_s^t \big[f(x) - f(x_0) \big] \right| \\ &\le \frac{1}{t-s} \int_s^t \big| f(x) - f(x_0) \big| \\ &< \epsilon. \end{aligned}$$

It follows that $F'(x_0) = f(x_0)$. □

Integration By Parts 5.3.5. *Suppose $u, v\colon [a,b] \to \mathbb{R}$ are functions such that*

(1) *u and v are continuous,*
(2) *$u, v\colon (a,b) \to \mathbb{R}$ are differentiable, and*
(3) *$u', v'\colon (a,b) \to \mathbb{R}$ are integrable (in the sense of Definition 5.3.2).*

Then:

$$\int_a^b uv' \ = \ u(b)v(b) - u(a)v(a) - \int_a^b u'v.$$

Proof. Define $f := uv$. Then on (a,b), $f' = uv' + u'v$, which is an integrable function by Exercise 5.5.9. Now note that

$$\begin{aligned} \int_a^b uv' + \int_a^b u'v &= \int_a^b f' \\ &= f(b) - f(a) \quad \text{by the First Fundamental Theorem 5.3.3} \\ &= u(b)v(b) - u(a)u(b). \end{aligned}$$

□

Taylor's Theorem 5.3.6. *Given $n \ge 0$, suppose that $f\colon (a,b) \to \mathbb{R}$ is a function such that $f, f', f'', \ldots, f^{(n+1)}$ all exist and are continuous on (a,b). Fix $x_0 \in (a,b)$. Then:*

(1) For each $x \in (a,b)$,

$$f(x) \;=\; \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) + \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\;dt,$$

(2) For each $x \in (a,b)$, *there is* c *between* x *and* x_0 *such that*

$$f(x) \;=\; \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

PROOF. We will prove (1) by induction on $n \geq 0$.

For the base case $n = 0$, we are assuming that f' is continuous on (a,b), hence bounded and integrable on $[x_0, x]$. We need to show that

$$f(x) \;=\; f(x_0) + \int_{x_0}^{x} f'(t)dt,$$

which is just a restatement of the First Fundamental Theorem of Calculus

Next, assume we know (1) holds for some $n \geq 0$. We will prove it for $n+1$, so assume $f, f', f'', \ldots, f^{(n+2)}$ all exist and are continuous. Then since it is true for n we have

$$\text{(A)} \qquad f(x) \;=\; \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) + \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\;dt$$

We will apply Integration by Parts to the remainder term, using $u = f^{(n+1)}(t)$, $u' = f^{(n+2)}(t)$, $v = -(x-t)^{n+1}/(n+1)!$, $v' = (x-t)^n/n!$. We get:

$$\begin{aligned} \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t) \;&=\; -f^{(n+1)}(x)\frac{(x-x)^{n+1}}{(n+1)!} + \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(x_0) \\ &\quad + \int_{x_0}^{x} \frac{(x-t)^{n+1}}{(n+1)!} f^{(n+2)}(t)dt \\ &= \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(x_0) + \int_{x_0}^{x} \frac{(x-t)^{n+1}}{(n+1)!} f^{(n+2)}(t)dt \end{aligned}$$

Plugging this in for the remainder term in (A) yields:

$$f(x) \;=\; \sum_{k=0}^{n+1} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) + \int_{x_0}^{x} \frac{(x-t)^{n+1}}{(n+1)!} f^{(n+2)}(t)\;dt,$$

For (b) we need to show there is c between x and x_0 such that

$$\int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt \;=\; \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

Applying the Mean Value Theorem for Integrals 5.3.1, with $g(t) = (x-t)^n/n!$, we get there is c between x and x_0 such that

$$\int_{x}^{x_0} \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt \;=\; f^{(n+1)}(c) \int_{x}^{x_0} \frac{(x-t)^n}{n!} dt \;=\; \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}. \; \square$$

5.4. The exponential function

As an application of Taylor's Theorem 5.3.6 we can now show that all three definitions of the exponential function are the same:

Theorem 5.4.1. *For every $x \in \mathbb{R}$ we have:*

$$\exp(x) \;=\; \Phi(x) \;=\; \mathrm{e}^x$$

Proof. By Corollary 4.4.2 it remains to show that $\exp(x) = \Phi(x)$. Let $x \in \mathbb{R}$ be arbitrary. We already know that $\exp(0) = \Phi(0) = 1$, so assume $x \neq 0$. Let $\epsilon > 0$. Since the series $\sum_{k=0}^{\infty} x^k/k!$ converges, there is $N_0 \in \mathbb{N}$ such that for every $n \geq N_0$,

$$\left|\exp(x) - \sum_{k=0}^{n} \frac{x^k}{k!}\right| \;<\; \frac{\epsilon}{2}.$$

Next, since $\Phi\colon \mathbb{R} \to \mathbb{R}$ is continuous, its restriction to the interval $[0, x]$ (or $[x, 0]$ if $x < 0$) is also continuous, hence bounded on this interval. Pick $M > 0$ such that $\Phi\big([0, x]\big) \leq M$. Next, since $\sum_{k=0}^{\infty} x^k/k!$ converges, so by the Divergence Test 2.4.6, $|x^k/k!| \to 0$. Thus, we may take $N_1 \in \mathbb{N}$ such that for every $n \geq N_1$,

$$\left|\frac{x^{n+1}}{(n+1)!}\right| \;<\; \frac{\epsilon}{2M}.$$

Now, set $N := \max(N_0, N_1)$, and let $n \geq N$. Then by Taylor's Theorem there is c between 0 and x such that

$$\Phi(x) \;=\; \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{\Phi(c)}{(n+1)} x^{n+1},$$

so we have

$$\begin{aligned}
|\exp(x) - \Phi(x)| \;&=\; \left|\left(\exp(x) - \sum_{k=0}^{n} \frac{x^k}{k!}\right) - \left(\Phi(x) - \sum_{k=0}^{n} \frac{x^k}{k!}\right)\right| \\
&\leq\; \left|\exp(x) - \sum_{k=0}^{n} \frac{x^k}{k!}\right| + \left|\Phi(x) - \sum_{k=0}^{n} \frac{x^k}{k!}\right| \\
&<\; \frac{\epsilon}{2} + \left|\frac{\Phi(c)x^{n+1}}{(n+1)!}\right| \\
&\leq\; \frac{\epsilon}{2} + M\left|\frac{x^{n+1}}{(n+1)!}\right| \\
&\leq\; \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we conclude that $\Phi(x) = \exp(x)$. □

As a consequence of Theorem 5.4.1, we get an alternative formula for $\mathrm{e} := \Phi(1)$:

$$\mathrm{e} \;=\; \sum_{k=0}^{\infty} \frac{1}{k!}$$

The convergence in this infinite series is fairly rapid, and in some sense, this fast convergence forces e to be irrational:

Theorem 5.4.2. e *is irrational.*

Proof. Assume towards a contradiction that e is rational, i.e., there are $p, q \in \mathbb{N}$ such that $\mathrm{e} = p/q$. Pick $n > \max\{q, \Phi([0,1])\}$ (using that Φ is continuous, so it attains a maximum on $[0,1]$). Then by Taylor's Theorem 5.3.6 applied to Φ for this n, $x_0 := 0$ and $x := 1$, there is $c \in [0,1]$ such that

$$\Phi(1) \;=\; \frac{p}{q} \;=\; 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{\Phi(c)}{(n+1)!}.$$

Multiplying both sides by $n!$ yields

$$\underbrace{n!\frac{p}{q}}_{\text{integer}} \;=\; \underbrace{n!\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)}_{\text{integer}} + \frac{\Phi(c)}{(n+1)},$$

which gives a contradiction, since $0 < \Phi(c) < n+1$, so $\Phi(c)/(n+1) \notin \mathbb{Z}$. □

5.5. Exercises

Exercise 5.5.1 (Upper-Lower Darboux symmetry)**.** Suppose $f\colon [a,b] \to \mathbb{R}$ is a bounded function. Prove the following:

(1) If P is a partition of $[a,b]$, then $U(f,P) = -L(-f,P)$.
(2) $U(f) = -L(-f)$.
(3) If f is integrable, then so is $-f$, and $\int_a^b -f = -\int_a^b f$.

Exercise 5.5.2. Suppose $f, g\colon [a,b] \to \mathbb{R}$ are bounded functions which disagree on a nonempty finite set, i.e., there are points $x_1 < \cdots < x_n$ $(n \geq 1)$ in $[a,b]$ such that

$$\{x \in [a,b] : f(x) \neq g(x)\} \;=\; \{x_1, \ldots, x_n\}$$

Show the following:

(1) $U(f) = U(g)$. (Hint: carefully prove the case $n = 1$ first, then argue by induction. Note: most of the time for induction proofs "the base case is trivial and the inductive step is where the math happens". In this particular induction proof the math really happens in the base case and the inductive step is trivial.)
(2) $L(f) = L(g)$.
(3) f is integrable iff g is integrable. Furthermore, in case they are integrable, then $\int_a^b f = \int_a^b g$.

Exercise 5.5.3. Consider the function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) \;:=\; \begin{cases} \frac{1}{n} & \text{if } x \in \mathbb{Q}, \text{ and } x = m/n \text{ with } m \in \mathbb{Z},\ n \in \mathbb{N}, \text{ and } n \text{ minimal} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Prove that f is integrable and determine $\int_0^1 f$. [Hint: the previous exercise might be useful.]

Exercise 5.5.4. Suppose $f, g\colon [a,b] \to \mathbb{R}$ are bounded, and P is a partition of $[a,b]$. Show that

$$L(f+g,P) \;\geq\; L(f,P) + L(g,P) \quad \text{and} \quad U(f+g,P) \;\leq\; U(f,P) + U(g,P).$$

Exercise 5.5.5. Suppose $f\colon [a,b] \to \mathbb{R}$ is integrable and $S \subseteq [a,b]$ is such that $S \neq \emptyset$. Then

$$M\big(|f|,S\big) - m\big(|f|,S\big) \;\leq\; M(f,S) - m(f,S).$$

Exercise 5.5.6. Show that if $f\colon [a,b] \to \mathbb{R}$ is integrable, then the restriction of f to an interval $[c,d] \subseteq [a,b]$ is also integrable.

Exercise 5.5.7. For this exercise, recall that the **floor function** of $x \in \mathbb{R}$ is the unique integer $\lfloor x \rfloor \in \mathbb{Z}$ such that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. Suppose $0 < y < x$ and $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, f is differentiable on $\mathbb{R}$, and $f'\colon \mathbb{R} \to \mathbb{R}$ is continuous.

(1) Show that for $n \in \mathbb{Z}$ we have

$$\int_{n-1}^{n} \lfloor t \rfloor f'(t)\; dt \;=\; \big[nf(n) - (n-1)f(n-1)\big] - f(n)$$

(2) Let $m := \lfloor y \rfloor$, $k := \lfloor x \rfloor$, and show that

$$\int_{m}^{k} \lfloor t \rfloor f'(t)\; dt \;=\; kf(k) - mf(m) - \sum_{y<n\leq x} f(n)$$

where the n in the summation ranges over all integers n such that $y < n \leq x$.

(3) Show also that

$$\int_{y}^{x} f(t)\; dt \;=\; xf(x) - yf(y) - \int_{y}^{x} tf'(t)\; dt$$

(4) Derive **Euler's summation formula**:

$$\sum_{y<n\leq x} f(n) \;=\; \int_{y}^{x} f(t)\; dt + \int_{y}^{x} (t - \lfloor t \rfloor) f'(t)\; dt + f(x)(\lfloor x \rfloor - x) - f(y)(\lfloor y \rfloor - y).$$

Exercise 5.5.8. Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded by M, i.e., for every $x \in [a,b]$, $|f(x)| \leq M$.

(1) Suppose P is a partition of $[a,b]$. Show that

$$U(f^2, P) - L(f^2, P) \;\leq\; 2M\big[U(f,P) - L(f,P)\big].$$

(2) Show that if f is integrable on $[a,b]$, then f^2 is integrable on $[a,b]$.

Exercise 5.5.9. Suppose $f, g\colon [a,b] \to \mathbb{R}$ are integrable.

(1) Show that $fg\colon [a,b] \to \mathbb{R}$ is integrable.
(2) Show the functions $\max(f,g)$ and $\min(f,g)$ are integrable.

Exercise 5.5.10 (Integral Test)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that f is integrable on $[a,b]$ for every $b > a$ where a is fixed. Define

$$\int_{a}^{\infty} f(x)dx \;:=\; \lim_{b\to\infty} \int_{a}^{b} f(x)dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left **converges**. Assume that $f(x) \geq 0$ for all x and that f decreases monotonically on $[0, \infty)$. Prove that

$$\int_{1}^{\infty} f(x)dx$$

converges if and only if

$$\sum_{n=1}^{\infty} f(n)$$

converges.

Exercise 5.5.11. Suppose $f, g : [a, b] \to \mathbb{R}$ are bounded functions. We further assume that $(x_n)_{n\geq 1}$ is a strictly increasing sequence in $[a, b]$ such that f and g agree everywhere except on the sequence $(x_n)_{n\geq 1}$. In other words,

$$\big\{x \in [a, b] : f(x) \neq g(x)\big\} \;=\; \{x_n : n \geq 1\}.$$

Show the following:

(1) $U(f) = U(g)$. (Note: try to take advantage of previously proven results from the chapter and previous exercises).
(2) $L(f) = L(g)$.
(3) f is integrable iff g is integrable. Furthermore, in case they are integrable, then $\int_a^b f = \int_a^b g$.

APPENDIX A

Formulas, Inequalities, and Identities

A.1. Formulas involving binomial coefficients

Definition A.1.1. For nonnegative integers n we define the **factorial** $n!$ of n recursively by setting

$$0! := 1 \quad \text{and} \quad n! := n \cdot (n-1)! \quad \text{if } n \geq 1.$$

In other words, for $n \geq 1$ we have

$$n! = n \cdot (n-1) \cdots 2 \cdot 1.$$

Given nonnegative integers n and k such that $0 \leq k \leq n$, we define the **binomial coefficient** $\binom{n}{k}$ via the formula

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}.$$

Factorials and binomial coefficients are fundamental in *combinatorics*, the field of mathematics devoted to counting. In analysis they show up quite naturally in many formulas and expansions (e.g., *Binomial Theorem* and *Taylor expansions*). The following easy identity relates adjacent binomial coefficients:

Pascal's Rule A.1.2. *For $1 \leq k, n$ we have*

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}.$$

PROOF. If $n < k$, then all three binomial coefficients are 0 so the identity is true. If $n = k$, then $\binom{n-1}{k-1} = 1 = \binom{n}{k}$, which is also true. Now assume that $n > k$. Then

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= (n-1)! \left[\frac{n-k}{k!(n-k)!} + \frac{k}{k!(n-k)!} \right] \\
&= (n-1)! \frac{n}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}.
\end{aligned}$$

□

Notice how both "$n-1$" and "n" occurs in the statement of Pascal's Rule. This property makes Pascal's Rule useful in inductive proofs involving binomial coefficients, for instance, in the proof of the important *Binomial Theorem*:

Binomial Theorem A.1.3. *Suppose $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. Then*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

PROOF. The statement we will prove by induction is:

$$P(n): \quad \text{“For every } a, b \in \mathbb{R},\ (a+b)^n = \textstyle\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.\text{”}$$

Base Case We will verify $P(1)$ is true. Note that

$$\sum_{k=0}^{1} \binom{1}{k} a^{1-k} b^k = \binom{1}{0} a + \binom{1}{b} = a + b = (a+b)^1.$$

Inductive step: We assume that $P(n)$ is true for some $n \in \mathbb{N}$. We will use this to prove that $P(n+1)$ is true. Note that

$$(a+b)^{n+1} = (a+b)(a+b)^n = (a+b)\left(\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\right),$$

where the last equality uses the inductive assumption $P(n)$. Next, we distribute the $(a+b)$:

$$\begin{aligned}(a+b)\left(\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\right) &= a\left(\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\right) + b\left(\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}\end{aligned}$$

Then we separate out the first term from the first sum and the last term from the second sum:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}$$

$$= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k} b^{k+1} + b^{n+1}$$

Then we reindex the second sum so that it starts at $k=1$:

$$= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=1}^{n} \binom{n}{k-1} a^{n-k+1} b^k + b^{n+1}$$

Then we combine the two sums:

$$= a^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k} a^{n-k+1} b^k + \binom{n}{k-1} a^{n-k+1} b^k\right] + b^{n+1}$$

$$= a^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1}\right] a^{n-k+1} b^k + b^{n+1}$$

Then apply Pascal's Rule and also note that $\binom{n+1}{0} = \binom{n+1}{n+1} = 0$:

$$= \binom{n+1}{0} a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^{(n+1)-k} b^k + \binom{n+1}{n+1} b^{n+1}$$

Finally, reincorporate the first and last term into the summation:

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{(n+1)-k} b^k.$$

Thus we have shown $P(n+1)$ holds (see the first and last term in the sequence of equalities.) □

A.2. Formulas involving summations

Triangular Number Formula A.2.1. *For every natural number $n \in \mathbb{N}$,*

$$1 + 2 + \cdots + n \;=\; \sum_{k=1}^{n} k \;=\; \frac{n(n+1)}{2} \;=\; \binom{n+1}{2}.$$

PROOF. See Example 1.1.5. □

Sum Of Squares Formula A.2.2. *For every $n \in \mathbb{N}$,*

$$1^2 + 2^2 + \cdots + n^2 \;=\; \sum_{k=1}^{n} k^2 \;=\; \frac{n(n+1)(2n+1)}{6}.$$

PROOF. See Exercise 1.9.1. □

The following gives life to an entire phylum of computations:

Difference of Powers Formula A.2.3. *For any $n \in \mathbb{N}$ such that $n \geq 2$ and $a, b \in \mathbb{R}$,*

$$\begin{aligned} a^n - b^n \;&=\; (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) \\ &=\; (a-b)\sum_{k=0}^{n-1} a^{n-1-k}b^k \end{aligned}$$

In particular, if $0 < a < b$, then

$$(b-a)na^{n-1} \;<\; b^n - a^n \;<\; (b-a)nb^{n-1}.$$

PROOF. We can prove this directly (without using induction). Start with the summation version of the righthand side:

$$(a-b)\sum_{k=0}^{n-1} a^{n-1-k}b^k$$

distribute the $(a-b)$:

$$= \; a\left(\sum_{k=0}^{n-1} a^{n-1-k}b^k\right) - b\left(\sum_{k=0}^{n-1} a^{n-1-k}b^k\right) \;=\; \sum_{k=0}^{n-1} a^{n-k}b^k - \sum_{k=0}^{n-1} a^{n-1-k}b^{k+1}$$

Pull out the first term from the first sum and the last term from the second sum:

$$= \; a^n + \left(\sum_{k=1}^{n-1} a^{n-k}b^k - \sum_{k=0}^{n-2} a^{n-1-k}b^{k+1}\right) - b^n$$

Reindex the second sum so that it starts at $k = 1$:

$$= \; a^n + \left(\sum_{k=1}^{n-1} a^{n-k}b^k - \sum_{k=1}^{n-1} a^{n-k}b^k\right) - b^n \;=\; a^n - b^n,$$

which is the desired lefthand side of the formula. □

Geometric Sum Formula A.2.4. *For any $n \in \mathbb{N}$ and $r \in \mathbb{R}$ such that $r \neq 1$,*

$$\sum_{k=0}^{n} r^k \ = \ 1 + r + r^2 + \cdots + r^n \ = \ \frac{1 - r^{n+1}}{1 - r}.$$

Proof. Setting $a = 1$ and $b = r$ in the Difference of Powers Formula A.2.3 gives for $n + 1$:

$$1 - r^{n+1} \ = \ (1 - r)(1 + r + \cdots + r^n).$$

The Geometric Sum Formula follows from dividing both sides by $1 - r$, which is permitted since $1 - r \neq 0$. □

A.3. Inequalities

Bernoulli's Inequality A.3.1. *For every $x \in \mathbb{R}$ and $n \in \mathbb{N}$, if $1 + x > 0$, then*

$$(1 + x)^n \ \geq \ 1 + nx.$$

Proof. We will prove this by induction on n. The specific statement we will prove by induction is:

$$P(n) : \text{“for every } x \in \mathbb{R}, \text{ if } 1 + x > 0, \text{ then } (1 + x)^n \geq 1 + nx\text{”}$$

Base Case: We will show $P(1)$. The righthand side is $(1 + x)$, the lefthand side is $1 + 1 \cdot x$. We have $1 + x = 1 + x$, so in particular, $1 + x \geq 1 + x$.

Induction Step: Suppose $P(n)$ holds for a specific $n \in \mathbb{N}$. We will show $P(n + 1)$ holds. Let $x \in \mathbb{R}$ be such that $1 + x > 0$. Note that

$$\begin{aligned} (1 + x)^{n+1} \ &= \ (1 + x)^n(1 + x) \\ &\geq \ (1 + nx)(1 + x) \quad \text{(using inductive hypothesis)} \\ &= \ 1 + (n + 1)x + nx^2 \\ &\geq \ 1 + (n + 1)x \quad (\text{using } x^2 \geq 0) \end{aligned}$$

and so $P(n + 1)$ holds.

We conclude by the Principle of Induction that $P(n)$ holds for all $n \in \mathbb{N}$. □

APPENDIX B

Ordered Fields

B.1. Fields

A *field* is an abstract generalization of familiar number systems such as $(\mathbb{Q};+,\cdot)$, $(\mathbb{R};+,\cdot)$ and $(\mathbb{C};+,\cdot)$. We give the definition of a field and then a discussion:

Definition B.1.1. A **field** is a set F equipped with two binary operations:

$$+\colon F\times F\to F(\text{addition})$$
$$\cdot\colon F\times F\to F(\text{multiplication})$$

such that the following axioms are satisfied:

(A1) For all $x,y\in F$, $x+y=y+x$ (commutativity)
(A2) For all $x,y,z\in F$, $(x+y)+z=x+(y+z)$ (associativity)
(A3) There is an element $0\in F$ such that $0+x=x$ for every $x\in F$ (additivity identity)
(A4) For every $x\in F$ there exists an element $-x\in F$ such that $x+(-x)=0$ (additivity inverse)
(M1) For all $x,y\in F$, $x\cdot y=y\cdot x$ (commutativity)
(M2) For all $x,y,z\in F$, $(x\cdot y)\cdot z=x\cdot(y\cdot z)$ (associativity)
(M3) There is an element $1\in F$ such that $1\neq 0$ and $1\cdot x=x$ for every $x\in F$ (multiplicative identity)
(M4) For every $x\in F$ such that $x\neq 0$, there exists an element $x^{-1}\in F$ such that $x\cdot x^{-1}=1$ (multiplicative inverse)
(D) For all $x,y,z\in F$, $x\cdot(y+z)=x\cdot y+x\cdot z$ (distributivity)

We will often denote a field as a tuple $(F;+,\cdot)$ in order to indicate that the operations "+" and "$\cdot$" are part of the data of the structure.

You probably already have experience dealing with certain specific fields:

Example B.1.2. Here are some fields you may have encountered before:

(1) The collection of rational numbers $(\mathbb{Q};+,\cdot)$ (here + and $\cdot$ refer to the "usual" addition and multiplication on the rational numbers)
(2) The collection of real numbers $(\mathbb{R};+,\cdot)$
(3) The collection of complex numbers $(\mathbb{C};+,\cdot)$, where $\mathbb{C}=\{a+bi:a,b\in\mathbb{R}\}$ and $i^2=-1$
(4) Let $\mathbb{Q}[\sqrt{2}]$ denote the subset $\{p+q\sqrt{2}:p,q\in\mathbb{Q}\}$ of $\mathbb{R}$. Then $(\mathbb{Q}[\sqrt{2}];+,\cdot)$ is a field
(5) The set:

$$\mathbb{R}(x)\ =\ \{p(x)/q(x):p(x),q(x)\text{ are polynomials with coefficients in }\mathbb{R}\text{ and }q\neq 0\}$$

of rational functions with real coefficients, equipped with usual addition and multiplication of rational functions forms a field $(\mathbb{R}(x);+,\cdot)$.

For this class, you do not need to have the field axioms above memorized. The most consequential of the axioms is probably (M4) which says that you can take reciprocals of nonzero elements. In other words, division is allowed in fields. This is not the case in every number system:

Example B.1.3. The collection of integers $(\mathbb{Z}; +, \cdot)$ is *not* a field. Indeed, the number $2 \in \mathbb{Z}$ does not have a reciprocal in $\mathbb{Z}$, since $1/2 \in \mathbb{Q} \setminus \mathbb{Z}$.

B.2. Ordered fields

Another useful feature of both the rational numbers and the real numbers is that they come equipped with an ordering. The abstract generalization of this is the notion of an *ordered field*:

Definition B.2.1. An **ordered field** is a field $(F; +, \cdot)$ equipped also with a binary relation $<$ (pronounced "less than") which satisfies the following axioms:

(O1) If $x, y \in F$, then one and only one of the statements

$$x < y, \quad x = y, \quad y < x$$

holds (trichotomy)

(O2) For all $x, y, z \in F$ if $x < y$ and $y < z$, then $x < z$ (transitivity)

(O3) For all $x, y, z \in F$ if $y < z$, then $x + y < x + z$ (additive invariance)

(O4) For all $x, y \in F$ if $x > 0$ and $y > 0$, then $x \cdot y > 0$ (multiplicative invariance)

We will often denote an ordered field as a tuple $(F; +, \cdot, <)$ to indicate that an ordered field consists of two operations $+$ and $\cdot$, together with an ordering $<$. If $(F; +, \cdot, <)$ is an ordered field and $x \in F$, then we say x is **positive** if $x > 0$ and **negative** if $x < 0$. Furthermore, given $x, y \in F$, then we also write

$$x \leq y \;:\Longleftrightarrow\; x < y \text{ or } x = y,$$

where $\leq$ is pronounced "less than or equal to".

Example B.2.2. The following are our main examples of ordered fields:

1. The rational numbers $(\mathbb{Q}; +, \cdot, <)$ is an ordered field. In some sense, this is the "smallest" example of an ordered field.
2. The real numbers $(\mathbb{R}; +, \cdot, <)$ is an ordered field. Note that for the real numbers, the ordering $<$ can be defined completely from the field structure $(\mathbb{R}; +, \cdot)$ without the ordering, i.e.,

$$x < y \iff \text{there exists } z \in \mathbb{R} \text{ such that } z \neq 0 \text{ and } z^2 = y - x$$

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