Lecture notes for Math33B: Differential Equations
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Abstract

The objective of this class is to experience an introduction to the rich, complex, and powerful subject of *Ordinary Differential Equations (ODEs)*. Specifically:

(1) Develop a working familiarity with linear algebra to the extent we need it for the differential equations we shall consider. Linear algebra serves us as a very robust backend for handling all higher-dimensional linear issues which will arise.
(2) Learn how to solve a reasonably large class of differential equations. Most differential equations cannot be solved (the solutions can only be approximated with computers, which is a story for a different math class), but we will teach you many of the differential equations for which we can find exact solutions.
(3) Observe and investigate real-world applications which are governed by differential equations.
(4) Study qualitative properties of both the differential equations we can solve and those we cannot.

The textbook for the course is *Differential Equations* Second Edition, by John Polking, Albert Boggess, and David Arnold [**1**]. These notes are based on this textbook, except for the sake of time we only include a select curated portion of the textbook material in these notes. Any and all comments, typos, errors, questions, suggestions are enthusiastically welcome!

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Introduction

The prerequisite for this course is *Math31B: Integration and Infinite Series*. Consequently, we will assume you have a working familiarity with the basic properties of differentiation and integration of common elementary functions (although we will review the tools which are most relevant for us). In this class we will put these existing tools to work to help us solve so-called *differential equations*. We begin with a simple example of a differential equation:

Question 0.0.1. *Find a differentiable function $y : \mathbb{R} \to \mathbb{R}$ which satisfies the following:*

(1) $y'(t) = \exp(t)$ *for all* $t \in \mathbb{R}$*, and*
(2) $y(0) = 10$.

Answer. From (1) we know that the function $y(t)$ must be of the form $y(t) = \exp(t) + C$ for some fixed $C \in \mathbb{R}$. By (2) we know that $y(0) = \exp(0) + C = 1 + C = 10$. Thus $C = 9$ and so $y(t) = \exp(t) + 9$. □

Question 0.0.1 illustrates a paradigm for differential equations in general. Namely, we will often be given the following information:

(1) Information about an unknown function y's derivative (or second derivative, etc.), for instance, saying "$y'(t) = \exp(t)$"
(2) Information about specific function values of y (or y', y'', etc.), for instance, saying "$y(0) = 10$".

Then the game will then be to use this information to determine the unknown function y as specifically as we can. Before we go any further, we make the following declaration:

You will not be able to solve most differential equations.

This is by no means a commentary on anyone's mathematical abilities, we simply want to bring you up to speed with a cold hard fact of life: *most differential equations are impossible (for anyone) to solve exactly.* However, we will study in detail many simple differential equations which we can solve exactly. Fortunately, the differential equations we will study also have many practical real-world applications.

What about the non-solvable differential equations? Not all hope is lost in this case. Indeed, for practical real-world applications you generally only need a sufficiently accurate approximation of a solution. Luckily this is something that computers are very good at and this is a very active area of applied mathematics. We will not go down this rabbit-hole in this class, but it helps to be aware of this remedy so you are not too discouraged if and when you encounter an impossible differential equation.

Algebraic equations

In this section we will review the state of affairs for one-variable algebraic equations. Recall that a one-variable algebraic equation is an equation of the form:

$$p(X) \;=\; 0,$$

where p is a polynomial and X is a variable. A **solution** to this equation is a specific real number $x \in \mathbb{R}$ which has the property that $p(x) = 0$ (i.e., when we plug in the number x into p, it evaluates to the number 0).

We also hope to make a general point in this section: that even for algebraic equations (i.e., a differential equation with *no* derivatives), things become very complicated and eventually impossible very quickly.

Linear equations. A **linear equation** (in one variable) is an equation of the form:

$$a_1 X + a_0 \;=\; 0 \quad (\text{where } a_1, a_0 \in \mathbb{R})$$

If $a_1 \neq 0$, then this has exactly one solution, namely:

$$x \;:=\; -\frac{a_0}{a_1}.$$

If $a_1 = 0$, then this has either zero solutions (for instance, if $a_0 \neq 0$), or infinitely many solutions (for instance, if $a_0 = 0$ then every $x \in \mathbb{R}$ is a solution). These observations foreshadow various features of systems of linear equations in multiple variables which we will study in Chapter 1.

Quadratic equations. A **quadratic equation** is an equation of the form:

$$a_2 X^2 + a_1 X + a_0 \;=\; \qquad (\text{where } a_2, a_1, a_0 \in \mathbb{R})$$

If $a_2 \neq 0$, then the **quadratic formula** yields solutions:

$$x_1 \;:=\; \frac{-a_1 + \sqrt{a_1^2 - 4a_2a_0}}{2a_2} \quad \text{and} \quad x_2 \;:=\; \frac{-a_1 - \sqrt{a_1^2 - 4a_2a_0}}{2a_2}$$

Recall that three things can happen depending on the sign of the **discriminant** $a_1^2 - 4a_2a_0$:

(Case 1) If $a_1^2 - 4a_2a_0 > 0$, then $x_1 \neq x_2$ are two *real* solutions.
(Case 2) If $a_1^2 - 4a_2a_0 = 0$, then $x_1 = x_2$ is a single real solution (of multiplicity two).
(Case 3) If $a_1^2 - 4a_2a_0 < 0$, then $x_1 \neq x_2$ are two distinct solutions, however, they will be complex solutions and not real solutions.

You are expected to be able to use the quadratic formula to solve quadratic equations in this class.

Cubic equations. A **cubic equation** is an equation of the form:

$$a_3 X^3 + a_2 X^2 + a_1 X + a_0 \;=\; 0 \quad (\text{where } a_3, a_2, a_1, a_0 \in \mathbb{R})$$

You were probably never taught the formula for the cubic equation in school. This is for good reason: it's complicated! You do not need it for this class either, but in case you are curious, here it is: if $a_3 \neq 0$, then the three solutions are

$$x_k \;=\; -\frac{1}{3a_3}\left(a_2 + \xi^k C + \frac{\Delta_0}{\xi^k C}\right), \quad \text{for } k = 0, 1, 2$$

where

$$\begin{aligned} \xi &:= \frac{-1+\sqrt{-3}}{2} \\ \Delta_0 &:= a_2^2 - 3a_3a_1 \\ \Delta_1 &:= 2a_2^3 - 9a_3a_2a_1 + 27a_3^2a_4 \\ C &:= \sqrt[3]{\frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}} \\ &\quad \text{(choose either + or − provided } C \neq 0) \end{aligned}$$

Here there can either be three, two, or one distinct solution, and the solutions can be either real or complex, much like the quadratic equation.

Quartic equations. A **quartic equation** is an equation of the form:

$$a_4X^4 + a_3X^3 + a_2X^2 + a_1X + a_0 = 0 \quad (\text{where } a_4, a_3, a_2, a_1, a_0 \in \mathbb{R})$$

The general solution for the quartic equation is even more complicated than the equation for the cubic. You definitely do not need to know it, but in case you are curious here it is: if $a_4 \neq 0$, then the four solutions are:

$$\begin{aligned} x_{1,2} &:= -\frac{a_3}{4a_4} - S \pm \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}} \\ x_{3,4} &:= -\frac{a_3}{4a_4} + S \pm \frac{1}{2}\sqrt{-4S^2 - 2p - \frac{q}{S}} \end{aligned}$$

where

$$\begin{aligned} p &:= \frac{8a_4a_2 - 3a_3^2}{8a_4^2} \\ q &:= \frac{a_3^3 - 4a_4a_3a_2 + 8a_4^2a_1}{8a_4^3} \\ S &:= \frac{1}{2}\sqrt{-\frac{2}{3}p + \frac{1}{3a}\left(Q + \frac{\Delta_0}{Q}\right)} \\ Q &:= \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}} \\ \Delta_0 &:= a_2^2 - 3a_3a_1 + 12a_4a_0 \\ \Delta_1 &:= 2a_2^3 - 9a_3a_2a_1 + 27a_3^2a_0 + 27a_4a_1^2 - 72a_4a_2a_0 \\ &\quad \text{(with special cases if } S = 0 \text{ or } Q = 0) \end{aligned}$$

Quintic (and higher degree) equations. A **quintic equation** is an equation of the form:

$$a_5X^5 + a_4X^4 + a_3X^3 + a_2X^2 + a_1X + a_0 = 0 \quad (\text{where } a_5, a_4, a_3, a_2, a_1, a_0 \in \mathbb{R})$$

You might be expecting an even longer and more complicated formula for the five solutions to a quintic equation, but actually it is known that this is impossible. In fact, there is a theorem which tells us that this is impossible:

Theorem 0.0.2 (Galois)**.** *Suppose $n \geq 5$. Then there is no general formula using radicals ($\sqrt{}$, $\sqrt[3]{}$, $\sqrt[4]{}$, ...) which gives the solutions to*

$$a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0 \ = \ 0$$

in terms of the coefficients $a_n, \ldots, a_0$.

Of course, sometimes you will be able to solve for the solutions of a high-degree polynomial equation (for instance, $x := 1$ is a solution to $X^{100} - 1 = 0$), but this is usually because the polynomial is carefully chosen in order to admit solutions you can find exactly. This is an exceptional case. In general, the only polynomial equations you can expect a guaranteed solution for is degree 1 (linear) and degree 2 (quadratic). If we do encounter higher-degree polynomials in this class, they will be chosen so that it is possible to find exact solutions. However in general we will stick to degree 2 or lower.

Conventions and notation

In this class the natural numbers is the set $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ of nonnegative integers. In particular, we consider 0 to be a natural number.

Unless stated otherwise, the following convention will be in force throughout the entire course:

Global Convention 0.0.3. Throughout, m and n range over $\mathbb{N} = \{0, 1, 2, \ldots\}$.

CHAPTER 1

Linear algebra I

Before commencing with differential equations, we begin with the first of three chapters on linear algebra. This might seem initially unrelated to differential equations (like the one considered in Question 0.0.1) but we will soon find that linear algebra is intimately connected with many of the things we will do with differential equations and it is the best language to explain many different phenomena we will encounter.

1.1. Systems of equations

In this section we will give a crash course in the correct way to completely solve a system of equations (with any number of variables and any number of equations).

Systems of equations. Here is an example of a system of equations:

$$\begin{aligned} 2X + Y &= 1 \\ X - Y &= 1 \end{aligned} \tag{1.1}$$

This is a system of equations with two variables (X and Y) and two equations. A solution to (1.1) is a pair (x, y) of real numbers, such that when we plug in x for X and y for Y, both equations are satisfied. We will recall how one solves (1.1) using what we will call the *naive method*:

Solution to (1.1). First we will multiply the second equation by 2 so that the coefficients on "X" are the same:

$$\begin{aligned} 2X + Y &= 1 \\ 2X - 2Y &= 2 \end{aligned} \tag{1.2}$$

Next we will subtract the first equation from the second equation to eliminate the second "X":

$$\begin{aligned} 2X + Y &= 1 \\ -3Y &= 1 \end{aligned} \tag{1.3}$$

Now we see that $y := -1/3$ is the only value for Y which works. Plugging this into the top equation yields:

$$2X - 1/3 = 1 \quad \text{and thus} \quad X = 2/3.$$

Thus $x := 2/3$ is the only value for X that works. We conclude that $(x, y) = (2/3, -1/3)$ is the *only* solution to (1.1). □

We call this the *naive method* because it relies on observations and *ad hoc* computations. We include it here mainly to jog your memory of how you might have previously learned to solve systems of equations. However, this method quickly becomes burdensome when you consider more variables and more equations. In the

rest of this section, we will introduce the *correct method* you should use to solve these systems. At this point we make the following declaration:

You should never again use the naive method

to solve a system of equations.

Instead you should commit to learning and using the method introduced below. Before we proceed, we will make a few more definitions:

Definition 1.1.1. A **system of equations (with m equations and n variables)** is a system

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n &= b_1 \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n &= b_2 \\ &\vdots \\ a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n &= b_m \end{aligned} \tag{1.4}$$

where $b_i, a_{ij} \in \mathbb{R}$ for every $i = 1, \ldots, m$ and $j = 1, \ldots, n$. A **solution** to the system (1.4) is an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers such that when you plug x_i in for X_i (for each $i = 1, \ldots, n$), each equation is true.

Example 1.1.2. The following system has 3 equations and 4 variables:

$$\begin{aligned} X_1 + 2X_2 - 3X_3 + X_4 &= 6 \\ 2X_1 + X_2 - 2X_3 - X_4 &= 4 \\ 6X_2 + 4X_3 - X_4 &= 4 \end{aligned}$$

and it is easy to check that $(1/3, 4/3, -1, 0)$ is a solution (although there are other solutions as well).

In general the goal will be to find *all* solutions to a system of equations, not just one single solution.

Augmented matrices. Recall that in our solution to the system (1.1) above we first had the system

$$\begin{aligned} 2X + 1Y &= 1 \\ 2X - 2Y &= 2 \end{aligned}$$

which then we transformed into the system

$$\begin{aligned} 2X + 1Y &= 1 \\ 0X - 1Y &= 1. \end{aligned} \tag{1.5}$$

Note also that every symbol colored in red has nothing to do with the specific numbers; the presence and locations of "X", "Y" and "=" is always guaranteed to be exactly the same each time we transform the system. The only thing that matters for each system is what coefficients are in which spot.

This brings us to the first major innovation linear algebra has to offer us for systems of equations: *augmented matrices*. An **augmented matrix** for a system of m

equations in n variables (such as (1.4) above) is a rectangular array with m rows and $n+1$ columns which stores all the coefficients of the system:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Example 1.1.3. For example, the system

$$\begin{aligned} 3a + 4b + c &= 2 \\ a - 5c &= 3 \end{aligned}$$

has corresponding augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 4 & 1 & 2 \\ 1 & 0 & -5 & 3 \end{array}\right]$$

In other words an augmented matrix is nothing more than a *compact storage device for an entire system of equations.* Whenever you see a system of equations, you should also picture it's augmented matrix, and vice versa.

Henceforth, we will primarily use augmented matrices

for writing systems of equations.

Now we return to the main order of business which is to efficiently solve systems of equations (i.e., determine *all* solutions). Basically, we will learn how to play a game. The game is called **Gaussian Elimination**. The rules of the game are roughly as follows:

(I) There are three legal moves (so-called *elementary row operations*) which we can use to transform one augmented matrix into the next augmented matrix.
(II) When starting out, the first[1] goal is to transform your matrix into *Row Echelon Form.*
(III) After getting to Row Echelon Form, the next goal is to continue to transform your matrix into *Reduced Row Echelon Form.*
(IV) Once the matrix is in Reduced Row Echelon Form, it is very easy to read off all solutions to the original system.

We will study these four things separately in the remainder of this section.

Row operations. Suppose we have an augmented matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

The following **elementary row operations** are the only ways we are allowed to transform this augmented matrix:

(1) (*Row switching*) A row in the matrix can be switched with another row in the matrix. Notation: $R_i \leftrightarrow R_j$

[1]In some linear algebra books and classes, this step is skipped and the goal is to go directly to reduced row echelon form in (III). It's fine if you do it that way, although in general it will take the same amount of work and effort.

(2) (*Row multiplicaton*) A row can be multiplied by a non-zero constant. Notation: $\alpha R_i \to R_i$
(3) (*Row addition*) A row can be replaced with the sum of that row and a multiple of another row. Notation: $R_i + \alpha R_j \to R_i$.

Here is an example of a sequence of three applications of elementary row operations:

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 2 & 4 & 4 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 4 & 4 & 3 \\ 0 & 1 & 1 & 2 \end{array}\right] \quad \text{(row switch row 1 and row 2)}$$

$$\xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3/2 \\ 0 & 1 & 1 & 2 \end{array}\right] \quad \text{(multiply row 1 by 1/2)}$$

$$\xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5/2 \\ 0 & 1 & 1 & 2 \end{array}\right] \quad \text{(add } -2 \text{ times row 2 to row 1)}$$

Question 1.1.4. *Why are these the only operations allowed?*

Proof. These row operations have the property that they are *reversible*. This means that the set of solutions remains the same in each augmented matrix. Note that if we allowed "multiplication by 0" to be a row operation, then this would have the effect of deleting information in the system and it might introduce additional solutions which are not solutions of the original system (which would be very undesirable). □

Below we will explain how to use these row operations to achieve our objective of solving the original system of equations.

Row echelon form (REF). We will illustrate the entire process with the following example which we will occasionally check back in with:

Example 1.1.5. Find all solutions to the system

$$\begin{aligned} 3X_1 + 6X_2 + 6X_3 &= 24 \\ -6X_1 - 12X_2 - 12X_3 &= -48 \\ 6X_1 + 12X_2 + 10X_3 &= 42 \end{aligned} \tag{1.6}$$

Solution to Example 1.1.5, Part I. The first step is to rewrite the system (1.6) as an augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & 6 & 6 & 24 \\ -6 & -12 & -12 & -48 \\ 6 & 12 & 10 & 42 \end{array}\right] \qquad \square$$

Now we need to know how are we supposed to transform our augmented matrix using the three elementary row operations. First objective is to transform our augmented matrix into *row echelon form*:

Definition 1.1.6. An augmented matrix is in **row echelon form (REF)** if

(1) every row with nonzero entries is above every row with all zeroes (if there are any), and
(2) the leading coefficient of a nonzero row (i.e., the leftmost nonzero entry of that row) is to the right of the leading coefficient of the row above it.

Example 1.1.7. The following augmented matrices are in REF (with the leading coefficients underlined):

$$\left[\begin{array}{cc|c}\underline{4} & 3 & 1\\ 0 & \underline{1} & 2\end{array}\right] \quad \left[\begin{array}{ccc|c}0 & \underline{3} & 1 & 8\end{array}\right] \quad \left[\begin{array}{ccc|c}\underline{1} & 0 & 0 & 1\\ 0 & \underline{1} & 0 & 2\\ 0 & 0 & \underline{1} & 3\\ 0 & 0 & 0 & 0\end{array}\right] \quad \left[\begin{array}{ccc|c}\underline{2} & 3 & 0 & 0\\ 0 & 0 & \underline{1} & 0\end{array}\right] \quad \left[\begin{array}{cc|c}0 & 0 & \underline{2}\\ 0 & 0 & 0\end{array}\right]$$

The following augmented matrices are *not* in REF:

$$\left[\begin{array}{cc|c}0 & 0 & 0\\ 0 & \underline{1} & 1\end{array}\right] \quad \left[\begin{array}{ccc|c}\underline{1} & 0 & 0 & 1\\ 0 & 0 & \underline{1} & 2\\ 0 & \underline{1} & 0 & 3\end{array}\right] \quad \left[\begin{array}{cc|c}0 & \underline{1} & 0\\ 0 & 0 & 0\\ \underline{1} & 0 & 0\end{array}\right]$$

Solution to Example 1.1.5, Part II. Our augmented matrix is not in row echelon form. In particular, the leading coefficients of the second and third row are directly below the leading coefficient of the first row, which is not allowed:

$$\left[\begin{array}{ccc|c}3 & 6 & 6 & 24\\ \underline{-6} & -12 & -12 & -48\\ \underline{6} & 12 & 10 & 42\end{array}\right]$$

To fix this, we need to use row addition with the first row to turn the leading -6 and 6 of the second and third row into a zero:

$$\left[\begin{array}{ccc|c}3 & 6 & 6 & 24\\ -6 & -12 & -12 & -48\\ 6 & 12 & 10 & 42\end{array}\right] \xrightarrow{R_2+2R_1\to R_2} \left[\begin{array}{ccc|c}3 & 6 & 6 & 24\\ 0 & 0 & 0 & 0\\ 6 & 12 & 10 & 42\end{array}\right]$$

$$\xrightarrow{R_3-2R_1\to R_3} \left[\begin{array}{ccc|c}3 & 6 & 6 & 24\\ 0 & 0 & 0 & 0\\ 0 & 0 & -2 & -6\end{array}\right]$$

We are still not in row echelon form since we have a row of all zeros above a row with nonzero entries:

$$\left[\begin{array}{ccc|c}3 & 6 & 6 & 24\\ \underline{0} & \underline{0} & \underline{0} & \underline{0}\\ 0 & 0 & -2 & -6\end{array}\right]$$

To remedy this, we will switch rows 2 and 3:

$$\xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{ccc|c}3 & 6 & 6 & 24\\ 0 & 0 & -2 & -6\\ 0 & 0 & 0 & 0\end{array}\right]$$

We are now in row echelon form and we are done this step. □

Once our augmented matrix is in row echelon form, we can make the following definition:

Definition 1.1.8. Given an augmented in REF, a **pivot** is a leading coefficient in a nonzero row.

For instance, the augmented matrix we arrived at in Example 1.1.5 has two pivots, which we indicate in boxes:

$$\left[\begin{array}{ccc|c}\boxed{3} & 6 & 6 & 24\\ 0 & 0 & \boxed{-2} & -6\\ 0 & 0 & 0 & 0\end{array}\right]$$

Pivots play an important role in Gaussian Elimination. The next step is to take our augmented matrix a little bit further to *reduced row echelon form*.

Reduced row echelon form (RREF). The ultimate goal is to get our augmented matrix into *reduced row echelon form*:

Definition 1.1.9. An augmented matrix is in **reduced row echelon form (RREF)** if

(1) it is in row echelon form (REF),
(2) every pivot is 1, and
(3) every entry above a pivot is 0.

Example 1.1.10. The following augmented matrices are in RREF:

$$\left[\begin{array}{ccc|c} 0 & \boxed{1} & & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right] \quad \left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 5 \end{array}\right]$$

The following matrices are in REF but *not* RREF:

$$\left[\begin{array}{cc|c} \boxed{4} & 3 & 1 \\ 0 & \boxed{1} & 2 \end{array}\right] \quad \left[\begin{array}{ccc|c} 0 & \boxed{3} & 1 & 8 \end{array}\right] \quad \left[\begin{array}{ccc|c} \boxed{2} & 3 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

We now continue on with our main example:

Solution to Example 1.1.5, Part III. We see that the augmented matrix we left off with is not in RREF, only REF. This is because the pivots are 3 and -2, not 1 and 1, and also the underlined 6 should be a 0:

$$\left[\begin{array}{ccc|c} \boxed{3} & 6 & \underline{6} & 24 \\ 0 & 0 & \boxed{-2} & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

To remedy this, we use row multiplication to fix the pivot values, and then row addition to get rid of the 6:

$$\xrightarrow{\frac{1}{3}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 8 \\ 0 & 0 & -2 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 8 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Finally we arrive at RREF. □

Once our augmented matrix is in RREF, it is easy to read off all solutions of the original system.

Getting the final answer from RREF. We will describe how to get the final answer from RREF first in terms of our main example:

Solution to Example 1.1.5, Part IV. First recall that the first three columns correspond to the three variables X_1, X_2, and X_3:

$$\begin{array}{c} \begin{matrix} X_1 & X_2 & X_3 \end{matrix} \\ \left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 2 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Since X_1 and X_3 have pivots in their columns, X_1 and X_3 are called **pivot variables** and the first and third columns are called **pivot columns**. Since X_2 does not have a pivot, it is called a **free variable** and the second column is called a **free column**. Now we read off the solutions using the following steps:

(1) Each free variable is can be any arbitrary value. In this case, we will say that $X_2 = s$, where $s \in \mathbb{R}$ is any number we like.
(2) Next we rewrite the augmented matrix as a system and solve for the pivot variables:

$$\begin{aligned} X_1 + 2X_2 &= 2 \\ X_3 &= 3 \\ 0 &= 0 \end{aligned}$$

which simplifies to:

$$\begin{aligned} X_1 &= 2 - 2s \\ X_3 &= 3. \end{aligned}$$

We now have our final answer: every solution is of the form:

$$\begin{aligned} X_1 &= 2 - 2s \\ X_2 &= s \\ X_3 &= 3, \end{aligned}$$

where $s \in \mathbb{R}$ can be any number. We write the set of all solutions as follows:

$$\big\{(2 - 2s, s, 3) : s \in \mathbb{R}\big\}$$

This way of describing the set of solutions is often called **parametric form** because it describes the solutions in terms of the free parameter s. Notice that there are infinitely many solutions, since there are infinitely many values of s. To get specific solutions, you can just choose values of s. For instance, $s := 0$ yields the solution $(2, 0, 3)$, whereas $s := 10$ yields the solution $(-18, 10, 3)$. □

Example 1.1.11. In this example we will see what to do with 2 free variables. Suppose we are given some system which has the following RREF:

$$\begin{array}{c} \begin{matrix} X_1 & X_2 & X_3 & X_4 \end{matrix} \\ \left[\begin{array}{cccc|c} 0 & \boxed{1} & 2 & 0 & 7 \\ 0 & 0 & 0 & \boxed{1} & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Then we have two free variables X_1 and X_3, so we need to introduce two parameters $s, t \in \mathbb{R}$ and set $X_1 = s$ and $X_3 = t$. Then the system becomes:

$$\begin{aligned} X_2 + 2X_3 &= 7 \\ X_4 &= 8 \end{aligned}$$

and so the general solution is:

$$\begin{aligned} X_1 &= s \\ X_2 &= -2t + 7 \\ X_3 &= t \\ X_4 &= 8 \end{aligned}$$

where $s, t \in \mathbb{R}$ are arbitrary. We can write the set of solutions in parametric form as follows:

$$\big\{(s, -2t + 7, t, 8) : s, t \in \mathbb{R}\big\}$$

Note that to get a specific solution, we are free to choose any s and any t we like. For instance, $s = 1, t = 0$ gives the solution $(1, 7, 0, 8)$ whereas $s = 0, t = 1$ gives the solution $(0, 5, 1, 8)$.

Example 1.1.12. We will give an example of a system with no solutions. Suppose we are given a system with the following RREF:

$$\begin{array}{c} \begin{matrix} X_1 & X_2 & \end{matrix} \\ \left[\begin{array}{cc|c} \boxed{1} & 2 & 0 \\ 0 & 0 & \boxed{1} \end{array}\right] \end{array}$$

Converting this augmented matrix back to a system of equations yields:

$$\begin{aligned} 1X_1 + 2X_2 &= 0 \\ 0X_1 + 0X_2 &= 1 \end{aligned}$$

We claim there cannot be any solutions. Indeed, if say (x_1, x_2) is a solution, then this would mean it satisfies both equations, in particular, the bottom equation. Then $0x_1 + 0x_2 = 1$, i.e., $0 = 1$. However this is always false.

We conclude this section with some more terminology and some general facts:

Definition 1.1.13. We say that a system of equations is **consistent** if it has at least one solution, and we say a system of equations is **inconsistent** if it does not have any solutions.

Fact 1.1.14. Given a system of equations, exactly one of the following three things will happen:

(1) The system has zero solutions (i.e., it is inconsistent). This happens when the RREF contains a row of the form

$$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \end{array}\right]$$

because this corresponds to the equation $0 = 1$ which can never be true.
(2) The system has exactly one solution. This happens when the system is consistent and there are no free variables in the RREF.
(3) The system has infinitely many solutions. This happens when the system is consistent and there is at least one free variable in the RREF.

In fact, all 3 of the above cases can be determined once you're in REF. If you only care about *how many* solutions there are (and not what exactly they are), then you can just stop once you get to REF. This is one of the benefits of going through the REF on your way to RREF.

Here are some cardinal rules to always follow:

(1) Always recopy the entire augmented matrix in each step, even if you are copying a row of zeros. It is important that the size of the augmented matrix (3×4 in our example) does not change.
(2) Always denote which row operation you are performing in each step.
(3) Always do one row operation at a time, at least when you are starting out. If you attempt to do multiple row operations in one step then this can lead to errors.

Remark 1.1.15. Given a system of equations, we take it to RREF and obtain the set of solutions for the *original system we started out with*. However, this is actually the set of solutions for *every system we encountered along the way*. This is because the RREF of the original system also works as the RREF for every intermediate system.

Geometric interpretation. When you are solving systems of equations, it is good to keep in mind the underlying geometric interpretation. Recall that a linear equation in two variables:

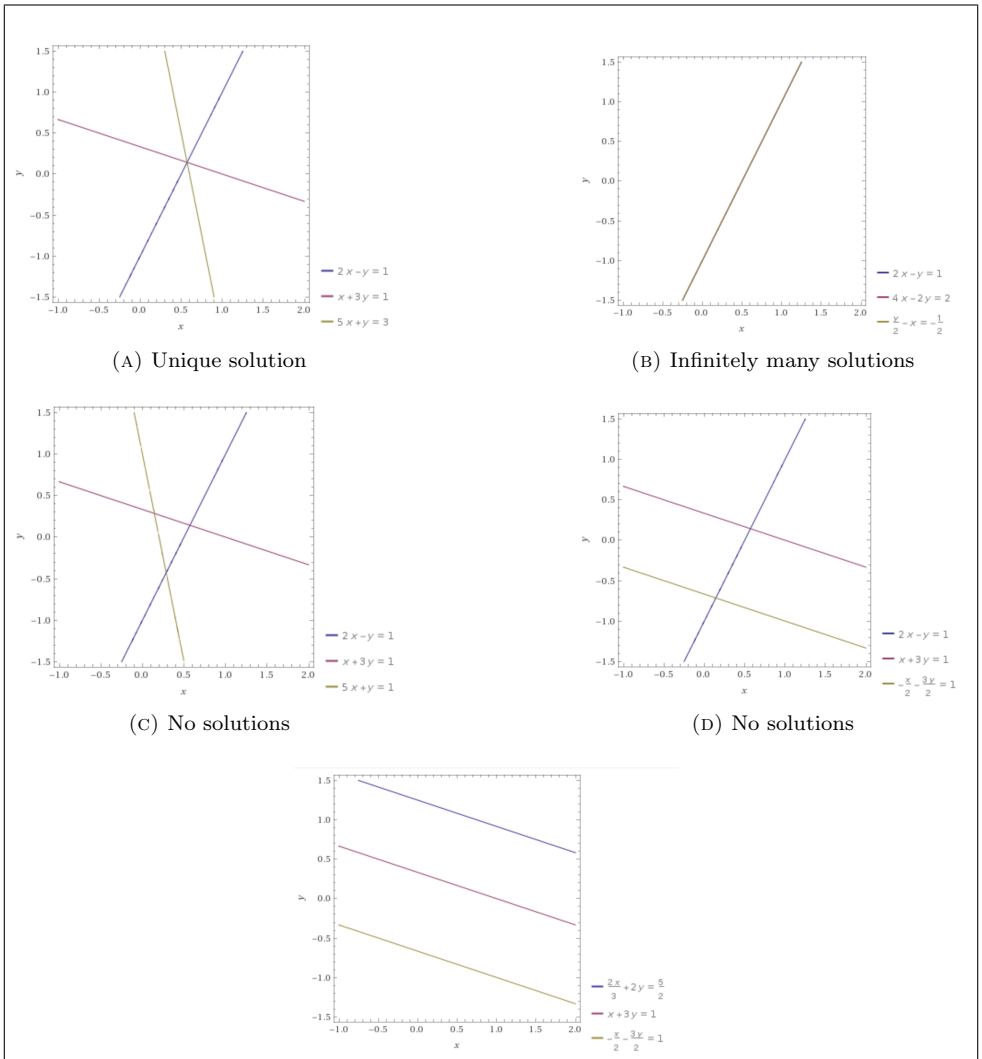
$$2x + 3y = 1$$

can also be viewed as an equation for a line in the plane ($y = -\frac{2}{3}x + \frac{1}{3}$). Thus, a system of linear equations:

$$\begin{aligned} 2x + 3y &= 1 \\ 5x + 7y &= 2 \\ x + y &= 3 \end{aligned}$$

is really asking us to find all points (x, y) in the plane which are part of all three lines, i.e., we want to know where do these three lines intersect, if at all. If we are consider three variables, then we are asking where do multiple planes simultaneously intersect, if at all. For more than 3 variables, we are asking where do higher-dimensional hyper-planes intersect in higher-dimensional euclidean space (something difficult to visualize).

In Figure 1.1, we consider five systems of equations, where each one has two variables and three equations. You can see that there are different ways that the cases *no solutions*, *exactly one solution*, and *infinitely many solutions* can arise.



(A) Unique solution

(B) Infinitely many solutions

(C) No solutions

(D) No solutions

(E) No solutions

FIGURE 1.1. Possible intersections of three lines in a plane

Some specifics about terminology. In this section, we have only been working with *augmented matrices*, for instance

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \tag{1.7}$$

An augmented matrix is just a special example of a *matrix* with a vertical bar which superficially separates the columns. A **matrix** (with m rows and n columns) is a rectangular array of numbers:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

For instance, the augmented matrix (1.7) is considered a 2×3 matrix. When discussing an augmented matrix, we will always consider every column as part of the augmented matrix. If we want to refer only to the entries to the left of the vertical bar:

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

this will be referred to as the **coefficient matrix** (of the linear system).

Definition 1.1.16. Here are some precise definitions summarized:

(1) Given a matrix, a **leading entry** of a row is the leftmost nonzero entry (if there is one). In the following matrices, we underline the leading entries:

$$\begin{bmatrix} \underline{2} & 3 & 0 \\ 0 & \underline{2} & 1 \\ \underline{1} & 0 & 2 \end{bmatrix} \quad \left[\begin{array}{cc|c} \underline{1} & 2 & 0 \\ 0 & 0 & \underline{1} \end{array}\right]$$

(2) If a matrix is in REF, then the leading entries are also called **pivots**. The following matrices are in REF and the pivots are in boxes:

$$\left[\begin{array}{ccc|c} \boxed{2} & 3 & 5 & 4 \\ 0 & 0 & \boxed{7} & 1 \\ 0 & 0 & 0 & \boxed{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(3) If a matrix is *not* in REF, then we choose not to define what a pivot is. In this class we will only discuss "pivots" in the context of Gaussian Elimination and only allow ourselves to refer to "the pivots of a matrix" if we know the matrix is already in REF. For all matrices, the expression "leading entry" will always make sense, regardless of whether the matrix is in REF or not.

(4) We define the **rank** of a matrix to be the number of pivots any REF of that matrix has (it will be the same number even though there could be many different REFs).

Question 1.1.17. *Why are we reluctant to call leading coefficients in a non-REF matrix "pivots"?*

Answer 1.1.18. In general, a *pivot* (noun) is something that you *pivot* (verb) around. Given a nonzero entry of a matrix, to **pivot** around that entry means to use elementary row operations to turn that entry into a 1 and then use it to turn the other entries in that column into 0. In the following example, we pivot around the boxed entry (for no particular reason other than to show an example of "pivoting"):

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & \boxed{2} & 2 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \boxed{1} & 1 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{R_1 - R_2 \to R_1}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & \boxed{1} & 1 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{R_3 - 3R_1 \to R_3} \begin{bmatrix} 0 & 0 & 0 \\ 1 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since this is what "pivoting" means, we define *pivots* so that in Gaussian Elimination we are essentially *pivoting around the pivots*. We do not pivot around the leading entries which are not pivots. Furthermore, there are other algorithms in linear algebra besides Gaussian Elimination (for instance, the *Simplex Algorithm*[2]) where you pivot around entries which are not leading coefficients. Thus, you shouldn't get too attached to the idea "pivot means leading entry".

Given the above discussion, we can now recast some of the above facts in more detail:

Fact 1.1.19. Suppose we are considering a system of equations which has augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

and coefficient matrix:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

(1) The following are equivalent:
   (a) the system has no solutions,
   (b) the system is inconsistent,
   (c) an REF of the augmented matrix has a row of the form
   $$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & \neq 0 \end{array}\right],$$
   (d) the RREF of the augmented matrix has a row of the form
   $$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \end{array}\right],$$
   (e) an REF of the augmented matrix has a pivot in the last column,
   (f) the RREF of the augmented matrix has a pivot in the last column,
   (g) the rank of the coefficient matrix is not equal to the rank of the entire augmented matrix.

(2) Suppose the system is consistent. Then the following are equivalent:
   (a) the system has exactly one solution,
   (b) every variable is a pivot variable,
   (c) there are no free variables,
   (d) the rank of the augmented matrix is equal to the number of columns in the coefficient matrix (= number of variables).

(3) Suppose the system is consistent. Then the following are equivalent:
   (a) the system has infinitely many solutions,
   (b) at least one variable is a free variable,
   (c) the rank of the augmented matrix is less than the number of columns in the coefficient matrix (i.e., less than the number of variables).

[2] `https://en.wikipedia.org/wiki/Simplex_algorithm`

1.2. Application: partial fractions

In this section, we revisit the powerful method of *partial fractions*, viewed as an application of linear systems.

Case I: distinct linear factors. Suppose we want to integrate the rational function:

$$\frac{3x+4}{x^3-3x^2+2x}$$

To do this, we must first factor the denominator polynomial: $x^3-3x^2+2x = (x-0)(x-1)(x-2)$. Since there are no (strictly) complex roots, this polynomial factors into linear factors (with real roots). Also, for this polynomial, every linear factor is distinct (occurs with multiplicity one). Thus, the general form of the partial fraction decomposition is:

$$\frac{3x+4}{x(x-1)(x-2)} \;=\; \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-2},$$

where $A, B, C \in \mathbb{R}$ are three unknown real numbers we need to solve for. Clearing denominators yields:

$$3x+4 \;=\; A(x-1)(x-2)+Bx(x-2)+C(x-1)(x-2)$$

This equality is to be interpreted as: for every possible real number $x \in \mathbb{R}$, when you plug x into both the lefthand side and the righthand side, you should get a true equality of two numbers. We will use this observation and plug in three carefully chosen numbers to see what they give us:

- $(x=0)$ In this case, the equation becomes $4 = 2A$
- $(x=1)$ In this case, the equation becomes $7 = -B$
- $(x=2)$ In this case, the equation becomes $10 = 2C$

Thus, we have arrived at a (easy) system of equations:

$$\begin{aligned} 2A &= 4 \\ -B &= 7 \\ 2C &= 10. \end{aligned}$$

We can solve this system using Gaussian Elimination:

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & -1 & 0 & 7 \\ 0 & 0 & 2 & 10 \end{array}\right] \xrightarrow{\frac{1}{2}R_1\to R_1,\,-R_2\to R_2,\,\frac{1}{2}R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

This gives us the unique solution $(A, B, C) = (2, -7, 5)$. We conclude that

$$\frac{3x+4}{x^3-3x^2+2x} \;=\; \frac{2}{x}-\frac{7}{x-1}+\frac{5}{x-2}$$

is our desired partial fraction decomposition. The rational function can now be integrated using the logarithm.

Case II: repeated linear factors. Suppose now we wish to decompose

$$\frac{5x^3+6x^2+7x+8}{x^4-2x^3+x^2}$$

We are able to factor the denominator as $x^4-2x^3+x^2=x^2(x-1)^2$. We see that there are two linear factors, each one with multiplicity two. Thus the general form of the partial fraction decomposition is

$$\frac{5x^3+6x^2+7x+8}{x^2(x-1)^2} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}$$

where $A,B,C,D\in\mathbb{R}$ are four unknown real numbers we need to solve for (the rule is, for each multiplicity of a linear factor, you get another term in the expansion and another variable). First we cross-multiply so that we have an equality of polynomials, then we rewrite the righthand side as a single polynomial:

$$\begin{aligned} 5x^3+6x^2+7x+8 &= Ax(x-1)^2+B(x-1)^2+Cx^2(x-1)+Dx^2 \\ &= A(x^3-2x^2+x)+B(x^2-2x+1)+C(x^3-x^2)+Dx^2 \\ &= (A+C)x^3+(-2A+B-C+D)x^2+(A-2B)x+B. \end{aligned}$$

Next, we use the important observation that two polynomials are the same if and only if they have the same degree and the corresponding coefficients are the same. Thus the above equality of polynomials yields the system:

$$\begin{aligned} A+C &= 5 \\ -2A+B-C+D &= 6 \\ A-2B &= 7 \\ B &= 8. \end{aligned}$$

We can now solve the system using Gaussian Elimination:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 5 \\ -2 & 1 & -1 & 1 & 6 \\ 1 & -2 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 8 \end{array}\right] \xrightarrow{\text{to RREF (steps omitted)}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 23 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & -18 \\ 0 & 0 & 0 & 1 & 26 \end{array}\right]$$

We find that the unique solution is $(A,B,C,D)=(23,8,-18,26)$. Thus the desired partial fraction decomposition is

$$\frac{5x^3+6x^2+7x+8}{x^4-2x^3+x^2} = \frac{23}{x}+\frac{8}{x^2}-\frac{18}{x-1}+\frac{26}{(x-1)^2}$$

Case III: irreducible quadratic factors. Technically speaking, if you are comfortable working with complex numbers and complex-valued functions, then you only ever have to consider factorizations of the denominator into linear factors. However, for various reasons it is convenient to have a method of partial fraction decomposition which does not require us to ever leave the realm of real numbers. For instance, for the following rational function

$$\frac{10x^2+11x+12}{(x^2+1)(x+1)}$$

we *could* factor the denominator into linear factors

$$(x^2+1)(x+1) = (x+i)(x-i)(x+1),$$

and then proceed as in Case I (which we'll do below just to prove a point). However, we can just as easily keep the quadratic factor x^2+1 as is in our computation. Since in general the number of unknowns in a partial fraction decomposition must be equal to the degree of the denominator polynomial, the quadratic factor has to contribute two unknowns to the general form:

$$\frac{10x^2+11x+12}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1}+\frac{C}{x+1}$$

We now proceed as in Case II by clearing denominators and getting an equality of two polynomials:

$$\begin{aligned} 10x^2+11x+12 &= (Ax+B)(x+1)+C(x^2+1) \\ &= (A+C)x^2+(A+B)x+(B+C) \end{aligned}$$

This gives us a system of equations:

$$\begin{aligned} A+C &= 10 \\ A+B &= 11 \\ B+C &= 12 \end{aligned}$$

which we can solve using Gaussian Elimination

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 10 \\ 1 & 1 & 0 & 11 \\ 0 & 1 & 1 & 12 \end{array}\right] \xrightarrow{\text{to RREF (steps omitted)}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 9/2 \\ 0 & 1 & 0 & 13/2 \\ 0 & 0 & 1 & 11/2 \end{array}\right]$$

This gives us the desired partial fraction expansion:

$$\frac{10x^2+11x+12}{(x^2+1)(x+1)} = \frac{9x+13}{2(x^2+1)}+\frac{11}{2(x+1)}$$

We can check our work by re-doing the decomposition with complex numbers:

$$\frac{10x^2+11x+12}{(x+i)(x-i)(x+1)} = \frac{A}{x+i}+\frac{B}{x-i}+\frac{C}{x+1}$$

Cross-multiplying yields

$$10x^2+11x+12 = A(x-i)(x+1)+B(x+i)(x+1)+C(x-i)(x+i)$$

Now we plug in the three denominator roots to get linear equations for the unknowns:

- $(x=-i)$ In this case, the equation becomes $2-11i=(-2-2i)A$
- $(x=i)$ In this case, the equation becomes $2+11i=(-2+2i)B$
- $(x=-1)$ In this case, the equation becomes $11=2C$

This yields the system:

$$\begin{aligned} (-2-2i)A &= 2-11i \\ (-2+2i)B &= 2+11i \\ 2C &= 11 \end{aligned}$$

which we can solve with Gaussian Elimination:

$$\left[\begin{array}{ccc|c} -2-2i & 0 & 0 & 2-11i \\ 0 & -2+2i & 0 & 2+11i \\ 0 & 0 & 2 & 11 \end{array}\right] \xrightarrow{\text{to RREF (steps omitted)}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & (9+13i)/4 \\ 0 & 1 & 0 & (9-13i)/4 \\ 0 & 0 & 1 & 11/2 \end{array}\right]$$

This yields the desired partial fraction decomposition:

$$\frac{10x^2+11x+12}{(x+i)(x-i)(x+1)} = \frac{9+13i}{4(x+i)} + \frac{9-13i}{4(x-i)} + \frac{11}{2(x+1)}$$

Finally, to pull this decomposition back into the realm of real numbers, we add the first two fractions together (since those two correspond to a conjugate pair of roots):

$$\begin{aligned}\frac{13-9i}{4(x+i)} + \frac{9-13i}{4(x-i)} + \frac{11}{2(x+1)} &= \frac{(9+13i)(x-i)+(9-13i)(x+i)}{4(x+i)(x-i)} + \frac{11}{2(x+1)} \\ &= \frac{9x+13}{2(x^2+1)} + \frac{11}{2(x+1)}\end{aligned}$$

This shows that working with complex numbers gives the same decomposition.

Case IV: repeated quadratic factors. Finally, we arrive at perhaps the most involved case: *repeated quadratic factors*. However, the method here is really just the same as the methods in Cases II and III provided you know the rule for the general form. Here is an example:

$$\frac{6x^3+7x^2+8x+9}{(x^2+x+1)^2}$$

Since the quadratic factor x^2+x+1 has multiplicity two, it has two show up twice in the decomposition. Since the total number of unknowns needs to be four (= degree of denominator polynomial), each occurrence of the quadratic factor has to have two unknowns:

$$\frac{6x^3+7x^2+8x+9}{(x^2+x+1)^2} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2}$$

Just as before, we cross-multiply and get an equality of polynomials:

$$\begin{aligned}6x^3+7x^2+8x+9 &= (Ax+B)(x^2+x+1)+Cx+D \\ &= Ax^3+(A+B)x^2+(A+B+C)x+(B+D)\end{aligned}$$

Equating the two polynomials gives us the system of equations:

$$\begin{aligned}A &= 6 \\ A+B &= 7 \\ A+B+C &= 8 \\ B+D &= 9\end{aligned}$$

which we can solve using Gaussian Elimination:

$$\left[\begin{array}{cccc|c}1&0&0&0&6\\1&1&0&0&7\\1&1&1&0&8\\0&1&0&1&9\end{array}\right] \xrightarrow{\text{to RREF (steps omitted)}} \left[\begin{array}{cccc|c}1&0&0&0&6\\0&1&0&0&1\\0&0&1&0&1\\0&0&0&1&8\end{array}\right]$$

This gives us the desired partial fraction decomposition:

$$\frac{6x^3+7x^2+8x+9}{(x^2+x+1)^2} = \frac{6x+1}{x^2+x+1} + \frac{x+8}{(x^2+x+1)^2}$$

CHAPTER 2

Calculus review

In this section we will summarize all the important definitions and results from calculus. In general we will state these results for arbitrary nice functions, for summary of calculus results pertaining to special elementary functions, see Appendix A. First, some terminology which will simplify some things. Given the set of real numbers $\mathbb{R}$, we artificially adjoin two new symbols $+\infty$ and $-\infty$ to serve as convenient bookends of the ordering. More specifically:

Definition 2.0.1. Define the **extended real numbers** to be the set $\mathbb{R}_{\pm\infty} := \mathbb{R} \cup \{-\infty, +\infty\}$. We extend the ordering on $\mathbb{R}$ to all of $\mathbb{R}_{\pm\infty}$ by declaring:

$$-\infty \leq a \leq +\infty \quad \text{for every } a \in \mathbb{R}_{\pm\infty}.$$

Unless we state otherwise, we do not extend the arithmetic operations $+, \cdot$ on $\mathbb{R}$ to include $\pm\infty$. It is important to realize the new elements $\pm\infty$ are *not* numbers and there is not supposed to be anything super deep or special about adjoining $\pm\infty$ to our real line. We primarily introduce it because it makes certain commonly occurring statements and expressions shorter.

For instance, we can define *bounded intervals* and *unbounded intervals* with uniform notation. Given $a, b \in \mathbb{R}$ such that $a < b$, an **interval** is a set of one of the following forms:

$$\begin{aligned}
(a,b) &:= \{x \in \mathbb{R} : a < x < b\} \\
[a,b) &:= \{x \in \mathbb{R} : a \leq x < b\} \\
(a,b] &:= \{x \in \mathbb{R} : a < x \leq b\} \\
[a,b] &:= \{x \in \mathbb{R} : a \leq x \leq b\} \\
(a,+\infty) &:= \{x \in \mathbb{R} : a < x\} \\
[a,+\infty) &:= \{x \in \mathbb{R} : a \leq x\} \\
(-\infty,b) &:= \{x \in \mathbb{R} : x < b\} \\
(-\infty,b] &:= \{x \in \mathbb{R} : x \leq b\} \\
(-\infty,+\infty) &:= \mathbb{R}
\end{aligned}$$

Intervals of the form (a,b), $[a,b)$, $(a,b]$, $[a,b]$ are called **bounded intervals**. Intervals of the form $(a,+\infty)$, $[a,+\infty)$, $(-\infty,b)$, $(-\infty,b]$, $(-\infty,+\infty)$ are called **unbounded intervals**. Intervals of the form (a,b), $(a,+\infty)$, $(-\infty,b)$, $(-\infty,+\infty)$ are call **open intervals**. Intervals of the form $[a,b]$, $[a,+\infty)$, $(-\infty,b]$, $(-\infty,+\infty)$ are called **closed intervals**.

Of course, intervals are not the only types of subsets of $\mathbb{R}$ which naturally arise in this class. For instance, the natural domain of the tangent function is not an

interval, but instead a union of intervals:

$$\begin{aligned}\text{domain}(\tan t) &= \{t \in \mathbb{R} : t \neq \pi/2 + \pi k \text{ for every } k \in \mathbb{Z}\} \\ &= \bigcup_{k\in\mathbb{Z}} \Big(\frac{\pi}{2} + \pi k, \frac{\pi}{2} + \pi(k+1)\Big)\end{aligned}$$

In order to avoid too many technicalities, we will consider a subset $D \subseteq \mathbb{R}$ to be *nice* if can show up as the true domain of some function one would encounter in freshman calculus. To be specific:

Definition 2.0.2. We call a set $D \subseteq \mathbb{R}$ **nice** if it is an interval or a union of a sequence of intervals, i.e., if there exists a sequence of intervals $I_0, I_1, I_2, \ldots$ such that

$$D = \bigcup_{n\geq 0} I_n$$

In general we will always restrict our attention to functions with nice domains, with the domain of the tangent function being representative of the worst type of nice domain. If you find the definition of *nice* too technical, then surprisingly very little is lost if you just interpret the adjective *nice* in the colloquial sense. Really, these things won't matter too much for this class (since you're being graded primarily on learning how to do calculations), but we introduce this terminology anyway so that way in these notes we can still restrict ourselves to making statements which are literally true in a mathematical sense, without being overly abstract and technical.

In the exposition we will occasionally refer to *elementary functions*. We don't mean anything too precise by this, although you can take the following as a rough definition:

Rough Definition 2.0.3. An **elementary function** $f : D \to \mathbb{R}$ is any function constructed from the following operations:

(1) arithmetic operations: $+, -, \cdot, /$
(2) algebraic operations such as taking nth roots
(3) composition of functions
(4) the exponential $\exp : \mathbb{R} \to \mathbb{R}$ and logarithm $\ln : [0, +\infty) \to \mathbb{R}$,
(5) the trigonometric functions $\sin, \cos, \tan$
(6) the inverse trigonometric functions $\arcsin, \arccos, \arctan$

In other words, an *elementary function* is the type of function which shows up in freshman calculus.

2.1. Limits

In this section D is a nice set. We will review the definition and rules for computing limits. Recall that sometimes, even if a function $f : (a, b) \to \mathbb{R}$ is defined on an open interval (a, b), it sometimes still makes sense to ask what is the limit of $f(x)$ as $x \to a$, i.e., $\lim_{x\to a} f(x)$, even though f is not defined at a. This makes sense because a is an endpoint of (a, b), so there are points in (a, b) which are arbitrarily closed to a. In general we will consider functions $f : D \to \mathbb{R}$ where the domain D is a nice set. Before we define *limit*, it first makes sense to define what is the set of all points which it might make sense to take the limit to.

Definition 2.1.1. Define the **closure of** D to be the slightly larger set $\mathrm{cl}(D) \supseteq D$ defined such that for every $\alpha \in \mathbb{R}_{\pm\infty}$, we say that $\alpha \in \mathrm{cl}(D)$ if there exists $x \in \mathbb{R}$ such that either:

(1) $x < \alpha$ and $(x, \alpha) \subseteq D$, or
(2) $\alpha < x$ and $(\alpha, x) \subseteq D$.

In particular, if $\alpha \in D$, then $\alpha \in \mathrm{cl}(D)$. In other words, $\mathrm{cl}(D)$ is the same thing as D plus all the endpoints of the intervals which define D. For example:

$$\begin{aligned} \mathrm{cl}\big((1,2]\big) &= [1,2] \\ \mathrm{cl}\big((-1,0)\cup(0,1]\big) &= [-1,1] \\ \mathrm{cl}\big(\mathrm{domain}(\tan t)\big) &= \mathbb{R} \end{aligned}$$

We can now define in one definition every type of limit of a function encountered in freshman calculus:

Definition 2.1.2. Suppose $f : D \to \mathbb{R}$ is a function with nice domain D. Suppose $\alpha \in \mathrm{cl}(D)$ and $L \in \mathbb{R}_{\pm\infty}$. We say the limit of f as x approaches α exists and is equal to L, notation:

$$\lim_{x\to\alpha} f(x) = L$$

if one of the following is satisfied (depending on whether $\alpha, L = \pm\infty$ or not):

(1) $(\alpha, L \in \mathbb{R})$ for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in D$, if $0 < |x - \alpha| < \delta$, then $\big|f(x) - L\big| < \epsilon$.
(2) $(\alpha = +\infty, L \in \mathbb{R})$ for every $\epsilon > 0$, there exists $M \in \mathbb{R}$ such that for all $x \in D$, if $M < x$, then $\big|f(x) - L\big| < \epsilon$.
(3) $(\alpha = -\infty, L \in \mathbb{R})$ for every $\epsilon > 0$, there exists $M \in \mathbb{R}$ such that for all $x \in D$, if $x < M$, then $\big|f(x) - L\big| < \epsilon$.
(4) $(\alpha \in \mathbb{R}, L = +\infty)$ for every $M \in \mathbb{R}$, there exists $\delta > 0$ such that for all $x \in D$, if $0 < |x - \alpha| < \delta$, then $M < f(x)$.
(5) $(\alpha = L = +\infty)$ for every $M \in \mathbb{R}$, there exists $N \in \mathbb{R}$ such that for all $x \in D$, if $N < x$, then $M < f(x)$.
(6) $(\alpha = -\infty, L = +\infty)$ for every $M \in \mathbb{R}$, there exists $N \in \mathbb{R}$ such that for all $x \in D$, if $x < N$, then $M < f(x)$.
(7) $(\alpha \in \mathbb{R}, L = -\infty)$ for every $M \in \mathbb{R}$, there exists $\delta > 0$ such that for all $x \in D$, if $0 < |x - \alpha| < \delta$, then $f(x) < M$.
(8) $(\alpha = +\infty, L = -\infty)$ for every $M \in \mathbb{R}$, there exists $N \in \mathbb{R}$ such that for all $x \in D$, if $N < x$, then $f(x) < M$.
(9) $(\alpha = L = -\infty)$ for every $M \in \mathbb{R}$, there exists $N \in \mathbb{R}$ such that for all $x \in D$, if $x < N$, then $f(x) < M$.

In general, for this class if and when we compute limits, we will not use directly Definition 2.1.2. Instead we will use known formulas for limits of special functions (see Appendix A) along with various limit laws, including facts about continuity.

Here is the general limit law for sums of limits:

Addition Limit Law 2.1.3. *Suppose $f, g : D \to \mathbb{R}$ are functions where D is a nice domain. Further suppose $\alpha \in \mathrm{cl}(D)$ and the limits*

$$\lim_{x\to\alpha} f(x) = L_f \quad \textit{and} \quad \lim_{x\to\alpha} g(x) = L_g$$

exist with $L_f, L_g \in \mathbb{R}_{\pm\infty}$. Then:

(1) *if* $L_f, L_g \in \mathbb{R}$, *then*
$$\lim_{x\to\alpha}(f+g)(x) \;=\; L_f + L_g$$
(2) *if* $L_f = +\infty$ *and* $L_g \neq -\infty$, *or* $L_g = +\infty$ *and* $L_f \neq -\infty$, *then*
$$\lim_{x\to\alpha}(f+g)(x) \;=\; +\infty$$
(3) *if* $L_f = -\infty$ *and* $L_g \neq +\infty$, *or* $L_g = -\infty$ *and* $L_f \neq +\infty$, *then*
$$\lim_{x\to\alpha}(f+g)(x) \;=\; -\infty$$
(4) *if* $L_f = +\infty$ *and* $L_g = -\infty$, *or* $L_f = -\infty$ *and* $L_g = +\infty$, *then more subtle investigation is needed (l'Hôpital's rule).*

Here is the general limit law for products of limits:

Product Limit Law 2.1.4. *Suppose* $f, g : D \to \mathbb{R}$ *are functions where* D *is a nice domain. Further suppose* $\alpha \in \mathrm{cl}(D)$ *and the limits*
$$\lim_{x\to\alpha} f(x) \;=\; L_f \quad \textit{and} \quad \lim_{x\to\alpha} g(x) \;=\; L_g$$
exist with $L_f, L_g \in \mathbb{R}_{\pm\infty}$. *Then:*

(1) *if* $L_f, L_g \in \mathbb{R}$, *then*
$$\lim_{x\to\alpha}(f\cdot g)(x) \;=\; L_f \cdot L_g$$
(2) *if one of the following is true:*
   (a) $L_f = +\infty$ *and* $L_g > 0$
   (b) $L_f = -\infty$ *and* $L_g < 0$
   (c) $L_f < 0$ *and* $L_g = -\infty$
   (d) $L_f > 0$ *and* $L_g = +\infty$

   then
$$\lim_{x\to\alpha}(f\cdot g)(x) \;=\; +\infty$$
(3) *if one of the following is true:*
   (a) $L_f = -\infty$ *and* $L_g > 0$
   (b) $L_f = +\infty$ *and* $L_g < 0$
   (c) $L_f < 0$ *and* $L_g = +\infty$
   (d) $L_f > 0$ *and* $L_g = -\infty$

   then
$$\lim_{x\to\alpha}(f\cdot g)(x) \;=\; -\infty$$
(4) *if one of the following is true:*
   (a) $L_f = 0$ *and* $L_g = \pm\infty$
   (b) $L_f = \pm\infty$ *and* $L_g = 0$,

   then more subtle investigation is needed (l'Hôpital's rule).

Finally, here is the general limit law for quotients of functions:

Quotient Limit Law 2.1.5. *Suppose* $f, g : D \to \mathbb{R}$ *are functions where* D *is a nice domain. Define the set:*
$$D' \;:=\; \big\{x \in D : g(x) \neq 0\big\} \;\subseteq\; \mathbb{R}.$$
Assume that D' *is also nice (for us it always will be) and suppose for* $\alpha \in \mathrm{cl}(D') \subseteq \mathrm{cl}(D)$ *the limits*
$$\lim_{x\to\alpha} f(x) \;=\; L_f \quad \textit{and} \quad \lim_{x\to\alpha} g(x) \;=\; L_g$$

exist with $L_f, L_g \in \mathbb{R}_{\pm\infty}$. Then for the quotient function:

$$\frac{f}{g} : D' \to \mathbb{R}$$

we have:

(1) if $L_f \in \mathbb{R}$, and $L_g \in \mathbb{R}$ and $L_g \neq 0$, we have

$$\lim_{x\in\alpha} \left(\frac{f}{g}\right)(x) \;=\; \frac{L_f}{L_g}$$

(2) if $L_f \neq \pm\infty$ and $L_g = \pm\infty$, we have

$$\lim_{x\in\alpha} \left(\frac{f}{g}\right)(x) \;=\; 0$$

(3) if $L_f = +\infty$ and $L_g > 0$, or $L_f = -\infty$ and $L_g < 0$, then

$$\lim_{x\in\alpha} \left(\frac{f}{g}\right)(x) \;=\; +\infty$$

(4) if $L_f = +\infty$ and $L_g < 0$, or $L_f = -\infty$ and $L_g > 0$, then

$$\lim_{x\in\alpha} \left(\frac{f}{g}\right)(x) \;=\; +\infty$$

(5) otherwise a more subtle investigation is needed (l'Hôpital's rule).

2.2. Continuity

The most basic property we might wish for a function $f : D \to \mathbb{R}$ to have is that it is *continuous*. Here is the definition:

Definition 2.2.1. Suppose $f : D \to \mathbb{R}$ is a function with nice domain $D \subseteq \mathbb{R}$. We say that f is **continuous** if for every $\alpha \in D$,

$$\lim_{x\to\alpha} f(x) \;=\; f(\alpha).$$

Example 2.2.2. Here are some continuous functions:

(1) Every constant function $x \mapsto c\colon \mathbb{R} \to \mathbb{R}$ (where $c \in \mathbb{R}$) is continuous.
(2) The identity function $x \mapsto x\colon \mathbb{R} \to \mathbb{R}$ is continuous.
(3) The absolute value function $x \mapsto |x| := \sqrt{x^2}\colon \mathbb{R} \to \mathbb{R}$ is continuous.
(4) The square root function $x \mapsto \sqrt{x}\colon [0, +\infty) \to \mathbb{R}$ is also continuous.

The following shows how continuity is preserved under the basic arithmetic operations:

Proposition 2.2.3. *Suppose $f, g\colon D \to \mathbb{R}$ are continuous functions on a nice domain D. Then the following functions are also continuous on D:*

(1) $f + g\colon D \to \mathbb{R}$,
(2) $f \cdot g\colon D \to \mathbb{R}$

Furthermore, define the set

$$D' \;:=\; \big\{x \in D : g(x) \neq 0\big\}$$

and assume that D' is nice (for us it always will be). Then

(3) $f/g\colon D' \to \mathbb{R}$ is continuous.

The following tells us that continuity is preserved when you compose two composable continuous functions:

Proposition 2.2.4 (Composition and continuity)**.** *Suppose $f\colon D \to \mathbb{R}$ is continuous with nice domain D and $g\colon E \to \mathbb{R}$ is continuous with nice domain E such that $f(D) \subseteq E$. Then $g \circ f\colon D \to \mathbb{R}$ is continuous.*

Combining Example 2.2.2(3) with Proposition 2.2.4 gives us:

Corollary 2.2.5. *If $f\colon D \to \mathbb{R}$ is continuous with nice domain D, then so is $|f|\colon D \to \mathbb{R}$, given by*

$$|f|(x) := |f(x)|, \quad \text{for } x \in D.$$

The following is an important theorem about continuous functions:

Intermediate Value Theorem 2.2.6. *Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous, with $a < b \in \mathbb{R}$. Let y be a number strictly between $f(a)$ and $f(b)$, i.e.,*

$$f(a) < y < f(b) \quad \text{or} \quad f(b) < y < f(a).$$

Then there is $x_0 \in (a,b)$ such that $f(x_0) = y$.

2.3. Differentiation

In this section $D \subseteq \mathbb{R}$ is a nice set. Given a function $f : D \to \mathbb{R}$, if it is differentiable at a point in its domain, then that means the function f can be approximated suspiciously well by a linear tangent line at that point. The following proposition gives three equivalent ways of saying exactly this:

Proposition 2.3.1. *Suppose $f\colon D \to \mathbb{R}$ is a function and $\alpha \in D$. The following are equivalent:*

(1) (Standard definition) The limit

$$\lim_{x\to\alpha} \frac{f(x) - f(\alpha)}{x - \alpha} \;=\; \ell$$

exists and is finite (i.e., $\ell \in \mathbb{R}$).

(2) (Taylor definition) There exists a number $d \in \mathbb{R}$ and a function $R\colon D \to \mathbb{R}$ such that

$$f(x) \;=\; f(\alpha) + d(x-\alpha) + R(x) \quad \text{and} \quad \lim_{x\to\alpha} \frac{R(x)}{x-\alpha} \;=\; 0.$$

(3) (Carathéodory definition) There exists a function $q\colon D \to \mathbb{R}$ which is continuous at α such that

$$f(x) \;=\; f(\alpha) + q(x)(x-\alpha).$$

Furthermore, if any (equivalently all) of (1), (2), and (3) holds, then

(4) $\ell = d = q(\alpha)$, and
(5) f is continuous at α.

Definition 2.3.2. We say that function $f : D \to \mathbb{R}$ is **differentiable on** D, if for every $\alpha \in D$, the equivalent conditions of Proposition 2.3.1 hold. In this case, we define the **derivative of f at** α to be

$$f'(\alpha) \;:=\; \lim_{x\to\alpha} \frac{f(x) - f(\alpha)}{x - \alpha}.$$

In this class, since we will be working with special elementary functions and not arbitrary differentiable functions, we generally will not have to use the formal definition when computing derivatives. In general we will be able to compute all relevant derivatives by employing the following rules as well as the known formulas (see Appendix A) for the derivatives of the functions we care about.

Example 2.3.3. (1) Constant functions are differentiable with derivative 0.

(2) Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that $f(x) = x^n$. Then f is differentiable, and and for every $\alpha \in \mathbb{R}$ $f'(\alpha) = n\alpha^{n-1}$. To see this, note by The Difference of Powers Formula,

$$f(x)-f(\alpha) \;=\; x^n-\alpha^n \;=\; (x-\alpha)\cdot(x^{n-1}+\alpha x^{n-2}+\alpha^2x^{n-3}+\cdots+\alpha^{n-2}x+\alpha^{n-1}),$$

thus for $x \neq \alpha$, we have

$$\frac{f(x)-f(\alpha)}{x-\alpha} \;=\; x^{n-1}+\alpha x^{n-2}+\alpha^2x^{n-3}+\cdots+\alpha^{n-2}x+\alpha^{n-1},$$

and so

$$\lim_{x\to\alpha}\frac{f(x)-f(\alpha)}{x-\alpha} \;=\; n\cdot\alpha^{n-1}.$$

The following rules show how computing the derivative interacts with the basic arithmetic operations:

Proposition 2.3.4. *Suppose $f, g\colon D \to \mathbb{R}$ are differentiable on D. Then*

$$f+g, f\cdot g\colon D\to\mathbb{R}$$

are differentiable on D, and for every $\alpha \in D$

(1) $(f+g)'(\alpha) = f'(\alpha)+g'(\alpha)$,
(2) (product rule) $(f\cdot g)'(a) = f(a)g'(a)+f'(a)g(a)$,

Furthermore, with $D' := \{x \in D : g(x) \neq 0\} \subseteq D$, if D' is nice, then the function

$$\frac{f}{g} : D' \to \mathbb{R}$$

is differentiable and

(3) (quotient rule) for every $\alpha \in D'$

$$\left(\frac{f}{g}\right)'(\alpha) \;=\; \frac{g(\alpha)f'(\alpha)-f(\alpha)g'(\alpha)}{g^2(\alpha)}$$

Remark 2.3.5. An immediate consequence of Proposition 2.3.4(1) and (2) is that if we have constants $c, d \in \mathbb{R}$ and differentiable functions $f, g\colon D \to \mathbb{R}$, then

$$(cf+dg)' \;=\; cf'+dg'.$$

In linear algebra terms, differentiation is $\mathbb{R}$-*linear* (i.e., it is a linear transformation on the $\mathbb{R}$-vector space of differentiable functions $D \to \mathbb{R}$).

Differentiation also behaves well with *composition* of differentiable functions:

Chain Rule 2.3.6. *Suppose $f\colon D \to \mathbb{R}$, $g\colon E \to \mathbb{R}$ are differentiable functions such that $f(D) \subseteq E$. Then $g\circ f : D \to \mathbb{R}$ is differentiable, and for every $\alpha \in D$*

$$(g\circ f)'(\alpha) \;=\; g'\big(f(\alpha)\big)\cdot f'(\alpha).$$

In theory, you should be able capable of computing the derivative of any elementary function provided you know the rules 2.3.4 and 2.3.6 as well as the formulas for the derivatives of the primitive functions of interest given in Appendix A. Of course, this should not be news to you.

The following is a very useful consequence of the so-called *Mean Value Theorem for Derivatives*. Note that Corollary 2.3.7 and Identity Criterion 2.3.8 are only true when the domain is an interval.

Corollary 2.3.7. *Suppose D is an interval and $f\colon D \to \mathbb{R}$ is differentiable. Then f is a constant function iff $f'(x) = 0$ for all $x \in I$.*

A common question we might ask when it comes to *uniqueness* of solutions of ODEs is: when are two functions $f, g\colon I \to \mathbb{R}$ the same? If f and g are differentiable (which pretty much all of our functions will be), the following makes this question easier to answer:

Identity Criterion 2.3.8. *Suppose D is an interval and $f, g\colon D \to \mathbb{R}$ are differentiable such that $f'(\alpha) = g'(\alpha)$ for every $\alpha \in D$. Then there exists a constant $C \in \mathbb{R}$ such that $f(x) = g(x) + C$ for all $x \in D$. Furthermore, if there is a point $x_0 \in D$ such that $f(x_0) = g(x_0)$, then $f(x) = g(x)$ for all $x \in D$.*

Proof. The function $f - g\colon D \to \mathbb{R}$ is differentiable by Proposition 2.3.4, and $(f-g)'(x) = f'(x) - g'(x) = 0$ for all $x \in D$. By Corollary 2.3.7, there is a constant $C \in \mathbb{R}$ such that $(f - g)(x) = C$ for all $x \in D$, i.e., $f(x) = g(x) + C$ for all $x \in D$.

Now, suppose there is $x_0 \in D$ such that $f(x_0) = g(x_0)$. Then also $f(x_0) = g(x_0) + C$, so we can conclude that $C = 0$. Thus $f(x) = g(x)$ for all $x \in D$. □

2.4. Integration

Definite integrals. When it comes to integration, the most fundamental notion is to define the following: given a function $f : [a, b] \to \mathbb{R}$, what does it mean for the function f to be *integrable* on $[a, b]$ and how do you define $\int_a^b f(t)\,dt$ if this integral is to exist? We will not dive into this question and instead assume you have a working understanding of what this means to you. In particular, we define:

Definition 2.4.1. Suppose $a < b \in \mathbb{R}$. We say that the function $f : [a, b] \to \mathbb{R}$ is **integrable** if the definite integral

$$\int_a^b f(t)\,dt$$

exists and is finite (i.e., it equals a real number from $\mathbb{R}$). If $f : [a, b] \to \mathbb{R}$ is integrable, then we also define:

$$\int_b^a f(t)\,dt \;:=\; -\int_a^b f(t)\,dt$$

Given any function $g : D \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, we define:

$$\int_\alpha^\alpha g(t)\,dt \;:=\; 0$$

Here are some basic facts about what types of functions are integrable:

Fact 2.4.2. Suppose $f : [a, b] \to \mathbb{R}$ is a function. Then:

(1) if f is continuous, then f is integrable,

(2) if f is piecewise continuous, then f is integrable,
(3) if f is integrable and $\tilde{f} : [a, b] \to \mathbb{R}$ is a function such that the set:

$$\{x \in [a, b] : f(x) \neq \tilde{f}(x)\}$$

is finite, then $\tilde{f}$ is also integrable and

$$\int_a^b f(t)\, dt \;=\; \int_a^b \tilde{f}(t)\, dt$$

Fact 2.4.2 tells us that basically every function $f : [a, b] \to \mathbb{R}$ we come across in this class will be integrable. Furthermore, 2.4.2(3) tells us that as far as computing integrals are concerned, we can safely change finitely many values of the function and still arrive at the same answer (for instance, if you are integrating a step function and you're not sure about the values at the endpoints).

The following law for computing definite integrals is used all the time:

Lemma 2.4.3 (Linearity of Integration)**.** *Let $f, g\colon [a, b] \to \mathbb{R}$ be integrable functions, and let $\alpha \in \mathbb{R}$. Then*

(1) $\alpha f\colon [a, b] \to \mathbb{R}$ is integrable, and $\int_a^b \alpha f(t)\, dt = \alpha \int_a^b f(t)\, dt$,
(2) $f + g\colon [a, b] \to \mathbb{R}$ is integrable, and $\int_a^b (f + g)(t)\, dt = \int_a^b f(t)\, dt + \int_a^b g(t)\, dt$.

The following is also very useful, especially if the behavior of a function changes on different intervals:

Lemma 2.4.4 (Additivity over intervals)**.** *Suppose $f\colon [a, b] \to \mathbb{R}$ is a function and $c \in (a, b)$. Then f is integrable on $[a, b]$ iff f is integrable on $[a, c]$ and $[c, b]$. In this case, we have*

$$\int_a^b f(t)\; dt \;=\; \int_a^c f(t)\; dt + \int_c^b f(t)\; dt.$$

The following two theorems tell us that *integration* and *differentiation* are inverse operations, which is what makes integration so useful when it comes to solving differential equations. First a definition:

Definition 2.4.5. Suppose $f : D \to \mathbb{R}$ is a continuous function with a nice domain $D \subseteq \mathbb{R}$. A function $F : D \to \mathbb{R}$ is called an **antiderivative** of f if:

(i) F is differentiable, and
(ii) for every $t \in D$, $F'(t) = f(t)$.

The so-called *first fundamental theorem of calculus* provides us a method of computing the exact value of the definite integral of a function provided we have available to us an antiderivative of that function:

First Fundamental Theorem of Calculus 2.4.6. *Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function on $[a, b]$ and differentiable on (a, b). Then:*

$$\int_a^b f'(t)\, dt \;=\; f(a) - f(b).$$

The so-called *second fundamental theorem of calculus* provides us a method of using definite integrals to construct an antiderivative of a continuous function:

Second Fundamental Theorem of Calculus 2.4.7. *Suppose $f : D \to \mathbb{R}$ is a continuous function with a nice domain $D \subseteq \mathbb{R}$, and fix $t_0 \in D$. Let $I \subseteq D$ be the largest interval such that $t_0 \in I$. Consider the function $F : I \to \mathbb{R}$ defined by*

$$F(t) \;:=\; \int_{t_0}^{t} f(s)\,ds$$

for every $t \in I$. Then

1. *F is differentiable on I, and*
2. *$F'(t) = f(t)$ for every $t \in I$, i.e., F is an antiderivative of f on the interval I.*

Indefinite integrals. When we later determine the general solution of a differential equation, we need to be able to find (and parametrize) *all* solutions of the differential equation, not just a particular one. In terms of antiderivatives, this means we need to be able to find (and parametrize) *all* antiderivatives of a particular function, not just one antiderivative. This is taken care of by the notion of *indefinite integral*:

Definition 2.4.8. Suppose $f : D \to \mathbb{R}$ is a continuous function with a nice domain $D \subseteq \mathbb{R}$. The **indefinite integral** of f is an infinite family of functions:

$$F(t; C) \;=\; F(t) + C$$

where $C \in \mathbb{R}$ and $F : D \to \mathbb{R}$ is a particular antiderivative of f. This situation is often denoted by writing:

$$\int f(t)\,dt \;=\; F(t) + C.$$

Remark 2.4.9. Technically speaking, the indefinite integral of f really should be the family of *all* antiderivatives of f. In particular, each so-called *connected component* of the domain of f requires its own constant of integration. For instance, for the function $f(t) = 1/t$ viewed as a function $(-\infty, 0) \cup (0, +\infty) \to \mathbb{R}$, the indefinite integral really should be:

$$\int \frac{dt}{t} \;=\; \begin{cases} \ln(t) + C_1 & \text{if } t > 0 \\ \ln(-t) + C_2 & \text{if } t < 0 \end{cases}$$

where $C_1, C_2 \in \mathbb{R}$ could be the same number, or could be different. Simply writing:

$$\int \frac{dt}{t} \;=\; \ln|t| + C$$

does not actually give us every possible antiderivative of $1/t$ on the domain $(-\infty, 0) \cup (0, +\infty)$ because it requires us to use the same constant of integration on both "connected components" $(-\infty, 0)$ and $(0, +\infty)$. This is a very minor issue which we are happy to ignore since the particular solutions to initial value problems (which we hope to be unique) will have intervals as their domain.

We also have the second fundamental theorem of calculus for indefinite integrals:

Second Fundamental Theorem of Calculus 2.4.10 (Indefinite version)**.** *Suppose $f : D \to \mathbb{R}$ is a continuous function with a nice domain $D \subseteq \mathbb{R}$. Then*

$$\frac{d}{dt} \int f(t)\,dt \;=\; f(t).$$

Theorem 2.4.10 is to be interpreted as: for every antiderivative $F(t) + C$ of $f(t)$,

$$\frac{d}{dt}\big(F(t) + C\big) \;=\; f(t).$$

CHAPTER 3

First-order differential equations

3.1. Implicit differential equations

In this course we will be primarily concerned with first-order differential equations, as well as higher-order *linear* differential equations. This begs the question:

What is a differential equation and what is the order of a differential equation?

We will answer this question by first giving a very general definition of *differential equation* which will encompass nearly all differential equations we will encounter in this Chapter and in Chapter 5:

Definition 3.1.1. An **implicit differential equation (of order r)** is an equation which can be written in the form

$$F(t, y, y', y'', \dots, y^{(r)}) \;=\; 0 \tag{$\dagger$}$$

where F is a real-valued function of $r+2$ variables. The **order** is the order r of the highest derivative $y^{(r)}$ of y which appears in the equation.

A **solution** to (†) is a function $y : I \to \mathbb{R}$ (where $I \subseteq \mathbb{R}$ is an interval) which is differentiable at least r times such that

$$F\big(t, y(t), y'(t), \dots, y^{(r)}(t)\big) \;=\; 0 \quad \text{for every } t \in I,$$

i.e., for every $t \in I$, when you plug $t, y(t), y'(t), \dots, y^{(r)}(t)$ into the function F the output is zero.

We now give some examples of implicit differential equations and some of their solutions, in increasing order of order.

Zeroth order. Here is an implicit differential equation of order 0:

$$y^5 + 2y^4 + 3y^3 + 4y^2 + 5y + 6 \;=\; 0 \tag{3.1}$$

Given a solution $\alpha \in \mathbb{R}$ of the polynomial equation

$$X^5 + 2X^4 + 3X^3 + 4X^2 + 5X + 6 \;=\; 0,$$

the function $y : \mathbb{R} \to \mathbb{R}$ defined by $y(t) := \alpha$ for all $t \in \mathbb{R}$ (i.e., the function with constant value α) is a solution of (3.1). This example should convince you that the subject of differential equations already encompasses all of one- and two-variable polynomial equations. In particular, we shouldn't get our hopes up that we will be able to solve too many higher-order differential equations in general.

First order. We will give two examples of a first-order differential equation. The first one takes full advantage of the *implicit* part of the definition:

Example 3.1.2 (Clairaut)**.** The differential equation:

$$y - ty' + \exp y' = 0 \tag{3.2}$$

Every solution $y : \mathbb{R} \to \mathbb{R}$ of (3.2) has the form

$$y(t) = Ct + \exp C$$

where $C \in \mathbb{R}$ is some fixed constant. Note that even though (3.2) is complicated, it is actually pretty easy to check that the given solution is actually correct. Indeed, first compute the derivative of y:

$$y'(t) = C$$

and then plug $t, y(t), y'(t)$ into (3.2) and notice that everything cancels out:

$$y(t) - ty'(t) + \exp y'(t) = Ct + \exp C - tC + \exp C = 0.$$

This illustrates another important lesson:

Checking that a given function is/is not a solution to a differential equation is usually easy, even if the given differential equation is hard/impossible.

Indeed, it is simply a matter of computing r derivatives and then plugging them into the equation and seeing if everything cancels out. Of course, we will be more interested in solving differential equations than checking whether a candidate solution is correct or not. However, it is reassuring to know that at least one direction of the process is fairly easy.

The next differential equation is a more typical example of a differential equation which we will study:

Example 3.1.3 (Logistic equation)**.** Let $b, c > 0$ be fixed positive constants. Then the **logistic equation** is the differential equation:

$$y' - y(b - cy) = 0$$

For every nonzero constant $C \in \mathbb{R} \setminus \{0\}$ we have a solution $y : \mathbb{R} \to \mathbb{R}$ defined by:

$$y(t) = \frac{b}{c} \cdot \frac{1}{1 + C\exp(-bt)}$$

Furthermore, the constant functions $y = 0$ and $y = b/c$ are also solutions. (Exercise: check this!) We will study the logistic equation in more detail later, including how to derive these solutions.

Second order. Here is a typical example of a second-order differential equation we will study:

$$y'' - 3y' + 2y = 0 \tag{3.3}$$

Every solution $y : \mathbb{R} \to \mathbb{R}$ of (3.3) is of the form:

$$y(t) = C_1 \exp 2t + C_2 \exp t$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants. Generally speaking, for second-order differential equations there will be two constants of integration we need to find. This reflects the fact that the equation involves a first and second derivative (so

somewhere we are doing two integrals, each one with its own constant of integration). Equation (3.3) is an example of a *second-order linear differential equation with constant coefficients*, which will be one of the main equations of interest in Chapter 5.

3.2. Differential equations in normal form

Definition 3.1.1 casts a very wide net. In general most differential equations we will encounter can be put into a slightly simpler form: *normal form*.

Definition 3.2.1. A **differential equation of order r in normal form** (or an **explicit differential equation of order** r) is a differential equation which can be written in the form

$$y^{(r)} \;=\; F(t, y, y', y'', \dots, y^{(r-1)}) \tag{$\dagger$}$$

where F is a real-valued function of $r+1$ variables. A **solution** of ($\dagger$) is a function $y : I \to \mathbb{R}$ (where $I \subseteq \mathbb{R}$ is an interval) which is at least r times differentiable, such that for every $t \in I$:

$$y^{(r)}(t) \;=\; F\big(t, y(t), y'(t), \dots, y^{(r-1)}(t)\big)$$

In other words, an implicit differential equation of order r can be put into normal form if it is possible to solve for the highest derivative $y^{(r)}$ in terms of the lower derivative $y, y', \dots, y^{(r-1)}$ and t.

Example 3.2.2. (1) A zeroth-order differential equation in normal form is an equation of the form:

$$y \;=\; F(t)$$

Clearly, the function $y(t) := F(t)$ is a solution. We will never be interested in explicit zeroth-order differential equations.

(2) A first-order differential equation in normal form is an equation of the form:

$$y' \;=\; F(t, y)$$

The logistic equation from Example 3.1.3 can be put into normal form:

$$y' \;=\; y(b - cy)$$

It is not clear whether the equation from Example 3.1.2

$$y - ty' + \exp y' \;=\; 0$$

can be put into normal form since this would involve solving for y'. In general, for the equations we deal with there will be no issue with rewriting them in normal form.

(3) A second-order differential equation in normal form is an equation of the form:

$$y'' \;=\; F(t, y, y').$$

Equation (3.3) can be written in normal form:

$$y'' \;=\; 3y' - 2y$$

This concludes our discussion of general-order differential equations. For the rest of the chapter we will focus on first-order differential equations in normal form.

Explicit first-order differential equations. Recall that an explicit first-order differential equation is an equation which can be written in the form:

$$y' = F(t, y) \tag{3.4}$$

where F is a real-valued function of two variables. A **solution** to (3.4) is a differentiable function $y : I \to \mathbb{R}$ ($I \subseteq \mathbb{R}$ is an interval) such that for all $t \in I$,

$$y'(t) = F\big(t, y(t)\big)$$

Solutions are also referred to as **integral curves** or **solution curves**, especially when we want to emphasize the geometric properties of the solution.

We will often be interested in obtaining a specific solutions which passes through a given point $(t_0, y(t_0))$. The best way to do this is to first find *all* solutions of the differential equation, and then find the particular solution we are interested in.

Definition 3.2.3. The **general solution** of (3.4) is a family[1] of functions $y(t; C)$ which depends on a parameter $C \in \mathbb{R}$ such that:

(1) for every valid parameter C_0, the function $y(t; C_0)$ is a solution of (3.4), and
(2) every solution of (3.4) is of the form $y(t; C_1)$ for some valid parameter C_1.

A **particular solution** is a function of the form $y(t) = y(t; C_0)$ for some fixed value C_0.

Example 3.2.4. Consider the differential equation

$$y' = t \tag{3.5}$$

We wish to find the general solution to (3.5). Integrating both sides, we find that

$$y(t) = \frac{1}{2}t^2 + C$$

for some constant of integration $C \in \mathbb{R}$. We claim that the general solution is

$$y(t; C) = \frac{1}{2}t^2 + C$$

where C can be any real number. Indeed, for every specific $C_0 \in \mathbb{R}$, the function $y(t) = \frac{1}{2}t^2 + C_0$ is a solution. Furthermore, if $\bar{y}(t)$ is also a solution, then $\bar{y}'(t) = t$, and thus

$$\big(\bar{y}(t) - y(t; 0)\big)' = \big(\bar{y}(t) - \frac{1}{2}t^2\big)' = t - t = 0$$

which shows that $\bar{y}(t)$ and $y(t; 0)$ differ by a constant. Thus there exists $C_1 \in \mathbb{R}$ such that $\bar{y}(t) = y(t; C_1)$. We conclude that $y(t; C)$ is the general solution of (3.5). Here are some particular solutions:

$$y(t) = y(t; 3) = \frac{1}{2}t^2 + 3$$
$$y(t) = y(t; -10) = \frac{1}{2}t^2 - 10.$$

The problem of finding a specific particular solution will be formulated as an *initial value problem*:

[1]The notation $y(t; C)$ is meant to suggest that the function $y(t)$ depends also on the parameter C. Each time you choose a specific value C_0 for C, then you get a particular solution $y(t) := y(t; C_0)$.

Definition 3.2.5. An **initial value problem** is a pair of two conditions:

(i) a differential equation:
$$y' = F(t, y)$$

(ii) a specific point which the solution must pass through:
$$y(t_0) = y_0,$$
where $(t_0, y_0) \in \mathbb{R}^2$. This is called the **initial condition**.

Example 3.2.6. We wish to solve the following initial value problem:

(i) $y' = t$
(ii) $y(3) = 7$

We have already found that the general solution to (i) is
$$y(t; C) = \frac{1}{2}t^2 + C$$
We will use (ii) to solve for the exact value of C:
$$y(3) = 7 = \frac{1}{2} \cdot 3^2 + C$$
and so
$$C = 7 - \frac{9}{2} = \frac{5}{2}.$$
We conclude that the solution to the above initial value problem is:
$$y(t) = y(t; 5/2) = \frac{1}{2}t^2 + \frac{5}{2}.$$

Direction fields. One of the remarkable features of explicit first-order differential equations is that, even if some of them might be difficult to solve, it is usually pretty easy to make a rough sketch of the general solutions. This is because the equation
$$y' = F(t, y)$$
tells us what the derivative of the solution needs to be at each point (t, y) in the plane. We make this precise with the notion of a *direction field*.

Definition 3.2.7. A **direction field** for the equation
$$y' = F(t, y)$$
is a plot where at each point (t_0, y_0) you draw a tiny line segment with slope $F(t_0, y_0)$.

Of course in practice when you (or a computer) draw a direction field, you can't possibly draw such a line segment at every point in the plane (since there are infinitely many such points). Instead you draw enough tiny line segments (say, at integer or half-integer coordinates) in order to get a sense of the general behavior of the direction field. Once you have an accurate direction field, you can sketch an approximation of a solution by "following the direction of the direction field".

Example 3.2.8. Consider the logistic equation
$$y' = y(3 - y) \tag{3.6}$$
In Figure 3.1 we plot the direction field for (3.6). We also include four solution curves corresponding to four different initial conditions.

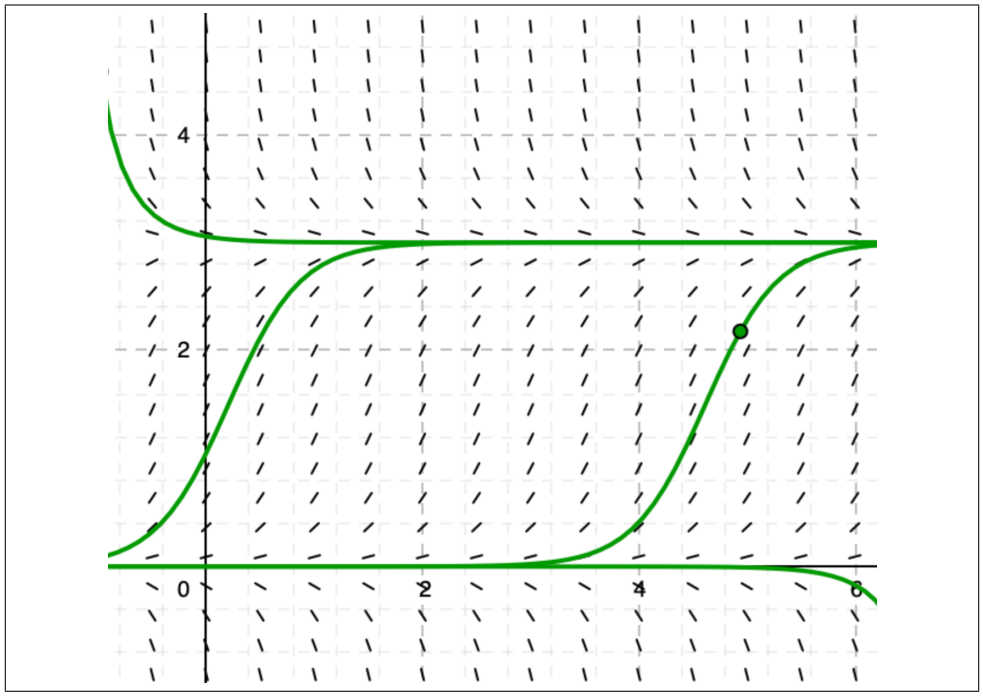


FIGURE 3.1. Direction field for the logistic equation $y' = y(3-y)$ and several solution curves.

We make the following observations:

(1) At each point (t_0, y_0), the slope only depends on y_0. This is because $y(3-y)$ only depends on y and not on t.
(2) This suggests that if $y(t)$ is a solution to (3.6), then so is $y(t+C)$ for any constant C.
(3) The direction field suggests that the constant functions
$$y(t) \;=\; 0 \quad \text{and} \quad y(t) \;=\; 3$$
are both solutions to (3.6). This is indeed the case, as can be easily verified.
(4) There are many other non-constant solutions as well, we will learn how to solve for them in Section 3.5.

Of course, by merely plotting a direction field and sketching a solution curve, you are not actually solving the differential equation yet. However, this procedure provides valuable insight into the nature of the solutions which can be very fruitful. In some sense, this is the starting point for the *qualitative study of differential equations*.

3.3. First-order linear differential equations

We now arrive at the first family of differential equations which we will study in detail, the so-called *first-order linear differential equations*.

Definition 3.3.1. A **first-order linear differential equation** is a differential equation which can be written in the form:
$$y' + f(t)y \;=\; g(t)$$

where f, g are real-valued functions of the variable t. The function $f(t)$ and $g(t)$ are called[2] the **coefficient functions**.

As we shall see, solving a first-order linear differential equation really boils down to performing an integration. We will work up to the general case (where both $f(t)$ and $g(t)$ are nonzero functions) in several steps.

Direct integration. Consider first the case where $f(t) = 0$ for all t. We call the resulting differential equation:

$$y' \;=\; g(t)$$

a **direct integration** differential equation. This is because you can directly solve this differential equation by integrating g and, if need be, solving for C with the initial condition. Here is an example:

Example 3.3.2. Consider the initial value problem:

(i) $y' = \sqrt{t}$,
(ii) $y(4) = 6$.

Integrating the differential equation we obtain

$$y(t) \;=\; 2/3t^{3/2} + C.$$

Using the initial condition we get

$$y(4) \;=\; 6 \;=\; 2/3(4)^{3/2} + C$$

and so $C = 6 - 16/3 = 2/3$. So the solution to the above initial value problem is

$$y(t) \;=\; 2/3t^{3/2} + 2/3.$$

In Figure 3.2 we plot the corresponding solution curve together with the direction field. Notice that the solution exists on the interval $[0, +\infty)$, and this is the possible interval on which the solution can exist and remain a solution because $g(t) = \sqrt{t}$ is only defined on $[0, +\infty)$.

We also remark that in Figure 3.2 we see that the direction field only depends on t and not on y. This observation allows us to guess (if we didn't know it already) that any two solutions of (i) differ by a vertical translation (i.e., adding a constant). This indeed is also the case for general *direct integration* differential equations.

[2]sometimes just $f(t)$ is called the **coefficient function** and $g(t)$ is called the **forcing function**.

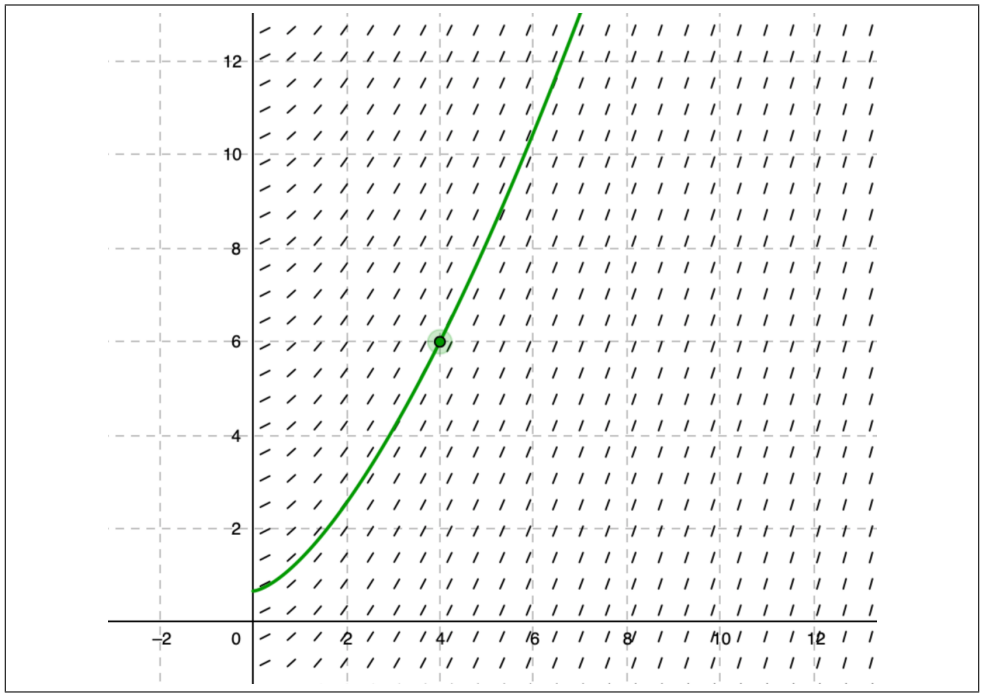


FIGURE 3.2. Direction field for the equation $y' = \sqrt{t}$ and the solution curve passing through the point $(4, 6)$.

Theorem 3.3.3 (Direct Integration)**.** *Suppose $g : D \to \mathbb{R}$ is a continuous function with nice domain $D \subseteq \mathbb{R}$. Consider the differential equation:*

(i) $y' = g(t)$

(1) The general solution of (i) is given by

$$y(t) \;=\; y(t; C) \;=\; \int g(t)\, dt + C$$

Furthermore, suppose we are also given an initial condition

(ii) $y(t_0) = y_0$*, where* $t_0 \in D$ *and* $y_0 \in \mathbb{R}$*.*

(2) Then the initial value problem (i)+(ii) has the unique solution:

$$y(t) \;=\; \int_{t_0}^{t} g(s)\, ds + y_0$$

(3) The ***interval of existence*** *of this solution (i.e., the largest interval containing t_0 for which this function remains a solution) is the largest interval $I \subseteq \mathbb{R}$ such that:*

(a) $t_0 \in I$*, and*

(b) $I \subseteq D$*.*

The homogeneous case. We next consider the case where $g(t)$ is the constant zero function and $f(t)$ is possibly nonzero.

Definition 3.3.4. A first-order linear differential equation is said to be **homogenous** if it is of the form:

$$y' + f(t)y \;=\; 0.$$

Solving the homogeneous case requires knowing a trick: multiplication by a so-called *integrating factor*. We illustrate this first with an example:

Example 3.3.5. Consider the homogeneous first-order linear differential equation:

$$y' + \frac{1}{t}y \;=\; 0 \tag{3.7}$$

Here we are regarding the coefficient function $1/t$ to have domain $(-\infty, 0) \cup (0, +\infty)$. First observe that if $\mu(t)$ is any function which is never zero, then the differential equation

$$\mu(t)\left(y' + \frac{1}{t}y\right) \;=\; 0$$

has the same solutions as equation (3.7). We will use the following choice of $\mu(t)$:

$$\mu(t) \;:=\; \exp\left(\int \frac{dt}{t}\right) \;=\; \exp \ln|t| \;=\; |t|$$

where the domain of $\mu(t)$ is also $(-\infty, 0) \cup (0, +\infty)$. Then we multiply the lefthand side of (3.7) by $\mu(t)$ to obtain:

$$|t|\left(y' + \frac{1}{t}y\right) \;=\; |t|y' + \operatorname{sgn}(t)y \;=\; \big(|t|y\big)' \;=\; 0.$$

In other words, multiplying through by the integrating factor $\mu(t)$ allows us to view the lefthand side as the derivative of a single function of t. Next we integrate both sides of

$$\big(|t|y\big)' \;=\; 0$$

to obtain

$$|t|y(t) \;=\; C,$$

or rather,

$$y(t) \;=\; \frac{C}{|t|}.$$

Here the function $y(t)$ also has domain $(-\infty, 0) \cup (0, +\infty)$.

Here is how to handle the general homogeneous case:

Theorem 3.3.6. *Suppose $f : D \to \mathbb{R}$ is a continuous function with nice domain $D \subseteq \mathbb{R}$ consider the differential equation:*

(i) $y' + f(t)y = 0$

(1) Define the ***integrating factor*** *to be the function $\mu : D \to \mathbb{R}$ given by:*

$$\mu(t) \;:=\; \exp\left(\int f(t)\,dt\right)$$

(here $\int f(t)\,dt$ can be any antiderivative of $f(t)$, the constant of integration does not matter). Then we can multiply (i) by μ to obtain:

$$\mu(t)\big(y' + f(t)y\big) \;=\; \big(\mu(t)y\big)' \;=\; 0.$$

(2) The general solution of (i) is given by:

$$y(t) \;=\; y(t; C) \;=\; \frac{C}{\mu(t)} \;=\; C\exp\left(-\int f(t)\,dt\right)$$

Furthermore, suppose we are also given an initial condition

(ii) $y(t_0) = y_0$, *where $t_0 \in D$ and $y_0 \in \mathbb{R}$.*

(3) Then the initial value problem (i)+(ii) has the unique solution:

$$y(t) \;=\; y_0 \exp\left(-\int_{t_0}^{t} f(s)\,ds\right) \;=\; \frac{y_0}{\mu(t)}$$

where $\mu(t) := \exp(\int_{t_0}^{t} f(s)\,ds)$.

(4) The interval of existence of this solution is the largest interval $I \subseteq \mathbb{R}$ *such that:*

(a) $t_0 \in I$, *and*

(b) $I \subseteq D$.

The general case. The general first-order linear case contains both the *direct integration case* and the *homogeneous case*. The trick with the integrating factor also works for the general case. We give an example first:

Example 3.3.7. Consider the first-order linear differential equation:

$$y' + \sin(t)y \;=\; \sin^3 t \tag{3.8}$$

The first thing to do is to compute the integrating factor:

$$\mu(t) \;=\; \exp\left(\int \sin t\,dt\right) \;=\; \exp(-\cos t)$$

Next we multiply both sides of (3.8) by $\mu(t)$ to obtain:

$$\mu(t)\big(y' + \sin(t)y\big) \;=\; \big(\exp(-\cos t)y\big)' \;=\; \sin^3 t \exp(-\cos t)$$

Integrating both sides yields:

$$\exp(-\cos t)y(t) \;=\; \int \sin^3 t \exp(-\cos t)\,dt \;=\; -4\exp(-\cos t)\cos^4(t/2) + C$$

Solving for $y(t)$ gives us the general solution:

$$y(t) \;=\; -4\cos^4(t/2) + C\exp\cos t$$

The general case works much the same way:

Theorem 3.3.8. *Suppose* $f : D \to \mathbb{R}$ *and* $g : E \to \mathbb{R}$ *are continuous functions with nice domains* $D, E \subseteq \mathbb{R}$ *and consider the differential equation*

(i) $y' + f(t)y = g(t)$

(1) Define the ***integrating factor*** *to be the function* $\mu : D \to \mathbb{R}$ *given by:*

$$\mu(t) \;:=\; \exp\left(\int f(t)\,dt\right)$$

(here $\int f(t)\,dt$ *can be any antiderivative of* $f(t)$*, the constant of integration does not matter). Then we can multiply (i) by* μ *to obtain:*

$$\mu(t)\big(y' + f(t)y\big) \;=\; \big(\mu(t)y\big)' \;=\; \mu(t)g(t).$$

(2) Then general solution of (i) is then given by:

$$y(t) \;=\; y(t;C) \;=\; \frac{1}{\mu(t)}\int \mu(t)g(t)\,dt + \frac{C}{\mu(t)}$$

Furthermore, suppose we are also given an initial condition

(ii) $y(t_0) = y_0$, *where* $t_0 \in D \cap E$ *and* $y_0 \in \mathbb{R}$.

(3) Then the initial value problem (i)+(ii) has the unique solution:

$$y(t) \;=\; \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s) g(s)\, ds + \frac{y_0}{\mu(t)}$$

where $\mu(t) := \exp(\int_{t_0}^{t} f(s)\, ds)$.

(4) The interval of existence of this solution is the largest interval $I \subseteq \mathbb{R}$ *such that:*

(a) $t_0 \in I$,

(b) $I \subseteq D$, *and*

(c) $I \subseteq E$.

PROOF. (1) First we will justify the key property of the integrating factor:

$$\mu(t)\big(y' + f(t)y\big) \;=\; \big(\mu(t)y\big)'$$

Note that:

$$\begin{aligned}
\big(\mu(t)y\big)' &= \mu(t)y' + \mu'(t)y \quad \text{by the product rule 2.3.4(2)} \\
&= \mu(t)y' + \frac{d}{dt}\left[\exp\left(\int f(t)\, dt\right)\right] y \\
&= \mu(t)y' + \exp\left(\int f(t)\, dt\right) \frac{d}{dt}\left[\int f(t)\, dt\right] y \quad \text{by the Chain Rule 2.3.6} \\
&= \mu(t)y' + \mu(t)f(t)y \quad \text{by Theorem 2.4.10} \\
&= \mu(t)\big(y' + f(t)y\big)
\end{aligned}$$

(2 part 1) Next, we will check that for every $C \in \mathbb{R}$, the function $y(t;C)$ is a solution. Since $\mu(t)$ is a function which is everywhere nonzero, it follows that $y(t;C)$ is a solution of

$$y' + f(t)y \;=\; g(t)$$

if and only if $y(t;C)$ is a solution of

$$(\dagger) \qquad \mu(t)\big(y' + f(t)y\big) \;=\; \mu(t)g(t).$$

We will verify that $y(t;C)$ is indeed a solution of (†). Note that:

$$\begin{aligned}
\mu(t)\big(y'(t;C) + f(t)y(t;C)\big) &= \big(\mu(t)y(t;C)\big)' \quad \text{by (1)} \\
&= \left(\int \mu(t)g(t) + C\right)' \\
&= \mu(t)g(t) \quad \text{by Theorem 2.4.10}
\end{aligned}$$

This verifies part (1) of Definition 3.2.3. We will return to verifying part (2) of the definition later.

(3 part 1) We now verify that

$$y(t) \;=\; \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s) g(s)\, ds + \frac{y_0}{\mu(t)}$$

is a solution to the initial value problem (i)+(ii). It is clear that $y(t)$ is a solution to (i) since it is a particular instance of the general solution in (2). To verify (ii),

we notice first that:

$$\begin{aligned}\mu(t_0) &= \exp\left(\int_{t_0}^{t_0} f(s)\,ds\right)\\ &= \exp(0) \quad \text{by Definition 2.4.1}\\ &= 1.\end{aligned}$$

Next, we observe:

$$\begin{aligned}y(t_0) &= \frac{1}{\mu(t_0)}\int_{t_0}^{t_0}\mu(s)g(s)\,ds + \frac{y_0}{\mu(t_0)}\\ &= \int_{t_0}^{t_0}\mu(s)g(s)\,ds + y_0\\ &= 0 + y_0 \quad \text{by Definition 2.4.1}\\ &= y_0.\end{aligned}$$

Thus $y(t)$ is a solution to the initial value problem (i)+(ii). We will prove uniqueness below.

(4) First observe that the interval $I \subseteq D$ is the largest possible interval which contains t_0 which we could hope to have as the domain of the solution. This is because the differential equation (i) is only defined on the set $D \cap E$ (the on which both coefficient functions f and g are defined).

(2 part 2) and (3 part 2) are taken care of by the next lemma. □

Lemma 3.3.9. *Suppose $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are continuous functions with nice domains $D, E \subseteq \mathbb{R}$. Suppose that $y_0, y_1 : I \to \mathbb{R}$ are two differentiable functions such that:*

(a) $I \subseteq \mathbb{R}$ is an interval contained in both D and E,
(b) for $i = 0, 1$, $y_i'(t) + f(t)y_i(t) = g(t)$ for every $t \in I$, i.e., y_0 and y_1 are both solutions to the differential equation:

$$y' + f(t)y = g(t)$$

Then:

(1) there exists a constant $C \in \mathbb{R}$ such that for every $t \in I$,

$$y_0(t) = y_1(t) + \frac{C}{\mu(t)}$$

where $\mu(t) = \exp(\int f(t)\,dt)$.
(2) Furthermore, if there is $t_0 \in I$ such that $y_0(t_0) = y_1(t_0)$, then $C = 0$ and so for every $t \in I$, $y_0(t) = y_1(t)$.

Proof. It follows from (b) that for every $t \in I$,

$$(y_0 - y_1)'(t) + f(t)(y_0 - y_1)(t) = 0.$$

Multiplying both sides by $\mu(t)$ yields for every $t \in I$:

$$\mu(t)\big((y_0 - y_1)'(t) + f(t)(y_0 - y_1)(t)\big) = 0$$

which we can rewrite as:

$$\big(\mu(t)(y_0 - y_1)(t)\big)' = 0$$

for every $t \in I$. Since I is an interval, by Corollary 2.3.7 there is a constant $C \in \mathbb{R}$ such that for every $t \in I$:

$$\mu(t)(y_0 - y_1)(t) \;=\; C.$$

Thus for every $t \in I$,

$$y_0(t) \;=\; y_1(t) + \frac{C}{\mu(t)}.$$

This establishes (1). For (2), suppose there is $t_0 \in I$ such that $y_0(t_0) = y_1(t_0)$. Plugging in t_0 into the above equation then yields:

$$y_0(t_0) \;=\; y_1(t_0) + \frac{C}{\mu(t_0)}$$

which simplifies to

$$0 \;=\; \frac{C}{\mu(t_0)}.$$

This gives us $C = 0$. In particular, for every $t \in I$, we have

$$y_0(t) \;=\; y_1(t).$$

This establishes (2). □

Remark about absolute values in the integrating factor. In this subsection we make a few remarks about the role of absolute values in the integrating factor $\mu(t)$ which appears when computing a solution of a first-order linear differential equation. We begin with a soft rule-of-thumb:

Rule of Thumb 3.3.10. *If there are absolute values which arise in*

$$\mu(t) \;=\; \exp\left(\int f(t)\,dt\right)$$

as a result of an expression $\ln|\cdots|$ *arising in* $\int f(t)\,dt$*, then these absolute values can be safely removed in the final expression for* $\mu(t)$*.*

TLDR EXPLANATION. Suppose we are looking at the first-order linear differential equation:

$$y' + f(t)y \;=\; g(t)$$

The only relevant property that we need an integrating factor $\mu(t)$ to satisfy is that it simplifies the lefthand side:

$$\mu(t)\big(y' + f(t)y\big) \;=\; \big(\mu(t)y\big)' \tag{$\dagger$}$$

However, if $\mu(t)$ satisfies (†), then so does $-\mu(t)$:

$$-\mu(t)\big(y' + f(t)y\big) \;=\; \big(-\mu(t)y\big)'$$

since this amounts to multiplying (†) through by -1. Now suppose that $\mu(t) = |u(t)|$ for some differentiable function $u(t)$. Then by definition,

$$\mu(t) \;=\; \begin{cases} u(t) & \text{if } u(t) > 0 \\ -u(t) & \text{if } -u(t) < 0 \end{cases}$$

The claim is that the function $u(t)$ (i.e., μ without the absolute values) can serve as an integrating factor. This is essentially because:

$$u(t) \;=\; \begin{cases} \mu(t) & \text{if } u(t) > 0 \\ -\mu(t) & \text{if } u(t) < 0 \end{cases}$$

Since both $\mu(t)$ and $-\mu(t)$ work perfectly well as integrating factors, it follows that in all cases, the function $u(t)$ works as an integrating factor. □

We hesitate to call 3.3.10 a "Fact" or "Theorem" because this would require a complete investigation into all possible ways that an absolute value could show up in a formula for an antiderivative of an elementary function. However, we will give a justification as to why dropping absolute value signs is allowed and what we are actually doing to the integrating factor when we do drop the absolute value signs. For this discussion, we first make more precise what we mean by an *integrating factor*:

Definition 3.3.11. Suppose $f : D \to \mathbb{R}$ is a continuous function with a nice domain $D \subseteq \mathbb{R}$ and $I \subseteq D$ is a nice subset of D. We call a differentiable function $\mu : I \to \mathbb{R}$ an **integrating factor for $y' + fy$ on I** if:

(1) $\mu(t) \neq 0$ for every $t \in I$, and
(2) for every differentiable function $y : I \to \mathbb{R}$, the following equality holds:

$$\mu(t)\big(y'(t) + f(t)y(t)\big) \;=\; \big(\mu(t)y(t)\big)'$$

for every $t \in I$.

Certainly, the integrating factors we've been using:

$$\mu(t) \;:=\; \exp\left(\int f(t)\,dt\right)$$

satisfy the definition of an *integrating factor* according to Definition 3.3.11. But an integrating factor is by no means unique. Indeed, we are free to multiply an integrating factor by any nonzero constant and it remains a perfectly valid integrating factor:

Observation 3.3.12. *Suppose $f : D \to \mathbb{R}$ is a continuous function with a nice domain $D \subseteq \mathbb{R}$, $I \subseteq D$ is a nice subset of D, and $\mu : I \to \mathbb{R}$ is an integrating factor for $y' + fy$ on I. Then for any nonzero constant $\alpha \in \mathbb{R}$ ($\alpha \neq 0$), the function $\alpha\mu : I \to \mathbb{R}$ is also an integrating factor for $y' + fy$ on I.*

However, we have a little bit more freedom in modifying our integrating factors than just multiplying everything through by nonzero constants. For instance, consider the differential equation:

$$y' + \frac{1}{t}y \;=\; 0$$

We find that an integrating factor is $\mu(t) = \exp(\int dt/t) = |t|$. However, 3.3.10 claims that we can switch to using $\tilde{\mu}(t) = t$ as an integrating factor. The modification from $\mu(t)$ to $\tilde{\mu}(t)$ is more involved than just scaling $\mu(t)$ by a nonzero constant. First, note that in this example, $f(t) = 1/t$ and so $f : (-\infty, 0) \cup (0, +\infty) \to \mathbb{R}$ does not have 0 in its domain, so we are also considering $\mu(t) = |t|$ also to be a function $\mu : (-\infty, 0) \cup (0, +\infty) \to \mathbb{R}$ without zero in its domain. Furthermore, note that:

$$\mu(t) \;=\; \begin{cases} t & \text{if } t > 0 \\ -t & \text{if } t < 0 \end{cases} \quad \text{and} \quad \tilde{\mu}(t) \;=\; \begin{cases} t & \text{if } t > 0 \\ t & \text{if } t < 0 \end{cases}$$

In other words, to change $\mu(t)$ into $\tilde{\mu}(t)$, we had to multiply $\mu(t)$ by -1 on the $(-\infty, 0)$ portion of its domain, and keep $\mu(t)$ the same on the $(0, +\infty)$ portion of its domain. The reason this type of "selective" multiplication of $\mu(t)$ is allowed is

because $(-\infty, 0)$ and $(0, +\infty)$ are not connected to each other, so we don't have to worry about the portion of $\tilde{\mu}$ on $(-\infty, 0)$ joining up nicely with the portion of $\tilde{\mu}$ on $(0, +\infty)$. This is an instance of the following general observation:

Observation 3.3.13. *Suppose $f : D \to \mathbb{R}$ is a continuous function with a nice domain $D \subseteq \mathbb{R}$, and suppose $\mu : D \to \mathbb{R}$ is an integrating factor for $y' + fy$ on D. Furthermore:*

(1) Suppose the domain $D = I_1 \cup I_2 \cup I_3 \cup \cdots$ is a union of disconnected intervals I_k (i.e., there is no $i \neq j$ and $a < b \in \mathbb{R}$ such that $[a, b] \subseteq I_i \cup I_j$), and

(2) Suppose $\alpha_1, \alpha_2, \alpha_3, \ldots$ is a sequence of nonzero constants from $\mathbb{R}$.

Then the function $\tilde{\mu} : D \to \mathbb{R}$ defined by:

$$\tilde{\mu}(t) \;:=\; \alpha_k \mu(t) \quad \text{if } t \in I_k$$

is also an integrating factor for $y' + fy$ on D.

We now arrive at a more precise version of 3.3.10:

Observation 3.3.14. *Suppose $f : D \to \mathbb{R}$ is a continuous function with a nice domain $D \subseteq \mathbb{R}$, and suppose*

$$\mu(t) \;:=\; \exp\left(\int f(t)\,dt\right) \;=\; \big|u(t)\big| \quad \text{for every } t \in D$$

where $u : D \to \mathbb{R}$ is some differentiable function. Then:

(1) for every $t \in D$, $u(t) \neq 0$,

(2) the sets,

$$D_1 \;:=\; \big\{t \in D : u(t) > 0\big\} \quad \text{and} \quad D_2 \;:=\; \big\{t \in D : u(t) < 0\big\}$$

are disconnected and $D = D_1 \cup D_2$, and thus

(3) the function $\tilde{\mu} : D \to \mathbb{R}$ defined by

$$\tilde{\mu}(t) \;:=\; u(t)$$

for every $t \in D$ is also an integrating factor of $y' + fy$.

JUSTIFICATION. (1) is clear because $\mu(t)$ is defined as an exponential of a certain function, and exp never takes the value zero.

(2) Suppose towards a contradiction that there is an interval $[a, b] \subseteq D$ such that $a \in D_1$ and $b \in D_2$ (the other case is similar). Then since $u : [a, b] \to \mathbb{R}$ is differentiable, and hence continuous, by the Intermediate Value Theorem 2.2.6 there is $y \in (a, b)$ such that $u(y) = 0$. This contradicts (1). Thus D_1 and D_2 are disconnected. The claim that $D = D_1 \cup D_2$ also follows from (1).

(3) is an application of Observation 3.3.13. In order to obtain $\tilde{\mu}$ from μ, on every interval $I \subseteq D_1$, we can keep μ the same, and on every interval $J \subseteq D_2$, we can multiply μ by -1. □

Remark 3.3.15. In general, you only need to worry about absolute value signs (and whether to drop them) when computing the general solution of a first-order linear differential equation. For an initial value problem, you use the precise integrating factor:

$$\mu(t) \;:=\; \exp\left(\int_{t_0}^{t} f(s)\,ds\right)$$

where t_0, t are both included in the same interval in the domain of f. Since your attention is restricted to this interval, the context should tell you, when faced with $|u(t)|$, whether to treat this as $u(t)$ or $-u(t)$ (depending on whether $u(t_0) > 0$ or $u(t_0) < 0$); only one of them can happen on an interval in the domain of f which contains t_0.

We now give a very carefully worked out example, where we show how to apply the above discussion on absolute values. In general, when you are doing computations, you are free to drop absolute values in this context without justification *provided that you still get the full correct answer.*

Example 3.3.16. Consider the following initial value problem:

(1) $y' + \tan(t)y = \sec(t)$
(2) $y(0) = 5$.

Find the general solution to (i) and the particular solution to (i)+(ii).

SOLUTION. First notice that the domain of $f(t) = \tan(t)$ and $g(t) = \sec(t)$ is

$$D \;:=\; \text{domain}(\tan t) \;=\; \text{domain}(\sec t) \;=\; \bigcup_{k\in\mathbb{Z}} \Big(\frac{\pi}{2} + \pi k, \frac{\pi}{2} + \pi(k+1)\Big)$$

i.e., the domain is all of $\mathbb{R}$ except points of the form $\pi/2 + \pi k$, where $k \in \mathbb{Z}$. Next we compute the usual integrating factor:

$$\mu(t) \;:=\; \exp\left(\int \tan t\, dt\right) \;=\; \exp \ln|\sec t| \;=\; |\sec t|.$$

The domain of $\mu(t)$ is the same as the domain of $\tan t$ and $\sec t$ above ($= D$). Furthermore, note that

$$D_1 \;:=\; \{t \in D : \sec t > 0\} \;=\; \bigcup_{k\in\mathbb{Z}, k \text{ odd}} \Big(\frac{\pi}{2} + \pi k, \frac{\pi}{2} + \pi(k+1)\Big)$$

$$D_2 \;:=\; \{t \in D : \sec t < 0\} \;=\; \bigcup_{k\in\mathbb{Z}, k \text{ even}} \Big(\frac{\pi}{2} + \pi k, \frac{\pi}{2} + \pi(k+1)\Big)$$

As we see, the intervals in D_1 are not connected to the intervals in D_2. Thus we can define $\tilde{\mu} : D \to \mathbb{R}$ by

$$\tilde{\mu}(t) \;:=\; \begin{cases} \mu(t) & \text{if } t \in D_1 \\ -\mu(t) & \text{if } t \in D_2 \end{cases} \;=\; \sec t$$

for every $t \in D$. By Observation 3.3.13, we know that $\tilde{\mu}(t) = \sec t$ also works as an integrating factor, so we will use that instead. Continuing on with the problem, we multiply (i) through by $\tilde{\mu}$ to obtain:

$$\big(\sec(t)y\big)' \;=\; \sec^2 t$$

Integrating both sides yields:

$$\sec(t)y \;=\; \tan t + C$$

where $C \in \mathbb{R}$ is an arbitrary constant. Thus the general solution is:

$$y(t) \;=\; y(t; C) \;=\; \frac{\tan t + C}{\sec t}$$

on the domain D.

Next, we will solve the initial value problem (i)+(ii) from scratch. Since $t_0 = 0$, we see that the interval of existence of the solution will be $(-\pi/2, \pi/2)$, so we can restrict our attention to this interval. First we compute the integrating factor (where $t \in (-\pi/2, \pi/2)$):

$$\begin{aligned}
\mu(t) &:= \exp\left(\int_0^t \tan s \, ds\right) \\
&= \exp\left(\ln|\sec s|\Big|_0^t\right) \\
&= \exp\left(\ln \sec s\Big|_0^t\right) \quad (*) \\
&= \exp\big(\ln\sec t - \ln\sec 0\big) \\
&= \exp\big(\ln\sec t - \ln 1\big) \\
&= \exp\big(\ln\sec t\big) \\
&= \sec t
\end{aligned}$$

where in step $(*)$ we removed the absolute value signs because $\sec s$ is positive at $s = 0$ (if the initial condition had $t_0 = \pi$ for instance, then we would have to replace $\ln|\sec s|$ with $\ln(-\sec s)$ in that step). Now that we have the integrating factor, we can proceed with the particular solution (which is only defined on the interval of existence $(-\pi/2, \pi/2)$):

$$\begin{aligned}
y(t) &= \frac{1}{\sec t}\int_0^t \sec^2 s \, dx + \frac{5}{\sec t} \quad \text{because } y_0 = 5 \\
&= \frac{\tan t}{\sec t} + \frac{5}{\sec t} \\
&= \frac{\tan t + 5}{\sec t}.
\end{aligned}$$

□

Mixing problems. We now discuss a practical application of first-order linear differential equations, the so-called *mixing problems*.

Variation of parameters.

3.4. Implicit equations and differential forms

3.5. Separable and exact differential equations

CHAPTER 4

Linear algebra II

CHAPTER 5

Second-order linear differential equations

CHAPTER 6

Linear algebra III

CHAPTER 7

Systems of differential equations

APPENDIX A

Special functions

In this appendix we will include an overview of relevant properties of common elementary functions which arise in calculus and differential equations. In general we will work within the realm of real numbers, although everything we say has an appropriate extension to the bigger world of complex numbers. However, we might occasionally have to refer to complex numbers every now and then.

A.1. Polynomials

A **polynomial** (in the single variable X) is an expression of the form:

$$p(X) \;=\; a_n X^n + a_{n-1}X^{n-1} + \cdots + a_2 X^2 + a_1 X + a_0 \quad (\text{where each } a_i \in \mathbb{R})$$

If $a_n \neq 0$, then we call n the **degree** of $p(X)$, denoted $\deg p = n$. We may also choose to write a polynomial in **summation notation**:

$$p(X) \;=\; \sum_{k=0}^{n} a_k X^k$$

We naturally construe a polynomial as a function $p : \mathbb{R} \to \mathbb{R}$ by declaring for $\alpha \in \mathbb{R}$:

$$p(\alpha) \;:=\; a_n \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_2 \alpha^2 + a_1 \alpha + a_0$$

Recall that given two polynomial $p(X) = \sum_{k=0}^{n} a_k X^k$ and $q(X) = \sum_{k=0}^{n} b_k X^k$, we can form their **sum**:

$$(p+q)(X) \;:=\; \sum_{k=0}^{n} (a_k + b_k) X^k$$

and their **product**:

$$(p \cdot q)(X) \;:=\; \sum_{k} \left(\sum_{i+j=k} a_i b_j \right) X^k$$

where the above sum ranges over all possible indices.

Polynomials are perhaps the most well-behaved type of function which shows up in calculus. Indeed:

Fact A.1.1. Suppose

$$p(X) \;=\; a_n X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0 \;=\; \sum_{k=0}^{n} a_k X^k$$

is a polynomial of degree n. Then the following facts are true about $p(X)$ as a function $p : \mathbb{R} \to \mathbb{R}$:

(1) p is continuous on all of $\mathbb{R}$. In particular, for every $\alpha \in \mathbb{R}$:

$$\lim_{x \to \alpha} p(x) \;=\; p(\alpha)$$

(2) The limits at infinity are computed as follows:
   (a) if $n = 0$, then
$$\lim_{x\to\infty} p(x) \;=\; \lim_{x\to-\infty} p(x) \;=\; a_0$$
   (b) if $n \geq 1$ is even, then
$$\lim_{x\to\infty} p(x) \;=\; \lim_{x\to-\infty} p(x) \;=\; \begin{cases}\infty & \text{if } a_n > 0\\ -\infty & \text{if } a_n < 0\end{cases}$$
   (c) if $n \geq 1$ is odd, then
$$\lim_{x\to\infty} p(x) \;=\; \begin{cases}\infty & \text{if } a_n > 0\\ -\infty & \text{if } a_n < 0\end{cases} \quad \text{and} \quad \lim_{x\to-\infty} p(x) \;=\; \begin{cases}-\infty & \text{if } a_n < 0\\ \infty & \text{if } a_n > 0\end{cases}$$
(3) p is differentiable on all of $\mathbb{R}$ with derivative
$$\frac{dp}{dX}(X) \;=\; na_nX^{n-1} + (n-1)a_{n-1}X^{n-2} + \cdots + 2a_2X + a_1$$
$$= \sum_{k=1}^{n} ka_kX^{k-1} \;=\; \sum_{k=0}^{n-1}(k+1)a_{k+1}X^k$$
(4) Since the derivative of a polynomial is again a polynomial, p is infinitely differentiable on all of $\mathbb{R}$,
(5) Define the degree $n+1$ polynomial:
$$P(X) \;:=\; \frac{a_n}{n+1}X^{n+1} + \frac{a_{n-1}}{n}X^n + \cdots + \frac{a_1}{2}X^2 + a_0X$$
$$= \sum_{k=1}^{n+1} \frac{a_{k-1}}{k}X^k \;=\; \sum_{k=0}^{n} \frac{a_k}{k+1}X^{k+1}$$
Then:
   (a) $P(X)$ is an antiderivative of $p(X)$, i.e.,
$$\frac{d}{dx}P(X) \;=\; p(X),$$
   (b) the indefinite integral of $p(X)$ is
$$\int p(X)\,dX \;=\; P(X) + C,$$
   (c) the definite integral of $p(X)$ is
$$\int_a^b p(X)\,dX \;=\; P(b) - P(a),$$
   for every $a, b \in \mathbb{R}$.

The following is an important theoretical tool for studying polynomials:

Fundamental Theorem of (Complex) Algebra A.1.2. *Suppose $n \geq 1$. Then for every polynomial*
$$p(X) \;=\; a_nX^n + a_{n-1}X^n - 1 + \cdots + a_1X + a_0$$
of degree n, there exists complex numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ such that
$$p(X) \;=\; a_n(X-\alpha_1)(X-\alpha_2)\cdots(X-\alpha_n).$$

The numbers $\alpha_1, \ldots, \alpha_n$ in A.1.2 need not be distinct. One (very minor) drawback of A.1.2 is that some of the roots might be complex numbers which are not real numbers. Since we usually want to stick to working entirely with real numbers, the following variant will be useful for us:

Fundamental Theorem of (Real) Algebra A.1.3. *Suppose $n \geq 1$. Then for every polynomial*

$$p(X) \;=\; a_n X^n + a_{n-1} X^n - 1 + \cdots + a_1 X + a_0$$

of degree n, there exists $r, s \in \mathbb{N}$ with $r + 2s = n$, and real numbers

$$\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s, \gamma_1, \ldots, \gamma_s \in \mathbb{R}$$

such that:

(1) p can be factored into linear and quadratic factors

$$p(X) \;=\; a_n \underbrace{(X - \alpha_1) \cdots (X - \alpha_r)}_{\textit{linear factors}} \underbrace{(X^2 + \beta_1 X + \gamma_1) \cdots (X^2 + \beta_s X + \gamma_s)}_{\textit{quadratic factors}},$$

and

(2) for each $i = 1, \ldots, s$, we have $\beta_i^2 - 4\gamma_i < 0$, i.e., the quadratic factor $X^2 + \beta_i X + \gamma_i$ does not have real roots.

Theorem A.1.3 is an easy consequence of Theorem A.1.2 since complex roots of polynomials occur in conjugate pairs. Combining these conjugate pairs together is what give rise to the quadratic factors.

When dealing with quadratic polynomials with no real roots, the following trick is essential:

Completing the Square A.1.4. *Suppose $a, b, c \in \mathbb{R}$ are arbitrary such that $a \neq 0$. Then*

$$aX^2 + bX + c \;=\; a\left(X + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \;=\; a\left[\left(X + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

If the discriminant $b^2 - 4ac < 0$ is negative, then the constant $(4ac - b^2)/4a^2 > 0$ is positive.

A.2. Rational functions

A **rational function** (in the single variable X) is an expression of the form

$$r(X) \;=\; \frac{a_m X^m + a_{m-1} X^{m-1} + \cdots + a_1 X + a_0}{b_n X^n + b_{n-1} X^{n-1} + \cdots + b_1 X + b_0} \quad (\text{where } a_i, b_j \in \mathbb{R})$$

i.e., a rational function is a quotient

$$r(X) \;=\; \frac{p(X)}{q(X)}$$

of polynomials, where $p(X) = a_m X^m + \cdots + a_0$ and $q(X) = b_n X^n + \cdots + b_0$.

Recall that given two rational functions $r_0(X) = p_0(X)/q_0(X)$ and $r_1(X) = p_1(X)/q_1(X)$, we can form their **sum**:

$$(r_0 + r_1)(X) \;:=\; \frac{p_0(X)q_1(X) + p_1(X)q_0(X)}{q_0(X)q_1(X)}$$

and their **product**:

$$(r_0 \cdot r_1)(X) \;:=\; \frac{p_0(X)p_1(X)}{q_0(X)q_1(X)}$$

Just as with polynomials, we naturally construe a rational function as a real-valued function. Since the denominator of a fraction is never allowed to be zero, the domain of $r(X) = p(X)/q(X)$ is:

$$\text{domain}(r) \;:=\; \big\{\alpha \in \mathbb{R} : q(\alpha) \neq 0\big\} \;\subseteq\; \mathbb{R}$$

Then we define the function $r : \text{domain}(r) \to \mathbb{R}$ by declaring for $\alpha \in \mathbb{R}$:

$$r(\alpha) \;:=\; \frac{p(\alpha)}{q(\alpha)}$$

Warning A.2.1. In general the domain of a rational function might exclude so-called *removable singularities*. For example, consider the following two rational functions:

$$r_0(X) \;:=\; \frac{(X+1)(X+2)}{(X+1)(X+3)} \quad \text{and} \quad r_1(X) \;:=\; \frac{X+2}{X+3}$$

Then as real-valued functions, we have

$$\text{domain}(r_0) \;=\; \mathbb{R} \setminus \{-1, -3\} \quad \text{and} \quad \text{domain}(r_1) \;=\; \mathbb{R} \setminus \{-3\}$$

i.e., r_0 is defined everywhere except -1 whereas r_1 is defined everywhere except -3. However, for every $\alpha \in \mathbb{R} \setminus \{-1, -3\}$, we have $r_0(\alpha) = r_1(\alpha)$. In other words, r_0 and r_1 are essentially the same real-valued function except that r_1 is defined at one more point than r_0 is. In some sense, the fact that r_0 does not have -1 in its domain is an artificial obstacle. It is due to the factor $x + 1$ occurring in both the numerator and denominator. Since this has no effect on the value of the function (since it contributes multiplication by 1), we can just cancel these factors out and gain an extra point where the function is defined. In practice, when working with rational functions, you always want to make sure that the numerator and the denominator have no common factors so that you can work with the largest possible "true" domain of the rational function.

In the rest of this section, we will ignore the issue of removable singularities. After polynomials, rational functions are the second best-behaved family of functions which show up in calculus:

Fact A.2.2. Suppose

$$r(X) \;=\; \frac{p(X)}{q(X)}$$

is a rational function with domain $D := \text{domain}(r)$. Then the following facts are true about $r(X)$ as a function $r : D \to \mathbb{R}$:

(1) r is continuous on all of D. In particular, for every $\alpha \in D$:

$$\lim_{x \to \alpha} r(x) \;=\; r(\alpha)$$

(2) r is differentiable on all of D with derivative

$$\frac{dr}{dX}(X) \;=\; \frac{q(X)\frac{dp}{dX}(X) - p(X)\frac{dq}{dX}(X)}{\big(q(X)\big)^2}$$

which is also a rational function with domain D.

(3) It follows that $r(X)$ is infinitely differentiable on D.

Integration of rational functions is a little bit more complicated and requires so-called *partial fraction decomposition*. First, some terminology:

Definition A.2.3. Suppose $r(X) = p(X)/q(X)$ is a rational function. We say that $r(X)$ is a **proper** rational function if $\deg p < \deg q$. Otherwise, we say that $r(X)$ is an **improper** rational function.

We have two versions of partial fraction decomposition, depending on whether every factor of the denominator is linear or not:

Partial Fraction Decomposition A.2.4 (Complex Case)**.** *Suppose*

$$r(X) \;=\; \frac{p(X)}{q(X)}$$

is a proper rational function with $\deg q = n$*. Then:*

(1) *By Theorem A.1.2 there exists a nonzero real number* $a \in \mathbb{R}$*, distinct complex numbers* $\alpha_1, \ldots, \alpha_r \in \mathbb{C}$*, and positive integers* $n_1, \ldots, n_r \in \mathbb{N}$ *such that*
   (a) $n_1 + \cdots + n_r = n$*, and*
   (b) $q(X) = a(X - \alpha_1)^{n_1} \cdots (X - \alpha_r)^{n_r}$
(2) *there exists a family of complex numbers* $(A_{i,j})_{1 \le i \le r, 1 \le j \le n_i}$ *such that*

$$r(X) \;=\; \frac{p(X)}{q(X)} \;=\; \sum_{i=1}^{r} \sum_{j=1}^{n_i} \frac{A_{i,j}}{(X - \alpha_i)^j}$$

You should use A.2.4 any time every root of $q(X)$ is real, or if you want to work with complex numbers. If not every root of $q(X)$ is real and you want to avoid using complex numbers, then you should use the following:

Partial Fraction Decomposition A.2.5 (Real Case)**.** *Suppose*

$$r(X) \;=\; \frac{p(X)}{q(X)}$$

is a proper rational function with $\deg q = n$*. Then:*

(1) *By Theorem A.1.3 there exists* $r, s \in \mathbb{N}$ *such that* $r + 2s = n$*, a nonzero real numbers* $a \in \mathbb{R}$*, distinct real numbers* $\alpha_1, \ldots, \alpha_t \in \mathbb{R}$*, positive integers* $n_1, \ldots, n_t$*, distinct pairs of real numbers* $(\beta_1, \gamma_1), \ldots, (\beta_u, \gamma_u) \in \mathbb{R}^2$ *and positive integers* $n'_1, \ldots, n'_u$ *such that:*
   (a) $n_1 + \cdots + n_t = r$*,*
   (b) $n'_1 + \cdots + n'_u = s$*,*
   (c) *the denominator factors as:*

$$q(X) \;=\; a(X - \alpha_1)_1^n \cdots (X - \alpha_r)^{n_r} (X^2 + \beta_1 X + \gamma_1)^{n'_1} \cdots (X^2 + \beta_u X + \gamma_u)^{n'_u}$$

   (d) *for every* $i = 1, \ldots, u$*, we have* $\beta_i^2 - 4\gamma_i < 0$*, i.e., the quadratic factor* $X^2 + \beta_i X + \gamma_i$ *does not have real roots.*
(2) *There exists families of real numbers* $(A_{i,j})_{1 \le i \le r, 1 \le j \le n_i}$*,* $(B_{i,j})_{1 \le i \le s, 1 \le j \le n'_i}$*,* $(C_{i,j})_{1 \le i \le s, 1 \le j \le n'_i}$ *such that*

$$r(X) \;=\; \frac{p(X)}{q(X)} \;=\; \sum_{i=1}^{r} \sum_{j=1}^{n_i} \frac{A_{i,j}}{(X - \alpha_i)^j} + \sum_{i=1}^{s} \sum_{j=1}^{n'_i} \frac{B_{i,j} X + C_{i,j}}{(X^2 + \beta_i X + \gamma_i)^j}$$

A.3. Algebraic functions

A.4. The exponential function

A.5. The logarithm

A.6. Power functions

A.7. Trigonometric functions

A.8. Inverse trigonometric functions

Bibliography

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