TOWARDS A MODEL THEORY OF LOGARITHMIC TRANSSERIES

BY

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DISSERTATION

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Abstract

The ordered valued differential field $\mathbb{T}_{\log}$ of logarithmic transseries is conjectured to have good model theoretic properties. This thesis records our progress in this direction and describes a strategy moving forward. As a first step, we turn our attention to the value group of $\mathbb{T}_{\log}$. The derivation on $\mathbb{T}_{\log}$ induces on its value group $\Gamma_{\log}$ a certain map ψ; together forming the pair $(\Gamma_{\log}, \psi)$, the *asymptotic couple of* $\mathbb{T}_{\log}$. We study the asymptotic couple $(\Gamma_{\log}, \psi)$ and show that it has a nice model theory. Among other things, we prove that $\mathrm{Th}(\Gamma_{\log}, \psi)$ has elimination of quantifiers in a natural language, is model complete, and has the non-independence property (NIP). As a byproduct of our work, we also give a complete characterization of when an H-field has exactly one or exactly two Liouville closures. Finally, we present an outline for proving a model completeness result for $\mathbb{T}_{\log}$ in a reasonable language. In particular, we introduce and study the notion of LD-*fields* and also the property of a differentially-valued field being Ψ-*closed*.

Dedicated to my advisor Lou, and to my parents Diane and Robert.

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Contents

Conventions and Notations

We include here conventions and notations which will be in force throughout the entire thesis. The most important convention is, of course, the following:

Global Convention 0.0.1. Throughout, m and n range over $\mathbb{N} = \{0, 1, 2, \dots\}$.

Citation conventions. In this thesis, we use many definitions and cite many results from [**6**]. As a general rule, any result taken directly from that reference is titled ADH instead of Lemma, Theorem, etc. In citing results in this way we do not imply that they are originally due to the authors of [**6**]; for instance, ADH 5.1.1 is actually a classical fact of valuation theory due to Kaplansky. Furthermore, in citations we omit qualifiers when no confusion should arise, writing, for example, [**14**, 3.2] instead of [**14**, Lemma 3.2].

Model theory conventions. We adopt the model theoretic conventions of Appendix B of [**6**]. In particular, $\mathcal{L}$ can be a many-sorted language. For a complete $\mathcal{L}$-theory T, we will sometimes consider a model $\mathbb{M} \models T$ and a cardinal $\kappa(\mathbb{M}) > |\mathcal{L}|$ such that $\mathbb{M}$ is $\kappa(\mathbb{M})$-saturated and every reduct of $\mathbb{M}$ is strongly $\kappa(\mathbb{M})$-homogeneous. Such a model is called a **monster model** of T. In particular, every model of T of size $\leqslant \kappa(\mathbb{M})$ has an elementary embedding into $\mathbb{M}$. "Small" will mean "of size $< \kappa(\mathbb{M})$". If M is a parameter set underlying an elementary submodel of $\mathbb{M}$, then we denote this elementary submodel also by M. For a parameter set A, we let $\langle A \rangle$ denote the $\mathcal{L}$-substructure of $\mathbb{M}$ generated by A. Similarly, if M is an elementary submodel of $\mathbb{M}$, we let $M\langle A \rangle$ denote $\langle M \cup A \rangle$. If $\mathcal{L}$ is one-sorted, then we let $S^n(A)$ denote the space of n-types over A.

Set theory conventions. We assume the reader is familiar with the basic concepts and definitions from set theory (for example, see [**25**] or [**20**]). Throughout, κ will denote an infinite cardinal and η, λ, ν will denote (possibly finite) ordinals. Given a linear order I and subset $J \subseteq I$, we say that J *is a dense subset of* I if the closure of J in the order topology on I is all of I (see [**6**, §2.1]). We define

$$\operatorname{ded}(\kappa) \ := \ \sup\{|\lambda| : \text{there is a linear order of size } |\lambda| \text{ which has a dense subset of size } \kappa\}.$$

In general $\kappa < \operatorname{ded}(\kappa) \leqslant \operatorname{ded}(\kappa)^{\aleph_0} \leqslant 2^\kappa$ for all κ with equality if $\kappa = \aleph_0$. Furthermore, $\operatorname{ded}(\kappa) \leqslant \operatorname{ded}(\lambda)$ if $\kappa \leqslant \lambda$.

Ordered set conventions. By "ordered set" we mean "totally ordered set".

Let S be an ordered set. Below, the ordering on S will be denoted by $\leqslant$, and a subset of S is viewed as ordered by the induced ordering. We put $S_\infty := S \cup \{\infty\}$, $\infty \notin S$, with the ordering on S extended to a (total) ordering on S_∞ by $S < \infty$. Occasionally, we even take two distinct elements $-\infty, \infty \notin S$, and extend the ordering on S to an ordering on $S \cup \{-\infty, \infty\}$ by $-\infty < S < \infty$. Suppose that B is a subset of S. We put $S^{>B} := \{s \in S : s > b \text{ for every } b \in B\}$ and we denote $S^{>\{a\}}$ as just $S^{>a}$; similarly for $\geqslant, <$, and $\leqslant$ instead of $>$. For $a, b \in S \cup \{-\infty, \infty\}$ and $B \subseteq S$ we put

$$[a, b]_B \ := \ \{x \in B : a \leqslant x \leqslant b\}.$$

If $B = S$, then we usually write $[a, b]$ instead of $[a, b]_S$. A subset C of S is said to be **convex** in S if for all $a, b \in C$ we have $[a, b] \subseteq C$. For $A \subseteq S$ we let

$$\operatorname{conv}(A) \ := \ \{x \in S : a \leqslant x \leqslant b \text{ for some } a, b \in A\}$$

be the **convex hull of** A in S, that is, the smallest convex subset of S containing A. A subset A of S is said to be a **cut** in S, or **downward closed** in S, if for all $a \in A$ and $s \in S$ we have $s < a \Rightarrow s \in A$. We say that an element x of an ordered set extending S **realizes** the cut A if $A < x < S \setminus A$. For $A \subseteq S$ we put

$$A^{\downarrow} \ := \ \{s \in S : s \leqslant a \text{ for some } a \in A\},$$

which is the smallest downward closed subset of S containing A.

We say that S is a **successor set** if every element $x \in S$ has an **immediate successor** $y \in S$, that is, $x < y$ and for all $z \in S$, if $x < z$, then $y \leqslant z$. For example, $\mathbb{N}$ and $\mathbb{Z}$ with their usual ordering are successor sets. We say that S is a **copy of** $\mathbb{Z}$ (respectively, **copy of** $\mathbb{N}$) if $(S, <)$ is isomorphic to $(\mathbb{Z}, <)$ (respectively, $(\mathbb{N}, <)$).

A **well-indexed sequence** is a sequence (a_ρ) whose terms a_ρ are indexed by the elements ρ of an infinite well-ordered set without a greatest element.

Algebra conventions. For an (additively written) abelian group G we set $G^{\neq} := G \setminus \{0\}$. We say a group G is **trivial** if $G = \{e\}$, where e is the identity element of G. We say a subgroup H of G is a **trivial subgroup (of** G**)** if H is a trivial group, otherwise we say that H is a **nontrivial subgroup (of** G**)**. For a field K we let $K^{\times} := K \setminus \{0\} = K^{\neq}$ be its multiplicative group of units. Let R be a commutative ring and M an R-module. When U and V are given as additive subgroups of R and M, respectively, then we set

$$UV \ := \ \left\{ \sum_{i=1}^{n} r_i x_i \in M : r_1, \ldots, r_n \in U, x_1, \ldots, x_n \in V \right\},$$

the additive subgroup of M generated by the products rx with $r \in U$ and $x \in V$.

Ordered abelian group conventions. Suppose that G is an ordered abelian group. Then we set $G^{<} := G^{<0}$; similarly for $\geqslant, \leqslant$, and $>$ instead of $<$. We define $|g| := \max(g, -g)$ for $g \in G$. For $a \in G$, the **archimedean class** of a is defined by

$$[a] \ := \ \{g \in G : |a| \leqslant n|g| \text{ and } |g| \leqslant n|a| \text{ for some } n \geqslant 1\}.$$

The archimedean classes partition G. Each archimedean class $[a]$ with $a \neq 0$ is the disjoint union of the two convex sets $[a] \cap G^{<}$ and $[a] \cap G^{>}$. We order the set $[G] := \{[a] : a \in G\}$ of archimedean classes by

$$[a] < [b] \ :\Longleftrightarrow \ n|a| < |b| \text{ for all } n \geqslant 1.$$

We have $[0] < [a]$ for all $a \in G^{\neq}$, and

$$[a] \leqslant [b] \ \Longleftrightarrow \ |a| \leqslant n|b| \text{ for some } n \geqslant 1.$$

The **rank** of G, denoted by $\operatorname{rank}(G)$, is defined to be n if there are exactly n nontrivial convex subgroups of G, and defined to be ∞ if there are infinitely many convex subgroups of G.

As a torsion-free abelian group, we will consider G as a subgroup of the divisible abelian group $\mathbb{Q}G := \mathbb{Q} \otimes_{\mathbb{Z}} G$ via the embedding $g \mapsto 1 \otimes g$. We also equip $\mathbb{Q}G$ with the unique linear order that makes it into an ordered abelian group containing G as an ordered subgroup. The ordered abelian group $\mathbb{Q}G$ is called the **divisible hull** of G.

CHAPTER 1

Introduction

1.1. Introduction

Consider the following function:

$$\Phi(x) \;=\; \frac{2}{1-(\log\log x)^{-1}} + \frac{1}{1-(\log x)^{-1}} + \frac{7}{x} + \frac{5}{x^2} - 3$$

Expanding $\Phi(x)$ formally as $x \to \infty$, we get a *logarithmic transseries*:

$$\frac{2}{\log\log x} + \frac{2}{(\log\log x)^2} + \frac{2}{(\log\log x)^3} + \cdots + \frac{1}{\log x} + \frac{1}{(\log x)^2} + \frac{1}{(\log x)^3} + \cdots + \frac{7}{x} + \frac{5}{x^2}$$

Transseries are formal transfinite series which provide a general setting for considering orders of growth which are different from the usual powers of x:

$$\ldots, x^{-3}, x^{-2}, x^{-1}, 1, x, x^2, x^3, \ldots$$

and they are an appropriate forum to give actual meaning to often divergent series that arise in nature. Transseries arise as solutions to algebraic differential equations, often where more classical methods break down (for instance, see [**42**]). Already, there are many important applications in dynamical systems with Écalle's and Il′yashenko's proofs of the Dulac conjecture [**11, 19, 41**] which is related to Hilbert's 16th Problem, as well as applications in model theory [**5, 9**] in connection with Tarski's problem on the real exponential field, and computer algebra [**13, 42**], allowing for the automation of solving differential equations by a computer. Logarithmic transseries also occur in combinatorics in the work of Loeb and Rota [**27, 28**] in connection with difference equations and certain generalizations of umbral calculus.

For us, the most compelling results on transseries can be found in the book *Asymptotic Differential Algebra and Model Theory of Transseries* by Matthias Aschenbrenner, Lou van den Dries and Joris van der Hoeven, [**6**]. In it, they study the model theory and algebra of the ordered valued differential field $\mathbb{T}$ of logarithmic exponential transseries. In particular, they prove a quantifier elimination result for $\mathbb{T}$ and they show that the theory of $\mathbb{T}$ is the model companion of a certain natural class of ordered valued differential fields: the so-called *H-fields*. This effectively anoints $\mathbb{T}$ as an appropriate universal domain for doing "ordered asymptotic differential algebra", in much the same way that the algebraically closed field $\mathbb{C}$ is a universal domain for algebraic geometry of characteristic 0.

In [**6**], they also isolate a particularly nice differential subfield of $\mathbb{T}$:

$$\mathbb{T}_{\log} : \text{the } \textit{ordered valued differential field of logarithmic transseries}$$

In this thesis, we study the algebra and model theory of $\mathbb{T}_{\log}$. It is our ultimate goal to accomplish for $\mathbb{T}_{\log}$ what is accomplished for $\mathbb{T}$ in [**6**]. This thesis records our progress in this direction and sheds light on our strategy moving forward.

1.2. Construction of $\mathbb{T}_{\log}$

In this section we give a construction of the ordered valued differential field $\mathbb{T}_{\log}$. In the rest of this thesis, the "official" construction of $\mathbb{T}_{\log}$ we use is the one given in Appendix A of [**6**], i.e., a construction of $\mathbb{T}_{\log}$ as a distinguished subfield of $\mathbb{T}$. However, the construction we give here is equivalent and perhaps a little more direct and transparent.

The iterated logarithms (ℓ_n). We set $\ell_0 := x$ and $\ell_{n+1} := \log \ell_n$ to obtain the formal sequence (ℓ_n) of iterated logarithms of x. At this point, the elements $\ell_0, \ell_1, \ell_2, \ell_3, \ldots$ have no meaning whatsoever, however they can (and should) be thought of as formal counterparts to the familiar functions from freshman calculus:

$$x,\ \log x,\ \log\log x,\ \log\log\log x, \ldots.$$

In particular, at no point will we ever talk about "branch cuts".

The ordered multiplicative group $\mathfrak{L}_n$ of logarithmic transmonomials. For each n, we construct the multiplicative group $\mathfrak{L}_n = (\mathfrak{L}_n, \cdot)$ of *logarithmic transmonomials of depth n*:

$$\mathfrak{L}_n \ := \ \ell_0^{\mathbb{R}} \cdots \ell_n^{\mathbb{R}} \ := \ \{\ell_0^{r_0} \cdots \ell_n^{r_n} : r_0, \ldots, r_n \in \mathbb{R}\}$$

with group multiplication given by:

$$(\ell_0^{r_0} \cdots \ell_n^{r_n}) \cdot (\ell_0^{s_0} \cdots \ell_n^{s_n}) \ := \ \ell_0^{r_0+s_0} \cdots \ell_n^{r_n+s_n}.$$

We further make $\mathfrak{L}_n$ into an ordered group $\mathfrak{L}_n = (\mathfrak{L}_n, \cdot, \prec)$ by requiring

$$\ell_0^{r_0} \cdots \ell_n^{r_n} \prec \ell_0^{s_0} \cdots \ell_n^{s_n} \quad \Longleftrightarrow \quad (r_0, \ldots, r_n) <_{\text{lex}} (s_0, \ldots, s_n)$$

where $<_{\text{lex}}$ is the usual lexicographical ordering on $\mathbb{R}^{1+n}$. Given $m < n$, we naturally view $\mathfrak{L}_m$ as an ordered abelian subgroup of $\mathfrak{L}_n$, and in particular, $\mathfrak{L}_m \subseteq \mathfrak{L}_n$.

Note that the ordering $\prec$ on $\mathfrak{L}_n$ respects the asymptotic behavior of the iterated logarithms as $x \to +\infty$ when viewed as real-valued functions. For example, the statement "for all m, $\ell_1^m \prec \ell_0$" about $\mathfrak{L}_1$ can be viewed as a formal counterpart to the asymptotic statement "for all m, $(\log x)^m = o(x)$ as $x \to +\infty$" about the real-valued functions x and $\log x$.

The Hahn field $\mathbb{R}[[\mathfrak{L}_n]]$. For each n, we construct the so-called *Hahn field* $\mathbb{R}[[\mathfrak{L}_n]]$, the field of formal series whose coefficients come from $\mathbb{R}$ and whose monomials come from the ordered group $\mathfrak{L}_n$. More specifically:

A set $\mathfrak{G} \subseteq \mathfrak{L}_n$ is said to be **well-based** if there is no strictly increasing sequence $\mathfrak{m}_0 \prec \mathfrak{m}_1 \prec \mathfrak{m}_2 \prec \cdots$ in $\mathfrak{G}$. Suppose we are given a function $f : \mathfrak{L}_n \to \mathbb{R}$; we may formally construe f as a series $\sum_{\mathfrak{m} \in \mathfrak{L}_n} f_{\mathfrak{m}} \mathfrak{m}$, with $f_{\mathfrak{m}} = f(\mathfrak{m})$, and we say the **support** of f is the set $\operatorname{supp} f := \{\mathfrak{m} \in \mathfrak{L}_n : f_{\mathfrak{m}} \neq 0\}$. Then we set

$$\mathbb{R}[[\mathfrak{L}_n]] \ := \ \{f : \mathfrak{L}_n \to \mathbb{R} : \operatorname{supp} f \subseteq \mathfrak{L}_n \text{ is well-based}\}.$$

We equip $\mathbb{R}[[\mathfrak{L}_n]]$ with pointwise addition, and multiplication given by

$$f \cdot g \ := \ \sum_{\mathfrak{m} \in \mathfrak{L}_n} \left(\sum_{\mathfrak{n}_1 \cdot \mathfrak{n}_2 = \mathfrak{m}} f_{\mathfrak{n}_1} g_{\mathfrak{n}_2} \right) \mathfrak{m}$$

making $\mathbb{R}[[\mathfrak{L}_n]]$ a field by [**6**, 3.1.3]. Here are some elements from $\mathbb{R}[[\mathfrak{L}_2]]$:

$$x^3 \log x + \sqrt{x} + 2 + \frac{1}{\log\log x} + \frac{1}{(\log\log x)^2} + \cdots$$

$$\frac{1}{\log\log x} + \frac{1}{(\log\log x)^2} + \cdots + \frac{1}{\log x} + \frac{1}{(\log x)^2} + \cdots + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots$$

Given $m < n$, we naturally view $\mathbb{R}[[\mathfrak{L}_m]]$ as a subfield of $\mathbb{R}[[\mathfrak{L}_n]]$. Furthermore, we identify $r \in \mathbb{R}$ with the series $r\ell_0^0 \in \mathbb{R}[[\mathfrak{L}_0]]$. In this way, we view $\mathbb{R}$ as a subfield of $\mathbb{R}[[\mathfrak{L}_n]]$ for every n.

The field $\mathbb{T}_{\log}$. We define the underlying field of $\mathbb{T}_{\log}$ to be the direct union $\bigcup_{n=0}^{\infty} \mathbb{R}[[\mathfrak{L}_n]]$. It is important to note that the construction of $\mathbb{T}_{\log}$ excludes formal series such as

$$\lambda \;=\; \frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \frac{1}{\ell_0\ell_1\ell_2} + \cdots + \frac{1}{\ell_0 \cdots \ell_n} + \cdots$$

or

$$\omega \;=\; \frac{1}{\ell_0^2} + \frac{1}{\ell_0^2\ell_1^2} + \frac{1}{\ell_0^2\ell_1^2\ell_2^2} + \cdots + \frac{1}{\ell_0^2 \cdots \ell_n^2} + \cdots$$

from being members of $\mathbb{T}_{\log}$, i.e., for each series in $\mathbb{T}_{\log}$, there is a bound on the iteration depth allowed in the logarithmic transmonomials which occur in the series. Constructing $\mathbb{T}_{\log}$ as an increasing union which avoids series such as λ or ω may seem a bit odd, but it is actually crucial that we do this.

The valued field $\mathbb{T}_{\log}$. We will now equip $\mathbb{T}_{\log}$ with the structure of a *valued field*:

First, we define the **value group** of $\mathbb{T}_{\log}$. Let $\bigoplus_n \mathbb{R}e_n$ be a vector space over $\mathbb{R}$ with basis (e_n). Then $\bigoplus_n \mathbb{R}e_n$ can be made into an ordered group using the usual lexicographic order, i.e., by requiring for nonzero $\sum_i r_i e_i$ that

$$\sum r_i e_i > 0 \iff r_n > 0 \text{ for the least } n \text{ such that } r_n \neq 0.$$

Let $\Gamma_{\log}$ be the above ordered abelian group $\bigoplus_n \mathbb{R}e_n$. It is often convenient to think of an element $\sum r_i e_i$ as the vector $(r_0, r_1, r_2, \ldots)$.

Next, to each nonzero transseries $f \in \mathbb{T}_{\log}^{\times}$ we associate

$$\mathfrak{d}(f) \;:=\; \max_{\preccurlyeq} \operatorname{supp} f,$$

the **dominant monomial** of f. In particular, for every $f \in \mathbb{T}_{\log}$ we have $\mathfrak{d}(f) = \ell_0^{r_0} \cdots \ell_n^{r_n}$ for some n and $r_0, \ldots, r_n \in \mathbb{R}$.

Now we can define the **valuation** $v : \mathbb{T}_{\log}^{\times} \to \Gamma_{\log}$ as the unique map such that

(1) $v(\ell_0^{r_0} \cdots \ell_n^{r_n}) = -r_0 e_0 - \cdots - r_n e_n$, and
(2) $v(f) = v\big(\mathfrak{d}(f)\big)$ for all $f \in \mathbb{T}_{\log}$.

Furthermore, we extend v to a map $\mathbb{T}_{\log} \to \Gamma_{\log,\infty}$ by setting $v(0) := \infty$. It is easy to see that v is indeed a valuation on $\mathbb{T}_{\log}$, i.e., for all $f, g \in \mathbb{T}_{\log}$:

(V1) $v(fg) = v(f) + v(g)$, and
(V2) $v(f+g) \geqslant \min\big(v(f), v(g)\big)$.

Here are some sample calculations using v:

$$v(5x^3 \log x + \sqrt{x} + 2 + \cdots) \;=\; (-3, -1, 0, 0, \ldots)$$
$$v\left(\frac{\pi}{\log\log x} + \frac{7}{(\log\log x)^2} + \cdots\right) \;=\; (0, 0, 1, 0, 0, \ldots)$$
$$v\left(\frac{1}{\ell_0\ell_1 \cdots \ell_n}\right) = v(\ell_0^{-1}\ell_1^{-1} \cdots \ell_n^{-1}) \;=\; (\underbrace{1, \ldots, 1}_{n \text{ times}}, 0, 0, \ldots)$$

With the valuation v, we also define the **valuation ring** $\mathcal{O}$ of $\mathbb{T}_{\log}$:

$$\mathcal{O} \ := \ \{f \in K : v(f) \geqslant 0\}$$

The valuation ring $\mathcal{O}$ is a local ring with maximal ideal:

$$\mathcal{o} \ := \ \{f \in K : v(f) > 0\}$$

Given $\mathcal{O}$ and $\mathcal{o}$, we also get the **residue field** $\operatorname{res}(\mathbb{T}_{\log}) := \mathcal{O}/\mathcal{o}$. In this case, the residue field can be identified with $\mathbb{R}$, the field of real numbers.

Finally, we equip $\mathbb{T}_{\log}$ with a **dominance relation** $\preccurlyeq$ by defining for all $f, g \in \mathbb{T}_{\log}$:

$$f \preccurlyeq g \ :\Longleftrightarrow \ vf \geqslant vg.$$

It is well known that the notions of *valuation*, *valuation ring*, and *dominance relation* in valuation theory are essentially equivalent (in the sense that given one of the three, you can recover the other two). Here, when we refer to $\mathbb{T}_{\log}$ as a valued field, we mean the field $\mathbb{T}_{\log}$ equipped with the valuation v, and/or the valuation ring $\mathcal{O}$, and/or the dominance relation $\preccurlyeq$. The distinction does not matter unless we need to decide on a first-order language for the sake of doing model theory.

The ordered valued field $\mathbb{T}_{\log}$. Next we equip the valued field $\mathbb{T}_{\log}$ with an ordering $<$ which makes it an ordered valued field. This ordering is defined in the natural way by looking at the sign of the leading coefficient, i.e., for $f = \sum_{\mathfrak{m} \in \mathfrak{L}_n} f_{\mathfrak{m}} \mathfrak{m} \in \mathbb{T}_{\log}^{\times}$ we define

$$f > 0 \ :\Longleftrightarrow \ f_{\mathfrak{d}(f)} > 0.$$

The ordered valued differential field $\mathbb{T}_{\log}$. To complete the construction of $\mathbb{T}_{\log}$, we finally equip it with a derivation ∂, making it an ordered valued differential field.

Define the **derivation** $\partial : \mathbb{T}_{\log} \to \mathbb{T}_{\log}$ as the unique map satisfying:

(1) ∂ is additive: $\partial(f+g) = \partial(f) + \partial(g)$ for all $f, g \in \mathbb{T}_{\log}$;
(2) ∂ satisfies the Leibniz rule: $\partial(fg) = f\partial(g) + \partial(f)g$ for all $f, g \in \mathbb{T}_{\log}$;
(3) ∂ is defined in the natural way on iterated logarithms: $\partial(\ell_0^r) = r\ell_0^{r-1}$ and $\partial(\ell_{n+1}^r) = r\ell_{n+1}^{r-1}(\ell_0 \cdots \ell_n)^{-1}$ for all $r \in \mathbb{R}$; and
(4) ∂ is *strongly linear*: $\partial(\sum_{\mathfrak{m} \in \mathfrak{L}_n} f_{\mathfrak{m}} \mathfrak{m}) = \sum_{\mathfrak{m} \in \mathfrak{L}_n} f_{\mathfrak{m}} \partial(\mathfrak{m})$ for $\sum_{\mathfrak{m} \in \mathfrak{L}_n} f_{\mathfrak{m}} \mathfrak{m} \in \mathbb{R}[[\mathfrak{L}_n]] \subseteq \mathbb{T}_{\log}$.

The reader should take on faith that such a map exists and is unique; any doubters would be better off just reading the official construction in [**6**, Appendix A].

We would like to emphasize that there is nothing fancy about the definition of ∂ on $\mathbb{T}_{\log}$; in fact, it is the most natural derivation one would define on logarithmic transseries. For example:

$$\begin{aligned} \partial(x^3 \log x + \sqrt{x} + 2 + \cdots) \ &= \ 3x^2 \log x + x^2 + \frac{1}{2x^{1/2}} + \cdots \\ \partial\left(\frac{1}{\log\log x} + \frac{1}{(\log\log x)^2} + \cdots\right) \ &= \ -\frac{1}{x \log x (\log\log x)^2} - \frac{2}{x \log x (\log\log x)^3} + \cdots \end{aligned}$$

Associated with the derivation on $\mathbb{T}_{\log}$ is its **constant field**:

$$C_{\mathbb{T}_{\log}} \ := \ \{f \in \mathbb{T}_{\log} : \partial(f) = 0\}$$

In this case, it turns out the constant field is just $\mathbb{R}$. Given the derivation ∂ on $\mathbb{T}_{\log}$, we also define the **logarithmic derivative**

$$f \mapsto f^{\dagger} \ := \ \frac{\partial(f)}{f} : \mathbb{T}_{\log}^{\times} \to \mathbb{T}_{\log}$$

and the additive abelian group of all logarithmic derivatives of $\mathbb{T}_{\log}$:

$$\mathbb{T}_{\log}^{\dagger} \ := \ \{f^{\dagger} : f \in \mathbb{T}_{\log}^{\times}\} \ = \ (\mathbb{T}_{\log}^{\times})^{\dagger}$$

The logarithmic derivative is interdefinable with the derivation, and it is of equal importance.

1.3. Properties of $\mathbb{T}_{\log}$

In this section, we survey various properties enjoyed by $\mathbb{T}_{\log}$ as an ordered valued differential field. In particular, we will draw attention to the special ways in which the ordering, the valuation, the derivation, and the field structure interact.

***H*-field.** Recall that the constant field of $\mathbb{T}_{\log}$ is the field $\mathbb{R}$ of real numbers. Above, we defined the valuation ring $\mathcal{O}$ from the valuation map v, which *a priori* had nothing to do with the derivation. Alternatively, we could have defined the same valuation ring as the convex hull of the constant field:

$$\mathcal{O} \ = \ \big\{f \in \mathbb{T}_{\log} : |f| \leqslant c \text{ for some } c \in \mathbb{R}\big\}$$

Furthermore, $\mathbb{T}_{\log}$ has the following two properties:

(H1) for all $f \in \mathbb{T}_{\log}$, if $f > \mathbb{R}$, then $\partial(f) > 0$;

(H2) $\mathcal{O} = \mathbb{R} + \scriptstyle\mathcal{O}$, where $\scriptstyle\mathcal{O}$ $= \{f \in \mathbb{T}_{\log} : |f| < c \text{ for all } c \in \mathbb{R}^{>}\}$ is the maximal ideal of the valuation ring $\mathcal{O}$.

(H1) and (H2) should be viewed as properties which describe an interaction between the derivation, valuation and ordering of $\mathbb{T}_{\log}$. These two properties already capture much of the "asymptotics" of logarithmic transseries.

More generally, an ***H*-field** is an ordered differential field K such that for the convex hull

$$\mathcal{O}_K \ = \ \{f \in K : |f| \leqslant c \text{ for some } c \in C_K\}$$

of the constant field C_K of K we have:

(H1) for all $f \in K$, if $f > C_K$, then $\partial(f) > 0$;

(H2) $\mathcal{O}_K = C_K + \scriptstyle\mathcal{O}_K$, where $\scriptstyle\mathcal{O}_K$ is the maximal ideal of the valuation ring $\mathcal{O}_K$ of K.

That $\mathbb{T}_{\log}$ satisfies (H1) and (H2) above means that $\mathbb{T}_{\log}$ is an H-field.

The asymptotic couple of $\mathbb{T}_{\log}$. It is a consequence of (H1) and (H2) that for $f \in \mathbb{T}_{\log}^{\times}$ such that $v(f) \neq 0$, the values of $v\big(\partial(f)\big)$ and $v(f^{\dagger})$ only depend on $v(f)$. Thus we follow Rosenlicht **[33]** in taking the function

$$\psi : \Gamma_{\log}^{\neq} \to \Gamma_{\log}$$

defined by

$$\psi(\gamma) \ := \ v(f^{\dagger}) \text{ for } f \in \mathbb{T}_{\log}^{\times} \text{ such that } \gamma = v(f) \neq 0$$

as a new primitive, calling the pair $(\Gamma_{\log}, \psi)$ an **asymptotic couple** (the asymptotic couple of $\mathbb{T}_{\log}$). On the level of vectors, the map ψ does the following:

$$(\underbrace{0,\ldots,0}_{n}, \underbrace{r_n}_{\neq 0}, r_{n+1}, \ldots) \ \mapsto \ (\underbrace{1,\ldots,1}_{n+1}, 0, 0, \ldots)$$

In Figure 1.1 we attempt to visualize the asymptotic couple $(\Gamma_{\log}, \psi)$. As with any dense linear order, we can picture the underlying divisible ordered abelian group $\Gamma_{\log}$ as an infinite line stretching from left to right. Additionally we include a distinguished vertical stick to indicate the location of $0 = (0, 0, 0, \ldots)$. To represent the important subset $\Psi_{\log} = \psi(\Gamma_{\log}^{\neq})$, we draw a collection of vertical sticks to the right of 0. The convergent and shrinking nature of this collection is intended to suggest that both

(1) the induced ordering $(\Psi_{\log}, <)$ is isomorphic to that of the natural numbers $(\mathbb{N}, <)$, and
(2) the distance between two adjacent sticks is much bigger than the distance between the next two adjacent sticks.

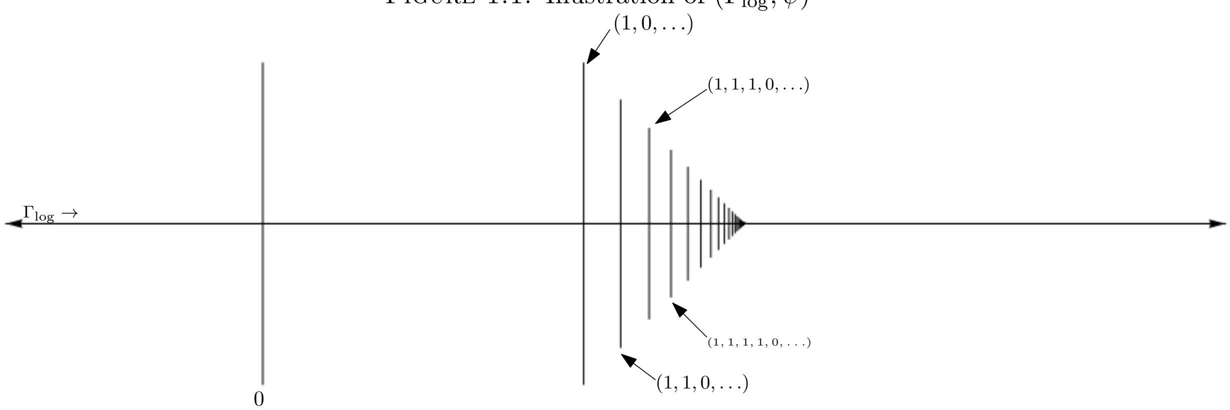
Indeed, the difference between, say, the first and second elements of $\Psi_{\log}$ is

$$(1, 1, 0, \ldots) - (1, 0, \ldots) \;=\; (0, 1, 0, \ldots)$$

which is infinitely larger (i.e., is a member of a larger archimedean class) than the difference between the second and third elements of $\Psi_{\log}$, which is

$$(1, 1, 1, 0, \ldots) - (1, 1, 0, \ldots) \;=\; (0, 0, 1, 0, \ldots).$$

FIGURE 1.1. Illustration of $(\Gamma_{\log}, \psi)$



Most of our intuition for this structure and its elementary extensions comes from drawing pictures of this form (for example, see Figure 4.1). Our choice of drawing the infinite set $\Psi_{\log}$ in this way was inspired by the illustrations from [**8**, Ch. 10].

Integration. Another feature of $\mathbb{T}_{\log}$ is that it has *integration*:

$$\text{For all } f \in \mathbb{T}_{\log}, \text{ there is } g \in \mathbb{T}_{\log} \text{ such that } \partial(g) = f$$

This is a consequence of the series construction of $\mathbb{T}_{\log}$, for example:

$$\frac{1}{\log x} \;=\; \partial\left(\frac{x}{\log x} + \frac{x}{(\log x)^2} + \frac{2x}{(\log x)^3} + \frac{6x}{(\log x)^4} + \cdots\right)$$

A slick way to prove that $\mathbb{T}_{\log}$ has integration is with [**6**, 15.2.4].

Partial exponential integration. In contrast to having full integration, $\mathbb{T}_{\log}$ only has *partial exponential integration* in the following sense:

$$\text{For } f \in \mathbb{T}_{\log}, \text{ there may or may not exist } g \in \mathbb{T}_{\log}^{\times} \text{ such that } g^{\dagger} = f$$

In other words, $\mathbb{T}_{\log}^{\dagger}$ is a proper subset of $\mathbb{T}_{\log}$. For example, there is no $g \in \mathbb{T}_{\log}^{\times}$ such that $g^{\dagger} = 1$. If there were, then $\partial(g) = g$, i.e., "$g = \exp(x)$". However, the derivation of $\mathbb{T}_{\log}$ does not contain any such fixed points.

The H-field $\mathbb{T}$ from [**6**] does have full exponential integration, i.e., $\mathbb{T}^{\dagger} = \mathbb{T}$. In many ways, this is the main difference between $\mathbb{T}_{\log}$ and $\mathbb{T}$. It is also seems to be the biggest hurdle when it comes to describing the first order theory of $\mathbb{T}_{\log}$.

Real closed field. As a field, $\mathbb{T}_{\log}$ is a real closed field. This is guaranteed by the construction of $\mathbb{T}_{\log}$: each $\mathbb{R}[[\mathfrak{L}_n]]$ is a real closed field because it is a spherically complete (hence henselian) valued subfield of the valued field $\mathbb{T}_{\log}$, with real closed residue field $\mathbb{R}$ and divisible value group $\mathfrak{L}_n$. As $\mathbb{T}_{\log}$ is the direct union of real closed fields, it is also real closed. This observation also gives an alternative way to define the same ordering $<$ on $\mathbb{T}_{\log}$ which makes it an ordered field, i.e., for $f \in \mathbb{T}_{\log}$:

$$f > 0 \quad \Longleftrightarrow \quad \text{there is } g \in \mathbb{T}_{\log}^{\times} \text{ such that } g^2 = f$$

ω-free and newtonian. Finally, there are two technical properties of H-fields which $\mathbb{T}_{\log}$ enjoys: it is *ω-free* and *newtonian*. We refer the reader to Section 5.3 for rigorous definitions and a fuller discussion of these properties. We will nevertheless make a few vague comments now:

For valued fields (with or without any derivation), the notion of *henselian* is very important. A valued field K with valuation ring $\mathcal{O}$ is *henselian* if every polynomial which "should" have a zero in $\mathcal{O}$ actually does have a zero in $\mathcal{O}$, where "should" has a precise technical meaning. *Newtonian* is a generalization of henselian for H-fields which involves also *differential*-polynomials, i.e., an H-field K is *newtonian* if every differential-polynomial which "should" have a zero in $\mathcal{O}$ actually does have a zero in $\mathcal{O}$.

The property of *ω-free* is a robust first-order property which prevents certain deviant behavior from occurring in differentially-algebraic extensions of an H-field. Unlike newtonian, there is no apparent valued field analogue for ω-free. $\mathbb{T}_{\log}$ being ω-free essentially boils down to the fact that its construction excludes the following formal series:

$$\omega \;=\; \frac{1}{\ell_0^2} + \frac{1}{\ell_0^2\ell_1^2} + \frac{1}{\ell_0^2\ell_1^2\ell_2^2} + \cdots + \frac{1}{\ell_0^2\cdots\ell_n^2} + \cdots$$

The H-field $\mathbb{T}$ is also ω-free and newtonian. In fact, these two properties do much of the heavy lifting in achieving the main model-theoretic results in [**6**].

1.4. Overview of results

We will briefly describe here the main results contained in this thesis. Most of these results are in [**14, 15, 16**].

Model theory of the asymptotic couple $(\Gamma_{\log}, \psi)$ of $\mathbb{T}_{\log}$. Much of Chapters 2, 3, and 4 work towards studying the first-order theory of the asymptotic couple $(\Gamma_{\log}, \psi)$ of $\mathbb{T}_{\log}$. In Chapter 4 we use the natural language of asymptotic couples:

$$\mathcal{L}_{AC} \;:=\; \{0, +, -, <, \psi, \infty\}$$

as well as introduce a larger language which is more specific to the structure $(\Gamma_{\log}, \psi)$:

$$\mathcal{L}_{AC,\log} \;=\; \{0, +, -, <, \psi, \infty, s, p, \delta_1, \delta_2, \delta_3, \ldots\}$$

In these languages, we axiomatize an $\mathcal{L}_{AC}$-theory T_{AC} and an $\mathcal{L}_{AC,\log}$-theory $T_{AC,\log}$ which extends T_{AC} by definitions, such that $T_{AC} = \mathrm{Th}_{\mathcal{L}_{AC}}(\Gamma_{\log}, \psi)$ and $T_{AC,\log} = \mathrm{Th}_{\mathcal{L}_{AC,\log}}(\Gamma_{\log}, \psi)$. Furthermore, we get:

Theorem (Theorem 4.2.2)**.** *$T_{AC,\log}$ has quantifier elimination.*

Theorem (Corollary 4.2.3)**.** *T_{AC} is model complete.*

The above two theorems, and the quantifier elimination in particular, open up the floodgates for obtaining further model theoretic results, for instance:

Theorem (Corollary 4.3.14)**.** *The subset $\Psi_{\log}$ of $\Gamma_{\log}$ is stably embedded in $(\Gamma_{\log}, \psi)$.*

Theorem (Theorem 4.5.3)**.** *T_{AC} has the non-independence property (NIP).*

Theorem (Theorem 4.6.3)**.** *T_{AC} is not strong. In particular, it is not strongly NIP, does not have finite* dp-*rank, and is not* dp-*minimal.*

Chapter 4 also contains several other minor model-theoretic results and observations about T_{AC} and $T_{AC,\log}$.

The number of Liouville closures of an H-field. Consider the classical ordinary differential equation

$$y' + fy \;=\; g \qquad (1.4.1)$$

where f and g are sufficiently nice real-valued functions. To solve (1.4.1), we first perform an *exponential integration* to obtain the so-called *integrating factor* $\mu = \exp(\int f)$. Then we perform an *integration* to obtain a solution $y = \mu^{-1}\int(g\mu)$. We can also solve equations of the form (1.4.1) in H-fields in a similar way, by first exponentially integrating and then by integrating.

One of the biggest differences between $\mathbb{T}_{\log}$ and $\mathbb{T}$ is that $\mathbb{T}_{\log}$ only has partial exponential integration. In particular, in $\mathbb{T}_{\log}$ some equations of the form (1.4.1) do *not* have nonzero solutions. However, in $\mathbb{T}$ all equations of the form (1.4.1) have nonzero solutions. In other words, we say that $\mathbb{T}$ is *Liouville closed*, whereas $\mathbb{T}_{\log}$ is not.

More generally, a real closed H-field in which every equation of the form (1.4.1) has a nonzero solution, with f and g ranging over K, is said to be *Liouville closed*. If K is an H-field, then a minimal Liouville closed H-field extension of K is called a *Liouville closure* of K. The main result of [**4**] is that for any H-field K, exactly one of the following occurs:

(I) K has exactly one Liouville closure up to isomorphism over K,
(II) K has exactly two Liouville closures up to isomorphism over K.

There are three distinct types of H-fields: an H-field K either is *grounded*, *has a gap*, or has *asymptotic integration*. According to [**4**], grounded H-fields fall into case (I) and H-fields with a gap fall into case (II). If an H-field has asymptotic integration, then it is either in case (I) or (II). However, the precise dividing line between (I) and (II) for K having asymptotic integration was not known.

The main result of Chapter 6 shows that this dividing line is exactly the property of an H-field being λ-*free*, which is a weakening of the property of being ω-free. Specifically we show:

Theorem. *(Theorem 6.7.1) Let K be an H-field. Then K has at least one and at most two Liouville closures up to isomorphism over K. In particular,*

(1) *K has exactly one Liouville closure up to isomorphism over K iff*
  (a) *K is grounded, or*
  (b) *K is λ-free.*

(2) *K has exactly two Liouville closures up to isomorphism over K iff*
   (c) *K has a gap, or*
   (d) *K has asymptotic integration and is not λ-free.*

A conjectured language and axiomatization for a model complete theory of $\mathbb{T}_{\log}$. The final part of this thesis (Chapters 7 and 8) outlines our strategy for proving model completeness for $\mathbb{T}_{\log}$ in a certain language. We direct the reader to the introduction of Chapter 7 and Section 8.3 for a more detailed introduction to our strategy.

In Chapter 7, we introduce two new classes of objects: LD-*fields* and LD-H-*fields*. Roughly speaking, an LD-H-field is a pair (K, LD) where K is an H-field and $\mathrm{LD} \subseteq K$ is a distinguished subset of K such that (K, LD) satisfies many of the same universal properties of the pair $(\mathbb{T}_{\log}, \mathbb{T}_{\log}^{\dagger})$. An LD-field is like an LD-H-field, except without a field ordering. In Chapter 7 we develop some of the general theory of LD-fields and LD-H-fields, although there is still much more work to be done.

In Chapter 8 we turn our attention to the big picture. Ultimately we show that model completeness of $\mathbb{T}_{\log}$ as an LD-H-field can be reduced to three precise conjectures about LD-H-fields.

CHAPTER 2

Asymptotic couples

Chapter 2 covers the basic properties and definitions for asymptotic couples. We also develop here some local machinery that we will need for doing the model theory of $(\Gamma_{\log},\psi)$ in Chapters 3 and 4, as well as for results in later chapters on valued differential fields.

In Section 2.1, we introduce some concepts in the more general setting of ordered abelian groups. In particular, given an ordered abelian group Γ and a subset $S\subseteq\Gamma$, we consider four properties the set S may or may not have. Such properties will be related to the "rate of pseudoconvergence" of pseudocauchy sequences we consider in later chapters. The properties of S to be considered are really properties of the cut $S^{\downarrow}$ determined by S in Γ, however for technical reasons it will be more convenient for us to define these notions for arbitrary S.

Section 2.2 reviews the basic theory of asymptotic couples. Most of this material comes from [**6**, §6.5 and §9.2]. We also introduce the asymptotic couple $(\Gamma^{\mathbb{Q}}_{\log},\psi)$ which will be important to us later.

In Section 2.3, we investigate further the theory of asymptotic couples of *H-type* with *asymptotic integration*. We introduce here the so-called *successor function* $s:\Gamma\to\Psi$ on such asymptotic couples. This function plays a very important role in the theory of $(\Gamma_{\log},\psi)$.

In Section 2.4, we introduce and study a new type of cut which can be found in an H-asymptotic couple with asymptotic integration: an *s-cut*. In contrast with the properties from Section 2.1, the notion of an s-cut requires the full asymptotic couple structure, not just the underlying ordered abelian group structure. We also introduce *(B,ε)-shifts*, a method of equipping an asymptotic couple (Γ,ψ) with a new ψ-map $\widetilde{\psi}:\Gamma^{\neq}\to\Gamma$ such that $(\Gamma,\widetilde{\psi})$ is also an asymptotic couple with the same *contraction map*.

Finally, in Section 2.5 we introduce the *yardstick property* and several variants. This property is also related to the "rate of pseudoconvergence" of pseudocauchy sequences to be considered in later chapters.

2.1. Ordered abelian groups

In this section Γ is an ordered abelian group, $S\subseteq\Gamma$, $\alpha\in\Gamma$ and $n\geqslant 1$. We define:

$$\alpha+nS \;:=\; \{\alpha+n\gamma:\gamma\in S\}$$

A set of the form $\alpha+nS$ is called an **affine transform** of S. Many qualitative properties of a set $S\subseteq\Gamma$ are preserved when passing to an affine transform, for instance:

Lemma 2.1.1. *S has a supremum in $\mathbb{Q}\Gamma$ iff $\alpha+nS$ does.*

Jammed sets. The notion of *jammed* appears in [**6**, §3.4] in connection with pseudocauchy sequences (see Section 5.1 below). However, it is ultimately a property of subsets of an ordered abelian group Γ. In this subsection we define *jammed* in this context and discuss some of its basic properties.

Definition 2.1.2. We say that S is **jammed (in Γ)** if $S \neq \emptyset$ does not have a greatest element and for every nontrivial convex subgroup Δ of Γ, there is $\gamma_0 \in S$ such that for every $\gamma_1 \in S^{>\gamma_0}$, $\gamma_1 - \gamma_0 \in \Delta$.

Example 2.1.3. Suppose $\Gamma \neq \{0\}$ is such that $\Gamma^{>}$ does not have a least element. Then $S := \Gamma^{<\beta}$ is jammed for every $\beta \in \Gamma$. In particular, $\Gamma^{<}$ is jammed.

Most $\Gamma \neq \{0\}$ we will deal with are either divisible or else $[\Gamma^{\neq}]$ does not have a least element and so Example 2.1.3 will provide a large collection of jammed subsets for such Γ. Of course, not all jammed sets are of the form $S^{\downarrow} = \Gamma^{<\beta}$.

Whether or not S is jammed in Γ depends on the archimedean classes of Γ in the following way:

Lemma 2.1.4. *Let Γ_1 be an ordered abelian group extension of Γ such that $[\Gamma^{\neq}]$ is coinitial in $[\Gamma_1^{\neq}]$. Then S is jammed in Γ iff S is jammed in Γ_1.*

Being jammed is also preserved by affine transforms:

Lemma 2.1.5. *S is jammed iff $\alpha + nS$ is jammed.*

Proof. ($\Rightarrow$) Assume S is jammed. Let Δ be a nontrivial convex subgroup of Γ. Let $\gamma_0 \in S$ be such that for every $\gamma_1 \in S^{>\gamma_0}$, $\gamma_1 - \gamma_0 \in \Delta$. Consider the element $\delta_0 := \alpha + n\gamma_0 \in \alpha + nS$. Let $\delta_1 \in (\alpha + nS)^{>\delta_0}$. Then $\delta_1 = \alpha + n\gamma_1$ with $\gamma_1 \in S^{>\gamma_0}$ and $\delta_1 - \delta_0 = n(\gamma_1 - \gamma_0) \in \Delta$. We conclude that $\alpha + nS$ is jammed.

($\Leftarrow$) Assume $\alpha + nS$ is jammed. Let Δ be a nontrivial convex subgroup of Γ. Let $\delta_0 = \alpha + n\gamma_0 \in \alpha + nS$ be such that $\delta_1 - \delta_0 \in \Delta$ for all $\delta_1 \in (\alpha + nS)^{>\delta_0}$. Then for $\gamma_1 \in S^{>\gamma_0}$ we have $\delta_1 := \alpha + n\gamma_1 \in (\alpha + nS)^{>\delta_0}$ and so $\delta_1 - \delta_0 = n(\gamma_1 - \gamma_0) \in \Delta$. As Δ is convex, it follows that $\gamma_1 - \gamma_0 \in \Delta$. We conclude that S is jammed. $\square$

Whether or not S is jammed depends only on the downward closure $S^{\downarrow}$ of S:

Lemma 2.1.6. *S is jammed iff $S^{\downarrow}$ is jammed.*

Example 2.1.7. Let $\Gamma_{\log}$ be the ordered divisible abelian group defined in Chapter 1. The important set

$$\Psi_{\log} := \{e_0, e_0 + e_1, e_0 + e_1 + e_2, \ldots\}$$

is a jammed subset of $\Gamma_{\log}$.

Decelerating sets. The next flavor of set we consider, a *decelerating set*, is similar in spirit to jammed sets; however, there is no requirement to exhaust all of the archimedean classes of Γ. There is no pseudocauchy sequence analogue of *decelerating set* in **[6]**.

Definition 2.1.8. We say that S **decelerates (in Γ)** (or that S is a **decelerating set**) if $S \neq \emptyset$ and for every $\gamma \in S$, there is $\delta_0 > 0$ such that $\gamma + \delta_0 \in S$ and for every $\delta_1 > 0$, if $\gamma + \delta_0 + \delta_1 \in S$, then $[\delta_1] < [\delta_0]$.

A set S decelerating is really a property of the cut that S induces in Γ, and is preserved under extensions:

Lemma 2.1.9. *S decelerates iff $S^{\downarrow}$ decelerates. Furthermore, if Γ_1 is an ordered abelian group extending Γ, then S decelerates in Γ iff S decelerates in Γ_1.*

Note that if $S \subseteq \Gamma$ has a largest element, then S does not decelerate.

The following lemma will be used in Proposition 5.5.4:

Lemma 2.1.10. *Let Δ be a finite rank ordered abelian group and let $S \subseteq \Delta$ be nonempty. Then S does not decelerate.*

Proof. This follows from the fact that ordered abelian groups of finite rank have only finitely many archimedean classes, whereas if S were to decelerate, then there would have to be infinitely many archimedean classes. $\square$

Example 2.1.11. Consider the abelian group

$$\Gamma \ := \ \bigoplus_{\rho<\omega+\omega} \mathbb{R}e_\rho$$

equipped with unique ordering such that $e_\rho > 0$ for all ρ, and $[e_\rho] > [e_\sigma]$ for all $\rho < \sigma < \omega+\omega$. Then the set

$$S \ := \ \{\textstyle\sum_{i=0}^{n} e_i : n < \omega\}$$

is not a jammed set, but it is a decelerating set.

Δ-fluent sets. *In the rest of this section Δ is a nontrivial convex subgroup of* Γ. Like *jammed*, the notion of Δ-*fluent* also occurs in [**6**, §3.4]. Here we extract what it means for subsets of Γ:

Definition 2.1.12. We say that S is **Δ-fluent** if $S \neq \emptyset$ and for every $\alpha \in S$, $\alpha + \Delta \subseteq S^{\downarrow}$.

It is clear that for nonempty S without a largest element, S is Δ-fluent for some Δ iff S is not jammed. Furthermore:

Lemma 2.1.13. *S is Δ-fluent iff $\alpha + nS$ is Δ-fluent.*

Proof. ($\Rightarrow$) Assume S is Δ-fluent. Then $S \neq \emptyset$, so $\alpha + nS \neq \emptyset$. Let $\alpha + n\beta \in \alpha + nS$ with $\beta \in S$. Then $\beta + \Delta \subseteq S^{\downarrow}$ by assumption. It suffices to show that $\alpha + n\beta + \Delta^{>} \subseteq (\alpha + nS)^{\downarrow}$. Let $\delta \in \Delta^{>}$ be arbitrary. Then we have $s \in S$ such that $\beta + \delta \leqslant s$. Thus

$$\alpha + n\beta + \delta \ \leqslant \ \alpha + n\beta + n\delta \ \leqslant \ \alpha + ns.$$

As δ was arbitrary, $\alpha + nS$ is Δ-fluent.

($\Leftarrow$) Assume $\alpha + nS$ is Δ-fluent. Then $\alpha + nS \neq \emptyset$ so $S \neq \emptyset$. Let $\beta \in S$ be arbitrary. Then $\alpha + n\beta + \Delta \subseteq (\alpha + nS)^{\downarrow}$ by assumption and it suffices to show that $\beta + \Delta^{>} \subseteq S^{\downarrow}$. Let $\delta \in \Delta^{>}$ be arbitrary. Then also $n\delta \in \Delta$ and we have $s \in S$ such that $\alpha + n\beta + n\delta \leqslant \alpha + ns$. Thus $\beta + \delta \leqslant s$. We conclude that S is Δ-fluent. $\square$

Almost Δ-special sets. Finally, we introduce here the notions of Δ-*special* and *almost* Δ-*special*, which are also related to concepts appearing in [**6**, §3.5].

Definition 2.1.14. We say that S is **almost Δ-special** if there is an $\alpha \in \Gamma$ such that $S^{\downarrow} = (\alpha + \Delta)^{\downarrow}$. We say that S is **Δ-special** if $S^{\downarrow} = \Delta^{\downarrow}$

Note that if S is Δ-special, then necessarily $S \neq \emptyset$ and S does not have a largest element. In some sense, almost Δ-special cuts in Γ are the most well-behaved because the existence of α and Δ gives us a very explicit description of the cut.

2.2. Asymptotic couples

An **asymptotic couple** is a pair (Γ, ψ) where Γ is an ordered abelian group and $\psi : \Gamma^{\neq} \to \Gamma$ satisfies for all $\alpha, \beta \in \Gamma^{\neq}$,

(AC1) $\alpha + \beta \neq 0 \Longrightarrow \psi(\alpha+\beta) \geqslant \min\big(\psi(\alpha), \psi(\beta)\big)$;
(AC2) $\psi(k\alpha) = \psi(\alpha)$ for all $k \in \mathbb{Z}^{\neq}$, in particular, $\psi(-\alpha) = \psi(\alpha)$;
(AC3) $\alpha > 0 \Longrightarrow \alpha + \psi(\alpha) > \psi(\beta)$.

If in addition for all $\alpha, \beta \in \Gamma$,

(HC) $0 < \alpha \leqslant \beta \Rightarrow \psi(\alpha) \geqslant \psi(\beta)$,

then (Γ, ψ) is said to be of **H-type**, or to be an **H-asymptotic couple**.

In the rest of this section, (Γ, ψ) is an asymptotic couple (not necessarily of H-type). By convention, we extend ψ to all of Γ by setting $\psi(0) := \infty$. Then $\psi(\alpha+\beta) \geqslant \min\big(\psi(\alpha), \psi(\beta)\big)$ holds for all $\alpha, \beta \in \Gamma$, and $\psi : \Gamma \to \Gamma_\infty$ is a (non-surjective) valuation on the abelian group Γ. In particular, the following is immediate:

Fact 2.2.1. *If $\alpha, \beta \in \Gamma$ and $\psi(\alpha) < \psi(\beta)$, then $\psi(\alpha+\beta) = \psi(\alpha)$.*

For $\alpha \in \Gamma^{\neq}$ we shall also use the following notation:

$$\alpha^\dagger := \psi(\alpha), \quad \alpha' := \alpha + \psi(\alpha).$$

Convention 2.2.2. *Given $\alpha \in \Gamma$ and a function f whose domain contains $\alpha^\dagger$, respectively α', expressions of the form $f\alpha^\dagger$, respectively $f\alpha'$, are abbreviations for $f(\alpha^\dagger)$, respectively $f(\alpha')$.*

The following subsets of Γ play special roles:

$$(\Gamma^{\neq})' \ := \ \{\gamma' : \gamma \in \Gamma^{\neq}\}, \quad (\Gamma^{>})' \ := \ \{\gamma' : \gamma \in \Gamma^{>}\},$$
$$\Psi \ := \ \psi(\Gamma^{\neq}) \ = \ \{\gamma^\dagger : \gamma \in \Gamma^{\neq}\} \ = \ \{\gamma^\dagger : \gamma \in \Gamma^{>}\}.$$

Note that by (AC3) we have $\Psi < (\Gamma^{>})'$.

For an arbitrary asymptotic couple (Γ^*, ψ^*) we may occasionally refer to the set $\psi^*\big((\Gamma^*)^{\neq}\big)$ as *the Ψ-set of* (Γ^*, ψ^*).

We think of the map $\mathrm{id} + \psi : \Gamma^{\neq} \to \Gamma$ as *the derivative*. When antiderivatives exist, they are unique:

ADH 2.2.3. *The map $\gamma \mapsto \gamma' = \gamma + \psi(\gamma) : \Gamma^{\neq} \to \Gamma$ is strictly increasing. In particular:*

(1) *$(\Gamma^{<})' < (\Gamma^{>})'$, and*
(2) *for $\beta \in \Gamma$ there is at most one $\alpha \in \Gamma^{\neq}$ such that $\alpha' = \beta$.*

Proof. This follows from [**6**, 6.5.4(iii)]. □

In fact, most elements have antiderivatives:

ADH 2.2.4. [**6**, 9.2.1] *The set $\Gamma \setminus (\Gamma^{\neq})'$ has at most one element. If Ψ has a largest element* $\max \Psi$*, then* $\Gamma \setminus (\Gamma^{\neq})' = \{\max \Psi\}$.

ADH 2.2.5. [**6**, 9.2.4] *There is at most one β such that*

$$\Psi \;<\; \beta \;<\; (\Gamma^{>})'.$$

If Ψ has a largest element, there is no such β.

Definition 2.2.6. If $\Gamma = (\Gamma^{\neq})'$, then we say that (Γ, ψ) has **asymptotic integration**. If $\beta \in \Gamma$ is as in ADH 2.2.5, then we say that β is a **gap** in (Γ, ψ) and that (Γ, ψ) **has a gap**. Finally, we call (Γ, ψ) **grounded** if Ψ has a largest element, and **ungrounded** otherwise.

The notions of asymptotic integration, gaps and being grounded form an important trichotomy for H-asymptotic couples:

ADH 2.2.7. [**6**, 9.2.16] *Suppose (Γ, ψ) is of H-type. Then exactly one of the following is true:*

(1) (Γ, ψ) *has a gap, in particular, $\Gamma \setminus (\Gamma^{\neq})' = \{\beta\}$ where β is a gap in Γ;*
(2) (Γ, ψ) *is grounded, in particular, $\Gamma \setminus (\Gamma^{\neq})' = \{\max \Psi\}$;*
(3) (Γ, ψ) *has asymptotic integration.*

Note that if (Γ, ψ) is of H-type, then ψ is constant on archimedean classes of Γ: for $\alpha, \beta \in \Gamma^{\neq}$ with $[\alpha] = [\beta]$ we have $\psi(\alpha) = \psi(\beta)$. The function $\mathrm{id} + \psi$ enjoys the following remarkable intermediate value property:

ADH 2.2.8. [**6**, 9.2.14] *Suppose (Γ, ψ) is of H-type. Then the functions*

$$\gamma \mapsto \gamma' : \Gamma^{>} \to \Gamma, \quad \gamma \mapsto \gamma' : \Gamma^{<} \to \Gamma$$

have the intermediate value property.

It is very useful to think of H-asymptotic couples in terms of the following geography:

$$\Psi \;<\; \text{possible gap} \;<\; (\Gamma^{>})'$$

Let (Γ_1, ψ_1) be an asymptotic couple. An **embedding**

$$h : (\Gamma, \psi) \to (\Gamma_1, \psi_1)$$

is an embedding $h : \Gamma \to \Gamma_1$ of ordered abelian groups such that

$$h\big(\psi(\gamma)\big) \;=\; \psi_1\big(h(\gamma)\big) \text{ for } \gamma \in \Gamma^{\neq}.$$

If $\Gamma \subseteq \Gamma_1$ and the inclusion $\Gamma \to \Gamma_1$ is an embedding $(\Gamma, \psi) \to (\Gamma_1, \psi_1)$, then we call (Γ_1, ψ_1) an **extension** of (Γ, ψ).

Definition 2.2.9. Call an asymptotic couple (Γ, ψ) **divisible** if the abelian group Γ is divisible. If (Γ, ψ) is a divisible asymptotic couple, then we construe Γ as a vector space over $\mathbb{Q}$ in the obvious way.

By [**4**, Proposition 2.3(2)], ψ extends uniquely to a map $(\mathbb{Q}\Gamma)^{\neq} \to \mathbb{Q}\Gamma$, also denoted by ψ, such that $(\mathbb{Q}\Gamma, \psi)$ is an asymptotic couple. We say that $(\mathbb{Q}\Gamma, \psi)$ is the **divisible hull** of (Γ, ψ). The following summarizes many important properties of the divisible hull:

ADH 2.2.10. *Let (Γ, ψ) be an asymptotic couple. Then $(\mathbb{Q}\Gamma, \psi)$ is an extension of (Γ, ψ) such that*

(1) $(\mathbb{Q}\Gamma, \psi)$ *is divisible,*

(2) $\psi\big((\mathbb{Q}\Gamma)^{\neq}\big) = \Psi = \psi(\Gamma^{\neq})$,
(3) $[\mathbb{Q}\Gamma] = [\Gamma]$,
(4) *if* $\dim_{\mathbb{Q}} \mathbb{Q}\Gamma$ *is finite, then* $\Psi = \psi(\Gamma^{\neq})$ *is a finite set,*
(5) *if* (Γ,ψ) *is of* H*-type, then so is* $(\mathbb{Q}\Gamma,\psi)$,
(6) *if* (Γ,ψ) *is grounded, then so is* $(\mathbb{Q}\Gamma,\psi)$,
(7) *if* $\beta \in \Gamma$ *is a gap in* (Γ,ψ), *then it is a gap in* $(\mathbb{Q}\Gamma,\psi)$,
(8) $(\Gamma^{\neq})' = \big((\mathbb{Q}\Gamma)^{\neq}\big)' \cap \Gamma$, *and*
(9) *if* $i : (\Gamma,\psi) \to (\Gamma_1,\psi_1)$ *is an embedding and* (Γ_1,ψ_1) *is divisible, then* i *extends to a unique embedding* $j : (\mathbb{Q}\Gamma,\psi) \to (\Gamma_1,\psi_1)$.

Proof. For proofs of all these facts, see [**6**, §6.5 and §9.2]. □

We say that (Γ,ψ) has **rational asymptotic integration** if its divisible hull $(\mathbb{Q}\Gamma,\psi)$ has asymptotic integration.

Remark 2.2.11. It is entirely possible that (Γ,ψ) has asymptotic integration whereas $(\mathbb{Q}\Gamma,\psi)$ has a gap. For an example of this, see the remark after Corollary 2 in [**2**]. We avoid this pathology in Section 4.2 by considering only divisible asymptotic couples.

Example 2.2.12. In analogy with $(\Gamma_{\log},\psi)$ defined in Chapter 1, we now define $(\Gamma_{\log}^{\mathbb{Q}},\psi)$. Its underlying abelian group is $\bigoplus_n \mathbb{Q}e_n$, a vector space over $\mathbb{Q}$ with basis (e_n). We equip $\Gamma_{\log}^{\mathbb{Q}}$ with the unique ordering which makes $\Gamma_{\log}^{\mathbb{Q}} \subseteq \Gamma_{\log}$ an extension of ordered abelian groups, and we define $\psi : (\Gamma_{\log}^{\mathbb{Q}})^{\neq} \to \Gamma_{\log}^{\mathbb{Q}}$ in such a way as to make $(\Gamma_{\log}^{\mathbb{Q}},\psi) \subseteq (\Gamma_{\log},\psi)$ an extension of asymptotic couples: for $r_n \neq 0$,

$$\alpha = (\underbrace{0,\ldots,0}_{n}, r_n, r_{n+1},\ldots) \;\mapsto\; \psi(\alpha) = (\underbrace{1,\ldots,1}_{n+1}, 0, 0, \ldots).$$

$(\Gamma_{\log}^{\mathbb{Q}},\psi)$ is a divisible H-asymptotic couple with (rational) asymptotic integration. In fact, $(\Gamma_{\log}^{\mathbb{Q}},\psi)$ will be a prime model for the theory $T_{AC} = \mathrm{Th}_{\mathcal{L}_{AC}}(\Gamma_{\log},\psi)$ to be introduced in Chapter 4.

2.3. Asymptotic integration

In this section (Γ,ψ) *is an* H*-asymptotic couple with asymptotic integration and we let* α,β *range over* Γ. By ADH 2.2.10 we may assume that (Γ,ψ) is given as a substructure of some divisible H-asymptotic couple. Doing this allows us to multiply by $1/n$ in the proofs, for $n \geqslant 1$.

Definition 2.3.1. Given α we let $\int\alpha$ denote the unique $\beta \neq 0$ such that $\beta' = \alpha$ and we call $\beta = \int\alpha$ the **integral** of α. This gives us a function $\int : \Gamma \to \Gamma^{\neq}$ which is the inverse of $\gamma \mapsto \gamma' : \Gamma^{\neq} \to \Gamma$. We sometimes refer to the act of applying the function $\int$ as **integrating**. Note that $\int\alpha < 0$ if $\alpha \in \Psi^{\downarrow} = (\Gamma^{<})'$.

We define the **successor function** $s : \Gamma \to \Psi$ by $\alpha \mapsto \psi(\int\alpha)$. The successor function gets its name from the observation that in many cases of interest, such as the asymptotic couple of $\mathbb{T}_{\log}$, the ordered subset Ψ of Γ is a successor set, and for $\alpha \in \Psi$, the immediate successor of α in Ψ is $s(\alpha)$. However in general, Ψ as an ordered subset of Γ need not be a successor set; for example, if (Γ,ψ) is a so-called *closed asymptotic couple* considered in [**3**], then Ψ, as an ordered set, is a dense linear order without endpoints, and hence not a successor set.

We also define the **contraction map** $\chi : \Gamma^{\neq} \to \Gamma^{<}$ by $\alpha \mapsto \int \psi(\alpha)$. We extend χ to a function $\Gamma \to \Gamma^{\leqslant}$ by setting $\chi(0) := 0$. The contraction map gets its name from the connection between asymptotic couples and contraction groups (for instance, see [**21, 22, 2**]). Since χ can be defined in terms of ψ and $\int$, and $\int$ can be defined in terms of s as we will see in Lemma 2.3.3, we choose to focus most of our attention on the function s.

Example 2.3.2. In this example (Γ, ψ) is either $(\Gamma_{\log}, \psi)$ or $(\Gamma_{\log}^{\mathbb{Q}}, \psi)$. The functions $\int$, s, and χ are given by the following formulas:

(1) (Integral) For $\alpha = (r_0, r_1, r_2, \ldots) \in \Gamma$, take the unique n such that $r_n \neq 1$ and $r_m = 1$ for $m < n$. Then

$$\alpha = (\underbrace{1, \ldots, 1}_{n}, \underbrace{r_n}_{\neq 1}, r_{n+1}, r_{n+2} \ldots) \;\mapsto\; \int \alpha = (\underbrace{0, \ldots, 0}_{n}, r_n - 1, r_{n+1}, r_{n+2}, \ldots)$$

(2) (Successor) For $\alpha = (r_0, r_1, r_2, \ldots) \in \Gamma$, take the unique n such that $r_n \neq 1$ and $r_m = 1$ for $m < n$. Then

$$\alpha = (\underbrace{1, \ldots, 1}_{n}, \underbrace{r_n}_{\neq 1}, r_{n+1}, r_{n+1} \ldots) \;\mapsto\; s(\alpha) = (\underbrace{1, \ldots, 1}_{n+1}, 0, 0, \ldots)$$

(3) (Contraction) If $\alpha = 0$, then $\chi(\alpha) = 0$. Otherwise, for $\alpha = (r_0, r_1, r_2, \ldots) \in \Gamma^{\neq}$, take the unique n such that $r_n \neq 0$ and $r_m = 0$ for $m < n$. Then

$$\alpha = (\underbrace{0, \ldots, 0}_{n}, \underbrace{r_n}_{\neq 0}, r_{n+1}, \ldots) \;\mapsto\; \chi(\alpha) = (\underbrace{0, \ldots, 0}_{n+1}, -1, 0, 0, \ldots)$$

In particular, note that for elements in $\Psi_{\log}$, s acts as follows:

$$\begin{array}{rcl}
s(1,0,0,0,0,\ldots) & = & (1,1,0,0,0,\ldots) \\
s(1,1,0,0,0,\ldots) & = & (1,1,1,0,0,\ldots) \\
s(1,1,1,0,0,\ldots) & = & (1,1,1,1,0,\ldots) \\
& \vdots & \\
s(\underbrace{1,\ldots,1}_{n},0,0,\ldots) & = & (\underbrace{1,\ldots,1}_{n+1},0,0,\ldots)
\end{array}$$

Furthermore, $s0 = (1, 0, 0, \ldots) = \min \Psi_{\log}$ and $s0 > 0$. It is clear that the function $\gamma \mapsto s\gamma : \Psi_{\log} \to \Psi_{\log}^{>s0}$ is a bijection and $(\Psi_{\log}; <)$ is a successor set such that for $\alpha < \beta \in \Psi_{\log}$ we have $s\alpha \leqslant \beta$.

Lemma 2.3.3 (Integral Identity)**.** $\int \alpha = \alpha - s\alpha$.

Proof. Note that $(\int \alpha)' = \alpha$. Expanding this out gives $\psi(\int \alpha) + \int \alpha = s\alpha + \int \alpha = \alpha$. $\square$

The next lemma tells us, among other things, that for each α, we get an increasing sequence

$$s\alpha \;<\; s^2\alpha \;<\; s^3\alpha \;<\; s^4\alpha \;<\; \cdots$$

in Ψ.

Lemma 2.3.4. *If $\alpha \in (\Gamma^{<})'$, then $\alpha < s(\alpha)$, and if $\alpha \in (\Gamma^{>})'$, then $\alpha > s(\alpha)$. In particular, if $\alpha \in \Psi$, then $\alpha < s(\alpha)$.*

Proof. If $\alpha \in (\Gamma^{>})'$, then $\alpha > \psi(\int \alpha)$ by (AC3). Thus assume that $\alpha \in (\Gamma^{<})'$ and let $\alpha = \beta'$ with $\beta < 0$. Then

$$\begin{aligned} \alpha < s(\alpha) \quad &\Leftrightarrow \quad \alpha < \psi(\textstyle\int \alpha) \\ &\Leftrightarrow \quad \alpha < \psi(\beta) \\ &\Leftrightarrow \quad \alpha - \psi(\beta) < 0 \end{aligned}$$

and the latter is true since $\alpha - \psi(\beta) = \beta' - \psi(\beta) = \beta$. □

By (HC), if $[\alpha] > [\beta]$, then $\psi(\beta - \alpha) = \psi(\alpha)$. In the case where $[\alpha] = [\beta]$ and α and β are both sufficiently far up the set $(\Gamma^{<})'$, the following lemma can be very useful:

Lemma 2.3.5 (Successor Identity)**.** *If $s\alpha < s\beta$, then $\psi(\beta - \alpha) = s\alpha$.*

Proof. Assume $s\alpha < s\beta$. We will prove that $[\beta - s\alpha] < [s\alpha - \alpha]$, and so $\psi(\beta - \alpha) = \psi(s\alpha - \alpha) = \psi(-\int \alpha) = s\alpha$. From $s\alpha < s\beta$ we get $\psi(\int \alpha) < \psi(\int \beta)$, which gives $[\int \beta] < [\int \alpha]$. First consider the case where $\alpha \in (\Gamma^{<})'$ and $s\alpha < \beta$. Then $\int \alpha < 0$ and $s\alpha - \alpha > 0$. Note that

$$\begin{aligned} [\beta - s\alpha] < [s\alpha - \alpha] \quad &\Leftrightarrow \quad \beta - s\alpha < \tfrac{1}{n}(s\alpha - \alpha) \text{ for all } n \geqslant 1 \\ &\Leftrightarrow \quad \beta < s\alpha + \tfrac{1}{n}(s\alpha - \alpha) \text{ for all } n \geqslant 1 \\ &\Leftrightarrow \quad \beta < \psi(\textstyle\int \alpha) + \tfrac{1}{n}(-\textstyle\int \alpha) \text{ for all } n \geqslant 1 \\ &\Leftrightarrow \quad \beta < \psi(-\tfrac{1}{n}\textstyle\int \alpha) + (-\tfrac{1}{n}\textstyle\int \alpha) \text{ for all } n \geqslant 1 \\ &\Leftrightarrow \quad \beta < (-\tfrac{1}{n}\textstyle\int \alpha)' \text{ for all } n \geqslant 1 \\ &\Leftrightarrow \quad \textstyle\int \beta < \tfrac{1}{n}(-\textstyle\int \alpha) \text{ for all } n \geqslant 1, \end{aligned}$$

and the latter holds because $[\int \beta] < [\int \alpha]$. All other cases are similar. □

It follows that s can be defined in terms of ψ if we allow a suitable "external parameter":

Corollary 2.3.6. *Let (Γ^*, ψ^*) be an H-asymptotic couple with asymptotic integration that extends (Γ, ψ). Suppose $\gamma^* \in \Psi^*$ is such that $\Psi < \gamma^*$. Then $s(\alpha) = \psi^*(\alpha - \gamma^*)$ for all $\alpha \in \Gamma$.*

Since Ψ has no largest element, compactness yields an extension (Γ^*, ψ^*) of (Γ, ψ) with an element γ^* as in Corollary 2.3.6. In Chapter 3 below we also give explicit constructions for extensions with this property in Lemma 3.2.3 and Lemma 3.2.4.

Since (Γ, ψ) has asymptotic integration, ADH 2.2.7 tells us that (Γ, ψ) most definitely does not have a gap. However, it is fun (also useful) to summarize Corollary 2.3.6 with the following slogan:

$$\text{“}s(x) \;=\; \psi(x - \text{gap that does not exist})\text{”}$$

This fact is essential for Corollary 4.3.7 and a variant of this device allows the proof of Lemma 3.2.5 to be carried out. The following is immediate from Corollary 2.3.6 and (HC):

Corollary 2.3.7. *The function s has the following properties:*

(1) *s is increasing on $(\Gamma^{<})'$ and decreasing on $(\Gamma^{>})'$,*
(2) *if $\alpha \in s(\Gamma)$, then $s^{-1}(\alpha) \cap (\Gamma^{>})'$ and $s^{-1}(\alpha) \cap (\Gamma^{<})'$ are convex in Γ,*

(3) *if s is injective on Ψ, then s is strictly increasing on Ψ.*

The following lemma is also useful in understanding s in terms of ψ.

Lemma 2.3.8 (Fixed Point Identity)**.** $\beta = \psi(\alpha - \beta)$ *iff* $\beta = s(\alpha)$.

Proof. Applying ψ to $\int \alpha = \alpha - s\alpha$ gives $s\alpha = \psi(\alpha - s\alpha)$. Next, suppose that $\beta = \psi(\alpha - \beta)$. Then $\alpha = (\alpha - \beta) + \beta = (\alpha - \beta) + \psi(\alpha - \beta)$ and so $\int \alpha = \alpha - \beta$. Applying ψ yields $s\alpha = \psi(\alpha - \beta) = \beta$. □

The following lemma is a more constructive version of [**3**, Lemma 4.6] and [**6**, 9.2.11 and 9.2.13]. It shows that our function s is the same as the function $\alpha \mapsto \psi(* - \alpha)$ from [**6**].

Lemma 2.3.9 (Limit Lemma)**.** *Let $\alpha \in \Gamma$. Then $\gamma_0 := s^2\alpha \in \Psi$ and $\delta_0 := s^2\alpha - \int s\alpha \in (\Gamma^>)'$ and the map*

$$\gamma \ \mapsto \ \psi(\gamma - \alpha) : \Gamma \to \Gamma_\infty$$

takes the constant value $s\alpha$ on the set $[\gamma_0, \delta_0] := \{\gamma : \gamma_0 \leqslant \gamma \leqslant \delta_0\}$.

Proof. Define $\beta_0 := -\int \psi \int \alpha = -\int s(\alpha) > 0$. Then $\gamma_0 = \psi(\beta_0) = s^2(\alpha) \in \Psi$ and $\delta_0 = s^2\alpha - \int s\alpha = s^2\alpha + \beta_0 = \psi(\beta_0) + \beta_0 = \beta_0' \in (\Gamma^>)'$. First we calculate the values of $\psi(\gamma_0 - \alpha)$ and $\psi(\delta_0 - \alpha)$:

$$\begin{aligned}
\psi(\gamma_0 - \alpha) &= \psi(s^2\alpha - \alpha) \\
&= s\alpha \quad \text{(by Lemma 2.3.5)} \\
\psi(\delta_0 - \alpha) &= \psi(s^2\alpha - \textstyle\int s\alpha - \alpha) \\
&= \psi\big((s^2\alpha - \alpha) - \textstyle\int s\alpha\big) \\
&= s\alpha \quad \big(\text{because } \psi(s^2\alpha - \alpha) = s\alpha \text{ and } \psi(\textstyle\int s\alpha) = s^2\alpha > \alpha\big)
\end{aligned}$$

Finally, we must show that $\psi(\gamma - \alpha)$ is constant as a function of $\gamma \in [\gamma_0, \delta_0]$. By (HC), it is sufficient to show that either $\alpha < \gamma_0 < \delta_0$ or $\gamma_0 < \delta_0 < \alpha$. First suppose $\alpha \in (\Gamma^<)'$. By Lemma 2.3.4 it follows that $\alpha < s\alpha < s^2\alpha = \gamma_0 < \delta_0$. Next, suppose $\alpha \in (\Gamma^>)'$. Then $\gamma_0 < \delta_0$ and

$$\begin{aligned}
\delta_0 < \alpha \quad &\Leftrightarrow \quad s^2\alpha - \textstyle\int s\alpha < \alpha \\
&\Leftrightarrow \quad -\textstyle\int s\alpha < \alpha - s^2\alpha.
\end{aligned}$$

The last inequality holds by (HC) and the observation that both $-\int s\alpha$ and $\alpha - s^2\alpha$ are positive. □

Lemma 2.3.10. $s0 \neq 0$ *and $s0$ is the unique element $x \in \Gamma^{\neq}$ for which $\psi(x) = x$.*

Proof. By Lemma 2.3.4 we have $s0 \neq 0$, and by the Integral Identity $\int 0 = -s0$ and so $s0 = \psi(\int 0) = \psi(-s0) = \psi(s0)$. If $x \in \Gamma^{\neq}$ and $\psi(x) = x$, then $x = \psi(0 - x)$, so $x = s0$ by the Fixed Point Identity. □

Lemma 2.3.10 tells us that H-asymptotic couples with asymptotic integration come in two flavors: those with $s0 > 0$ and those with $s0 < 0$. The asymptotic couples $(\Gamma_{\log}, \psi)$ and $(\Gamma_{\log}^{\mathbb{Q}}, \psi)$ are both of type "$s0 > 0$". When $s0 > 0$, then the element $s0$ is often denoted by "1" and (Γ, ψ) is said to **have a** 1. We will not use this notation since we have the function s at our disposal and we already will be making use of the rational number $1 \in \mathbb{Q}$.

The following lemma further clarifies the geographic relationship between $s0$, 0, and Ψ:

Lemma 2.3.11. *For every $q \in \mathbb{Q}^>$, if $s0 < 0$, then $\Psi < (1-q)s0$, and if $0 < s0$, then $\Psi < (1+q)s0$.*

Proof. Let $q \in \mathbb{Q}^>$. If $s0 < 0$, then

$$(-qs0)' \;=\; -qs0 + \psi(-qs0) \;=\; -qs0 + \psi(s0) \;=\; (1-q)s0 \in (\mathbb{Q}\Gamma^>)'$$

and thus $\Psi < (1-q)s0$ by (AC3). If $0 < s0$, then

$$(qs0)' \;=\; qs0 + \psi(qs0) \;=\; (1+q)s0 \in (\mathbb{Q}\Gamma^>)',$$

and likewise $\Psi < (1+q)s0$. □

The following application of Corollary 2.3.6 says that the functions s and ψ agree on a large portion of Γ:

Corollary 2.3.12. *For all $q \in \mathbb{Q}^>$ and $\alpha \in \Gamma$ with $|\alpha| > (1+q)|s0|$ we have $s(\alpha) = \psi(\alpha)$.*

Proof. Suppose $q \in \mathbb{Q}^>$ and $\alpha \in \Gamma$ are such that $|\alpha| > (1+q)|s0|$. Let (Γ^*, ψ^*) be an elementary extension of (Γ, ψ) which contains an element $\gamma^* \in \Psi^*$ such that $\Psi < \gamma^*$. If $s0 < 0$, then $s0 < \gamma^* < 0$ and thus $|\alpha| \geqslant (1+q)|\gamma^*|$. If $s0 > 0$, then $s0 < \gamma^* < (1+q')s0$ for every $q' \in \mathbb{Q}^>$ and so $|\alpha| \geqslant (1+q)|\gamma^*|$ as well. In both cases, $[\alpha - \gamma^*] = [\alpha]$, so $s(\alpha) = \psi^*(\alpha - \gamma^*) = \psi^*(\alpha) = \psi(\alpha)$ by Corollary 2.3.6. □

We conclude with some facts about the contraction mapping χ:

Lemma 2.3.13. *For all $\alpha, \beta \in \Gamma$ and $\gamma \in \Gamma^{\neq}$:*

(1) $\big[\chi(\gamma)\big] < [\gamma]$.
(2) $\alpha \neq \beta \Longrightarrow \big[\chi(\alpha) - \chi(\beta)\big] < [\alpha - \beta]$.
(3) $\alpha < \beta \Longrightarrow \alpha - \chi(\alpha) < \beta - \chi(\beta)$.

Proof. (1) and (2) follow easily from [**6**, 9.2.18 (iii,iv)]. (3) follows from (2). □

2.4. s-cuts

In this section (Γ, ψ) is an H-asymptotic couple with asymptotic integration.

Definition 2.4.1. We say $B \subseteq \Psi$ is an ***s*-cut of** Ψ if B is an downward closed subset of Ψ such that $s(B) \subseteq B$. Let $\operatorname{sded}(\Psi)$ be the collection of all s-cuts of Ψ. We define a linear ordering $\leqslant$ on $\operatorname{sded}(\Psi)$ by $B_0 \leqslant B_1$ iff $B_0 \subseteq B_1$.

For notational convenience in Lemma 3.3.2 we also define the linear order $\operatorname{sded}^{op}(\Psi)$ as follows: The elements of $\operatorname{sded}^{op}(\Psi)$ are subsets $B \subseteq \Psi$ such that $\Psi \setminus B$ is an s-cut, i.e., B is upward closed and $s(\Psi \setminus B) \subseteq \Psi \setminus B$. Furthermore, the linear ordering $\leqslant$ on $\operatorname{sded}^{op}(\Psi)$ is defined by $B_0 \leqslant B_1$ iff $B_0 \supseteq B_1$.

We identify the ordered sets $\operatorname{sded}(\Psi)$ and $\operatorname{sded}^{op}(\Psi)$ via the isomorphism $B \mapsto \Psi \setminus B$, and refer to elements of either set as an "s-cut of Ψ".

Definition 2.4.2. For $\alpha, \beta \in \Psi$, we define $\alpha \ll \beta$ to mean $s^n\alpha < \beta$ for all n, and define $\alpha \gg \beta$ to mean $\beta \ll \alpha$. It follows that if $\alpha \ll \beta$, then there is a $B \in \operatorname{sded}(\Psi)$ such that $\alpha \in B < \beta$. Finally, we define the equivalence relation $\sim_s$ on Ψ:

$$\alpha \sim_s \beta \;:\Longleftrightarrow\; \alpha \not\ll \beta \text{ and } \beta \not\ll \alpha$$

and we call the equivalence class $\alpha/\sim_s$ of α the ***s*-class** of α.

For H-asymptotic couples with asymptotic integration, it is useful to have the following stratification in mind:

$$\begin{array}{ccc} \Gamma^{\neq} & & \\ \big\downarrow{\scriptstyle \gamma\mapsto[\gamma]} & & \\ [\Gamma^{\neq}] & & \text{archimedean classes} \\ \big\downarrow{\scriptstyle [\gamma]\mapsto\psi(\gamma)} & & \\ \Psi & & \Psi\text{-set} \\ \big\downarrow{\scriptstyle \psi(\gamma)\mapsto\psi(\gamma)/\sim_s} & & \\ \Psi/\sim_s & & s\text{-classes on the } \Psi\text{-set} \end{array}$$

Perhaps the most important s-cut of Ψ is $B := \Psi$. The following two lemmas highlight some of the interesting behavior of this s-cut.

Lemma 2.4.3. *Suppose $\alpha \in (\Gamma^{<})'$ and $q \in \mathbb{Q}^{>}$. Then $\alpha + (1+q)(s\alpha - \alpha) \in (\mathbb{Q}\Gamma^{>})'$*

Proof. Suppose $\alpha \in (\Gamma^{<})'$. Then

$$\begin{aligned} \alpha + (q+1)(s\alpha - \alpha) &= s\alpha + qs\alpha - q\alpha \\ &= \psi(\textstyle\int \alpha) + q\psi(\int \alpha) - q(\int \alpha)' \\ &= \psi(\textstyle\int \alpha) + q\psi(\int \alpha) - q(\int \alpha) - q\psi(\int \alpha) \\ &= \psi(\textstyle\int \alpha) - q\int \alpha \\ &= (-q\textstyle\int \alpha)' \in (\mathbb{Q}\Gamma^{>})'. \end{aligned}$$

The last part follows because $\alpha \in (\Gamma^{<})'$ iff $\int \alpha \in \Gamma^{<}$ iff $-q\int \alpha \in \mathbb{Q}\Gamma^{>}$ iff $(-q\int \alpha)' \in (\mathbb{Q}\Gamma^{>})'$. □

Lemma 2.4.4. *The sets Ψ and $\Psi^{\downarrow}$ are jammed.*

Proof. By Lemma 2.1.6 it suffices to show that $\Psi^{\downarrow} = (\Gamma^{<})'$ is jammed. By asymptotic integration and ADH 2.2.7, $(\Gamma^{<})'$ is nonempty and does not have a largest element. Let Δ be a nontrivial convex subgroup of Γ. Take $\delta \in \Delta^{>}$ and set $\gamma_0 := (-\delta)' \in (\Gamma^{<})'$. Then

$$\begin{aligned} \gamma_0 + 2\delta &= \gamma_0 + 2\big(-\textstyle\int(-\delta)'\big) \\ &= \gamma_0 + 2(-\textstyle\int \gamma_0) \\ &= \gamma_0 + 2(s\gamma_0 - \gamma_0) \quad \text{(Lemma 2.3.3)}. \end{aligned}$$

Thus $\gamma_0 + 2\delta \in (\Gamma^{>})'$ by Lemma 2.4.3. In particular, for every $\gamma_1 \in (\Gamma^{<})'$ with $\gamma_1 > \gamma_0$ we have $\gamma_1 - \gamma_0 < 2\delta \in \Delta$. We conclude that $(\Gamma^{<})'$ is jammed. □

The next two corollaries of Lemma 2.4.3 tell us something about the shape of arbitrary nonempty s-cuts.

Corollary 2.4.5. *Suppose $B \in \mathrm{sded}(\Psi)$ is nonempty. Then B decelerates in Γ.*

Proof. To see this, take $\alpha \in B$ arbitrary. Then $\delta_0 := s\alpha - \alpha > 0$ and $\alpha + \delta_0 \in B$. However, for every $q \in \mathbb{Q}^{>}$, $\alpha + \delta_0 + q\delta_0 \in (\Gamma^{>})' > B \subseteq \Psi$ by Lemma 2.4.3. Thus if $\delta_1 \in \Gamma^{>}$ is such that $\alpha + \delta_0 + \delta_1 \in B$, then necessarily, $\delta_1 < q\delta_0$ for every $q \in \mathbb{Q}^{>}$, i.e., $[\delta_1] < [\delta_0]$. □

Corollary 2.4.6. *Suppose $B \in \operatorname{sded}(\Psi)$ is nonempty. Then for every $\alpha \in \Gamma$ and nontrivial convex subgroup Δ of Γ we have $B^{\downarrow} \neq (\alpha+\Delta)^{\downarrow}$. In particular, B is not almost Δ-special for any such Δ.*

Proof. Let Δ be a nontrivial convex subgroup of Γ. Assume towards a contradiction that $B^{\downarrow} = (\alpha+\Delta)^{\downarrow}$. Since $\alpha+\Delta$ is convex, we have $B \cap (\alpha+\Delta) \neq \emptyset$ and so we arrange $\alpha \in B$. Then $\alpha < s\alpha \in B$ and so $s\alpha \in \alpha+\Delta$ and thus $\alpha + 2(s\alpha-\alpha) \in \alpha+\Delta$. However, $\alpha+2(s\alpha-\alpha) \in (\Gamma^{>})' > B$ by Lemma 2.4.3, a contradiction. ☐

(B,ε)-shifts. In this subsection we give a method of equipping Γ with a new ψ-map $\widetilde{\psi} : \Gamma^{\neq} \to \Gamma$ such that $(\Gamma, \widetilde{\psi})$ is also an H-asymptotic couple with asymptotic integration, and the contraction map $\widetilde{\chi}$ which corresponds to $\widetilde{\psi}$ is identical to the original contraction map χ associated to ψ. This construction is a generalization of one done in [**2**, §5].

Lemma 2.4.7. *Let $B \in \operatorname{sded}(\Psi)$ and $\varepsilon \in \Gamma$ be such that $\psi(\varepsilon) > B$. Define the function $\widetilde{\psi} : \Gamma_{\infty} \to \Gamma_{\infty}$ by*

$$\widetilde{\psi}(\alpha) \;:=\; \begin{cases} \psi(\alpha) & \text{if } \psi(\alpha) \in B \\ \psi(\alpha)+\varepsilon & \text{if } \psi(\alpha) > B \\ \infty & \text{if } \alpha = 0. \end{cases}$$

Then $(\Gamma, \widetilde{\psi})$ is an H-asymptotic couple with asymptotic integration such that $\widetilde{\chi} = \chi$ on Γ.

Proof. We will first show (HC). Suppose $0 < \alpha < \beta$ and $\psi(\alpha) > B$ and $\psi(\beta) \in B$. Then by Lemma 2.3.5, $\psi\big(\psi(\beta)-\psi(\alpha)\big) = s\psi(\beta) \in B$. It follows that $[\varepsilon] < \big[\psi(\alpha)-\psi(\beta)\big]$ and thus $\psi(\alpha)-\psi(\beta) \geqslant -\varepsilon$ since $\psi(\alpha)-\psi(\beta) > 0$. From this we get $\widetilde{\psi}(\alpha) = \psi(\alpha)+\varepsilon \geqslant \psi(\beta) = \widetilde{\psi}(\beta)$. All other cases follow immediately from (HC) for (Γ,ψ).

(AC2) is clear from the definition of $\widetilde{\psi}$.

For (AC1), first suppose that α, β are such that $[\alpha] > [\beta]$. Then $\widetilde{\psi}(\alpha+\beta) = \widetilde{\psi}(\alpha) \geqslant \min\big(\widetilde{\psi}(\alpha), \widetilde{\psi}(\beta)\big)$ by (HC) and (AC2). Otherwise, assume that $[\alpha] = [\beta]$, $\psi(\alpha) = \psi(\beta) \in B$ and $\psi(\alpha+\beta) > B$. Then by a similar argument as for (HC) using $[\varepsilon] < \big[\psi(\alpha+\beta)-\psi(\alpha)\big]$, we can show that $\widetilde{\psi}(\alpha+\beta) = \psi(\alpha+\beta)+\varepsilon \geqslant \psi(\alpha) = \min\big(\widetilde{\psi}(\alpha), \widetilde{\psi}(\beta)\big)$. All other cases are trivial.

Instead of verifying (AC3), by [**6**, 6.5.5] it is sufficient to show that the map $\gamma \mapsto \gamma + \widetilde{\psi}(\gamma) : \Gamma^{>} \to \Gamma$ is strictly increasing. The main case to consider is $0 < \alpha < \beta$ where $\psi(\alpha) > B$ and $\psi(\beta) \in B$. In this case, $[\beta] > [\alpha], [\varepsilon]$ and so

$$\psi(\alpha) \;<\; (\beta-\alpha-\varepsilon)' \;=\; \beta-\alpha-\varepsilon+\psi(\beta-\alpha-\varepsilon) \;=\; \beta-\alpha-\varepsilon+\psi(\beta)$$

by (HC) and (AC3) for (Γ,ψ). Rearranging terms gives us $\alpha+\psi(\alpha)+\varepsilon < \beta+\psi(\beta)$, or rather $\alpha+\widetilde{\psi}(\alpha) < \beta+\widetilde{\psi}(\beta)$.

We will explicitly show that $(\Gamma, \widetilde{\psi})$ has asymptotic integration. Define the function $\iota : \Gamma \to \Gamma^{\neq}$ by:

$$\iota(\alpha) \;:=\; \begin{cases} \int \alpha & \text{if } s\alpha \in B \\ \int(\alpha-\varepsilon) & \text{if } s\alpha > B \end{cases}$$

We claim that $\iota\alpha + \widetilde{\psi}(\iota\alpha) = \alpha$ for all $\alpha \in \Gamma$. If $s\alpha \in B$, then $\psi \int \alpha = \widetilde{\psi} \int \alpha$, and so

$$\iota(\alpha) + \widetilde{\psi}\big(\iota(\alpha)\big) \;=\; \int \alpha + \psi \int \alpha \;=\; \alpha,$$

i.e., $\iota\alpha = \widetilde{\int}\alpha$. Otherwise, suppose $s\alpha > B$. Take an elementary extension (Γ^*, ψ^*) of (Γ, ψ) with an element $\gamma^* \in \Psi^*$ such that $\gamma^* > \Psi$. Then

$$s(\alpha - \varepsilon) \ \geqslant\ \psi(\alpha - \gamma^* - \varepsilon) \ \geqslant\ \min\big(s\alpha, \psi(\varepsilon)\big) \ >\ B$$

by Corollary 2.3.6 and (AC1). In particular, $\widetilde{\psi}\int(\alpha - \varepsilon) = \psi\int(\alpha - \varepsilon) + \varepsilon$. Thus

$$\iota\alpha + \widetilde{\psi}(\iota\alpha) \ =\ \textstyle\int(\alpha - \varepsilon) + \psi\int(\alpha - \varepsilon) + \varepsilon \ =\ (\alpha - \varepsilon) + \varepsilon \ =\ \varepsilon.$$

We conclude that $(\Gamma, \widetilde{\psi})$ has asymptotic integration and that $\widetilde{\int} = \iota$.

The claim about the contraction mapping follows from checking that $\widetilde{\int}\widetilde{\psi}(\alpha) = \int\psi(\alpha)$ for all $\alpha \in \Gamma^{\neq}$. $\square$

Definition 2.4.8. Let $B \in \mathrm{sded}(\Psi)$, $\varepsilon \in \Gamma$ be such that $\psi(\varepsilon) > B$, and let $\widetilde{\psi} : \Gamma_\infty \to \Gamma_\infty$ be as in Lemma 2.4.7 above. Then we call $(\Gamma, \widetilde{\psi})$ the (B, ε)**-shift** of (Γ, ψ).

As a special case of Lemma 2.4.7, we note that the $(\emptyset, \varepsilon)$-shift of ψ is just a shift $(\Gamma, \psi + \varepsilon)$ in the sense of [**33**, Pg. 978, Lemma(2)]. See also [**6**, §6.5].

2.5. Yardstick inequalities

In this section (Γ, ψ) is an H-asymptotic couple with asymptotic integration and $\alpha, \beta, \gamma, \varepsilon$ range over Γ. Recall that by Convention 2.2.2, expressions of the form $\int\gamma', s\gamma^\dagger, s(\gamma - \beta)^\dagger$, etc. are abbreviations for $\int(\gamma'), s(\gamma^\dagger), s\big((\gamma - \beta)^\dagger\big)$, etc.

To motivate Calculation 2.5.1 below, suppose $\gamma \in \Gamma^{\neq}$. Then clearly $\int\gamma' = \gamma$. Now, suppose $\varepsilon \in \Gamma$ is sufficiently small, so that $\gamma' + \varepsilon$ is a perturbation of γ'. Then we want to think of $\int(\gamma' + \varepsilon)$ as being a small perturbation of γ. Calculation 2.5.1 is an identity of this form and is rather important for later results, for instance, in Chapter 6.

Calculation 2.5.1. *Suppose $\gamma \neq \beta$. Then*

$$\textstyle\int\big((\gamma - \beta)' - \int s((\gamma - \beta)')\big) \ =\ (\gamma - \beta) + \big(s(\gamma - \beta)^\dagger - (\gamma - \beta)^\dagger\big) \ =\ (\gamma - \beta) - \chi(\gamma - \beta).$$

Proof. Renaming $\gamma - \beta$ as γ, we arrange that $\beta = 0$ and $\gamma \neq 0$. Then it suffices to prove

$$\textstyle\int(\gamma' - \int s\gamma') \ =\ \gamma + (s\gamma^\dagger - \gamma^\dagger) \ =\ \gamma - \chi(\gamma).$$

We begin by showing

(A) $$s(\gamma + s\gamma^\dagger) \ =\ \gamma^\dagger.$$

By Lemmas 2.3.5 and 2.3.4 we have

$$\psi(-\gamma) \ =\ \gamma^\dagger < s\gamma^\dagger \ =\ \psi(\gamma^\dagger - s\gamma^\dagger),$$

so $\psi(\gamma^\dagger - \gamma - s\gamma^\dagger) \ =\ \gamma^\dagger$. Now (A) follows by Lemma 2.3.8.

We now proceed with our main calculation:

$$\begin{aligned}
\textstyle\int(\gamma' - \int s\gamma') &= (\gamma' - \textstyle\int s\gamma') - s(\gamma' - \int s\gamma') \quad \text{(Lemma 2.3.3)}\\
&= (\gamma' - s\gamma' + s^2\gamma') - s(\gamma' - s\gamma' + s^2\gamma') \quad \text{(Lemma 2.3.3)}\\
&= (\gamma + \gamma^\dagger - \gamma^\dagger + s\gamma^\dagger) - s(\gamma + \gamma^\dagger - \gamma^\dagger + s\gamma^\dagger) \quad \text{(Def. of } s \text{ and } {}'\text{)}\\
&= \gamma + s\gamma^\dagger - s(\gamma + s\gamma^\dagger)\\
&= \gamma + (s\gamma^\dagger - \gamma^\dagger) \quad \text{(by (A))}
\end{aligned}$$

Finally, $-\chi(\gamma) = s\gamma^\dagger - \gamma^\dagger$ follows from applying Lemma 2.3.3 to $\gamma^\dagger$ and the definition of χ. $\square$

Lemma 2.5.2. *Let $\gamma \in (\Gamma^>)'$. Then*

$$\textstyle\int\gamma \;>\; -\int s\gamma \;=\; -\chi\int\gamma > 0.$$

Furthermore, if $\gamma_0, \gamma_1 \in (\Gamma^>)'$, then

$$\gamma_0 \;\leqslant\; \gamma_1 \quad \textit{implies} \quad -\textstyle\int s\gamma_0 \;\leqslant\; -\int s\gamma_1.$$

Proof. We have $s\gamma \in (\Gamma^<)'$, so $-\int s\gamma > 0$, which gives the second part of the first inequality. For the first part we note that

$$\begin{aligned}
\textstyle\int\gamma \;>\; -\int s\gamma &\iff \textstyle\int\gamma + \int s\gamma \;>\; 0\\
&\iff \textstyle\int\gamma + \chi\int\gamma \;>\; 0,
\end{aligned}$$

this last equivalence being true because $\int\gamma > 0$ and $[\chi\int\gamma] < [\int\gamma]$ by Lemma 2.3.13(1).

For the second inequality, we have for $\gamma_0, \gamma_1 \in (\Gamma^>)'$,

$$\begin{aligned}
\gamma_0 \;\leqslant\; \gamma_1 &\implies s\gamma_0 \;\geqslant\; s\gamma_1 \quad \text{since } \gamma_0, \gamma_1 \in (\Gamma^>)'\\
&\iff \textstyle\int s\gamma_0 \;\geqslant\; \int s\gamma_1 \quad \text{by ADH 2.2.3}\\
&\iff -\textstyle\int s\gamma_0 \;\leqslant\; -\int s\gamma_1.
\end{aligned}$$

$\square$

Lemma 2.5.3. *Let S be a nonempty subset of Γ without a greatest element. Given $\beta \in \Gamma$, the following conditions on S are equivalent:*

(1) *for cofinally many $\gamma \in S$, $\gamma - \chi(\gamma - \beta) \in S^\downarrow$;*
(2) *for all $\gamma \in S^\downarrow$, $\gamma - \chi(\gamma - \beta) \in S^\downarrow$.*

Proof. (1) $\Rightarrow$ (2) follows from Lemma 2.3.13(2). (2) $\Rightarrow$ (1) is clear. $\square$

Definition 2.5.4. Let S be a nonempty subset of Γ without a greatest element. We say that S has the **β-yardstick property** if it satisfies one of the equivalent conditions of Lemma 2.5.3. If $\beta = 0$, then we also say that S has the **yardstick property**.

Note that if S is a nonempty subset of Γ without a greatest element, then S has the β-yardstick property iff $S^\downarrow$ has the β-yardstick property.

Remark 2.5.5. The β-yardstick property says that if you have an element $\gamma \in S$, then you can increase upwards at least a distance of $-\chi(\gamma - \beta)$ and still remain in S. Similar to the property *jammed* from Section 2.1, this is a qualitative property concerning the top of the set S. Unlike *jammed*, the β-yardstick property requires the asymptotic couple structure of (Γ, ψ), and the contraction map χ in particular.

The β-yardstick property and being jammed are incompatible properties, except in the following case:

Lemma 2.5.6. *Let S be a nonempty subset of Γ without a greatest element with the β-yardstick property. Then S is jammed iff $S^{\downarrow} = \Gamma^{<\beta}$.*

Proof. If $S^{\downarrow} = \Gamma^{<\beta}$, then S is jammed by Example 2.1.3. Now suppose that $S^{\downarrow} \neq \Gamma^{<\beta}$. We will show that S is not jammed. First, assume that $S \cap \Gamma^{>\beta} \neq \emptyset$ and take $\gamma \in S \cap \Gamma^{>\beta}$. Let Δ be a nontrivial convex subgroup of Γ such that $[\Delta] < \big[\chi(\gamma - \beta)\big]$. Now let $\gamma_0, \gamma_1 \in S$ be such that $\gamma < \gamma_0 < \gamma_0 - \chi(\gamma_0 - \beta) < \gamma_1$. Note that

$$\gamma_1 - \gamma \ > \ \gamma_1 - \gamma_0 \ > \ -\chi(\gamma_0 - \beta) \ \geqslant \ -\chi(\gamma - \beta) \ > \ \Delta.$$

We conclude that S is not jammed.

Next, suppose δ is such that $S < \delta < \beta$. Let Δ be a nontrivial convex subgroup of Γ such that $[\beta - \delta] > \big[\chi(\beta - \delta)\big] > [\Delta]$. Let $\gamma_0 \in S$ be arbitrary. Then $\gamma_0 - \chi(\gamma_0 - \beta) \in S^{\downarrow}$ and we can take $\gamma_1 \in S$ such that $\gamma_1 > \gamma_0 - \chi(\gamma_0 - \beta)$. Thus

$$\gamma_1 - \gamma_0 \ > \ \big(\gamma_0 - \chi(\gamma_0 - \beta)\big) - \gamma_0 \ = \ -\chi(\gamma_0 - \beta) \geqslant -\chi(\beta - \delta) \ > \ \Delta.$$

We conclude that S is not jammed since $\gamma_0 \in S$ was arbitrary. $\square$

The following technical variant of the yardstick property will come in handy in Sections 6.2, 6.3, and 6.4:

Definition 2.5.7. Let $S \subseteq \Gamma$ be a nonempty convex set without a greatest element such that either $S \subseteq (\Gamma^{>})'$ or $S \subseteq (\Gamma^{<})'$. We say that S has the **derived yardstick property** if there is $\beta \in S$ such that for every $\gamma \in S^{>\beta}$,

$$\gamma - \textstyle\int s\gamma \in S^{>\beta}.$$

Proposition 2.5.8. *Suppose $S \subseteq \Gamma$ is a nonempty convex set without a greatest element such that either $S \subseteq (\Gamma^{>})'$ or $S \subseteq (\Gamma^{<})'$ and S has the derived yardstick property. Then $\int S := \{\int s : s \in S\} \subseteq \Gamma$ is nonempty, convex, does not have a greatest element, and has the yardstick property.*

Proof. By ADH 2.2.3, $\int S$ is nonempty, convex, and does not have a greatest element. Let $\beta \in S$ be such that for every $\gamma \in S^{>\beta}$, $\gamma - \int s\gamma \in S$. Now let $\gamma \in (\int S)^{>\int \beta}$. Then $\gamma' \in S^{>\beta}$, so $\gamma' - \int s\gamma' \in S^{>\beta}$. Thus

$$\textstyle\int(\gamma' - \int s\gamma') \in (\int S)^{>\int \beta}.$$

By Calculation 2.5.1,

$$\gamma - \chi(\gamma) \in (\textstyle\int S)^{>\int \beta}.$$

We conclude that $\int S$ has the yardstick property. $\square$

Example 2.5.9. (The yardstick property in $(\Gamma_{\log}, \psi)$) To get a feel for what the yardstick property says, suppose $S \subseteq \Gamma_{\log}$ is nonempty, downward closed, and has the yardstick property. Then, given an element $\alpha \neq 0$ in S we have

$$\alpha \ = \ (\underbrace{0, \ldots, 0}_{n}, \underbrace{r_n}_{\neq 0}, r_{n+1}, \ldots)$$

and then the yardstick property says that the following larger element is also in S:

$$\alpha - \chi(\alpha) \ = \ (\underbrace{0, \ldots, 0}_{n}, \underbrace{r_n}_{\neq 0}, r_{n+1}) - (\underbrace{0, \ldots, 0}_{n+1}, -1, 0, 0, \ldots) \ = \ (\underbrace{0, \ldots, 0}_{n}, \underbrace{r_n}_{\neq 0}, r_{n+1} + 1, \ldots) \in S$$

In fact, by iterating the yardstick property, we find that for *any* m, the following element is in S:

$$(\underbrace{0,\ldots,0}_{n},\underbrace{r_n}_{\neq 0},r_{n+1}+m,\ldots)\in S$$

Thus if Δ is the convex subgroup generated by $-\chi(\alpha)$, then $\alpha+\Delta\subseteq S$.

The following lemma gets used in Proposition 5.6.8:

Lemma 2.5.10. *Suppose $\varepsilon>0$ and $\gamma\neq\beta$. Then*

$$\gamma\ <\ \gamma+s(\gamma-\beta)^\dagger-(\gamma-\beta)^\dagger\ =\ \gamma-\chi(\gamma-\beta)\ <\ \textstyle\int(\gamma-\beta+\varepsilon')+\beta.$$

Additionally, suppose that $\psi(\gamma-\beta)>\alpha$. Then

$$\gamma+s(\gamma-\beta)^\dagger-(\gamma-\beta)^\dagger\ <\ \gamma+\varepsilon'-\alpha.$$

Proof. The first inequality and equality are clear. For use in the fourth implication below, note that for $\alpha\neq 0$ we have $\big(\alpha-\chi(\alpha)\big)^\dagger=\alpha^\dagger$ and thus $\big(\alpha-\chi(\alpha)\big)'=\alpha-\chi(\alpha)+\big(\alpha-\chi(\alpha)\big)^\dagger$. To get the second inequality, note that

$$\begin{aligned}
s(\gamma-\beta)^\dagger<\varepsilon' &\implies (\gamma-\beta)^\dagger-\textstyle\int(\gamma-\beta)^\dagger<\varepsilon'\\
&\implies (\gamma-\beta)^\dagger-\chi(\gamma-\beta)<\varepsilon'\\
&\implies (\gamma-\beta)+(\gamma-\beta)^\dagger-\chi(\gamma-\beta)<\gamma-\beta+\varepsilon'\\
&\implies (\gamma-\beta)-\chi(\gamma-\beta)<\textstyle\int(\gamma-\beta+\varepsilon')\quad\text{(integrating both sides)}\\
&\implies \gamma-\textstyle\int(\gamma-\beta)^\dagger<\textstyle\int(\gamma-\beta+\varepsilon')+\beta\\
&\implies \gamma+s(\gamma-\beta)^\dagger-(\gamma-\beta)^\dagger<\textstyle\int(\gamma-\beta+\varepsilon')+\beta.
\end{aligned}$$

Finally, we get the last inequality by adding together

$$s(\gamma-\beta)^\dagger\ <\ \varepsilon'\quad\text{and}\quad-(\gamma-\beta)^\dagger\ <\ -\alpha$$

and then adding γ to both sides. ☐

Lemma 2.5.11. *Suppose that $\Psi=\{s^n0:n\geqslant 1\}$ and $S\subseteq\Gamma$ is downward closed such that either*

(i) *$S=\Gamma^{<\alpha}$ for some $\alpha\in\Gamma$, or*
(ii) *S is Δ-fluent for some nontrivial convex subgroup Δ of Γ.*

Then exactly one of the following holds:

(1) *$S<\Psi$;*
(2) *the set $S\cap\Psi$ has a maximum;*
(3) *$S\not\subseteq\Psi^\downarrow$.*

Proof. It is clear that properties (1), (2) and (3) are mutually exclusive.

First consider the case that $S=\Gamma^{<\alpha}$ (so S is jammed). If $\alpha>\Psi^\downarrow=(\Gamma^<)'$, then we are in case (3) since $\alpha\in(\Gamma^>)'$ and $(\Gamma^>)'$ does not have a least element. If $\alpha\leqslant\Psi$, then we are in case (1). Otherwise, there is a unique $\beta\in\Psi$ such that $\beta<\alpha\leqslant s\beta$. In this case, $\beta=\max(S\cap\Psi)$.

Next, suppose that S is Δ-fluent where Δ is a nontrivial convex subgroup of Γ. Assume we are not in case (1). Let $N\geqslant 1$ be such that $s^{N+1}0-s^N0\in\Delta$ (here we use that the sequence $\big([s^{n+1}0-s^n0]\big)$ is coinitial in $[\Gamma^{\neq}]$, a consequence of our assumption on the Ψ-set). If we are not in case (2), then $s^N0\in S\cap\Psi$

(because it is the Nth element of the Ψ-set). Thus $s^N 0 + 2(s^{N+1}0 - s^N 0) \in S^{>\Psi}$ by Lemma 2.4.3, so we are in case (3). $\square$

CHAPTER 3

Embedding lemmas for asymptotic couples

In this chapter we (Γ,ψ) *is an* H*-asymptotic couple and* (Γ_1,ψ_1) *is an asymptotic couple, not necessarily of* H*-type.* We include here many embedding results of the following form:

Embedding Lemma Template. Suppose (Γ,ψ) has property P. Then there is an asymptotic couple (Γ',ψ') extending (Γ,ψ) such that:

(1) (Γ',ψ') has property Q;

(2) if $i:(\Gamma,\psi)\to(\Gamma_1,\psi_1)$ is an embedding such that (Γ_1,ψ_1) has property Q, then i extends uniquely to an embedding $j:(\Gamma',\psi')\to(\Gamma_1,\psi_1)$.

In Section 3.1, we survey various "small" embedding lemmas of asymptotic couples. By "small" here, we mostly mean embedding lemmas where the underlying abelian group of Γ grows by a summand of $\mathbb{Z}$ or $\mathbb{Q}$. These lemmas accomplish very specific things, and will serve as building blocks for the "bigger" embedding lemmas to come. Many of the embedding lemmas in Section 3.1 are from [**6**].

In Section 3.2, we prove various "big" embedding lemmas. These big embedding lemmas all deal with adjoining copies of $\mathbb{N}$ or $\mathbb{Z}$ to the Ψ-set and are important in our proof of quantifier elimination for $(\Gamma_{\log},\psi)$ (Theorem 4.2.2).

Finally, in Section 3.3 we prove a "very big" embedding lemma (Lemma 3.3.2). This allows us to adjoin ordinal-many copies of $\mathbb{Z}$ to the Ψ-set and is useful in showing that $(\Gamma_{\log},\psi)$ has the non-independence property (NIP) in Sections 4.4 and 4.5.

3.1. Small embedding lemmas

Adjoining integrals. The first two lemmas allow us to remove a gap by "adjoining an integral" for the gap. The first lemma shows that we can make the gap the derivative of a positive element; the lemma after that shows how to make the gap the derivative of a negative element.

ADH 3.1.1 (Removing a gap, positive version)**.** *Let* β *be a gap in* (Γ,ψ)*. Then there is an* H*-asymptotic couple* $(\Gamma+\mathbb{Z}\alpha,\psi^\alpha)$ *extending* (Γ,ψ) *such that:*

(1) $\alpha>0$ *and* $\alpha'=\beta$*;*

(2) *if* $i:(\Gamma,\psi)\to(\Gamma_1,\psi_1)$ *is an embedding and* $\alpha_1\in\Gamma_1$, $\alpha_1>0$, $\alpha_1'=i(\beta)$*, then* i *extends uniquely to an embedding* $j:(\Gamma+\mathbb{Z}\alpha,\psi^\alpha)\to(\Gamma_1,\psi_1)$ *with* $j(\alpha)=\alpha_1$*.*

Furthermore, $\psi^\alpha\big((\Gamma+\mathbb{Z}\alpha)^{\neq}\big)=\Psi\cup\{\beta-\alpha\}$ *with* $\Psi<\beta-\alpha$*.*

Proof. This is by [**6**, 9.8.2] and its proof. □

ADH 3.1.2 (Removing a gap, negative version)**.** *Let* β *be a gap in* (Γ,ψ)*. Then there is an* H*-asymptotic couple* $(\Gamma+\mathbb{Z}\alpha,\psi^\alpha)$ *extending* (Γ,ψ) *such that:*

(1) $\alpha < 0$ *and* $\alpha' = \beta$;

(2) *if* $i : (\Gamma, \psi) \to (\Gamma_1, \psi_1)$ *is an embedding and* $\alpha_1 \in \Gamma_1$, $\alpha_1 < 0$, $\alpha_1' = i(\beta)$, *then* i *extends uniquely to an embedding* $j : (\Gamma + \mathbb{Z}\alpha, \psi^\alpha) \to (\Gamma_1, \psi_1)$ *with* $j(\alpha) = \alpha_1$.

Furthermore, $\psi^\alpha\big((\Gamma + \mathbb{Z}\alpha)^{\neq}\big) = \Psi \cup \{\beta - \alpha\}$ *with* $\Psi < \beta - \alpha$.

Proof. This follows from remarks after the proof of [**6**, 9.8.2]. $\square$

Remark 3.1.3. ADH 3.1.1 and 3.1.2 show us that there are essentially two ways to remove a gap. These two ways are incompatible in the sense that given (Γ, ψ) with gap β, we can obtain (Γ^+, ψ^+) from ADH 3.1.1 and (Γ^-, ψ^-) from ADH 3.1.2 and there is no common extension (Γ', ψ') in which these two can be amalgamated, i.e., the following configuration of embeddings is impossible:

$$\begin{tikzcd}
& (\Gamma', \psi') & \\
(\Gamma^+, \psi^+) \arrow[ur] & & (\Gamma^-, \psi^-) \arrow[ul] \\
& (\Gamma, \psi) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

This issue is referred to as the "fork in the road" and is an obstruction to quantifier elimination. In [**3**] this issue is resolved by adding an additional predicate to the language that "decides" for a gap whether it is supposed to be the derivative of a positive or of a negative element. We avoid this obstacle in Section 4.2 by adding the function s to our language which ensures that all asymptotic couples considered already have asymptotic integration. The tradeoff in doing so is that we are forced to work entirely in the category of H-asymptotic couples with asymptotic integration.

If (Γ, ψ) has a largest element β in its Ψ-set, then ADH 2.2.4 tells us that there is no $\alpha \in \Gamma$ such that $\alpha' = \beta$. ADH 3.1.4 tells us how to "adjoin an integral" for such an element β. It is important to note that the extension of (Γ, ψ) constructed in Lemma 3.1.4 also has a Ψ-set with a largest element.

ADH 3.1.4 (Adjoining an integral for $\max \Psi$)**.** *Assume* Ψ *has a largest element* β. *Then there is an* H*-asymptotic couple* $(\Gamma + \mathbb{Z}\alpha, \psi^\alpha)$ *extending* (Γ, ψ) *with* $\alpha \neq 0$, $\alpha' = \beta$, *such that for any embedding* $i : (\Gamma, \psi) \to (\Gamma_1, \psi_1)$ *and any* $\alpha_1 \in \Gamma_1^{\neq}$ *with* $\alpha_1' = i(\beta)$ *there is a unique extension of* i *to an embedding* $j : (\Gamma + \mathbb{Z}\alpha, \psi^\alpha) \to (\Gamma_1, \psi_1)$ *with* $j(\alpha) = \alpha_1$. *Furthermore,* $\psi^\alpha\big((\Gamma + \mathbb{Z}\alpha)^{\neq}\big) = \Psi \cup \{\beta - \alpha\}$ *with* $\Psi < \beta - \alpha$.

Proof. This follows from [**6**, 9.8.3] and its proof. $\square$

Adding a gap. The next lemma allows us to add a gap to an asymptotic couple with asymptotic integration.

ADH 3.1.5 (Adding a gap)**.** *Suppose* (Γ, ψ) *is divisible and has asymptotic integration. Then there is a divisible* H*-asymptotic couple* $(\Gamma + \mathbb{Q}\beta, \psi_\beta)$ *extending* (Γ, ψ) *such that:*

(1) $\Psi < \beta < (\Gamma^>)'$;

(2) *for any divisible* (Γ_1, ψ_1) *extending* (Γ, ψ) *and* $\beta_1 \in \Gamma_1$ *with* $\Psi < \beta_1 < (\Gamma^>)'$ *there is a unique embedding* $(\Gamma + \mathbb{Q}\beta, \psi_\beta) \to (\Gamma_1, \psi_1)$ *that is the identity on* Γ *and sends* β *to* β_1;

(3) *the set* Γ *is dense in the ordered abelian group* $\Gamma + \mathbb{Q}\beta$, *so* $[\Gamma] = [\Gamma + \mathbb{Q}\beta]$, $\Psi = \psi_\beta\big((\Gamma + \mathbb{Q}\beta)^{\neq}\big)$ *and* β *is a gap in* $(\Gamma + \mathbb{Q}\beta, \psi_\beta)$.

PROOF. This is [**6**, Lemma 9.8.4]. The proof uses a compactness argument. $\square$

Adding archimedean classes and growing the Ψ-set. Recall that a cut in an ordered set S is simply a downward closed subset of S, and an element a of an ordered set extending S is said to realize the cut C in S if $C < a < S \setminus C$. The following Lemma 3.1.6 is useful because it enables us to either:

1. add an element α witnessing $\psi(\alpha) = \beta$, if β is not already in the Ψ-set, but is not disqualified from being in a larger Ψ-set by satisfying $\beta \in (\Gamma^>)'$, or
2. add an additional archimedean class to $\left[\psi^{-1}(\beta)\right]$, if β is already in the Ψ-set.

ADH 3.1.6. *Let C be a cut in $[\Gamma^{\neq}]$ and let $\beta \in \Gamma$ be such that $\beta < (\Gamma^>)'$, $\gamma^\dagger \leqslant \beta$ for all $\gamma \in \Gamma^{\neq}$ with $[\gamma] > C$, and $\beta \leqslant \delta^\dagger$ for all $\delta \in \Gamma^{\neq}$ with $[\delta] \in C$. Then there exists an H-asymptotic couple $(\Gamma \oplus \mathbb{Z}\alpha, \psi^\alpha)$ extending (Γ, ψ), with $\alpha > 0$, such that:*

1. *$[\alpha] \notin [\Gamma^{\neq}]$ realizes the cut C in $[\Gamma^{\neq}]$, $\psi^\alpha(\alpha) = \beta$;*
2. *given any embedding i of (Γ, ψ) into an H-asymptotic couple (Γ_1, ψ_1) and any element $\alpha_1 \in \Gamma_1^>$ such that $[\alpha_1] \notin \left[i(\Gamma^{\neq})\right]$ realizes the cut $\left\{[i(\delta)] : [\delta] \in C\right\}$ in $\left[i(\Gamma^{\neq})\right]$ and $\psi_1(\alpha_1) = i(\beta)$, there is a unique extension of i to an embedding $j : (\Gamma \oplus \mathbb{Z}\alpha, \psi^\alpha) \to (\Gamma_1, \psi_1)$ with $j(\alpha) = \alpha_1$.*

If (Γ, ψ) has asymptotic integration, then so does $(\Gamma \oplus \mathbb{Z}\alpha, \psi^\alpha)$. If (Γ, ψ) has rational asymptotic integration, then so does $(\Gamma \oplus \mathbb{Z}\alpha, \psi^\alpha)$.

We also have the following divisible version of ADH 3.1.6:

Lemma 3.1.7. *Suppose (Γ, ψ) is divisible and let C be a cut in $[\Gamma^{\neq}]$ and let $\beta \in \Gamma$ be such that $\beta < (\Gamma^>)'$, $\gamma^\dagger \leqslant \beta$ for all $\gamma \in \Gamma^{\neq}$ with $[\gamma] > C$, and $\beta \leqslant \delta^\dagger$ for all $\delta \in \Gamma^{\neq}$ with $[\delta] \in C$. Then there exists a divisible H-asymptotic couple $(\Gamma \oplus \mathbb{Q}\alpha, \psi^\alpha)$ extending (Γ, ψ), with $\alpha > 0$, such that:*

1. *$[\alpha] \notin [\Gamma^{\neq}]$ realizes the cut C in $[\Gamma^{\neq}]$, $\psi^\alpha(\alpha) = \beta$;*
2. *given any embedding i of (Γ, ψ) into a divisible H-asymptotic couple (Γ_1, ψ_1) and any element $\alpha_1 \in \Gamma_1^>$ such that $[\alpha_1] \notin \left[i(\Gamma^{\neq})\right]$ realizes the cut $\left\{[i(\delta)] : [\delta] \in C\right\}$ in $\left[i(\Gamma^{\neq})\right]$ and $\psi_1(\alpha_1) = i(\beta)$, there is a unique extension of i to an embedding $j : (\Gamma \oplus \mathbb{Q}\alpha, \psi^\alpha) \to (\Gamma_1, \psi_1)$ with $j(\alpha) = \alpha_1$.*

If (Γ, ψ) has asymptotic integration, then so does $(\Gamma \oplus \mathbb{Q}\alpha, \psi^\alpha)$.

PROOF. This follows from performing the extension as in ADH 3.1.6 and then taking the divisible hull. The universal property follows from respective universal property of those constructions. Also note that if (Γ, ψ) has asymptotic integration, then the $(\Gamma \oplus \mathbb{Z}\alpha, \psi^\alpha)$ constructed in ADH 3.1.6 will have rational asymptotic integration, and thus its divisible hull $(\Gamma \oplus \mathbb{Q}\alpha, \psi^\alpha)$ will have asymptotic integration. $\square$

For the special case of $C = \emptyset$ and β a gap in (Γ, ψ), ADH 3.1.6 gives [**6**, 9.8.8]:

ADH 3.1.8 (Making the gap become $\max \Psi$)**.** *Let $\beta \in \Gamma$ be a gap in (Γ, ψ). Then there exists an H-asymptotic couple $(\Gamma + \mathbb{Z}\alpha, \psi^\alpha)$ extending (Γ, ψ), such that:*

1. *$0 < n\alpha < \Gamma^>$ for all $n > 0$, and $\psi^\alpha(\alpha) = \beta$;*
2. *for any embedding i of (Γ, ψ) into an H-asymptotic couple (Γ_1, ψ_1) and any $\alpha_1 \in \Gamma_1^>$ with $\psi_1(\alpha_1) = i(\beta)$, there is a unique extension of i to an embedding $j : (\Gamma + \mathbb{Z}\alpha, \psi^\alpha) \to (\Gamma_1, \psi_1)$ with $j(\alpha) = \alpha_1$.*

Note that ADH 3.1.8 is compatible with ADH 3.1.2 and incompatible with ADH 3.1.1: if (Γ, ψ) has a gap β, then applying ADH 3.1.8 "decides" that β will be the derivative of a negative element in any extension with asymptotic integration.

Embedding lemmas concerning ordered abelian group structure. The following two embedding lemmas primarily involve the underlying ordered abelian group structure of the extension.

ADH 3.1.9. [**6**, 9.8.1] *Let $i : \Gamma \to G$ be an embedding of ordered abelian groups inducing a bijection $[\Gamma] \to [G]$. Then there is a unique function $\psi_G : G^{\neq} \to G$ such that (G, ψ_G) is an H-asymptotic couple and $i : (\Gamma, \psi) \to (G, \psi_G)$ is an embedding.*

Lemma 3.1.10. *Suppose $(\Gamma_0, \psi_0) \subseteq (\Gamma_1, \psi_1)$ and (Γ^*, ψ^*) are H-asymptotic couples, $i : (\Gamma_0, \psi_0) \to (\Gamma^*, \psi^*)$ is an embedding and $j : \Gamma_1 \to \Gamma^*$ is an ordered group embedding. Furthermore, suppose that $i = j|_{\Gamma_0}$ and $[\Gamma_0] = [\Gamma_1]$. Then j is also an embedding of asymptotic couples, i.e., $j\big(\psi_1(\gamma)\big) = \psi^*\big(j(\gamma)\big)$ for all $\gamma \in \Gamma_1^{\neq}$.*

Proof. Let $\gamma \in \Gamma_1^{\neq}$. Since $[\Gamma_0] = [\Gamma_1]$, there is $\gamma_0 \in \Gamma_0^{\neq}$ such that $[\gamma] = [\gamma_0]$. By (HC) $\psi_1(\gamma) = \psi_1(\gamma_0)$ and so $j\big(\psi_1(\gamma)\big) = j\big(\psi_1(\gamma_0)\big) = i\big(\psi_0(\gamma_0)\big)$. Since j is an ordered group embedding, it also follows that $\big[j(\gamma)\big] = \big[i(\gamma_0)\big]$ in $[\Gamma^*]$. Thus $\psi^*\big(j(\gamma)\big) = \psi^*\big(i(\gamma_0)\big)$. Since i is an embedding of asymptotic couples, $i\big(\psi_0(\gamma_0)\big) = \psi^*\big(i(\gamma_0)\big)$ and we are done. $\square$

Tournant Dangereux. Suppose that $(\Gamma \oplus \mathbb{Q}\alpha, \psi^\alpha)$ and $(\Gamma \oplus \mathbb{Q}\beta, \psi^\beta)$ are two H-asymptotic couple extensions of (Γ, ψ). In this case, it may be tempting to conclude the following:

> The unique abelian group isomorphism $\Gamma \oplus \mathbb{Q}\alpha \to (\Gamma \oplus \mathbb{Q}\beta)$ over Γ which sends α to β is an isomorphism of asymptotic couple $(\Gamma \oplus \mathbb{Q}\alpha, \psi^\alpha) \to (\Gamma \oplus \mathbb{Q}\beta, \psi^\beta)$ over (Γ, ψ) *if and only if α and β realize the same cut over Γ.*

However, this is *not* true in general. Consider the following scenario:

Suppose $\Psi = \Psi_{\Gamma \oplus \mathbb{Q}\alpha} = \Psi_{\Gamma \oplus \mathbb{Q}\beta}$ is a successor set such that for each $\alpha \in \Psi$, the successor is given by $s(\alpha)$. Let $\delta, s\delta \in \Psi$ be two adjacent members of the common Ψ-set. Consider the following sets of archimedean classes of Γ:

$$C_0 \ := \ \big\{[\gamma] : \gamma \in \Gamma \text{ and } \psi(\gamma) = s\delta\big\} \ < \ C_1 \ := \ \big\{[\gamma] : \gamma \in \Gamma \text{ and } \psi(\gamma) = \delta\big\}.$$

It could be the case that both $\alpha, \beta > 0$ and $C_0 < [\alpha], [\beta] < C_1$, which would guarantee that they realize the same cut over Γ. However, it is possible that $\psi(\alpha) = \delta$ whereas $\psi(\beta) = s\delta$ and in this case we would not have an isomorphism of asymptotic couples over (Γ, ψ) which sends α to β. To account for this, we will determine which extra information about α and β is needed in order to get an isomorphism of asymptotic couples.

We first have the following scenario where the cuts alone determine an isomorphism of asymptotic couples:

Corollary 3.1.11. *Suppose $(\Gamma \oplus \mathbb{Q}\alpha, \psi^\alpha)$ and $(\Gamma \oplus \mathbb{Q}\beta, \psi^\beta)$ are two H-asymptotic couple extensions of (Γ, ψ) such that*

1. $[\Gamma \oplus \mathbb{Q}\alpha] = [\Gamma]$, *and*
2. *α and β realize the same cut over Γ.*

Then the isomorphism $i : \Gamma \oplus \mathbb{Q}\alpha \to \Gamma \oplus \mathbb{Q}\beta$ of ordered abelian groups over Γ which sends α to β is also an isomorphism $i : (\Gamma \oplus \mathbb{Q}\alpha, \psi^\alpha) \to (\Gamma \oplus \mathbb{Q}\beta, \psi^\beta)$ of asymptotic couples over (Γ, ψ).

Proof. By (1) we have that $\psi^\alpha\big((\Gamma \oplus \mathbb{Q}\alpha)^{\neq}\big) = \Psi$ and by (2) that $[\Gamma \oplus \mathbb{Q}\beta] = [\Gamma]$. Given $\gamma_0 + q\alpha \in \Gamma \oplus \mathbb{Q}\alpha^{\neq}$, let $\gamma_1 \in \Gamma^{\neq}$ be such that $[\gamma_0 + q\alpha] = [\gamma_1]$. It follows from condition (2) that $[\gamma_1] = [\gamma_0 + q\beta]$. Thus $i\big(\psi^\alpha(\gamma_0 + i\alpha)\big) = \psi^\alpha(\gamma_0 + i\alpha) = \psi(\gamma_1) = \psi^\beta(\gamma_0 + q\beta) = \psi^\beta\big(i(\gamma_0 + q\alpha)\big)$, using ADH 3.1.10 for $(\Gamma \oplus \mathbb{Q}\beta, \psi^\beta)$. $\square$

The following shows what information about α and β is needed in the case where the Ψ-set doesn't grow. This follows easily from ADH 3.1.6.

Corollary 3.1.12. *Suppose $(\Gamma \oplus \mathbb{Q}\alpha, \psi^\alpha)$ and $(\Gamma \oplus \mathbb{Q}\beta, \psi^\beta)$ are two H-asymptotic couple extensions of (Γ, ψ) such that:*

(1) $\psi^\alpha\big((\Gamma \oplus \mathbb{Q}\alpha)^{\neq}\big) = \Psi = \psi^\beta\big((\Gamma \oplus \mathbb{Q}\beta)^{\neq}\big)$,
(2) $\alpha > 0$ *and* $\beta > 0$,
(3) $\psi^\alpha(\alpha) = \psi^\beta(\beta)$, *and*
(4) $[\alpha] \notin [\Gamma]$, $[\beta] \notin [\Gamma]$, *and* $[\alpha]$ *and* $[\beta]$ *realize the same cut over* $[\Gamma]$;

then necessarily α and β realize the same cut over Γ and the isomorphism $i : \Gamma \oplus \mathbb{Q}\alpha \to \Gamma \oplus \mathbb{Q}\beta$ of ordered abelian groups over Γ which sends α to β is also an isomorphism $i : (\Gamma \oplus \mathbb{Q}\alpha, \psi^\alpha) \to (\Gamma \oplus \mathbb{Q}\beta, \psi^\beta)$ of asymptotic couples over (Γ, ψ).

3.2. Big embedding lemmas

In the next embedding lemma, we show how to extend a grounded divisible H-asymptotic couple to a divisible H-asymptotic couple with asymptotic integration:

Lemma 3.2.1 (Divisible asymptotic integration closure)**.** *Let (Γ_0, ψ_0) be a divisible H-asymptotic couple such that Ψ has a largest element β_0. Then there exists a divisible H-asymptotic couple*

$$(\Gamma, \psi) \;=\; (\Gamma_0 \oplus \bigoplus_n \mathbb{Q}\alpha_{n+1}, \psi) \;=\; (\Gamma_0 \oplus \bigoplus_n \mathbb{Q}\beta_{n+1}, \psi)$$

extending (Γ_0, ψ) such that:

(1) (Γ, ψ) *has asymptotic integration;*
(2) $s(\beta_n) = \beta_{n+1}$ *and* $\int \beta_n = \alpha_{n+1}$ *for all n;*
(3) *for any embedding i of (Γ_0, ψ_0) into a divisible H-asymptotic couple (Γ^*, ψ^*) with asymptotic integration, there is a unique extension of i to an embedding $(\Gamma, \psi) \to (\Gamma^*, \psi^*)$.*

Proof. For $n \geqslant 0$, define $(\Gamma_{n+1}, \psi_{n+1})$ to be the asymptotic couple $(\Gamma_n + \mathbb{Q}\alpha_{n+1}, \psi_n^{\alpha_{n+1}})$, the divisible hull of the asymptotic couple $(\Gamma_n + \mathbb{Z}\alpha_{n+1}, \psi_n^{\alpha_{n+1}})$ constructed in ADH 3.1.4 as an extension of (Γ_n, ψ_n). With $\Psi_n := \psi_n(\Gamma_n^{\neq})$, we have $\Psi_{n+1} = \Psi_n \cup \{\beta_0 - \sum_{k=0}^n \alpha_{k+1}\}$ with $\max \Psi_{n+1} = \beta_0 - \sum_{k=0}^n \alpha_{k+1} =: \beta_{n+1}$. Let $(\Gamma, \psi) = \bigcup_n (\Gamma_n, \psi_n)$ and so $\Psi = \psi(\Gamma^{\neq}) = \bigcup_n \Psi_n$. Note that Ψ does not have a maximum element. Furthermore, (Γ, ψ) does not have a gap because it is the union of a chain of asymptotic couples which do not have gaps. Thus (Γ, ψ) has asymptotic integration.

For (3), assume by induction that we have an embedding $i_n : (\Gamma_n, \psi_n) \to (\Gamma^*, \psi^*)$. Since (Γ^*, ψ^*) has asymptotic integration, there is a unique extension of i_n to an embedding $i_{n+1} : (\Gamma_{n+1}, \psi_{n+1})$ such that $i_{n+1}(\alpha_{n+1}) = \int \big(i_n(\beta_n)\big)$ by the universal property from ADH 3.1.4 and 2.2.10. Thus there is a unique embedding $\cup_n i_n : (\Gamma, \psi) \to (\Gamma^*, \psi^*)$. $\square$

Given (Γ_0, ψ_0) as in Lemma 3.2.1, the extension (Γ, ψ) constructed in this lemma is the unique divisible H-asymptotic couple with asymptotic integration extending (Γ_0, ψ_0) which has the universal property (3) in Lemma 3.2.1. We call this extension the **divisible asymptotic integration closure** of (Γ_0, ψ_0). The following summarizes the relationship between the α's and β's in Lemma 3.2.1, with $\beta_0 = \max \Psi_0$.

$$
\begin{tikzcd}
\beta_0 \arrow[r, "s"] \arrow[dr, "\int"'] & \beta_1 \arrow[r, "s"] \arrow[dr, "\int"'] & \beta_2 \arrow[r, "s"] \arrow[dr, "\int"'] & \beta_3 \arrow[r, "s"] \arrow[dr, "\int"'] & \cdots \\
& \alpha_1 \arrow[u, "\psi"] \arrow[r, "\chi"'] & \alpha_2 \arrow[u, "\psi"] \arrow[r, "\chi"'] & \alpha_3 \arrow[u, "\psi"] \arrow[r, "\chi"'] & \cdots
\end{tikzcd}
$$

The diagram illustrates the manner in which we adjoined integrals at each stage of the construction.

Example 3.2.2. Let $(\Gamma_0, \psi_0) \subseteq (\Gamma_{\log}^{\mathbb{Q}}, \psi)$ be such that $\Gamma_0 = \mathbb{Q}e_0$. Then $e_0 = \max \psi(\Gamma_0^{\neq})$, and by the construction in Lemma 3.2.1, $(\Gamma_{\log}^{\mathbb{Q}} \psi)$ is the divisible asymptotic integration closure of (Γ_0, ψ_0). Recall that $e_0 = s0$ in $(\Gamma_{\log}^{\mathbb{Q}}, \psi)$. Thus if (Γ^*, ψ^*) is any divisible H-asymptotic couple with asymptotic integration such that $s0 > 0$, then there is a unique embedding

$$i : (\Gamma_{\log}^{\mathbb{Q}}, \psi) \to (\Gamma^*, \psi^*).$$

Lemma 3.2.3. *Let (Γ_0, ψ_0) be a divisible H-asymptotic couple with asymptotic integration. Then there exists a divisible H-asymptotic couple $(\Gamma, \psi) = (\Gamma_0 \oplus \mathbb{Q}\alpha_0 \oplus \bigoplus_n \mathbb{Q}\beta_n, \psi)$ extending (Γ_0, ψ_0), such that:*

1. *(Γ, ψ) has asymptotic integration;*
2. *$\psi_0(\Gamma_0^{\neq}) < \beta_0 < (\Gamma_0^{>})'$, $\beta_0 = \psi(\alpha_0)$, $\beta_{n+1} = s(\beta_n)$ for all n;*
3. *for any embedding i of (Γ_0, ψ_0) into a divisible H-asymptotic couple (Γ^*, ψ^*) with asymptotic integration and any $\alpha^* \in (\Gamma^*)^{<}$ such that $i\big(\psi_0(\Gamma_0)\big) < \psi^*(\alpha^*) < \big(i(\Gamma_0)^{>}\big)'$, there is a unique extension of i to an embedding $j : (\Gamma, \psi) \to (\Gamma^*, \psi^*)$ such that $j(\alpha_0) = \alpha^*$, $j(\beta_0) = \psi^*(\alpha^*)$ and $j(\beta_{k+1}) = s^k\big(\psi^*(\alpha^*)\big)$.*

Proof. By ADH 3.1.5, we can extend (Γ_0, ψ_0) to an asymptotic couple $(\Gamma_0 \oplus \mathbb{Q}\beta_0, \psi)$ such that β_0 is a gap. Then by ADH 3.1.8 and passing to the divisible hull, we can extend $(\Gamma_0 \oplus \mathbb{Q}\beta_0, \psi)$ to an asymptotic couple $(\Gamma_0 \oplus \mathbb{Q}\beta_0 \oplus \mathbb{Q}\alpha_0, \psi)$ such that $\psi(\alpha_0) = \beta_0$ and $\alpha_0 < 0$. Thus $\beta_0 = \max \psi\big((\Gamma_0 \oplus \mathbb{Q}\beta_0 \oplus \mathbb{Q}\alpha_0)^{\neq}\big)$. Finally, we apply Lemma 3.2.1 to this last asymptotic couple to obtain an asymptotic couple $(\Gamma, \psi) = (\Gamma_0 \oplus \mathbb{Q}\beta_0 \oplus \mathbb{Q}\alpha_0 \oplus \bigoplus_n \mathbb{Q}\beta_{n+1}, \psi)$ with the desired properties. $\square$

Setting $\alpha_{n+1} := \int \beta_n = \chi\alpha_n$ in Lemma 3.2.3, we have the following configuration of the elements we adjoined to (Γ_0, ψ_0):

$$
\begin{tikzcd}
\beta_0 \arrow[r, "s"] \arrow[dr, "\int"'] & \beta_1 \arrow[r, "s"] \arrow[dr, "\int"'] & \beta_2 \arrow[r, "s"] \arrow[dr, "\int"'] & \beta_3 \arrow[r, "s"] \arrow[dr, "\int"'] & \cdots \\
\alpha_0 \arrow[u, "\psi"] \arrow[r, "\chi"'] & \alpha_1 \arrow[u, "\psi"] \arrow[r, "\chi"'] & \alpha_2 \arrow[u, "\psi"] \arrow[r, "\chi"'] & \alpha_3 \arrow[u, "\psi"] \arrow[r, "\chi"'] & \cdots
\end{tikzcd}
$$

The top row of the above diagram is a "copy of $\mathbb{N}$" that has been added to the top of Ψ_0, i.e., $\Psi = \Psi_0 \cup \{\beta_0, \beta_1, \ldots\}$ with $\Psi_0 < \beta_0 < \beta_1 < \beta_2 < \cdots$. The bottom row is a sequence of increasingly smaller and smaller elements (in the sense that $[\Gamma_0^{\neq}] > [\alpha_0] > [\alpha_1] > \cdots$) which serve as "witnesses to the top row".

In the next lemma, we iterate the construction given by Lemma 3.2.3 to add a "copy of $\mathbb{Z}$" to the top of the Ψ-set.

Lemma 3.2.4. *Suppose (Γ, ψ) is divisible. Then there is a divisible H-asymptotic couple $(\Gamma_\diamond, \psi_\diamond) \supseteq (\Gamma, \psi)$ with a family $(\beta_k)_{k \in \mathbb{Z}}$ in $\Psi_\diamond$ such that:*

(1) *$(\Gamma_\diamond, \psi_\diamond)$ has asymptotic integration;*
(2) *$\Psi < \beta_0$, and $s(\beta_k) = \beta_{k+1}$ for all k;*
(3) *for any embedding $i : (\Gamma, \psi) \to (\Gamma^*, \psi^*)$ into a divisible H-asymptotic couple with asymptotic integration and any family $(\beta_k^*)_{k \in \mathbb{Z}}$ in Ψ^* with $i(\Psi) < \beta_0^*$, and $s(\beta_k^*) = \beta_{k+1}^*$ for all k, there is a unique extension of i to an embedding $j : (\Gamma_\diamond, \psi_\diamond) \to (\Gamma^*, \psi^*)$ sending β_k to β_k^* for all k.*

Proof. For each $k \geqslant 0$, let $(\Gamma, \psi) \subseteq (\Gamma_k, \psi_k)$ be the extension given by Lemma 3.2.3. In the terms of the diagram below the proof of Lemma 3.2.3, label the sequence of β's and α's in (Γ_k, ψ_k) as $\beta_0^k, \beta_1^k, \ldots$ and $\alpha_0^k, \alpha_1^k, \ldots$. By the universal property of Lemma 3.2.3, there is a unique embedding $j_k : (\Gamma_k, \psi_k) \to (\Gamma_{k+1}, \psi_{k+1})$ such that $\alpha_0^k \mapsto \alpha_1^{k+1}$.

$$\begin{tikzcd}
(\Gamma_k, \psi_k) \arrow[r, "j_k"] & (\Gamma_{k+1}, \psi_{k+1}) \\
(\Gamma, \psi) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

This embedding results in identifications $\beta_l^k = \beta_{l+1}^{k+1}$ and $\alpha_l^k = \alpha_{l+1}^{k+1}$ for all $l \geqslant 0$. Thus we may define $(\Gamma_\diamond, \psi_\diamond)$ as the union of the increasing chain

$$(\Gamma, \psi) \;\subseteq\; (\Gamma_0, \psi_0) \;\subseteq\; (\Gamma_1, \psi_1) \;\subseteq\; (\Gamma_2, \psi_2) \;\subseteq\; \cdots$$

In $(\Gamma_\diamond, \psi_\diamond)$, we define $\beta_k := \beta_k^0$ for $k \geqslant 0$ and $\beta_k := \beta_0^{-k}$ for $k < 0$. Furthermore we also define $\alpha_k := \int \beta_{k-1}$ for all k. The following table illustrates the identifications of the β's in this increasing union, with elements in the same column being identified:

$$\begin{array}{lccccccc}
\text{in } (\Gamma_\diamond, \psi_\diamond): & \cdots & \beta_{-2} & \beta_{-1} & \beta_0 & \beta_1 & \beta_2 & \cdots \\
\quad\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\text{in } (\Gamma_2, \psi_2): & & \beta_0^2 & \beta_1^2 & \beta_2^2 & \beta_3^2 & \beta_4^2 & \cdots \\
\text{in } (\Gamma_1, \psi_1): & & & \beta_0^1 & \beta_1^1 & \beta_2^1 & \beta_3^1 & \cdots \\
\text{in } (\Gamma_0, \psi_0): & & & & \beta_0^0 & \beta_1^0 & \beta_2^0 & \cdots
\end{array}$$

The asymptotic couple $(\Gamma_\diamond, \psi_\diamond)$ has asymptotic integration since each (Γ_k, ψ_k) has asymptotic integration. Furthermore, $\beta_0 = \beta_0^0 > \Psi$ by Lemma 3.2.3. Also $s(\beta_l) = \beta_{l+1}$ for all $l \in \mathbb{Z}$. Indeed, if $l \geqslant 0$, then this is evident already in (Γ_0, ψ_0). If $l < 0$, then this can be observed in (Γ_{-l}, ψ_{-l}) as $s(\beta_0^{-l}) = \beta_1^{-l}$.

Next, suppose $i : (\Gamma, \psi) \to (\Gamma^*, \psi^*)$ is an embedding into a divisible H-asymptotic couple with asymptotic integration and there is a family $(\beta_k^*)_{k \in \mathbb{Z}}$ in Ψ^* with $i(\Psi) < \beta_0^*$ and $s(\beta_k^*) = \beta_{k+1}^*$ for all k. Since $si(\Psi) \subseteq i(\Psi)$, we have $i(\Psi) < \beta_k^*$ for all $k \in \mathbb{Z}$. Define the auxiliary $(\alpha_k^*)_{k \in \mathbb{Z}}$ in Γ^* by $\alpha_k^* := \int \beta_{k-1}^*$. Then we have $\psi(\alpha_k^*) = \beta_k^*$.

Next, for each $k \geqslant 0$, let $i_k : (\Gamma_k, \psi_k) \to (\Gamma^*, \psi^*)$ be the embedding given by Lemma 3.2.3 with $i_k(\alpha_0^k) = \alpha^*_{-k}$.

$$\begin{tikzcd}
(\Gamma_k, \psi_k) \arrow[r, "i_k"] & (\Gamma^*, \psi^*) \\
(\Gamma, \psi) \arrow[u] \arrow[ur, "i"'] &
\end{tikzcd}$$

In order to show that this embedding extends to an embedding of $(\Gamma_\diamond, \psi_\diamond)$, we must show that $i_k \subseteq i_{k+1}$. By the universal property of i_k, it suffices to show that $i_{k+1}(\alpha_0^k) = \alpha^*_{-k}$, which follows from a routine diagram chase. Thus we get an embedding $j = \cup_k i_k : (\Gamma_\diamond, \psi_\diamond) \to (\Gamma^*, \psi^*)$.

$$\begin{tikzcd}
(\Gamma_\diamond, \psi_\diamond) \arrow[dr, "\cup_k i_k"] & \\
\vdots \arrow[u] & (\Gamma^*, \psi^*) \\
(\Gamma_1, \psi_1) \arrow[u] \arrow[ur, "i_1"] & \\
(\Gamma_0, \psi_0) \arrow[u] \arrow[uur, "i_0"] & \\
(\Gamma, \psi) \arrow[u] \arrow[uuur, "i"'] &
\end{tikzcd}$$

It remains to prove uniqueness of j. Suppose $j' : (\Gamma_\diamond, \psi_\diamond) \to (\Gamma^*, \psi^*)$ is an arbitrary embedding such that $j'(\beta_k) = \beta^*_k$ for all $k \in \mathbb{Z}$. It suffices to show that $j'|_{\Gamma_k} = i_k$ for all $k \geqslant 0$. I.e., $j'(\alpha_{-k}) = j'(\alpha_0^k) = i_k(\alpha_0^k) = \alpha^*_{-k}$. Integrating the expression $j'(\beta_{k-1}) = \beta^*_{k-1}$ yields

$$\alpha^*_k \ = \ \int j'(\beta_{k-1}) \ = \ j'(\int \beta_{k-1}) \ = \ j'(\alpha_k). \qquad \square$$

The following lemma allows us to insert a "copy of $\mathbb{Z}$" into the middle of the Ψ-set of a divisible H-asymptotic couple with asymptotic integration in a canonical way. The decisive point in the proof is to pick an element $a^\star \in \Psi \setminus B$ and use it as in that proof.

Lemma 3.2.5. *Suppose (Γ, ψ) is divisible. Let $B \in \operatorname{sded}(\Psi)$ be nonempty such that $B \neq \Psi$. Then there is a divisible H-asymptotic couple $(\Gamma_B, \psi_B) \supseteq (\Gamma, \psi)$ with a family $(\beta_k)_{k\in\mathbb{Z}}$ in Ψ_B satisfying the following conditions:*

1. *(Γ_B, ψ_B) has asymptotic integration;*
2. *$B < \beta_k < \Gamma^{>B}$, and $s(\beta_k) = \beta_{k+1}$ for all k;*
3. *for any embedding $i : (\Gamma, \psi) \to (\Gamma^*, \psi^*)$ into a divisible H-asymptotic couple with asymptotic integration and any family $(\beta^*_k)_{k\in\mathbb{Z}}$ in Ψ^* such that $i(B) < \beta^*_k < i(\Gamma^{>B})$ and $s(\beta^*_k) = \beta^*_{k+1}$ for all k, there is a unique extension of i to an embedding $(\Gamma_B, \psi_B) \to (\Gamma^*, \psi^*)$ sending β_k to β^*_k for all k.*

Proof. To motivate the construction of (Γ_B, ψ_B) as required, suppose $(\Gamma_B, \psi_B) \supseteq (\Gamma, \psi)$ is an H-asymptotic couple with asymptotic integration and $(\beta_k)_{k\in\mathbb{Z}}$ a family in Ψ_B such that $B < \beta_k < \Gamma^{>B}$ and $s(\beta_k) = \beta_{k+1}$

for all k. Fix any $a^\star \in \Psi \setminus B$. Let $k \in \mathbb{Z}$ and note that by Corollary 2.3.5,

$$\psi_B(a^\star - \beta_k) \ = \ s(\beta_k) \ = \ \beta_{k+1}.$$

Therefore setting $\alpha_k := \beta_{k-1} - a^\star$, we have $\psi_B(\alpha_k) = \beta_k$. Then $\alpha_k < 0$ and

$$[\gamma_1] \ < \ [\alpha_k] \ < \ [\alpha_{k+1}] \ < \ [\gamma_2]$$

whenever $\gamma_1, \gamma_2 \in \Gamma$ such that $\psi(\gamma_1) \in B$ and $\psi(\gamma_2) \in \Psi \setminus B$. Thus for $\gamma \in \Gamma$, $i_1 < \cdots < i_n$ and $q_1, \ldots, q_n \in \mathbb{Q}^{\neq}$, we have

$$\gamma + q_1\alpha_{i_1} + \cdots + q_n\alpha_{i_n} > 0 \iff \begin{cases} \gamma > 0 & \text{if } \psi(\gamma) \in B \text{ or } n = 0, \\ q_1 < 0 & \text{if } \psi(\gamma) \notin B \text{ and } n \geqslant 1. \end{cases}$$

and the ψ_B-value of such an element is uniquely determined:

$$\psi_B(\gamma + q_1\alpha_{i_1} + \cdots + q_n\alpha_{i_n}) \ = \ \begin{cases} \psi(\gamma) & \text{if } \psi(\gamma) \in B \text{ or } n = 0, \\ \beta_{i_1} & \text{if } \psi(\gamma) \notin B \text{ and } n \geqslant 1. \end{cases}$$

Furthermore, $\alpha_k + \psi(\alpha_k) = \alpha_k + \beta_k = \alpha_k + a^\star + \alpha_{k+1}$. Rearranging terms gives us $\beta_{k-1} = \alpha_k + a^\star = \alpha_k - \alpha_{k+1} + \psi(\alpha_k)$. Since $[\alpha_k] > [\alpha_{k+1}]$, we have $\psi(\alpha_k - \alpha_{k+1}) = \psi(\alpha_k)$. Thus $\int \beta_{k-1} = \alpha_k - \alpha_{k+1}$, and $s(\beta_{k-1}) = \psi(\alpha_k - \alpha_{k+1}) = \psi(\alpha_k) = \beta_k$. Also, $\int \psi(\alpha_k - \alpha_{k+1}) = \int \psi(\alpha_k) = \alpha_{k+1} - \alpha_{k+2}$ and thus $\chi(\alpha_k - \alpha_{k+1}) = \chi(\alpha_k) = \alpha_{k+1} - \alpha_{k+2}$. Here is a picture of what is going on:

$$\begin{tikzcd}
\beta_{k-1} \arrow[rr, "s"] \arrow[ddr, "\int"] & & \beta_k \arrow[rr, "s"] \arrow[ddr, "\int"] & & \beta_{k+1} \\
\alpha_{k-1} \arrow[u, "\psi"] \arrow[dr, "\chi"'] & & \alpha_k \arrow[u, "\psi"] \arrow[dr, "\chi"'] & & \alpha_{k+1} \arrow[u, "\psi"'] \\
& \alpha_k - \alpha_{k+1} \arrow[uur, "\psi"] \arrow[rr, "\chi"] & & \alpha_{k+1} - \alpha_{k+2} \arrow[uur, "\psi"] &
\end{tikzcd}$$

Note that the α_k's depend on the choice of $a^\star$ whereas the elements of the form $\alpha_k - \alpha_{k+1}$ do not depend on this choice.

Next, to actually obtain (Γ_B, ψ_B), by compactness we take an elementary extension $(\Gamma_\star, \psi_\star)$ of (Γ, ψ) with a family $(\beta_k)_{k\in\mathbb{Z}}$ in $\Psi_\star$ such that $B < \beta_k < \Gamma^{>B}$ and $s(\beta_k) = \beta_{k+1}$ for all k (this uses the assumption that $B \neq \emptyset$). Take $a^* \in \Psi \setminus B$ and define $\alpha_k := \beta_{k-1} - a^\star$. Set $\Gamma_B := \Gamma + \sum_k \mathbb{Q}\alpha_k$. By the above observations, $(\Gamma_B, \psi_\star|_{\Gamma_B})$ is a divisible H-asymptotic couple with the desired properties. $\square$

It follows from the proof of Lemma 3.2.5 that $\Psi_B = \Psi \cup \{\beta_k : k \in \mathbb{Z}\}$.

3.3. A very big embedding lemma

In this section (Γ, ψ) is a divisible H-asymptotic couple with asymptotic integration. We can combine and restate Lemmas 3.2.4 and 3.2.5 using the $\mathrm{sded}^{op}(\Psi)$ terminology:

Lemma 3.3.1. *Let $B \in \mathrm{sded}^{op}(\Psi)$ be such that $B \neq \Psi$. Then there is a divisible H-asymptotic couple $(\Gamma_B, \psi_B) \supseteq (\Gamma, \psi)$ with a family $(\beta_k)_{k\in\mathbb{Z}}$ in Ψ_B satisfying the following conditions:*

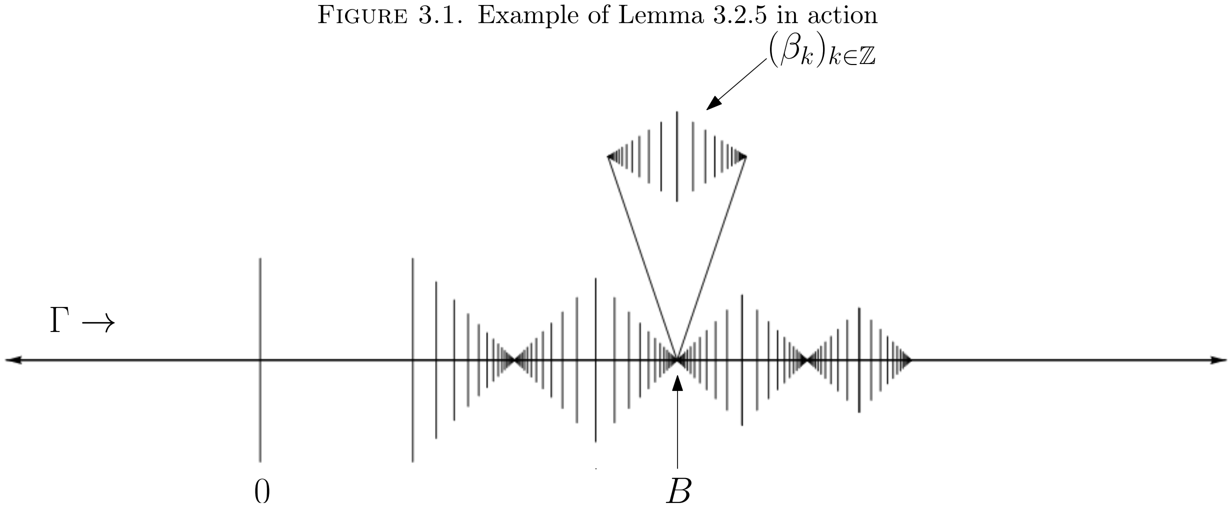
(1) *(Γ_B, ψ_B) has asymptotic integration;*
(2) *$\Gamma^{<B} < \beta_k < B$, and $s_B(\beta_k) = \beta_{k+1}$ for all k;*

(3) $\Psi_B = \Psi \cup \{\beta_k : k \in \mathbb{Z}\}$;

(4) *for any embedding $i : (\Gamma, \psi) \to (\Gamma^*, \psi^*)$ into a divisible H-asymptotic couple with asymptotic integration and any family $(\beta_k^*)_{k\in\mathbb{Z}}$ in Ψ^* such that $i(\Gamma^{<B}) < \beta_k^* < i(B)$ and $s^*(\beta_k^*) = \beta_{k+1}^*$ for all k, there is a unique extension of i to an embedding $(\Gamma_B, \psi_B) \to (\Gamma^*, \psi^*)$ sending β_k to β_k^* for all k.*

In Figure 3.1, we illustrate an instance of the construction that is done in Lemma 3.2.5 (over an elementary extension of $(\Gamma_{\log}, \psi)$, see Chapter 4). Technically speaking, here B (as a set) is the two rightmost copies of $\mathbb{Z}$, however, we think of B as indicating the cut between existing copies of $\mathbb{Z}$ where a new copy of $\mathbb{Z}$ (namely, $(\beta_k)_{k\in\mathbb{Z}}$) is to be added.

Figure 3.1. Example of Lemma 3.2.5 in action



As for the universal property, suppose $i : (\Gamma, \psi) \to (\Gamma^*, \psi^*)$ is an embedding as in (4) from Lemma 3.3.1 above. The uniqueness of the extension of i to an embedding $(\Gamma_B, \psi_B) \to (\Gamma^*, \psi^*)$ depends heavily on the specification of the family $(\beta_k^*)_{k\in\mathbb{Z}}$ in Ψ^* and in particular the requirement that $\beta_k \mapsto \beta_k^*$ for all k:

$$\begin{tikzcd}
 & (\Gamma^*, \psi^*) \textbf{ and } (\beta_k^*)_{k\in\mathbb{Z}} \\
(\Gamma_B, \psi_B) \textbf{ and } (\beta_k)_{k\in\mathbb{Z}} \arrow[ur, dotted, "\exists!"] & \\
(\Gamma, \psi) \arrow[u, dash] \arrow[uur, "i"'] &
\end{tikzcd}$$

In fact, if we were to drop the requirement that the extension of i to an embedding $(\Gamma_B, \psi_B) \to (\Gamma^*, \psi^*)$ has the property that $\beta_k \mapsto \beta_k^*$ for all $k \in \mathbb{Z}$, then there would always be infinitely many distinct extensions of i

to embeddings $(\Gamma_B, \psi_B) \to (\Gamma^*, \psi^*)$:

$$
\begin{tikzcd}
 & (\Gamma^*, \psi^*) \\
(\Gamma_B, \psi_B) \arrow[ur, dotted, "\exists^\infty"] & \\
(\Gamma, \psi) \arrow[u, dash] \arrow[uur, "i"'] &
\end{tikzcd}
$$

This follows from Lemma 3.2.5 by considering the reindexing $(\beta^*_{k+l})_{k \in \mathbb{Z}}$ of the family $(\beta^*_k)_{k \in \mathbb{Z}}$ by an arbitrary $l \in \mathbb{Z}$.

In the lemma below we add transfinitely many copies of $\mathbb{Z}$ to Ψ. We think of the extension (Γ_ρ, ψ_ρ) of (Γ, ψ) constructed in that lemma as adding ν-many copies of $\mathbb{Z}$ to Ψ in the s-cuts specified by ρ, where ν is a (possibly finite) ordinal.

Lemma 3.3.2. *Let $\rho : \nu \to \operatorname{sded}^{op}(\Psi) \setminus \{\Psi\}$ be an increasing function. Then there is a divisible H-asymptotic couple $(\Gamma_\rho, \psi_\rho) \supseteq (\Gamma, \psi)$ with a family $(\beta_{k,\eta})_{k \in \mathbb{Z}, \eta < \nu}$ in Ψ_ρ satisfying the following conditions:*

(1) *(Γ_ρ, ψ_ρ) has asymptotic integration;*
(2) *$\Gamma^{<\rho(\eta)} < \beta_{k,\eta} < \rho(\eta)$, and $s_\rho(\beta_{k,\eta}) = \beta_{k+1,\eta}$ for all $k \in \mathbb{Z}$ and $\eta < \nu$;*
(3) *$\beta_{k,\eta_0} < \beta_{l,\eta_1}$ for all $k, l \in \mathbb{Z}$ and $\eta_0 < \eta_1 < \nu$;*
(4) *$\Psi_\rho = \Psi \cup \{\beta_{k,\eta} : k \in \mathbb{Z}, \eta < \nu\}$;*
(5) *for any embedding $i : (\Gamma, \psi) \to (\Gamma^*, \psi^*)$ into a divisible H-asymptotic couple with asymptotic integration and any family $(\beta^*_{k,\eta})_{k \in \mathbb{Z}, \eta < \nu}$ in Ψ^* such that $i\big(\Gamma^{<\rho(\eta)}\big) < \beta^*_{k,\eta} < i\big(\rho(\eta)\big)$ and $s^*(\beta^*_{k,\eta}) = \beta^*_{k+1,\eta}$ for all $k \in \mathbb{Z}$ and $\eta < \nu$, and $\beta^*_{k,\eta_0} < \beta^*_{l,\eta_1}$ for all $k, l \in \mathbb{Z}$ and $\eta_0 < \eta_1 < \nu$, then there is a unique extension of i to an embedding $(\Gamma_\rho, \psi_\rho) \to (\Gamma^*, \psi^*)$ sending $\beta_{k,\eta}$ to $\beta^*_{k,\eta}$ for all $k \in \mathbb{Z}$ and $\eta < \nu$;*
(6) *if (Γ, ψ) is a model of T_0, then so is (Γ_ρ, ψ_ρ).*

Proof. We will prove this by transfinite induction on ν.

($\nu = 0$) In this case we set $(\Gamma_\rho, \psi_\rho) := (\Gamma, \psi)$ and we are done.

($\nu = \eta + 1$) By the inductive hypothesis, we can construct an extension $(\Gamma_{\rho\restriction\eta}, \psi_{\rho\restriction\eta})$ of (Γ, ψ) which satisfies properties (1)-(5) for the function $\rho \restriction \eta : \eta \to \operatorname{sded}(\Psi)$.

Claim 3.3.3. *$\rho(\eta)$ is an s-cut in $\Psi_{\rho\restriction\eta}$, i.e., $\rho(\eta) \in \operatorname{sded}^{op}(\Psi_{\rho\restriction\eta})$.*

Proof of claim. By the inductive hypothesis, $\Psi_{\rho\restriction\eta} = \Psi \cup \{\beta_{k,\eta_0} : k \in \mathbb{Z}, \eta_0 < \eta\}$, so it suffices to prove that $\beta_{k,\eta_0} < \rho(\eta)$ for all $k \in \mathbb{Z}$ and $\eta_0 < \eta$. This is clear because $\beta_{k,\eta_0} < \rho(\eta_0)$ by (3) for $(\Gamma_{\rho\restriction\eta}, \psi_{\rho\restriction\eta})$ and $\rho(\eta_0) \leqslant \rho(\eta)$ because ρ is increasing. $\square$

Since $\rho(\eta)$ is also an s-cut in $\Psi_{\rho\restriction\eta}$, we can use Lemma 3.2.5 to add a copy of $\mathbb{Z}$ to $(\Gamma_{\rho\restriction\eta}, \psi_{\rho\restriction\eta})$ at $\rho(\eta)$. Thus we set $(\Gamma_\rho, \psi_\rho) := \big((\Gamma_{\rho\restriction\eta})_{\rho(\eta)}, (\psi_{\rho\restriction\eta})_{\rho(\eta)}\big)$. As an extension of (Γ, ψ), it is clear that (Γ_ρ, ψ_ρ) satisfies properties (1)-(4). Property (5) is satisfied because $(\Gamma_{\rho\restriction\eta}, \psi_{\rho\restriction\eta})$ satisfies property (5) over (Γ, ψ) and (Γ_ρ, ψ_ρ) satisfies the universal property of Lemma 3.2.5 over $(\Gamma_{\rho\restriction\eta}, \psi_{\rho\restriction\eta})$.

(ν limit ordinal) By the inductive hypothesis, for all $\eta_0 < \eta_1 < \nu$ we can construct extensions $(\Gamma_{\rho\restriction\eta_i}, \psi_{\rho\restriction\eta_i})$ of (Γ, ψ) $(i = 1, 2)$ such that there is a unique embedding $i_{\eta_0,\eta_1} : (\Gamma_{\rho\restriction\eta_0}, \psi_{\rho\restriction\eta_0}) \to (\Gamma_{\rho\restriction\eta_1}, \psi_{\rho\restriction\eta_1})$ over (Γ, ψ) such that $\beta_{k,\eta} \mapsto \beta_{k,\eta}$ for all $k \in \mathbb{Z}$ and $\eta < \eta_0$.

$$\begin{tikzcd}
(\Gamma_{\rho\restriction\eta_0}, \psi_{\rho\restriction\eta_0}) \arrow[r, "i_{\eta_0,\eta_1}"] & (\Gamma_{\rho\restriction\eta_1}, \psi_{\rho\restriction\eta_1}) \\
(\Gamma, \psi) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Thus without loss of generality we may assume that for all $\eta_0 < \eta_1 < \nu$ we have an increasing chain:

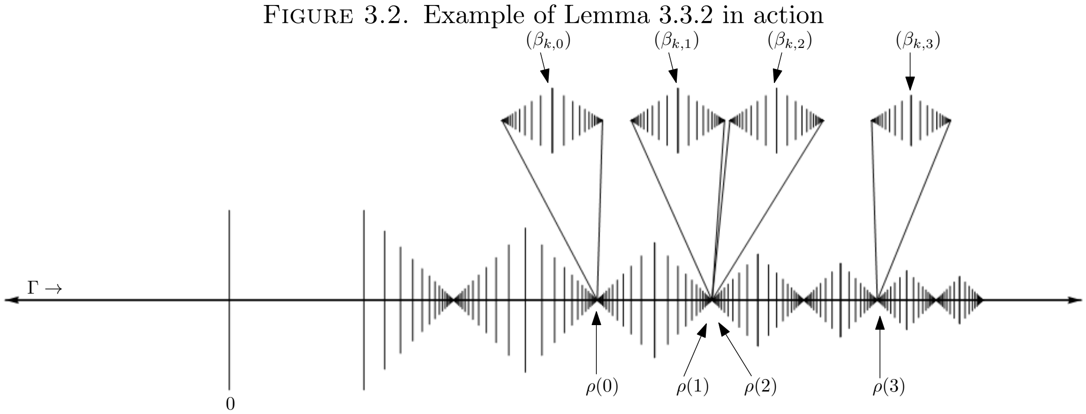
$$(\Gamma, \psi) \ \subseteq \ (\Gamma_{\rho\restriction\eta_0}, \psi_{\rho\restriction\eta_0}) \ \subseteq \ (\Gamma_{\rho\restriction\eta_1}, \psi_{\rho\restriction\eta_1})$$

Therefore we may set $(\Gamma_\rho, \psi_\rho) := \left(\bigcup_{\eta<\nu} \Gamma_{\rho\restriction\eta}, \bigcup_{\eta<\nu} \psi_{\rho\restriction\eta}\right)$ and it is clear that this extension satisfies properties (1)-(4). Suppose that $i : (\Gamma, \psi) \to (\Gamma^*, \psi^*)$ is an embedding such that (Γ^*, ψ^*) is a divisible H-asymptotic couple with asymptotic integration and there is a family $(\beta^*_{k,\eta})_{k\in\mathbb{Z},\eta<\nu}$ in Ψ^* satisfying the properties listed in (5). Then for each $\eta < \nu$ there is a unique extension of i to an embedding $i_\eta : (\Gamma, \psi) \subseteq (\Gamma_{\rho\restriction\eta}, \psi_{\rho\restriction\eta}) \to (\Gamma^*, \psi^*)$ sending β_{k,η_0} to β^*_{k,η_0} for all $k \in \mathbb{Z}$ and $\eta_0 < \eta$. Thus it is clear that $i_\nu := \cup_{\eta<\nu} i_\eta : (\Gamma_\rho, \psi_\rho) \to (\Gamma^*, \psi^*)$ is an extension of i sending $\beta_{k,\eta}$ to $\beta^*_{k,\eta}$ for all $k \in \mathbb{Z}$ and $\eta < \nu$. Uniqueness of i_ν follows from the observation that the restriction of i_ν to each $(\Gamma_{\rho\restriction\eta}, \psi_{\rho\restriction\eta})$ is uniquely determined by the universal property that each $(\Gamma_{\rho\restriction\eta}, \psi_{\rho\restriction\eta})$ enjoys (by induction).

Finally, (6) is immediate from the above construction. $\square$

In Figure 3.2, we illustrate an instance of the construction done in Lemma 3.3.2 (over a model of T_0). Here we have the increasing function $\rho : 4 \to \mathrm{sded}(\Psi)$ where $\rho(0) < \rho(1) = \rho(2) < \rho(3)$. Since $\rho(1) = \rho(2)$, $(\beta_{k,1})$, the copy of $\mathbb{Z}$ corresponding to $\rho(1)$, gets added to the same cut in Ψ as $(\beta_{k,2})$ the copy of $\mathbb{Z}$ corresponding to $\rho(2)$. However, the construction ensures that $(\beta_{k,1})$ gets added entirely to the left of $(\beta_{k,2})$.

Figure 3.2. Example of Lemma 3.3.2 in action



CHAPTER 4

Model theory of the asymptotic couple of the field of logarithmic transseries

In this chapter we collect our results on the model theory of the asymptotic couple $(\Gamma_{\log}, \psi)$. Most of this material is from [**14**] and [**15**].

After reviewing relevant definitions from model theory in Section 4.1, we prove the main quantifier elimination and model completeness results for $(\Gamma_{\log}, \psi)$ in Section 4.2. Specifically, we identify first order languages $\mathcal{L}_{AC} \subseteq \mathcal{L}_{AC,\log}$, axiomatize complete theories $T_{AC} = \mathrm{Th}_{\mathcal{L}_{AC}}(\Gamma, \psi)$ and its extension by definitions $T_{AC,\log} = \mathrm{Th}_{\mathcal{L}_{AC,\log}}(\Gamma_{\log}, \psi)$, and prove that $T_{AC,\log}$ has QE (Theorem 4.2.2) and T_{AC} is model complete (Corollary 4.2.3).

In Section 4.3 we analyze in detail the Ψ-set of models $T_{AC,\log}$. In particular, we characterize all definable functions $\Psi \to \Gamma$ (Theorem 4.3.3) and show that the definable set Ψ is stably embedded (Corollary 4.3.14).

In Sections 4.4 and 4.5, we characterize all 1-types over arbitrary parameter sets in models of $T_{AC,\log}$ (Theorem 4.4.6), resulting in a proof that T_{AC} has the *non-independence property*, or *NIP*, (Theorem 4.5.3).

We conclude the chapter with a few more minor model-theoretic results and observations in Section 4.6 concerning T_{AC} and $T_{AC,\log}$, and a section on the relationship between models of T_{AC} and the so-called *precontraction groups* (Section 4.7).

4.1. Quantifier elimination and model completeness

In this section we recall the definitions of quantifier elimination and model completeness for first order theories. We also state the relevant criteria to be used later for obtaining quantifier elimination and model completeness results.

In the rest of this section $\mathcal{L}$ is a first-order language such that for each sort s there is a constant symbol of sort s, $\boldsymbol{M}$ is an $\mathcal{L}$-structure, and Σ is a set of $\mathcal{L}$-sentences.

Definition 4.1.1. We say that Σ **admits quantifier elimination** (QE) if each formula $\varphi(x)$ with finite x is Σ-equivalent to a quantifier-free formula $\varphi'(x)$. We also express this by saying "Σ has QE" or "Σ eliminates quantifiers". We say that $\boldsymbol{M}$ admits QE if $\mathrm{Th}(\boldsymbol{M})$ admits QE.

There are a variety of ways to prove that a theory T admits QE. The one which we will use in Section 4.2 below is the following:

ADH 4.1.2. [**6**, B.11.10] *Suppose that for every $\boldsymbol{M} \models \Sigma$ and substructure $\boldsymbol{A}$ of $\boldsymbol{M}$ with $A \neq M$ and every embedding i of $\boldsymbol{A}$ into an $|A|^+$-saturated model $\boldsymbol{N}$ of Σ there exist $s \in S$ and $b \in M_s \setminus A_s$ such that i extends to an embedding $\boldsymbol{A}\langle b\rangle \to \boldsymbol{N}$. Then Σ has QE.*

Example 4.1.3 (Quantifier elimination for divisible ordered abelian groups)**.** Consider the one-sorted language $\mathcal{L}_{OA} = \{\leqslant, 0, -, +\}$ of ordered abelian groups and the $\mathcal{L}_{OA}$-theory T_{DOAG} whose models are the nontrivial divisible ordered abelian groups. It is well-known that T_{DOAG} admits QE (see [**6**, B.11.12], for example). It follows from the various embedding properties associated with QE from [**6**, §B.11] that if $\Gamma_0 \subseteq \Gamma$ and Γ^* are models of T_{DOAG} such that Γ^* is $|\Gamma_1|^+$-saturated, and $i : \Gamma_0 \to \Gamma^*$ is an embedding of $\mathcal{L}_{OA}$-structures, then there is an embedding $j : \Gamma_1 \to \Gamma^*$ of $\mathcal{L}_{OA}$-structures which extends i:

$$\begin{tikzcd}
& & \Gamma^* \\
\Gamma_1 \arrow[urr, dotted, "\exists j"] & & \\
& \Gamma_0 \arrow[ul] \arrow[uur, "i"'] &
\end{tikzcd}$$

From QE we can often conclude that Σ is complete:

ADH 4.1.4. [**6**, B.11.7] *Suppose Σ admits QE and has a model. Then Σ is complete iff some $\mathcal{L}$-structure (not necessarily a model of Σ) embeds into every model of Σ.*

Associated with QE is the weaker notion of *model complete*:

Definition 4.1.5. Σ is said to be **model complete** if every formula is Σ-equivalent to an existential formula. We say that $\boldsymbol{M}$ is model complete if $\mathrm{Th}(\boldsymbol{M})$ is.

It is immediate that if Σ admits QE, then Σ is model complete. We also have an embedding criterion for proving that a certain theory is model complete:

ADH 4.1.6. [**6**, B.10.4] *The following are equivalent:*

(1) *Σ is model complete;*
(2) *for all models $\boldsymbol{M}$, $\boldsymbol{N}$ of Σ with $\boldsymbol{M} \subseteq \boldsymbol{N}$ and every elementary extension $\boldsymbol{M}^*$ of $\boldsymbol{M}$ that is κ-saturated for some $\kappa > |N|$, there is an embedding $\boldsymbol{N} \to \boldsymbol{M}^*$ that extends the natural inclusion $\boldsymbol{M} \to \boldsymbol{M}^*$.*

Model completeness can also be used to show that Σ is complete.

Definition 4.1.7. A **prime model** of Σ is a model of Σ that embeds elementarily into every model of Σ.

ADH 4.1.8. [**6**, B.10] *If Σ is model complete and $\boldsymbol{M}$ is a model of Σ that embeds into every model of Σ, then $\boldsymbol{M}$ is a prime model of Σ and therefore Σ is complete.*

A consequence of a theory of being model complete is that it is *inductive*:

Definition 4.1.9. We call Σ **inductive** if the direct union of any directed family of models of Σ is a model of Σ.

ADH 4.1.10. [**6**, B.10.7] *If Σ is model complete, then Σ is inductive.*

Definition 4.1.11. Given $\mathcal{L}$-structures $\boldsymbol{M}$ and $\boldsymbol{N}$, we say that $\boldsymbol{M}$ is **existentially closed in $\boldsymbol{N}$** if $\boldsymbol{M} \subseteq \boldsymbol{N}$ and every existential $\mathcal{L}_M$-sentence true in $\boldsymbol{N}$ is true in $\boldsymbol{M}$. An **existentially closed model of** Σ is a model $\boldsymbol{M}$ of Σ that is existentially closed in every extension $\boldsymbol{N} \models \Sigma$ of $\boldsymbol{M}$.

In the rest of this section T is an inductive $\mathcal{L}$-theory.

Definition 4.1.12. A **model companion** of T is a model complete $\mathcal{L}$-theory $T^* \supseteq T$ such that every model of T embeds into a model of T^*.

ADH 4.1.13. [**6**, B.10.10] *Let T^* be an $\mathcal{L}$-theory. Then T^* is a model companion of T iff the models of T^* are exactly the existentially closed models of T.*

4.2. Quantifier elimination and model completeness for $(\Gamma_{\log}, \psi)$

Let $\mathcal{L}_{AC}$ be the "natural" language of asymptotic couples; $\mathcal{L}_{AC} = \{0, +, -, <, \psi, \infty\}$ where $0, \infty$ are constant symbols, $+$ is a binary function symbol, $-$ and ψ are unary function symbols and $<$ is a binary relation symbol. We consider an asymptotic couple (Γ, ψ) as an $\mathcal{L}_{AC}$-structure with underlying set Γ_∞ and the obvious interpretation of the symbols of $\mathcal{L}_{AC}$, with ∞ as a default value:

$$-\infty \;=\; \gamma + \infty \;=\; \infty + \gamma \;=\; \infty + \infty \;=\; \psi(0) \;=\; \psi(\infty) \;=\; \infty$$

for all $\gamma \in \Gamma$.

Let T_{AC} be the $\mathcal{L}_{AC}$-theory whose models are the divisible H-asymptotic couples with asymptotic integration such that

(1) Ψ as an ordered subset of Γ has a least element $s0$,
(2) $s0 > 0$,
(3) Ψ as an ordered subset of Γ is a successor set,
(4) for each $\alpha \in \Psi$, the immediate successor of α in Ψ is $s\alpha$, and
(5) $\gamma \mapsto s\gamma : \Psi \to \Psi^{>s0}$ is a bijection.

It is clear that $(\Gamma_{\log}, \psi)$ and $(\Gamma_{\log}^{\mathbb{Q}}, \psi)$ are models of T_{AC}. For a model (Γ, ψ) of T_{AC}, we define the function $p : \Psi^{>s0} \to \Psi$ to be the inverse to the function $\gamma \mapsto s\gamma : \Psi \to \Psi^{>s0}$. We extend p to a function $\Gamma_\infty \to \Gamma_\infty$ by setting $p(\alpha) := \infty$ for $\alpha \in \Gamma_\infty \setminus \Psi^{>s0}$.

Next, let $\mathcal{L}_{AC,\log} = \mathcal{L}_{AC} \cup \{s, p, \delta_1, \delta_2, \delta_3, \ldots\}$ where s, p and δ_n for $n \geqslant 1$ are unary function symbols. All models of T_{AC} are considered as $\mathcal{L}$-structures in the obvious way, again with ∞ as a default value, and with δ_n interpreted as division by n.

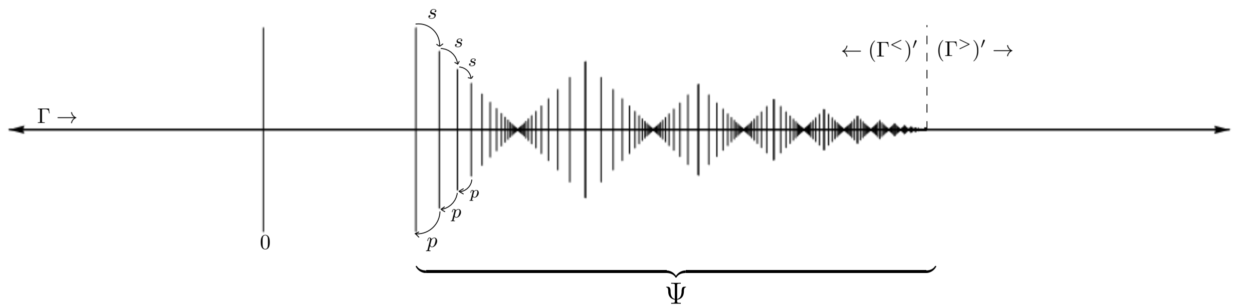
We let $T_{AC,\log}$ be the $\mathcal{L}_{AC,\log}$-theory whose models are the models of T_{AC}. In Figure 4.1 we have attempted to illustrate what we picture as a "typical" model of $T_{AC,\log}$, although in general the copies of $\mathbb{Z}$ beyond the initial copy of $\mathbb{N}$ in the Ψ-set may be indexed by any arbitrary linear order.
By adding function symbols $s, p, \delta_1, \delta_2, \ldots$ we have guaranteed the following:

Lemma 4.2.1. *$T_{AC,\log}$ has a universal axiomatization.*

Since $T_{AC,\log}$ has a universal axiomatization, if $(\Gamma_1, \psi_1) \models T_{AC,\log}$ and (Γ_0, ψ_0) is an $\mathcal{L}_{AC,\log}$-substructure of (Γ_1, ψ_1), then $(\Gamma_0, \psi_0) \models T_{AC,\log}$. This fact is very convenient for our proof of QE in Theorem 4.2.2 below. In Theorem 4.2.2, we actually use the embedding criteria ADH 4.1.2 which implies QE for $T_{AC,\log}$.

Theorem 4.2.2 (QE for $T_{AC,\log}$)**.** *Suppose that $(\Gamma_0, \psi_0) \subsetneq (\Gamma_1, \psi_1)$ and (Γ^*, ψ^*) are models of $T_{AC,\log}$ such that (Γ^*, ψ^*) is $|\Gamma_1|^+$-saturated, and $i : (\Gamma_0, \psi_0) \to (\Gamma^*, \psi^*)$ is an embedding of $\mathcal{L}_{AC,\log}$-structures. Then there is an element $\alpha \in \Gamma_1 \setminus \Gamma_0$ such that i extends to an embedding $(\Gamma, \psi) \to (\Gamma^*, \psi^*)$ where $(\Gamma_0, \psi_0) \subseteq (\Gamma, \psi) \subseteq (\Gamma_1, \psi_1)$ and $\alpha \in \Gamma$.*

FIGURE 4.1. A “typical” model of $T_{AC,\log}$



PROOF. The general picture to keep in mind for this proof is the following:

$$\begin{tikzcd}
 & (\Gamma^*, \psi^*) \arrow[dd, dash] \\
(\Gamma_1, \psi_1) \arrow[d, dash] & \\
(\Gamma, \psi) \arrow[r, dotted, "\exists ?", "\sim"'] \arrow[d, dash] & (i\Gamma, \psi^*|_{i\Gamma}) \arrow[d, dash] \\
(\Gamma_0, \psi_0) \arrow[r, "i", "\sim"'] & (i\Gamma_0, \psi^*|_{i\Gamma_0})
\end{tikzcd}$$

Let $\Psi_1 := \psi_1(\Gamma_1^{\neq})$, $\Psi_0 := \psi_0(\Gamma_0^{\neq})$ and $\Psi^* := \psi^*\big((\Gamma^*)^{\neq}\big)$. In particular, $\Psi_0 \subseteq \Psi_1$ and the first two cases deal with the situation that $\Psi_0 \neq \Psi_1$.

Case 1: *There is* $\beta \in \Psi_1 \setminus \Psi_0$ *such that* $\Psi_0 < \beta$. Take such β, and define the family $(\beta_k)_{k\in\mathbb{Z}}$ by $\beta_0 := \beta$, $\beta_n := s^n\beta$, and $\beta_{-n} := p^n\beta$ for $n \geqslant 1$. Note that $s\beta_k = \beta_{k+1}$ for all $k \in \mathbb{Z}$. By Lemma 3.2.4 we may assume that $(\Gamma_0, \psi_0) \subseteq (\Gamma_\diamond, \psi_\diamond) \subseteq (\Gamma_1, \psi_1)$ with $\beta \in \Gamma_\diamond$. By saturation of (Γ^*, ψ^*), there is a family $(\beta_k^*)_{k\in\mathbb{Z}}$ in Γ^* such that $i(\Psi_0) < \beta_0^*$ and $s(\beta_k^*) = \beta_{k+1}^*$ for all $k \in \mathbb{Z}$, and so there is a unique extension of i to an embedding $(\Gamma_\diamond, \psi_\diamond) \to (\Gamma^*, \psi^*)$ sending β_k to β_k^* for all $k \in \mathbb{Z}$.

Case 2: $\Psi_1 \neq \Psi_0$ *and we are not in Case 1.* Take $\beta \in \Psi_1 \setminus \Psi_0$, and define the set $B := \{\alpha \in \Psi_0 : \alpha < \beta\}$ and the family $(\beta_k)_{k\in\mathbb{Z}}$ by $\beta_0 := \beta$, $\beta_n = s^n(\beta_0)$, and $\beta_{-n} = p^n(\beta_0)$ for $n \geqslant 1$. Then $s(B) \subseteq B$, $B < \beta_k < \Gamma_0^{>B}$ and $s(\beta_k) = \beta_{k+1}$ for all $k \in \mathbb{Z}$. Thus by Lemma 3.2.5 we may assume that $(\Gamma_0, \psi_0) \subseteq (\Gamma_{0,B}, \psi_{0,B}) \subseteq (\Gamma_1, \psi_1)$. Again, by Lemma 3.2.5 and saturation of (Γ^*, ψ^*), there is a family $(\beta_k^*)_{k\in\mathbb{Z}}$ in Γ^* such that $i(B) < \beta_0^* < i(\Gamma_0^{>B})$ and $s\beta_k^* = \beta_{k+1}^*$, and so there is a unique extension of $i : (\Gamma_0, \psi_0) \to (\Gamma^*, \psi^*)$ to an embedding $(\Gamma_{0,B}, \psi_{0,B}) \to (\Gamma^*, \psi^*)$ that sends β_k to β_k^* for all $k \in \mathbb{Z}$.

Case 3: $\Psi_0 = \Psi_1$ *but* $[\Gamma_0] \neq [\Gamma_1]$. Take $\alpha \in \Gamma_1$ such that $[\alpha] \notin [\Gamma_0]$. Let $\beta = \psi_1(\alpha) \in \Gamma_0$. Define C to be the cut in $[\Gamma_0]$ which is realized by $[\alpha]$ in $[\Gamma_1]$. By Lemma 3.1.7 there is an asymptotic couple $(\Gamma, \psi) = (\Gamma_0 + \mathbb{Q}\alpha, \psi_1|_{\Gamma+\mathbb{Q}\alpha})$ extending (Γ_0, ψ_0) inside (Γ_1, ψ_1), and by saturation we can extend i to an embedding $(\Gamma, \psi) \to (\Gamma^*, \psi^*)$.

Case 4: $[\Gamma_0] = [\Gamma_1]$. By QE for nontrivial divisible ordered abelian groups, Example 4.1.3, we get an extension $j : \Gamma_1 \to \Gamma^*$ of i as an embedding of ordered abelian groups:

$$\begin{tikzcd}
 & & \Gamma^* \\
\Gamma_1 \arrow[urr, "j"] & & \\
 & \Gamma_0 \arrow[ul] \arrow[uur, "i"'] &
\end{tikzcd}$$

By Lemma 3.1.10, j is actually an embedding of divisible H-asymptotic couples with asymptotic integration. Since $\Psi_0 = \Psi_1$ and i is an embedding of $\mathcal{L}_{AC,\log}$-structures, j is as well. □

In the next corollary we collect the usual consequences of a quantifier elimination result:

Corollary 4.2.3. *$T_{AC,\log}$ and T_{AC} are complete, decidable, model complete, and inductive.*

Proof. Model completeness of $T_{AC,\log}$ follows immediately from QE, and completeness follows from ADH 4.1.4 and the observation in Example 3.2.2 that the $\mathcal{L}_{AC,\log}$-structure $(\Gamma_{\log}^{\mathbb{Q}}, \psi)$ embeds into every model of $T_{AC,\log}$.

Model completeness of T_{AC} follows from ADH 4.1.6. To verify (2) of ADH 4.1.6 we use the same proof as in Theorem 4.2.2 above, except that all the asymptotic couples considered in the proof are construed as $\mathcal{L}_{AC}$-structures.

Completeness for T_{AC} follows from ADH 4.1.8 using that the $\mathcal{L}_{AC}$-structure $(\Gamma_{\log}^{\mathbb{Q}}, \psi)$ is a model of T_{AC} which embeds into every model of T_{AC} by Example 3.2.2.

Decidability for both theories follows from completeness and the fact that these theories are formulated in recursive languages (a finite language in the case of T_{AC}) and that they have recursively enumerable axiomatizations.

Finally, both theories are inductive by model completeness and ADH 4.1.10. □

Example 4.2.4. Below are some quantifier-free definitions of several definable sets in models of $T_{AC,\log}$:

(1) The set Ψ can be defined by the formula:

$$x = ps(x) \wedge x \neq \infty$$

(2) The set $(\Psi - \Psi)^{>0} := \{\alpha_1 - \alpha_2 : \alpha_1, \alpha_2 \in \Psi \text{ and } \alpha_1 > \alpha_2\}$ can be defined by the formula:

$$x = -p\psi(x) + ps\big(-(x - p\psi(x))\big) \wedge x \neq \infty$$

It follows from results in Section 4.3 that this last formula does indeed define the set $(\Psi - \Psi)^{>0}$.

The following example will be useful later in Section 8.3.

Example 4.2.5. Below are some existential definitions of several definable sets in models of T_{AC}:

(1) The set Ψ can be defined by the formula:

$$\phi_0(x) \;:=\; \exists y\big(x = \psi(y) \wedge x \neq \infty\big)$$

(2) The set

$$\ker s \;:=\; \{(x,y) \in \Gamma_\infty^2 : sx = sy\} \;\subseteq\; \Gamma_\infty \times \Gamma_\infty$$

can be defined by the formula:

$$\phi_1(x,y) \;:=\; \exists z\big(x = y = \infty \vee \big(z \neq \infty \wedge z = \psi(x-z) = \psi(y-z)\big)\big)$$

(3) The set $\Gamma^{<s0}$ can be defined by the formula:

$$\phi_2(x) \;:=\; \exists y\big(y \neq \infty \wedge \psi(y) = y \wedge x < y\big)$$

(4) Finally, we will define the set $\Gamma_\infty \setminus \Psi$ with an existential formula. First, we have for $x \in \Gamma_\infty$:

$$x \notin \Psi \iff \big[x < s0\big] \text{ or } \big[x = \infty\big] \text{ or } \big[x \in (\Gamma^{>})'\big] \text{ or } \big[\exists y(y \in \Psi \wedge x \neq y \wedge sx = sy)\big]$$

Thus the following formula defines $\Gamma_\infty \setminus \Psi$:

$$\phi_3(x) \;:=\; \big[\phi_2(x)\big] \;\vee\; \big[x = \infty\big] \;\vee\; \big[\exists y(y > 0 \wedge y + \psi(y) = x)\big] \;\vee\; \big[\exists y(\phi_0(y) \wedge x \neq y \wedge \phi_1(x,y))\big]$$

By renaming variables which are being quantified over, and moving all quantifiers out to the front, we obtain an existential formula which is equivalent to ϕ_3.

4.3. Definable functions on, and subsets of, Ψ

For a language $\mathcal{L}$ and an $\mathcal{L}$-structure $\boldsymbol{M}$ with underlying set M, we say that a set $D \subseteq M_x$ is **definable** if it is "definable with parameters", i.e., there is an $\mathcal{L}$-formula $\varphi(x,y)$ and tuple $b \in M_x$ such that

$$D \;=\; \big\{a \in M_x : \boldsymbol{M} \models \varphi(a,b)\big\}.$$

Given $A \subseteq M$, a subset of M_x is **definable over** A if the parameter b above can be taken from A_x. A function $X \to M_y$ ($X \subseteq M_x$) is definable (over A) if its graph is definable (over A).

In this section (Γ,ψ) *is a model of* $T_{AC,\log}$.

Definition 4.3.1. For $k < 0$ and $x \in \Psi$, we set $s^k(x) := p^{-k}(x) \in \Psi_\infty$. Also, $s^0(x) := x$ for all $x \in \Psi$. A function $F : \Psi \to \Gamma_\infty$ is an s**-function** if it is constant, or there are $n \geqslant 1, k_1 < \cdots < k_n$ in $\mathbb{Z}$, $q_1,\ldots,q_n \in \mathbb{Q}^{\neq}$ and $\beta \in \Gamma$ such that $F(x) = \sum_{j=1}^n q_j s^{k_j}(x) - \beta$ for all $x \in \Psi$.

For an s-function $F(x) = \sum_{j=1}^n q_j s^{k_j}(x) - \beta$ as above with $n \geqslant 1$, define the set $D_F \subseteq \Psi$ to be

$$D_F \;=\; \begin{cases} [s^{-k_1+1}0, \infty)_\Psi & \text{if } k_1 < 0 \\ \Psi & \text{if } k_1 \geqslant 0 \end{cases}$$

and the set $I_F \subseteq \Psi$ to be $\Psi \setminus D_F$.

Note that $I_F < D_F$, and $I_F \cup D_F = \Psi$. Furthermore, F takes the constant value ∞ on I_F and takes only values in Γ on D_F. It is also useful to observe that for $x \in D_F$ and $l \in \mathbb{Z}$, if $l \geqslant k_1$, then $s^l(x) \in \Psi$.

By convention, if we refer to an s-function $F(x) = \sum_{j=1}^n q_j s^{k_j}(x) - \beta$, it is understood that $n \geqslant 1$, $k_1 < \cdots < k_n$ in $\mathbb{Z}$, $q_1,\ldots,q_n \in \mathbb{Q}^{\neq}$, and $\beta \in \Gamma$.

In general, the s-functions are rather well-behaved. To begin with, we get the following:

Lemma 4.3.2. *Let* $F : \Psi \to \Gamma_\infty$ *be the s-function* $F(x) = \sum_{j=1}^n q_j s^{k_j}(x) - \beta$. *If* $q_1 > 0$, *then* F *is strictly increasing on* D_F, *otherwise* F *is strictly decreasing on* D_F. *In particular, the restriction of* F *to* D_F *is injective. Furthermore, if* F *changes sign on* D_F, *then there is* $\alpha \in D_F$ *such that* $\operatorname{sign}\big(F(\alpha)\big) \neq \operatorname{sign}\big(F(s\alpha)\big)$.

Proof. Let $\alpha_0, \alpha_1 \in D_F$ be such that $\alpha_0 < \alpha_1$. Then

$$F(\alpha_1) - F(\alpha_0) \;=\; q_1(s^{k_1}\alpha_1 - s^{k_1}\alpha_0) + q_2(s^{k_2}\alpha_1 - s^{k_2}\alpha_0) + \cdots + q_n(s^{k_n}\alpha_1 - s^{k_n}\alpha_0).$$

By Lemma 2.3.5, we compute $\psi(s^{k_j}\alpha_1 - s^{k_j}\alpha_0) = s^{k_j+1}\alpha_0$ for $j = 1, \ldots, n$, and thus

$$[s^{k_1}\alpha_1 - s^{k_1}\alpha_0] \;>\; [s^{k_2}\alpha_1 - s^{k_2}\alpha_0] \;>\; \cdots \;>\; [s^{k_n}\alpha_1 - s^{k_n}\alpha_0].$$

Since $s^{k_1}\alpha_1 > s^{k_1}\alpha_0$, we get that

$$\operatorname{sign}\big(F(\alpha_1) - F(\alpha_0)\big) \;=\; \operatorname{sign}(q_1).$$

The second statement follows from an appeal to completeness of $T_{\log}$ and the observation that it is obviously true in $(\Gamma_{\log}^{\mathbb{Q}}, \psi)$. $\square$

The following theorem is one of the main results of this section. It says that all definable functions $\Psi \to \Gamma_\infty$ are given piecewise by s-functions.

Theorem 4.3.3. *Let $F : \Psi \to \Gamma_\infty$ be a definable function. Then there is an increasing sequence $s0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \alpha_n = \infty$ in Ψ_∞ such that for $k = 0, \ldots, n-1$, the restriction of F to $[\alpha_k, \alpha_{k+1})_\Psi$ is given by an s-function.*

We first prove that for an s-function $F(x)$, the compositions $\psi\big(F(x)\big)$, $s\big(F(x)\big)$ are given piecewise by s-functions.

The following lemma is a step in this direction:

Lemma 4.3.4. *Let $n \geqslant 1$, $q_1, \ldots, q_n \in \mathbb{Q}^{\neq}$, $\alpha_1, \ldots, \alpha_n \in \Psi$, $\alpha_1 < \cdots < \alpha_n$, and $\alpha = \sum_{j=1}^n q_j\alpha_j$. Then*
(1) $\sum_{j=1}^n q_j = 0 \Longrightarrow \psi(\alpha) = s(\alpha_1)$,
(2) $\sum_{j=1}^n q_j \neq 0 \Longrightarrow \psi(\alpha) = s0$.

Proof. By completeness of $T_{AC,\log}$, the lemma will follow from its validity for the case $(\Gamma, \psi) = (\Gamma_{\log}^{\mathbb{Q}}, \psi)$. In $(\Gamma_{\log}^{\mathbb{Q}}, \psi)$, we may take integers $0 \leqslant m_1 < \cdots < m_n$ such that $\alpha_j = \sum_{i=0}^{m_j} e_i$, for $j = 1, \ldots, n$. Then

$$\alpha \;=\; \sum_{j=1}^{n} q_j \left(\sum_{i=0}^{m_j} e_i\right) \;=\; \sum_{i=0}^{m_1}\left(\sum_{j=1}^{n} q_j\right) e_i + \sum_{i=m_1+1}^{m_2}\left(\sum_{j=2}^{n} q_j\right) e_i + \cdots + \sum_{i=m_{n-1}+1}^{m_n} q_n e_i,$$

i.e., as an infinite tuple, α has the form:

$$\alpha \;=\; (\underbrace{\textstyle\sum_{j=1}^n q_j, \ldots, \sum_{j=1}^n q_j}_{m_1+1}, \underbrace{\textstyle\sum_{j=2}^n q_j, \ldots, \sum_{j=2}^n q_j}_{m_2-m_1}, \ldots).$$

From this it is clear that if $\sum_{j=1}^n q_j \neq 0$, then $\psi(\alpha) = e_0 = s0$. Otherwise, if $\sum_{j=1}^n q_j = 0$, then $q_1 = -\sum_{j=2}^n q_j \neq 0$, and so

$$\psi(\alpha) \;=\; \sum_{i=1}^{m_1+1} e_i \;=\; \alpha_1 + e_{m_1+1} \;=\; s(\alpha_1). \qquad \square$$

The Fixed Point Identity (Lemma 2.3.8) which relates ψ and s immediately gives us an s-analogue of Lemma 4.3.4.

Corollary 4.3.5. *Let $n \geqslant 1$, $q_1, \ldots, q_n \in \mathbb{Q}^{\neq}$, $\alpha_1, \ldots, \alpha_n \in \Psi$, $\alpha_1 < \cdots < \alpha_n$, and $\alpha = \sum_{j=1}^n q_j\alpha_j$. Then*

(1) $\sum_{j=1}^{n} q_j \neq 1 \Longrightarrow s(\alpha) = s0$,
(2) $\sum_{j=1}^{n} q_j = 1 \Longrightarrow s(\alpha) = s(\alpha_1)$.

Proof. Suppose that $\sum_{j=1}^{n} q_j \neq 1$. Then $\sum_{j=1}^{n} q_j - 1 \neq 0$, so by Lemma 4.3.4, $\psi(\alpha - s0) = s0$. Thus $s\alpha = s0$ by Lemma 2.3.8.

Next, suppose that $\sum_{j=1}^{n} q_j = 1$. Then $\sum_{j=1}^{n} q_j - 1 = 0$, so by Lemma 4.3.4, $\psi(\alpha - s\alpha_1) = s\alpha_1$. Thus $s\alpha = s\alpha_1$ by Lemma 2.3.8. □

In Theorem 4.3.6 below we give an explicit description of how compositions $\psi\big(F(x)\big)$ behave in all possible cases.

Theorem 4.3.6. *Let $F : \Psi \to \Gamma_\infty$ be the s-function $F(x) = \sum_{j=1}^{n} q_j s^{k_j}(x) - \beta$. Define the function $G : \Psi \to \Gamma_\infty$ by $G(x) = \psi\big(F(x)\big)$. If $x \in I_F$, then $G(x) = \infty$. Otherwise, if $x \in D_F$, then the values $G(x)$ are given in the following table (with $q := \sum_{j=1}^{n} q_j$):*

β	$G(x) = \psi\big(\sum_{j=1}^{n} q_j s^{k_j}(x) - \beta\big)$ *(assuming $x \in D_F$)*
if $\psi(\beta) > s0$	$G(x) = \begin{cases} s0 & \text{if } q \neq 0 \\ s^{k_1+1}(x) & \text{if } q = 0 \text{ and } s^{k_1+1}(x) < \psi(\beta) \\ G\big(s^{-k_1-1}(\psi(\beta))\big) & \text{if } q = 0 \text{ and } s^{k_1+1}(x) = \psi(\beta) \\ \psi(\beta) & \text{if } q = 0 \text{ and } s^{k_1+1}(x) > \psi(\beta) \end{cases}$
if $\psi(\beta) = s0$	$G(x) = \begin{cases} G(s^{-k_1}s0) & \text{if } s^{k_1}(x) = s0 \\ s0 & \text{if } s^{k_1}(x) > s0 \text{ and } q = 0 \\ s^{k_1+1}(x) & \text{if } s^{k_1}(x) > s0,\ q \neq 0, \\ & \quad \text{and } s^{k_1+1}(x) < s(q^{-1}\beta) \\ G\big(s^{-k_1-1}s(q^{-1}\beta)\big) & \text{if } s^{k_1}(x) > s0,\ q \neq 0, \\ & \quad \text{and } s^{k_1+1}(x) = s(q^{-1}\beta) \\ s(q^{-1}\beta) & \text{if } s^{k_1}(x) > s0,\ q \neq 0, \\ & \quad \text{and } s^{k_1+1}(x) > s(q^{-1}\beta) \end{cases}$

Proof. In the third, fifth and eighth cases in the table the computation is immediate since we are able to solve for x in terms of β. For example, in the third case the assumption $s^{k_1+1}(x) = \psi(\beta)$ gives $x = s^{-k_1-1}\big(\psi(\beta)\big)$ and so the function takes the value $G\big(s^{-k_1-1}(\psi(\beta))\big)$.

Otherwise, the idea is to do a computation of the form $\psi(\alpha - \beta)$ where $\alpha = \sum_{j=1}^{n} q_j s^{k_j}(x)$. In the first, second, fourth and sixth cases, we can compute the ψ-value of α by Lemma 4.3.4 and the assumptions are such that the ψ-values of α and β will be different so the ψ-value of their difference is immediate from Fact 2.2.1.

For the seventh and ninth case, we have to compute

$$\psi\big(\underbrace{q_1 s^{k_1}(x) + \cdots + q_n s^{k_n}(x)}_{\alpha} - \beta\big)$$

where by assumption $\psi(\beta) = \psi(\alpha) = s0$ since $q \neq 0$. Using (AC2), we can pivot to a situation where we can use Lemma 2.3.5 and Corollary 4.3.5 to do the computation. I.e., by dividing by q we reduce to computing

$$\psi\big(\underbrace{q^{-1}(q_1 s^{k_1}(x) + \cdots + q_n s^{k_n}(x))}_{q^{-1}\alpha} - q^{-1}\beta\big).$$

By Corollary 4.3.5, we know that $s(q^{-1}\alpha) = s^{k_1+1}(x)$. Our assumptions in cases seven and nine say precisely that the s-values of $q^{-1}\alpha$ and $q^{-1}\beta$ are different. From that point, it suffices to just use Lemma 2.3.5. $\square$

Corollary 2.3.6 allows us to easily transform Theorem 4.3.6 into an s-analogue. In the proof of Corollary 4.3.7 below we perform this transformation.

Corollary 4.3.7. *Let $F : \Psi \to \Gamma_\infty$ be the s-function $F(x) = \sum_{j=1}^n q_j s^{k_j}(x) - \beta$. Define the function $G : \Psi \to \Gamma_\infty$ by $G(x) = s\big(F(x)\big)$. If $x \in I_F$, then $G(x) = \infty$. Otherwise, if $x \in D_F$, then the values $G(x)$ are given in the following table:*

β	$G(x) = s\big(\sum_{j=1}^n q_j s^{k_j}(x) - \beta\big)$ *(assuming $x \in D_F$)*
if $s(-\beta) > s0$	$G(x) = \begin{cases} s0 & \text{if } q \neq 0 \\ s^{k_1+1}(x) & \text{if } q = 0 \text{ and } s^{k_1+1}(x) < s(-\beta) \\ G\big(s^{-k_1-1}(s(-\beta))\big) & \text{if } q = 0 \text{ and } s^{k_1+1}(x) = s(-\beta) \\ s(-\beta) & \text{if } q = 0 \text{ and } s^{k_1+1}(x) > s(-\beta) \end{cases}$
if $s(-\beta) = s0$	$G(x) = \begin{cases} G(s^{-k_1}s0) & \text{if } s^{k_1}(x) = s0 \\ s0 & \text{if } s^{k_1}(x) > s0 \text{ and } q = 0 \\ s^{k_1+1}(x) & \text{if } s^{k_1}(x) > s0,\ q \neq 0 \text{ and } s^{k_1+1}(x) < \gamma_0 \\ G(s^{-k_1-1}\gamma_0) & \text{if } s^{k_1}(x) > s0,\ q \neq 0 \text{ and } s^{k_1+1}(x) = \gamma_0 \\ \gamma_0 & \text{if } s^{k_1}(x) > s0,\ q \neq 0 \text{ and } s^{k_1+1}(x) > \gamma_0 \end{cases}$

where $q := \sum_{j=1}^n q_j$ and for $q \neq 0$,

$$\gamma_0 := \begin{cases} s0 & \text{if } \beta = 0, q \neq 1 \\ \infty & \text{if } \beta = 0, q = 1 \\ s\big((1-q)^{-1}\beta\big) & \text{if } \beta \neq 0, q \neq 1 \\ \psi(\beta) & \text{if } \beta \neq 0, q = 1. \end{cases}$$

Proof. It is clear that if $x \in I_F$, then $s^{k_1}(x) = \infty$ and so $G(x) = \infty$ as a result. Thus from now on we will assume that $x \in D_F$ and we think of D_F as a fixed subset of the Ψ-set of Γ. Next we will take an elementary extension (Γ^*, ψ) of (Γ, ψ) with an element $\gamma \in \Psi_{\Gamma^*}$ such that $\gamma > \Psi$. Now, if we take the table from Theorem 4.3.6, but we replace β with $\beta + \gamma$ and have x range over $D_F \subseteq \Gamma$, then we get the following table, computed in (Γ^*, ψ):

β	$G(x) = \psi\big(\sum_{j=1}^{n} q_j s^{k_j}(x) - \beta - \gamma\big)$ (assuming $x \in D_F \subseteq \Gamma$)
if $\psi(\beta+\gamma) > s0$	$G(x) = \begin{cases} s0 & \text{if } q \neq 0 \\ s^{k_1+1}(x) & \text{if } q = 0 \text{ and } s^{k_1+1}(x) < \psi(\beta+\gamma) \\ G\big(s^{-k_1-1}(\psi(\beta+\gamma))\big) & \text{if } q = 0 \text{ and } s^{k_1+1}(x) = \psi(\beta+\gamma) \\ \psi(\beta+\gamma) & \text{if } q = 0 \text{ and } s^{k_1+1}(x) > \psi(\beta+\gamma) \end{cases}$
if $\psi(\beta+\gamma) = s0$	$G(x) = \begin{cases} G(s^{-k_1}s0) & \text{if } s^{k_1}(x) = s0 \\ s0 & \text{if } s^{k_1}(x) > s0 \text{ and } q = 0 \\ s^{k_1+1}(x) & \text{if } s^{k_1}(x) > s0,\ q \neq 0, \\ & \quad \text{and } s^{k_1+1}(x) < s\big(q^{-1}(\beta+\gamma)\big) \\ G\big(s^{-k_1-1}s(q^{-1}(\beta+\gamma))\big) & \text{if } s^{k_1}(x) > s0,\ q \neq 0, \\ & \quad \text{and } s^{k_1+1}(x) = s\big(q^{-1}(\beta+\gamma)\big) \\ s\big(q^{-1}(\beta+\gamma)\big) & \text{if } s^{k_1}(x) > s0,\ q \neq 0, \\ & \quad \text{and } s^{k_1+1}(x) > s\big(q^{-1}(\beta+\gamma)\big) \end{cases}$

Since we are assuming that $x \in D_F \subseteq \Gamma$, we can apply Corollary 2.3.6 to replace $\psi\big(\sum_{j=1}^{n} q_j s^{k_j}(x) - \beta - \gamma\big)$ with $s\big(\sum_{j=1}^{n} q_j s^{k_j}(x) - \beta\big)$ and also $\psi(\beta+\gamma) = \psi(-\beta-\gamma)$ with $s(-\beta)$. Finally, we set $\gamma_0 := s\big(q^{-1}(\beta+\gamma)\big)$ when $q \neq 0$. This gives us the desired table:

β	$G(x) = s\big(\sum_{j=1}^{n} q_j s^{k_j}(x) - \beta\big)$ (assuming $x \in D_F \subseteq \Gamma$)
if $s(-\beta) > s0$	$G(x) = \begin{cases} s0 & \text{if } q \neq 0 \\ s^{k_1+1}(x) & \text{if } q = 0 \text{ and } s^{k_1+1}(x) < s(-\beta) \\ G\big(s^{-k_1-1}(s(-\beta))\big) & \text{if } q = 0 \text{ and } s^{k_1+1}(x) = s(-\beta) \\ s(-\beta) & \text{if } q = 0 \text{ and } s^{k_1+1}(x) > s(-\beta) \end{cases}$
if $s(-\beta) = s0$	$G(x) = \begin{cases} G(s^{-k_1}s0) & \text{if } s^{k_1}(x) = s0 \\ s0 & \text{if } s^{k_1}(x) > s0 \text{ and } q = 0 \\ s^{k_1+1}(x) & \text{if } s^{k_1}(x) > s0,\ q \neq 0, \text{ and } s^{k_1+1}(x) < \gamma_0 \\ G(s^{-k_1-1}\gamma_0) & \text{if } s^{k_1}(x) > s0,\ q \neq 0, \text{ and } s^{k_1+1}(x) = \gamma_0 \\ \gamma_0 & \text{if } s^{k_1}(x) > s0,\ q \neq 0, \text{ and } s^{k_1+1}(x) > \gamma_0 \end{cases}$

However we are not done yet; currently γ_0 is still an external parameter. We will show (or arrange) that $\gamma_0 \in \Psi_\infty$ (and give an explicit formula for it), which will then yield the corollary. First, we assume that $\beta = 0$. If $q \neq 1$, then $\gamma_0 = s(q^{-1}\gamma) = s0$ by Corollary 4.3.5. If $q = 1$, then $\gamma_0 = s(\gamma) > \Psi$ and so $\gamma_0 \notin \Gamma$. However, in this case, $s^{k_1+1}(x) \star \gamma_0$ iff $s^{k_1+1}(x) \star \infty$ for $\star \in \{<, =, >\}$ because $s^{k_1+1}(x) \in \Psi$ and both $s(\gamma)$ and ∞ are $> \Psi$. Thus we redefine $\gamma_0 := \infty$ if $\beta = 0$ and $q = 1$. Now we assume that $\beta \neq 0$ and we take yet another elementary extension (Γ^{**}, ψ) of (Γ^*, ψ) with an element $\widetilde{\gamma} \in \Psi_{\Gamma^{**}}$ such that $\widetilde{\gamma} > \Psi_{\Gamma^*}$. If $q = 1$, then we have

$$\gamma_0 \;=\; s(\beta+\gamma) \;=\; \psi(\beta+\gamma-\widetilde{\gamma}) \;=\; \psi(\beta)$$

by Fact 2.2.1 and Lemma 2.3.5 because $\psi(\gamma - \widetilde{\gamma}) = s\gamma > \psi(\beta) \in \Psi$. Otherwise, assume that $q \neq 1$. Then we can multiply on the inside by $(q^{-1} - 1)^{-1}$ to compute

$$\begin{aligned} \gamma_0 \ &= \ s\big(q^{-1}(\beta+\gamma)\big) \ = \ \psi\big(q^{-1}(\beta+\gamma) - \widetilde{\gamma}\big) \ = \ \psi(q^{-1}\beta + q^{-1}\gamma - \widetilde{\gamma}) \\ &= \ \psi\left(\frac{1}{1-q}\beta + \frac{q}{1-q}(q^{-1}\gamma - \widetilde{\gamma})\right). \end{aligned}$$

Next note that $s\big((1-q)^{-1}\beta\big) \in \Psi$ whereas $s\big(q(1-q)^{-1}(q^{-1}\gamma - \widetilde{\gamma})\big) = s\gamma > \Psi$ by Corollary 4.3.5. Thus $\gamma_0 = s\big((1-q)^{-1}\beta\big)$ by Lemma 2.3.5. □

Theorem 4.3.6 and Corollary 4.3.7 are the heart of the proof of Theorem 4.3.3. To round things out, we need to make a few more minor observations before proceeding with our proof of Theorem 4.3.3.

Lemma 4.3.8. *Ψ is a linearly independent subset of Γ as a vector space over $\mathbb{Q}$.*

PROOF. Let $n \geqslant 1$, $q_1, \ldots, q_n \in \mathbb{Q}^{\neq}$, $\alpha_1, \ldots, \alpha_n \in \Psi$, $\alpha_1 < \cdots < \alpha_n$, and $\alpha = \sum_{j=1}^{n} q_j\alpha_j$. By Lemma 4.3.4, either $\psi(\alpha) = s0$, or $\psi(\alpha) = s\alpha_1$, and so $\alpha \neq 0$. □

Lemma 4.3.9 describes the values an s-function can take in the set Ψ:

Lemma 4.3.9. *Let an s-function $F(x) = \sum_{j=1}^{n} q_j s^{k_j}(x) - \beta$ be given and let F^* be its restriction to D_F. Then exactly one of the following is true:*

(1) $\operatorname{image} F^* \subseteq \Psi$, $\beta = 0$, $n = 1$, *and* $q_1 = 1$,
(2) $\big|(\operatorname{image} F^*) \cap \Psi\big| = 2$,
(3) $\big|(\operatorname{image} F^*) \cap \Psi\big| = 1$,
(4) $\big|(\operatorname{image} F^*) \cap \Psi\big| = 0$.

PROOF. If $\beta \notin \operatorname{span}_{\mathbb{Q}} \Psi \subseteq \Gamma$, then $(\operatorname{image} F^*) \cap \Psi = \emptyset$. Thus assume for the rest of the proof that

$$\beta \ = \ q_1'\alpha_1 + \cdots + q_m'\alpha_m$$

where $q_1', \ldots, q_m' \in \mathbb{Q}^{\neq}$, $\alpha_1, \ldots, \alpha_m \in \Psi$ and $\alpha_1 < \cdots < \alpha_m$.

The idea is that we are interested in which values of x will put the expression

$$\underbrace{q_1 s^{k_1}(x) + \cdots + q_n s^{k_n}(x)}_{\alpha(x)} - \underbrace{(q_1'\alpha_1 + \cdots + q_m'\alpha_m)}_{\beta}$$

into the set Ψ. By the $\mathbb{Q}$-linear independence of Ψ, it is necessary that nearly all of the components of $\alpha(x)$ and β will cancel. We will do this by a case distinction.

If $m > n+1$ or $m < n-1$, then for all $x \in D_F$, the value of $F^*(x)$ is a linear combination of two or more elements of Ψ with nonzero coefficients so $(\operatorname{image} F^*) \cap \Psi = \emptyset$. Thus further assume that $n-1 \leqslant m \leqslant n+1$. If $m = 0$ (so $\beta = 0$), then $F^*(x) \in \Psi$ iff $n = 1$ and $q_1 = 1$, by the linear independence of Ψ. So further assume that $m > 0$. Now we look at three subcases:

Case 1: $m = n-1$ *and* $m > 0$. In this case we can expand out $F^*(x)$ as follows:

$$F^*(x) \ = \ q_1 s^{k_1}(x) + \cdots + q_n s^{k_n}(x) - q_1'\alpha_1 - \cdots - q_{n-1}'\alpha_{n-1}.$$

In order for $F^*(x)$ above to be an element of Ψ, it is necessary that either $s^{k_1}(x) = \alpha_1$ or $s^{k_n}(x) = \alpha_{n-1}$, otherwise the value of $F(x)$ is a linear combination of two or more elements of Ψ. Thus $\big|(\operatorname{image} F^*) \cap \Psi\big| \leqslant 2$ in this case.

Case 2: $m = n+1$ *and* $m > 0$. This case is similar to Case 1 and $\big|(\operatorname{image} F^*) \cap \Psi\big| \leqslant 2$.

Case 3: $m = n$. We can expand $F^*(x)$ as follows:

$$F^*(x) \;=\; q_1 s^{k_1}(x) + \cdots + q_n s^{k_n}(x) - q_1'\alpha_1 - \cdots - q_n'\alpha_n.$$

In order for $F^*(x) \in \Psi$, it is necessary that $s^{k_j}(x) = \alpha_j$ for $j = 1, \ldots, n$. Otherwise the value of $F^*(x)$ is a linear combination of two or more elements of Ψ. Thus $\big|(\operatorname{image} F^*) \cap \Psi\big| \leqslant 1$ in this case. $\square$

Lemma 4.3.10. *Let $t(x) : \Gamma_\infty \to \Gamma_\infty$ be an $\mathcal{L}_{AC,\log}$-term and let $F : \Psi \to \Gamma_\infty$ be the restriction $t|\Psi$ of t to Ψ. Then there is an increasing sequence $s0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \alpha_n = \infty$ in Ψ_∞ such that for $k = 0, \ldots, n-1$, the restriction of F to $[\alpha_k, \alpha_{k+1})_\Psi$ is given by an s-function.*

Proof. We do this by induction on the complexity of the $\mathcal{L}_{AC,\log}$-terms.

Easy Cases: By definition the constant term β for $\beta \in \Gamma_\infty$ is an s-function, and it is clear that the set of s-functions is closed under $+, -$ and δ_n for $n \geqslant 1$.

ψ **Case:** Let $F(x) = \sum_{j=1}^n q_j s^{k_j}(x) - \beta$ be an s-function. Then we can determine the value of $\psi\big(F(x)\big)$ from Theorem 4.3.6. Note that whenever the expression $s^l(x) < \delta$ is not vacuous in the table of Theorem 4.3.6, then it is equivalent to $x < s^{-l}\delta$ (similarly for $=$ and $>$).

s **Case:** This is similar to the ψ case, except we use Corollary 4.3.7.

p **Case:** Let $F(x) = \sum_{j=1}^n q_j s^{k_j}(x) - \beta$ be an s-function. By Lemma 4.3.9, if $\beta = 0, n = 1, q_1 = 1$, then F^* is of the form $s^k(x)$ and so

$$p\big(F(x)\big) \;=\; \begin{cases} \infty & \text{if } x \in I_F \\ \infty & \text{if } x = \min D_F \text{ and } k \leqslant 0 \\ s^{k-1}(x) & \text{if } x > \min D_F \text{ or } k > 0. \end{cases}$$

Otherwise, $F(x) \in \Psi$ for $0, 1$ or 2 values of x, so $p\big(F(x)\big) = \infty$ for all $x \in \Psi$ with at most 0, 1 or 2 exceptions. $\square$

We say that a set $I \subseteq \Psi$ is an **interval in** Ψ if there are $\alpha, \beta \in \Psi_\infty$ with $\alpha < \beta$ such that $I = [\alpha, \beta)_\Psi$. The following is immediate from Theorem 4.2.2, and Lemmas 4.3.2 and 4.3.10:

Corollary 4.3.11. *Every definable $A \subseteq \Psi$ is a finite union of intervals in Ψ and singletons.*

Proof of Theorem 4.3.3. It follows from quantifier elimination and the fact that $T_{AC,\log}$ has a universal axiomatization that there are $\mathcal{L}_{AC,\log}$-terms $t_1(x), \ldots, t_n(x)$ such that on Ψ we have $F(x) = t_k(x)$, for some $i \in \{1, \ldots, k\}$. By Corollary 4.3.11, the set

$$D_i \;:=\; \big\{x \in \Psi : F(x) = t_k(x)\big\} \subseteq \Psi$$

is a finite union of intervals and singletons. Furthermore, by Lemma 4.3.10, the restriction of $F(x)$ to D_i is given piecewise by s-functions in the desired way. $\square$

Corollary 4.3.12 (Characterization of definable functions $\Psi \to \Psi$)**.** *Let $F : \Psi \to \Psi$ be definable in (Γ, ψ). Then there is an increasing sequence $s0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \alpha_n = \infty$ in Ψ_∞ such that for $k = 1, \ldots, n$, the restriction of F to $[\alpha_{k-1}, \alpha_k)_\Psi$ is either constant or of the form $x \mapsto s^l(x)$ for some $l \in \mathbb{Z}$.*

Proof. This follows from Theorem 4.3.3 and Lemma 4.3.9. $\square$

It is clear from Corollary 4.3.11 that each nonempty definable $A \subseteq \Psi$ has a least element. This gives us definable Skolem functions for definable subsets of Ψ^n (see, for example, [**40**, p. 94]).

The following Theorem 4.3.13 follows immediately from Corollary 4.3.11 and the main result of [**31**]. For the reader's convenience we supply a more direct and self-contained proof. It is a variant of [**39**, Lemma 4.7], which itself is a variant of [**18**, Lemma 1].

Theorem 4.3.13. *Let $n \geqslant 1$ and suppose that $f : \Psi^n \to \Psi$ is a definable function. Then f is definable in the structure $(\Psi; <)$.*

Proof. We can arrange that (Γ, ψ) is $\aleph_0$-saturated.

The case $n = 1$ follows from Corollary 4.3.12.

Let $n > 1$. Let A be the finite set of parameters from Γ used to define f. For each $a \in \Psi$ we can define the function $f_a : x \mapsto f(a, x) : \Psi^{n-1} \to \Psi$. By induction, f_a is definable in the structure $(\Psi; <)$, so we have $c_a \in \Psi^{N_a}$ and a set $\Phi_a \subseteq \Psi^{N_a+(n-1)+1}$ definable in $(\Psi; <)$ such that $\Phi_a(c_a) = \operatorname{graph}(f_a)$. We can arrange that Φ_a is the graph of a function $F_a : \Psi^{N_a+(n-1)} \to \Psi$ such that $F_a(c_a, x) = f_a(x)$ for all $x \in \Psi$. Next let $\Delta_a \subseteq \Psi$ be the A-definable set of all $b \in \Psi$ such that the function $f_b : \Psi^{n-1} \to \Psi$ occurs as a section of F_a. Note that $a \in \Delta_a$ since $F_a(c_a, x) = f(a, x)$. Thus

$$\Psi \;=\; \bigcup_{a \in \Psi} \Delta_a$$

By saturation there are $a_1, \ldots, a_k \in \Psi$ such that:

$$\Psi \;=\; \bigcup_{j=1}^{k} \Delta_j$$

where $\Delta_j := \Delta_{a_j}$ for $j = 1, \ldots, k$. Let $F_j := F_{a_j}$, $\Phi_j := \Phi_{a_j}$, $c_j := c_{a_j}$ and $N_j := N_{a_j}$ for $j = 1, \ldots, k$ and let $N = \max_{1 \leqslant j \leqslant k} N_j$. Extend each function $F_j : \Psi^{N_j+(n-1)} \to \Psi$ to a function $F'_j : \Psi^{N+(n-1)} \to \Psi$ by setting

$$F'_j(w_1, \ldots, w_N, x) \;:=\; F_j(w_1, \ldots, w_{N_j}, x) \text{ for all } (w_1, \ldots, w_N, x) \in \Psi^{N+(n-1)}$$

so the last $N - N_j$ variables before x are dummy variables. Next define a function $F : \Psi^{1+N+(n-1)} \to \Psi$ by

$$F(v, w_1, \ldots, w_N, x) \;=\; \begin{cases} F'_1(w_1, \ldots, w_N, x) & \text{if } v = s0 \\ F'_2(w_1, \ldots, w_N, x) & \text{if } v = s^2 0 \\ & \vdots \\ F'_{k-1}(w_1, \ldots, w_N, x) & \text{if } v = s^{k-1}0 \\ F'_k(w_1, \ldots, w_N, x) & \text{if } v \geqslant s^k 0 \end{cases}$$

Finally, we note the following:

$$\begin{aligned}
\Psi = \bigcup_{j=1}^{n} \Delta_j \quad &\Rightarrow \quad \text{for every } a \in \Psi \text{ there is } j \in \{1,\dots,k\} \text{ such that } a \in \Delta_j \\
&\Rightarrow \quad \text{for every } a \in \Psi \text{ there is } j \in \{1,\dots,k\} \text{ and } c \in \Psi^{N_j} \\
&\qquad\qquad \text{such that } f(a,x) = F_j(c,x) \text{ for every } x \in \Psi \\
&\Rightarrow \quad \text{for every } a \in \Psi \text{ there is } j \in \{1,\dots,k\} \text{ and } c \in \Psi^{N} \\
&\qquad\qquad \text{such that } f(a,x) = F_j'(c,x) \text{ for every } x \in \Psi \\
&\Rightarrow \quad \text{for every } a \in \Psi \text{ there is } v \in \Psi \text{ and } c \in \Psi^{N} \\
&\qquad\qquad \text{such that } f(a,x) = F(v,c,x) \text{ for every } x \in \Psi \\
&\Rightarrow \quad \text{for every } a \in \Psi \text{ there is } c \in \Psi^{1+N} \\
&\qquad\qquad \text{such that } f(a,x) = F(c,x) \text{ for every } x \in \Psi \\
&\Rightarrow \quad \forall a \in \Psi\ \exists c \in \Psi^{1+N}\ \forall x \in \Psi\ \big(f(a,x) = F(c,x)\big)
\end{aligned}$$

By definability of Skolem functions, there is a definable function $c = (c_0, \dots, c_N) : \Psi \to \Psi^{1+N}$ such that

$$\forall a \in \Psi\ \forall x \in \Psi\ \big(f(a,x) = F(c(a),x)\big)$$

From the base case of this lemma, $c_i : \Psi \to \Psi$ is definable in $(\Psi; <)$ for $i = 0, \dots, N$. Thus $f(z,x) : \Psi^n \to \Psi$ agrees with the function $F\big(c(z),x\big) : \Psi^n \to \Psi$, which is definable in $(\Psi; <)$. This concludes the proof of the induction step. $\square$

Corollary 4.3.14. *The subset Ψ of Γ is stably embedded in (Γ, ψ).*

4.4. Simple extensions and examples

In this section $\mathbb{M} = (\mathbb{M}, \psi, s, p, \dots)$ is a monster model of $T_{AC,\log}$. All other models considered will be small submodels of $\mathbb{M}$.

Simple extensions. In this subsection we consider an arbitrary small submodel $\Gamma = (\Gamma, \psi, s, p, \dots)$ of cardinality $\leqslant \kappa < \kappa(\mathbb{M})$. The element α will range over $\mathbb{M}$ and we will assume $\alpha \notin \Gamma$ to avoid some trivial cases. Note that the set $\Psi = \Psi_\Gamma$ will always contain the initial copy of $\mathbb{N}$ together with at most κ-many copies of $\mathbb{Z}$, whereas the set $\Psi_\mathbb{M} \setminus \Psi$ is the union of all copies of $\mathbb{Z}$ in $\Psi_\mathbb{M}$ that are not part of Ψ.

When considering simple extensions $\Gamma\langle\alpha\rangle$ of Γ (in the language $\mathcal{L}_{AC,\log}$), it is useful to know whether the ordered abelian group $\Gamma \oplus \mathbb{Q}\alpha$ is already closed under the primitives ψ and s. If it is not closed, then we want to know how badly $\Gamma \oplus \mathbb{Q}\alpha$ fails to be closed under ψ and s. This motivates defining

$$\mathbb{Q}^{\neq}\alpha - \Gamma \ := \ \{q\alpha - \gamma : q \in \mathbb{Q}^{\neq} \text{ and } \gamma \in \Gamma\}$$

as well as the following subsets of $\Psi_\mathbb{M}$:

$$\begin{aligned}
\psi(\mathbb{Q}^{\neq}\alpha - \Gamma) \ &:= \ \big\{\psi(q\alpha - \gamma) : q \in \mathbb{Q}^{\neq} \text{ and } \gamma \in \Gamma\big\} \\
s(\mathbb{Q}^{\neq}\alpha - \Gamma) \ &:= \ \big\{s(q\alpha - \gamma) : q \in \mathbb{Q}^{\neq} \text{ and } \gamma \in \Gamma\big\} \\
T_\Gamma(\alpha) \ &:= \ \psi(\mathbb{Q}^{\neq}\alpha - \Gamma) \cup s(\mathbb{Q}^{\neq}\alpha - \Gamma).
\end{aligned}$$

Note that $\psi(\mathbb{Q}^{\neq}\alpha - \Gamma) = \psi(\alpha - \Gamma) := \big\{\psi(\alpha - \gamma) : \gamma \in \Gamma\big\}$ by (AC2).

Since $T_\Gamma(\alpha)$ is defined using the primitives ψ and s, and $\alpha \notin \Gamma$, it is clear that $T_\Gamma(\alpha) \subseteq \Psi_{\mathbb{M}}$. If $T_\Gamma(\alpha) \subseteq \Psi = \Psi_\Gamma$, then the ordered abelian group $\Gamma \oplus \mathbb{Q}\alpha$ is already closed under the primitives ψ and s. However, if $T_\Gamma(\alpha) \setminus \Psi$ is nonempty, then $\Gamma \oplus \mathbb{Q}\alpha$ is not closed under ψ and s and then we are interested in the possibilities of the set $T_\Gamma(\alpha) \setminus \Psi$.

As we will show below in Corollary 4.4.4, the set $T_\Gamma(\alpha) \setminus \Psi$ is either empty, or contains a single element in $\Psi_{\mathbb{M}} \setminus \Psi$. At any rate, since $T_\Gamma(\alpha) \subseteq \Gamma\langle\alpha\rangle$, all elements of $T_\Gamma(\alpha) \setminus \Psi$ must get added to Γ in order to have any chance at closing off under s and ψ.

Remark 4.4.1. In fact, $T_\Gamma(\alpha) \setminus \Psi$ also measures the failure of $\Gamma \oplus \mathbb{Q}\alpha$ to be closed under p in the following way: if $p(q\alpha - \gamma) \in \Psi_{\mathbb{M}} \setminus \Psi$, then $q\alpha - \gamma \in \Psi_{\mathbb{M}} \setminus \Psi$ and in particular, $s(q\alpha - \gamma) \in \Psi_{\mathbb{M}} \setminus \Psi$. For such a $q\alpha - \gamma$, $p(q\alpha - \gamma)$ and $s(q\alpha - \gamma)$ are on the same copy of $\mathbb{Z}$ in $\Psi_{\mathbb{M}} \setminus \Psi$. Thus if $\Gamma \oplus \mathbb{Q}\alpha$ is not closed under p, then this failure is already detected by the fact that $\Gamma \oplus \mathbb{Q}\alpha$ is not closed under s.

In view of Corollary 2.3.6 which relates the functions ψ and s through a translation by an external parameter, it may come as no surprise that $\psi(\mathbb{Q}^{\neq}\alpha - \Gamma)$ and $s(\mathbb{Q}^{\neq}\alpha - \Gamma)$ are very similar as the following two lemmas show:

Lemma 4.4.2. *Let Δ be either $\psi(\mathbb{Q}^{\neq}\alpha - \Gamma)$ or $s(\mathbb{Q}^{\neq}\alpha - \Gamma)$. Then for $\beta_0 \in \mathbb{M}$, $\beta_1 \in \Delta$ such that $\beta_0 < \beta_1$, we have $\beta_0 \in \Psi$ iff $\beta_0 \in \Delta$. In particular, $\Delta \cap \Psi$ is a downward closed subset of Ψ and $\Delta \setminus \Psi$ consists of at most one element β; furthermore, such β realizes the cut $(\Delta \cap \Psi, \Psi \setminus \Delta)$ in Ψ.*

Proof. First, consider the case that $\Delta = \psi(\mathbb{Q}^{\neq}\alpha - \Gamma) = \psi(\alpha - \Gamma)$ and let $\beta_0 \in \mathbb{M}$ and $\beta_1 \in \Delta$ be arbitrary such that $\beta_0 < \beta_1$. Then $\beta_1 = \psi(\alpha - \gamma_1)$ for some $\gamma_1 \in \Gamma$. First suppose that $\beta_0 \in \Psi$. Then there is $\gamma_0 \in \Gamma$ such that $\beta_0 = \psi(\gamma_0) < \psi(\alpha - \gamma_1) = \beta_1$. Note that

$$\beta_0 \;=\; \psi(\gamma_0) \;=\; \psi\big(\gamma_0 - (\alpha - \gamma_1)\big) \;=\; \psi\big(\alpha - (\gamma_0 + \gamma_1)\big) \in \Delta.$$

Conversely, if $\beta_0 \in \Delta$, then $\beta_0 = \psi(\alpha - \gamma_0)$ for some $\gamma_0 \in \Gamma$. It then follows from $\beta_0 = \psi(\alpha - \gamma_0) < \psi(\alpha - \gamma_1) = \beta_1$ that

$$\beta_0 \;=\; \psi(\alpha - \gamma_0) \;=\; \psi\big((\alpha - \gamma_0) - (\alpha - \gamma_1)\big) \;=\; \psi(\gamma_1 - \gamma_0) \in \Psi.$$

Next, consider the case that $\Delta = s(\mathbb{Q}^{\neq}\alpha - \Gamma)$ and let $\beta_0 \in \mathbb{M}$ and $\beta_1 \in \Delta$ be arbitrary such that $\beta_0 < \beta_1$. Then $\beta_1 = s(q_1\alpha - \gamma_1)$ for some $q_1 \in \mathbb{Q}^{\neq}$ and $\gamma_1 \in \Gamma$. We will also take $\gamma^* \in \Psi_{\mathbb{M}}$ such that $\gamma^* > \Psi_{\Gamma\langle\alpha\rangle}$. First suppose that $\beta_0 \in \Psi$. Then $\beta_0 = \psi(\gamma_0)$ for some $\gamma_0 \in \Gamma$ and thus $\beta_0 = \psi(\gamma_0) < s(q_1\alpha - \gamma_1) = \beta_1$. Then by Corollary 2.3.6,

$$\begin{aligned}\beta_0 \;&=\; \psi(\gamma_0) \;=\; \min\big(s(q_1\alpha - \gamma_1), \psi(\gamma_0)\big) \;=\; \min\big(\psi(q_1\alpha - \gamma_1 - \gamma^*), \psi(\gamma_0)\big) \\ &=\; \psi(q_1\alpha - \gamma_1 - \gamma^* - \gamma_0) \;=\; s\big(q_1\alpha - (\gamma_1 + \gamma_0)\big) \in \Delta.\end{aligned}$$

Conversely, if $\beta_0 \in \Delta$, then $\beta_0 = s(q_0\alpha - \gamma_0)$ for some $q_0 \in \mathbb{Q}^{\neq}$ and $\gamma_0 \in \Gamma$. Then $\beta_0 = s(q_0\alpha - \gamma_0) < s(q_1\alpha - \gamma_1) = \beta_1$, and it follows that

$$\beta_0 \;=\; \psi(\alpha - q_0^{-1}\gamma_0 - q_0^{-1}\gamma^*) \;<\; \psi(\alpha - q_1^{-1}\gamma_1 - q_1^{-1}\gamma^*)$$

and so

$$\beta_0 \;=\; \psi\big(q_1^{-1}\gamma_1 - q_0^{-1}\gamma_0 + (q_1^{-1} - q_0^{-1})\gamma^*\big).$$

If $q_0 = q_1$, then $\beta_0 \in \Psi$. Otherwise,

$$\beta_0 \ = \ \psi\left(-\frac{q_1^{-1}}{q_1^{-1}-q_0^{-1}}\gamma_1 + \frac{q_0^{-1}}{q_1^{-1}-q_0^{-1}}\gamma_0 - \gamma^*\right) \ = \ s\left(-\frac{q_1^{-1}}{q_1^{-1}-q_0^{-1}}\gamma_1 + \frac{q_0^{-1}}{q_1^{-1}-q_0^{-1}}\gamma_0\right) \in \Psi. \qquad \square$$

Lemma 4.4.3. *$s(\mathbb{Q}^{\neq}\alpha - \Gamma) \cap \Psi = \psi(\mathbb{Q}^{\neq}\alpha - \Gamma) \cap \Psi$. Furthermore, $s(\mathbb{Q}^{\neq}\alpha - \Gamma) \triangle \psi(\mathbb{Q}^{\neq}\alpha - \Gamma)$ consists of at most one element.*

Proof. Suppose $\beta_0 \in s(\mathbb{Q}^{\neq}\alpha - \Gamma) \cap \Psi$. Let $q \in \mathbb{Q}^{\neq}$ and $\gamma_0 \in \Gamma$ be such that $\beta_0 = s(q\alpha - \gamma_0)$. Let $\gamma_1 \in \Gamma$ be such that $s(\gamma_1) > \beta_0 = s(q\alpha - \gamma_0)$. Then Lemma 2.3.5 gives

$$\psi\big(q\alpha - (\gamma_1 + \gamma_0)\big) \ = \ \psi\big(\gamma_1 - (q\alpha - \gamma_0)\big) \ = \ s(q\alpha - \gamma_0) = \beta_0$$

and so $\beta_0 \in \psi(\mathbb{Q}^{\neq}\alpha - \Gamma) \cap \Psi$.

Next we consider two cases. First suppose $s(\mathbb{Q}^{\neq}\alpha - \Gamma) \cap \Psi$ is cofinal in Ψ. Since it is also downward closed in Ψ, it is necessarily the case that $\Psi = s(\mathbb{Q}^{\neq}\alpha - \Gamma) \cap \Psi \subseteq \psi(\mathbb{Q}^{\neq}\alpha - \Gamma) \cap \Psi \subseteq \Psi$, so we get equality throughout.

Otherwise, by Lemma 4.4.2 we can take $\rho \in \Psi$ such that $s(\mathbb{Q}^{\neq}\alpha - \Gamma) < \rho$. Let $\gamma \in \Gamma$ be arbitrary such that $\psi(\alpha - \gamma) \in \psi(\mathbb{Q}^{\neq}\alpha - \Gamma) \cap \Psi$. Then by choice of $\rho \in \Psi$ we have

$$\underbrace{s(\alpha - \gamma + \rho)}_{\in s(\mathbb{Q}^{\neq}\alpha - \Gamma)} \ < \ \rho \ < \ s(\rho).$$

Thus by Lemma 2.3.5 we have

$$\psi(\alpha - \gamma) \ = \ \psi\big((\alpha - \gamma + \rho) - \rho\big) \ = \ s(\alpha - \gamma + \rho) \in s(\mathbb{Q}^{\neq}\alpha - \Gamma)$$

and we conclude that $\psi(\mathbb{Q}^{\neq}\alpha - \Gamma) \cap \Psi \subseteq s(\mathbb{Q}^{\neq}\alpha - \Gamma) \cap \Psi$.

Finally, suppose $s(\mathbb{Q}^{\neq}\alpha - \Gamma) \setminus \Psi = \{\beta\}$ where $\beta \in \Psi_{\mathbb{M}} \setminus \Psi$. Then we will show that $\psi(\mathbb{Q}^{\neq}\alpha - \Gamma) \subseteq \Psi \cup \{\beta\}$. Take $\gamma^* \in \Psi_{\mathbb{M}}$ such that $\gamma^* > \Psi_{\Gamma\langle\alpha\rangle}$ and take $q \in \mathbb{Q}^{\neq}$ and $\gamma \in \Gamma$ such that $\beta = s(q\alpha - \gamma) = \psi(q\alpha - \gamma - \gamma^*)$. Let $\delta \in \Gamma$ be arbitrary. Note that

$$\begin{aligned}
\psi(q\alpha - \gamma - \delta) \ &= \ \psi\big((q\alpha - \gamma - \gamma^*) - (\delta - \gamma^*)\big) \\
&\geqslant \ \min\big(\psi(q\alpha - \gamma - \gamma^*), \psi(\delta - \gamma^*)\big) \\
&= \ \min\big(s(q\alpha - \gamma), s(\delta)\big) \\
&= \ \min\big(\beta, s(\delta)\big).
\end{aligned}$$

But since $\beta \notin \Psi$ and $s(\delta) \in \Psi$, we actually get $\psi(q\alpha - \gamma - \delta) = \min\big(\beta, s(\delta)\big)$. Since $q \neq 0$ and as δ ranges over Γ, $\gamma + \delta$ will also range over Γ, together with (AC2) this argument shows that $\psi(\mathbb{Q}^{\neq}\alpha - \Gamma) \subseteq \Psi \cup \{\beta\}$. $\square$

It follows that $T_\Gamma(\alpha)$ occurs in only three different ways:

Corollary 4.4.4. *Exactly one of the following is true:*

(1) $T_\Gamma(\alpha) = [s0, \beta]_\Psi = \Psi^{\leqslant\beta} \subseteq \Psi$ *for some* $\beta \in \Psi$.

(2) $T_\Gamma(\alpha) = B$ *where* $B \subseteq \Psi$ *is nonempty, downward closed and is such that* $s(B) \subseteq B$ *(i.e.,* $\Psi \setminus B \in \operatorname{sded}^{op}(\Psi)$*).*

(3) $T_\Gamma(\alpha) = B \cup \{\beta\}$ *where* $B \subseteq \Psi$ *is nonempty, downward closed and is such that* $s(B) \subseteq B$ *and* $\beta \in \Psi_{\mathbb{M}} \setminus \Psi$ *with* $B < \beta < (\Psi \setminus B)$.

In particular, $\big|T_\Gamma(\alpha) \setminus \Psi\big| \leqslant 1$.

Note that if $T_{\Gamma}(\alpha) \subseteq \Psi$ for a particular Γ and $\alpha \in \mathbb{M}$, then $\Gamma \oplus \mathbb{Q}\alpha$ as an ordered abelian subgroup of $\mathbb{M}$ is closed under the functions ψ and s. In fact, it follows from Remark 4.4.1 that $\Gamma \oplus \mathbb{Q}\alpha$ is also closed under p. Thus $(\Gamma \oplus \mathbb{Q}\alpha, \psi, s, p, \ldots)$ is an $\mathcal{L}_{AC,\log}$-substructure of $\mathbb{M}$ which extends Γ and hence also is a model of $T_{AC,\log}$ since $T_{AC,\log}$ has a universal axiomatization. In this case, $\Gamma\langle\alpha\rangle = (\Gamma \oplus \mathbb{Q}\alpha, \psi, s, p, \ldots)$.

The following observation illustrates how the inductive step in Theorem 4.4.6 below will work:

Observation 4.4.5. Suppose that $\Gamma_0 \subseteq \Gamma_1 \subseteq \mathbb{M}$ are models of $T_{AC,\log}$ and that $\alpha \in \mathbb{M} \setminus \Gamma_1$. Then $T_{\Gamma_0}(\alpha) \subseteq T_{\Gamma_1}(\alpha)$. In particular, if $T_{\Gamma_0}(\alpha) = B_0 \cup \{\beta_0\}$ as in case (3) of Corollary 4.4.4 above, and if $\Gamma_1 = \Gamma_0\langle\beta_0\rangle = \Gamma_0 + \sum_n \mathbb{Q}s^n\beta_0 + \sum_n \mathbb{Q}p^n\beta_0$ also has the property that $T_{\Gamma_1}(\alpha) = B_1 \cup \{\beta_1\}$ as in case (3) of Corollary 4.4.4, then it must be the case that $\beta_0 \in B_1$ and thus $s^n\beta_0 < \beta_1$ for all n.

Theorem 4.4.6. *Let $\alpha \in \mathbb{M}$. Then $\Gamma\langle\alpha\rangle$ is isomorphic over Γ to one of the following:*

(1) *Γ_ρ for some increasing $\rho : n \to \mathrm{sded}^{op}(\Psi) \setminus \{\Psi\}$ and some n,*
(2) *$\Gamma_\rho \oplus \mathbb{Q}\alpha$ for some increasing $\rho : n \to \mathrm{sded}^{op}(\Psi) \setminus \{\Psi\}$ and some n,*
(3) *$\Gamma_\rho \oplus \mathbb{Q}\alpha$ for some increasing $\rho : \omega \to \mathrm{sded}^{op}(\Psi) \setminus \{\Psi\}$.*

Proof. We will recursively construct a sequence of extensions $\Gamma =: \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Gamma\langle\alpha\rangle$ of models of $T_{AC,\log}$ inside $\mathbb{M}$. This sequence will either be finite or have order type $\omega + 1$ and the last element of the sequence will be $\Gamma\langle\alpha\rangle$.

We will inductively assume that each Γ_n constructed so far is isomorphic to some Γ_ρ for some increasing $\rho : n \to \mathrm{sded}^{op}(\Psi) \setminus \{\Psi\}$. This is true for $n = 0$ since $\Gamma_0 = \Gamma = \Gamma_\rho$ for the empty increasing function $\rho : 0 \to \mathrm{sded}^{op}(\Psi) \setminus \{\Psi\}$. Given Γ_n for $n < \omega$, if $\alpha \in \Gamma_n$, then we are done, i.e., $\Gamma\langle\alpha\rangle = \Gamma_n$ and so $\Gamma\langle\alpha\rangle \cong \Gamma_\rho$ for some increasing $\rho : n \to \mathrm{sded}^{op}(\Psi) \setminus \{\Psi\}$. Otherwise, consider the set $T_{\Gamma_n}(\alpha)$. If $T_{\Gamma_n}(\alpha) \subseteq \Psi_{\Gamma_n}$, then we set $\Gamma_{n+1} := \Gamma_n \oplus \mathbb{Q}\alpha$ and we are done, i.e., $\Gamma\langle\alpha\rangle = \Gamma_{n+1} \cong \Gamma_\rho \oplus \mathbb{Q}\alpha$ for some increasing $\rho : n \to \mathrm{sded}^{op}(\Psi) \setminus \{\Psi\}$.

Otherwise, we are in the case where $T_{\Gamma_n}(\alpha) = B \cup \{\beta\}$ where $B \subseteq \Psi_n$ is nonempty, downward closed and is such that $s(B) \subseteq B$ and $\beta \in \Psi_{\mathbb{M}} \setminus \Psi_n$ and $B < \beta < (\Psi_n \setminus B) = \Psi \setminus B$. In this case we set $\Gamma_{n+1} := \Gamma_n\langle\beta\rangle$, i.e., we add to Γ_n the element β, and with it, the entire copy of $\mathbb{Z}$ that β lives on, so $\Gamma_{n+1} = \Gamma_n + \sum_n \mathbb{Q}p^n\beta + \sum_n \mathbb{Q}s^n\beta$. Thus $\Gamma_{n+1} \cong (\Gamma_n)_{(\Psi_n \setminus B)}$. By Observation 4.4.5 we actually have $\Gamma_{n+1} \cong \Gamma_{\rho'}$ for some increasing $\rho' : n+1 \to \mathrm{sded}^{op}(\Psi) \setminus \{\Psi\}$. Now that we've constructed Γ_{n+1}, we keep going.

Note that we either terminate the construction at a finite n or else $\bigcup_n \Gamma_n$ is isomorphic to Γ_ρ inside $\Gamma\langle\alpha\rangle$ for some increasing $\rho : \omega \to \mathrm{sded}^{op}(\Psi) \setminus \{\Psi\}$, by Observation 4.4.5. In the latter case, we note the ordered abelian group $\Gamma_\omega := (\bigcup_n \Gamma_n) \oplus \mathbb{Q}\alpha$ is automatically closed under ψ and s by construction and so we are done: $\Gamma\langle\alpha\rangle = \Gamma_\omega$ and so $\Gamma\langle\alpha\rangle \cong \Gamma_\rho \oplus \mathbb{Q}\alpha$ for some increasing $\rho : \omega \to \mathrm{sded}(\Psi) \setminus \{\Psi\}$. □

Examples. In this section we give explicit examples of extensions of models of $T_{AC,\log}$ which realize each type of simple extension in Theorem 4.4.6.

First, we recall the useful notion of *pseudocauchy sequences* and *pseudolimits* from valuation theory, given here only in the special context of asymptotic couples with valuation map ψ:

Definition 4.4.7. Let (Γ, ψ) be an asymptotic couple and let (α_ρ) be a well-indexed sequence in Γ. We say that (α_ρ) is a **pseudocauchy sequence** (or **pc-sequence in** (Γ, ψ)) if for some index ρ_0 we have

$$\rho_0 < \rho < \sigma < \tau \implies \psi(\alpha_\rho - \alpha_\sigma) < \psi(\alpha_\sigma - \alpha_\tau).$$

For $\alpha \in \Gamma$, the sequence (α_ρ) is said to **pseudoconverge to** α, and α is a **pseudolimit of** (α_ρ) if for some index ρ_0 we have

$$\rho_0 < \rho < \sigma < \nu \implies \psi(\alpha - \alpha_\rho) < \psi(\alpha - \alpha_\sigma).$$

The basic connection between pc-sequences and model theory is the following:

Lemma 4.4.8. *Let (Γ, ψ) be an asymptotic couple, and suppose (α_ρ) a pc-sequence in Γ. Then there is an elementary extension (Γ^*, ψ^*) of (Γ, ψ) and an element $\alpha \in \Gamma^*$ such that (α_ρ) pseudoconverges to α.*

Proof. Suppose (α_ρ) is a pc-sequence in Γ. Let ρ_0 be as in Definition 4.4.7. Consider the partial type given by all formulas of the form

$$\psi(x - \alpha_\rho) \ < \ \psi(x - \alpha_\sigma)$$

for $\rho_0 < \rho < \sigma$. Since every finite subset of this type is realized in (Γ, ψ), this type will be realized by an element α in an elementary extension (Γ^*, ψ^*) of (Γ, ψ). It easily follows that α is a pseudolimit of the pc-sequence (α_ρ). □

Example 1. Consider the $\mathcal{L}_{AC,\log}$-substructure $(\Gamma_{\log}^{\mathbb{Q}}, \psi)$ of $(\Gamma_{\log}, \psi)$ with underlying group $\Gamma_{\log}^{\mathbb{Q}} = \sum_n \mathbb{Q}e_n$. We have shown that $(\Gamma_{\log}^{\mathbb{Q}}, \psi) \models T_{AC,\log}$, and in fact, $(\Gamma_{\log}^{\mathbb{Q}}, \psi)$ is a prime model of $T_{AC,\log}$. Let α be the element

$$\alpha \ := \ \sqrt{2}e_2 \ = \ (0, 0, \sqrt{2}, 0, \ldots) \in \Gamma_{\log} \setminus \Gamma_{\log}^{\mathbb{Q}}.$$

An arbitrary element of $\mathbb{Q}^{\neq}\alpha - \Gamma_{\log}^{\mathbb{Q}}$ looks like

$$(q_0, q_1, \underbrace{q_2 + q\sqrt{2}}_{\neq 0,1}, q_3, \ldots)$$

where $q \in \mathbb{Q}^{\neq}$ and $q_n \in \mathbb{Q}$, where $q_n = 0$ for all but finitely many n. Since the third entry $q_2 + q\sqrt{2}$ can never be 0 or 1, a computation using Example 2.3.2 shows that

$$\psi(\mathbb{Q}^{\neq}\alpha - \Gamma_{\log}^{\mathbb{Q}}) \ = \ s(\mathbb{Q}^{\neq}\alpha - \Gamma_{\log}^{\mathbb{Q}}) \ = \ \{s0, s^2 0, s^3 0\}$$

and thus

$$T_{(\Gamma_{\log}^{\mathbb{Q}}, \psi)}(\alpha) \ = \ \{s0, s^2 0, s^3 0\} \ = \ [e_0, e_0 + e_1 + e_2]_{\Psi_{(\Gamma_{\log}^{\mathbb{Q}}, \psi)}} \subseteq \Psi_{(\Gamma_{\log}^{\mathbb{Q}}, \psi)}.$$

Therefore

$$(\Gamma_{\log}^{\mathbb{Q}}, \psi)\langle\alpha\rangle \ = \ (\Gamma_{\log}^{\mathbb{Q}} \oplus \mathbb{Q}\alpha, \psi)$$

where the direct sum is taken inside $\Gamma_{\log}$ and ψ is the restriction of the ψ-map of $(\Gamma_{\log}, \psi)$. This is an example of (2) from Theorem 4.4.6 with $n = 0$, and (1) from Corollary 4.4.4.

Example 2. The idea for this example is to adjoin the vector

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right)$$

to the asymptotic couple $(\Gamma_{\log}, \psi)$. This can be made precise using the notions of pc-sequences and pseudolimits as follows:

Consider the sequence $(\alpha_N)_{N<\omega} := \left(\sum_{i=0}^{N}(1+i)^{-1}e_i\right)_{N<\omega}$ in $(\Gamma_{\log}, \psi)$. If $N_0 < N_1 < \omega$, then

$$\alpha_{N_0} - \alpha_{N_1} \ = \ -\textstyle\sum_{i=N_0+1}^{N_1}(1+i)^{-1}e_i, \quad \text{and thus:}$$

(4.4.1) $$\psi(\alpha_{N_0} - \alpha_{N_1}) \ = \ \textstyle\sum_{i=0}^{N_0+1} e_i = s^{N_0+1}0, \quad \text{for all } N_0 < N_1 < \omega$$

This shows $(\alpha_N)_{N<\omega}$ is a pc-sequence in $(\Gamma_{\log}, \psi)$. By Lemma 4.4.8, we get an elementary extension (Γ^*, ψ^*) of $(\Gamma_{\log}, \psi)$ and an element $\alpha \in \Gamma^*$ such that α is a pseudolimit of $(\alpha_N)_{N<\omega}$. In some sense α can be thought of as the vector above, especially when it comes to doing calculations. It follows from (4.4.1) and the definition of pseudolimit that

$$\psi(\alpha - \alpha_N) \;=\; s^{N+1}0, \quad \text{for all } N < \omega. \tag{4.4.2}$$

Let $\gamma = \sum_n q_n e_n \in \Gamma_{\log}$ be arbitrary, where $q_n \in \mathbb{Q}$ for all n. Then take the unique $N < \omega$ such that $q_n = (1+n)^{-1}$ iff $n < N$. Next let $M < \omega$ be arbitrary and note that

$$\begin{aligned}
\psi(\gamma - \alpha_{N+M}) \;&=\; \psi\big(\textstyle\sum_n q_n e_n - \sum_{n=0}^{M+N}(1+n)^{-1}e_n\big) \\
&=\; \psi\big(\textstyle\sum_{n\geqslant N} q_n e_n - \sum_{n=N}^{N+M}(1+n)^{-1}e_n\big) \\
&=\; \psi\big(\underbrace{(q_N - (1+N)^{-1})}_{\neq 0}\, e_N + \textstyle\sum_{n>N} q_n^* e_n\big) \quad \text{(for some } q_n^* \in \mathbb{Q}\text{)} \\
&=\; \textstyle\sum_{n=0}^{N} e_n \;=\; s^N 0.
\end{aligned}$$

In light of (4.4.2), this computation shows that $\alpha \in \Gamma^* \setminus \Gamma_{\log}$. Using Fact 2.2.1 and the definition of pseudolimit, the above computation also shows that

$$\psi(\mathbb{Q}^{\neq}\alpha - \Gamma_{\log}) \;=\; \Psi_{\Gamma_{\log}}.$$

To compute $s(\mathbb{Q}^{\neq}\alpha - \Gamma_{\log})$, let $q \in \mathbb{Q}^{\neq}$ and $\gamma = \sum_n q_n e_n \in \Gamma_{\log}$ be arbitrary. Take the unique $N < \omega$ such that $q_n = q(1+n)^{-1} - 1$ iff $n < N$. Then we have

$$q\alpha_{N+1} - \gamma = e_0 + \cdots + e_{N-1} + \underbrace{\big(q(1+N)^{-1} - q_N\big)}_{\neq 1} e_N + \sum_{n>N} q_n^* e_n \quad \text{(for some } q_n^* \in \mathbb{Q}\text{)}$$

and so $s(q\alpha_{N+1} - \gamma) = s^N 0$. Furthermore, (4.4.2) gives

$$[q\alpha - q\alpha_{N+1}] \;<\; [e_N].$$

Thus, with $\widetilde{q} := \big|1 - q(1+N)^{-1} + q_N\big|/2 \in \mathbb{Q}^{>}$, we have that

$$q\alpha_{N+1} - \gamma - \widetilde{q}e_N \;<\; q\alpha - \gamma \;=\; (q\alpha - q\alpha_{N+1}) + (q\alpha_{N+1} - \gamma) \;<\; q\alpha_{N+1} - \gamma + \widetilde{q}e_N,$$

with all three quantities contained either entirely within $\big((\Gamma^*)^{<}\big)'$ or entirely within $\big((\Gamma^*)^{>}\big)'$. By Corollary 2.3.7, it follows that $s(q\alpha - \gamma) = s^N 0$. This computation shows that

$$s(\mathbb{Q}^{\neq}\alpha - \Gamma_{\log}) \;=\; \Psi_{\Gamma_{\log}}.$$

We conclude that

$$T_{(\Gamma_{\log},\psi)}(\alpha) \;=\; \Psi_{\Gamma_{\log}}$$

and so

$$(\Gamma_{\log}, \psi)\langle\alpha\rangle \;=\; (\Gamma_{\log} \oplus \mathbb{Q}\alpha, \psi)$$

where the direct sum is being taken inside Γ^* and ψ is the restriction of the ψ-map of (Γ^*, ψ^*). This is an example of (2) from Theorem 4.4.6 with $n = 0$, and (2) from Corollary 4.4.4.

Example 3. In this example, we let (Γ, ψ) be an arbitrary model of $T_{AC,\log}$ and we fix an extension (Γ_ρ, ψ_ρ) for some increasing $\rho : n \to \mathrm{sded}^{op}(\Psi) \setminus \{\Psi\}$ for some $n \geqslant 1$. Consider an element $\alpha \in \Gamma_\rho$ such that

$$\alpha \;:=\; \gamma + \sum_{j=0}^{n-1} \alpha_j$$

where $\gamma \in \Gamma$ and $\alpha_j \in \big(\mathrm{span}_{\mathbb{Q}}(\beta_{k,j})_{k\in\mathbb{Z}}\big)^{\neq}$, i.e., each α_j is constructed from a nontrivial linear combination of $\beta_{k,j}$'s from the jth copy of $\mathbb{Z}$ that was added to Γ in Γ_ρ. We will show that α has the property that $\Gamma\langle\alpha\rangle = \Gamma_\rho$, and so it is in some sense a "primitive element" for the extension Γ_ρ of Γ.

First, since $\Gamma\langle\alpha\rangle = \Gamma\langle\alpha - \gamma\rangle$, we replace α with $\alpha - \gamma$ to arrange $\alpha = \sum_{j=0}^{n-1} \alpha_j$. By the $\mathbb{Q}$-linear independence of the $(\beta_{k,j})_{k\in\mathbb{Z},j<n}$ (Lemma 4.3.8), we uniquely write $\alpha = \sum_{l=0}^{N} q_l\beta_l$ for some $N > 0$, with $q_0, \ldots, q_N \in \mathbb{Q}^{\neq}$ and $(\beta_l)_{l\leqslant N} \subseteq (\beta_{k,j})_{k\in\mathbb{Z},j<n}$ are such that $\beta_0 < \cdots < \beta_N$.

Next, if $\sum_{l=0}^{N} q_l = 0$, then $\psi(\alpha) = s\beta_0 \in \Gamma\langle\alpha\rangle$, otherwise $s\big((\sum_{l=0}^{N} q_l)^{-1}\alpha\big) = s\beta_0$ by Lemma 4.3.4 and Corollary 4.3.5. Thus $(s^k\beta_0)_{k\in\mathbb{Z}} \subseteq \Gamma\langle\alpha\rangle$ and $\alpha - q_0\beta_0 = \sum_{l=1}^{N} q_l\beta_l \in \Gamma\langle\alpha\rangle$. In this way, we have "stripped off" the least $\beta_{k,j}$ in α and we have recovered the first copy of $\mathbb{Z}$ in the construction of Γ_ρ. Continuing in this manner we can recover all the other copies of $\mathbb{Z}$.

It is also clear that all such "primitive elements" of Γ_ρ must take this form. This simple extension is an example of (1) in Theorem 4.4.6 with arbitrary n.

Example 4. Finally we give an example of a simple extension of type (3) from Theorem 4.4.6. Let (Γ, ψ) be an arbitrary model of $T_{AC,\log}$ and we fix an extension (Γ_ρ, ψ_ρ) for some increasing $\rho : \omega \to \mathrm{sded}^{op}(\Psi) \setminus \{\Psi\}$ inside $\mathbb{M}$. Let $(\beta_{k,j})_{k\in\mathbb{Z},j<\omega}$ be the elements from the copies of $\mathbb{Z}$'s that were added to Γ in Γ_ρ.

Next, define the element $\alpha_n := \sum_{j=0}^{n} \beta_{1,j} - \beta_{0,j} \in \Gamma_{\rho\restriction(n+1)} \subseteq \Gamma_\rho \subseteq \mathbb{M}$. Note that from Example 3 above we have $\Gamma\langle\alpha_n\rangle = \Gamma_{\rho\restriction(n+1)}$. Also note that by Lemma 4.3.4, we have that

$$\psi(\alpha_n - \alpha_m) \;=\; \psi(\textstyle\sum_{j=m+1}^{n} \beta_{1,j} - \beta_{0,j}) \;=\; s(\beta_{0,m+1}) = \beta_{1,m+1}, \quad \text{for all } m < n < \omega, \tag{4.4.3}$$

and so the sequence $(\alpha_n)_{n<\omega}$ is a pc-sequence. By saturation of $\mathbb{M}$, we can take an element α that is a pseudolimit of (α_n).

We claim that $\Gamma\langle\alpha\rangle$ is of the form $\Gamma_\rho \oplus \mathbb{Q}\alpha$. First, by (4.4.3) and the definition of pseudolimit,

$$\psi(\alpha - \alpha_n) \;=\; \beta_{1,n+1}, \quad \text{for all } n < \omega. \tag{4.4.4}$$

By Fact 2.2.1, (4.4.4), and Lemma 4.3.4, we get

$$\psi(\alpha) \;=\; \psi\big((\alpha - \alpha_0) + \alpha_0\big) \;=\; \min\big(\psi(\alpha - \alpha_0), \psi(\alpha_0)\big) \;=\; \min(\beta_{1,1}, \beta_{1,0}) \;=\; \beta_{1,0}.$$

From this it is clear that in fact $\alpha_0 = \beta_{1,0} - \beta_{0,0} = \beta_{1,0} - p\beta_{1,0} \in \Gamma_{\rho\restriction 1} \subseteq \Gamma\langle\alpha\rangle$. In general, if we show that $\alpha_0, \ldots, \alpha_m \in \Gamma_{\rho\restriction(m+1)} \subseteq \Gamma\langle\alpha\rangle$, then we may consider the pc-sequence $(\alpha_n - \sum_{j=0}^{m} \alpha_m)_{n\geqslant m+1}$ which pseudoconverges to $\alpha - \sum_{j=0}^{m} \alpha_m$ in $\Gamma\langle\alpha\rangle$. Then we can recover $\beta_{1,m+1}$ and thus also α_{m+1} similar to above by computing $\psi(\alpha - \sum_{j=0}^{m} \alpha_m)$.

We have shown $\Gamma_\rho \subseteq \Gamma\langle\alpha\rangle$, from which it follows from the proof of Theorem 4.4.6 that in fact $\Gamma\langle\alpha\rangle = \Gamma_\rho \oplus \mathbb{Q}\alpha$ inside $\mathbb{M}$.

4.5. Counting types and the non-independence property (NIP)

Counting Types in $T_{AC,\log}$. In this section we derive a consequence of Theorem 4.4.6 necessary for proving NIP for $T_{AC,\log}$ below:

Corollary 4.5.1. *If* $(\Gamma, \psi) \models T_{AC,\log}$, *then* $\left|S^1(\Gamma)\right| \leqslant \operatorname{ded}\left(|\Gamma|\right)^{\aleph_0}$.

Under the assumptions of Section 4.4, it follows from the quantifier elimination for $T_{AC,\log}$ that two elements $\alpha, \beta \in \mathbb{M} \setminus \Gamma$ have the same type over Γ iff α and β have the same isomorphism type over Γ, i.e., iff there is an isomorphism $\Gamma\langle\alpha\rangle \cong \Gamma\langle\beta\rangle$ over Γ which sends α to β. This is how Corollary 4.5.1 will follow from Theorem 4.4.6.

In the rest of this section $\mathbb{M}$ *is a monster model of* $T_{AC,\log}$ *and* Γ *is a small submodel of* $\mathbb{M}$ *of size* κ. As a warmup to proving Corollary 4.5.1, we first prove the following:

Lemma 4.5.2. *There are at most* $\operatorname{ded}(\kappa)$*-many types of the form* $\operatorname{tp}(\alpha|\Gamma)$ *where* $\alpha \in \mathbb{M} \setminus \Gamma$ *has the property that* $\Gamma\langle\alpha\rangle = \Gamma \oplus \mathbb{Q}\alpha$ *inside* $\mathbb{M}$.

Proof. We have to count the isomorphism types of elements $\alpha \in \mathbb{M} \setminus \Gamma$ that have the property that $\Gamma\langle\alpha\rangle = \Gamma \oplus \mathbb{Q}\alpha$. Let $\alpha \in \mathbb{M} \setminus \Gamma$ have this property. There are two cases to consider:

Case 1: $[\Gamma \oplus \mathbb{Q}\alpha] = [\Gamma]$. In this case the isomorphism type of α over Γ is determined completely by its cut over Γ by Corollary 3.1.11. In particular, there are at most $\operatorname{ded}(\kappa)$-many types that fall into this case.

Case 2: $[\Gamma \oplus \mathbb{Q}\alpha] \neq [\Gamma]$. In this case, there are $\gamma \in \Gamma$, $q \in \mathbb{Q}^{\neq}$ such that $\gamma + q\alpha > 0$ and $[\gamma + q\alpha] \notin [\Gamma]$. In this case, the isomorphism type of α over Γ is completely determined by this choice of $\gamma \in \Gamma$, $q \in \mathbb{Q}^{\neq}$, the cut that $[\gamma + q\alpha]$ realizes in $[\Gamma]$ and the element $\delta \in \Psi$ such that $\psi(\gamma + q\alpha) = \delta$, by Corollary 3.1.12. Thus there are at most $\kappa \cdot \aleph_0 \cdot \operatorname{ded}(\kappa) \cdot \kappa = \operatorname{ded}(\kappa)$-many types that fall into this case. $\square$

Proof of Corollary 4.5.1. Let $\alpha \in \mathbb{M} \setminus \Gamma$. Then by Theorem 4.4.6, we have three cases:

Case 1: $\Gamma\langle\alpha\rangle \cong \Gamma_\rho$ *for some increasing* $\rho : n \to \operatorname{sded}^{op}(\Psi) \setminus \{\Psi\}$, *for some* n. In this case, the isomorphism type of α over Γ is completely determined by the map ρ and the specific element of Γ_ρ which maps to α. Since $|\Gamma_\rho| = |\Gamma|$, for each n this gives $\operatorname{ded}(\kappa)^n \cdot \kappa = \operatorname{ded}(\kappa)$-many isomorphism types over Γ. In total, Case 1 gives $\sum_{n<\omega} \operatorname{ded}(\kappa) = \operatorname{ded}(\kappa)$-many types.

Case 2: $\Gamma\langle\alpha\rangle \cong \Gamma_\rho \oplus \mathbb{Q}\alpha$ *for some increasing* $\rho : n \to \operatorname{sded}^{op}(\Psi) \setminus \{\Psi\}$, *for some* n. In this case, the isomorphism type of α over Γ is determined by the map ρ and then the type of α over the image of Γ_ρ in $\mathbb{M}$. By Lemma 4.5.2, Case 2 gives $\sum_{n<\omega} \operatorname{ded}(\kappa)^n \cdot \operatorname{ded}(\kappa) = \operatorname{ded}(\kappa)$-many types.

Case 3: $\Gamma\langle\alpha\rangle \cong \Gamma_\rho \oplus \mathbb{Q}\alpha$ *for some increasing* $\rho : \omega \to \operatorname{sded}^{op}(\Psi) \setminus \{\Psi\}$. In this case, the isomorphism type of α over Γ is also determined by the map ρ and then the type of α over the image of Γ_ρ in $\mathbb{M}$. By Lemma 4.5.2, Case 3 gives $\operatorname{ded}(\kappa)^{\aleph_0} \cdot \operatorname{ded}(\kappa) = \operatorname{ded}(\kappa)^{\aleph_0}$-many types. $\square$

NIP. In this section we derive the main result of **[15]** as an immediate consequence of Corollary 4.5.1:

Theorem 4.5.3. $T_{AC,\log}$ *and* T_{AC} *have NIP.*

In the rest of this section T *is an arbitrary first-order theory with monster model* $\mathbb{M}$.

Definition 4.5.4. Let $R \subseteq \mathbb{M}^{m+n} = \mathbb{M}^m \times \mathbb{M}^n$ be a definable relation. We say that R, and any $L_{\mathbb{M}}$-formula $\phi(x, y)$ that defines R, has the **independence property** (or **IP**) if there are $(a_i)_{i\in\mathbb{N}} \subseteq \mathbb{M}^m$ and $(b_I)_{I\subseteq\mathbb{N}} \subseteq \mathbb{M}^n$ such that

$$R(a_i, b_I) \iff i \in I, \quad \text{for all } i \in \mathbb{N} \text{ and } I \subseteq \mathbb{N}.$$

Otherwise we say that R, and any $L_{\mathbb{M}}$-formula $\phi(x, y)$ that defines R, does not have the independence property (or **has NIP**).

We say that T **has NIP** if every definable relation $R \subseteq \mathbb{M}^{m+n}$ for every m, n has NIP.

Definition 4.5.5. Define the **stability function of** T to be the function

$$g_T(\kappa) \;=\; \sup_{M\models T, |M|=\kappa} \left| \bigcup_{n<\omega} S^n(M) \right| \;=\; \sup_{M\models T, |M|=\kappa} \left|S^1(M)\right|.$$

The main result concerning NIP and the function $g_T(\kappa)$ is the following:

Proposition 4.5.6. *If T has NIP, then*

$$g_T(\kappa) \;\leqslant\; \mathrm{ded}(\kappa)^{|T|} \quad \textit{for all } \kappa,$$

and if T has the independence property, then

$$g_T(\kappa) \;=\; 2^\kappa \quad \textit{for all } \kappa.$$

Proposition 4.5.6 is a global form of [**36**, Theorem 4.10]. For additional accounts, also see [**1**, §4] or [**37**, 2.3.4].

In the presence of the Generalized Continuum Hypothesis (GCH), we have $\mathrm{ded}(\kappa) = 2^\kappa$ for all κ and so we cannot get a converse to Proposition 4.5.6. However, if we dare to reject CH, then we have [**30**, Corollary 4.3] at our disposal:

Proposition 4.5.7. $\mathrm{Con}(\mathit{ZF}) \to \mathrm{Con}\big(\mathit{ZFC},\ 2^{\aleph_0} = \aleph_{\omega_1},\ 2^{\aleph_1} = \aleph_{\omega_1}^+,\ \textit{and}\ \mathrm{ded}(\aleph_1) < 2^{\aleph_1}\big)$.

Note that if we are in a model of ZFC where $2^{\aleph_0} = \aleph_{\omega_1}$, $2^{\aleph_1} = \aleph_{\omega_1}^+$ and $\mathrm{ded}(\aleph_1) < 2^{\aleph_1}$ are true, then it follows that $\mathrm{ded}(\aleph_1) \leqslant \aleph_{\omega_1}$ and so

$$\mathrm{ded}(\aleph_1)^{\aleph_0} \;\leqslant\; \aleph_{\omega_1}^{\aleph_0} \;=\; (2^{\aleph_0})^{\aleph_0} \;=\; 2^{\aleph_0\cdot\aleph_0} \;=\; 2^{\aleph_0} \;=\; \aleph_{\omega_1} < \aleph_{\omega_1}^+ \;=\; 2^{\aleph_1}.$$

In other words:

Corollary 4.5.8 (Mitchell)**.** $\mathrm{Con}(\mathit{ZF}) \to \mathrm{Con}\big(\mathit{ZFC}\ \textit{and}\ \mathrm{ded}(\aleph_1)^{\aleph_0} < 2^{\aleph_1}\big)$.

By absoluteness of NIP, Proposition 4.5.6 and Corollary 4.5.8, we get:

Proof of Theorem 4.5.3. Since $T_{AC,\log}$ is countable in a recursive language with a recursively enumerable axiomatization, the statement "$T_{AC,\log}$ has NIP" is an arithmetic statement, i.e., via Gödel numbering this statement is expressible by a sentence in Peano arithmetic. Any proof of such a sentence from ZFC $+\big(\mathrm{ded}(\aleph_1)^{\aleph_0} < 2^{\aleph_1}\big)$ can be converted into a (possibly much longer) proof from ZFC. Now, suppose we are in a model of ZFC $+\big(\mathrm{ded}(\aleph_1)^{\aleph_0} < 2^{\aleph_1}\big)$. Then in such a model it follows from Corollary 4.5.1 that $g_{T_{AC,\log}}(\aleph_1) \leqslant \mathrm{ded}(\aleph_1)^{\aleph_0} < 2^{\aleph_1}$. Then by Proposition 4.5.6, $T_{AC,\log}$ has NIP in that particular model, i.e.

$$\mathrm{ZFC} + \big(\mathrm{ded}(\aleph_1)^{\aleph_0} < 2^{\aleph_1}\big) \;\vdash\; T_{\log} \text{ has NIP}$$

and thus

$$\mathrm{ZFC} \;\vdash\; T_{AC,\log} \text{ has NIP},$$

or in other words, $T_{AC,\log}$ has NIP. It follows that T_{AC} also has NIP since every model of T_{AC} can be expanded into a model of $T_{AC,\log}$. □

4.6. Other model-theoretic results

In this section $\mathbb{M}$ is a monster model of $T_{AC,\log}$ and (Γ,ψ) is a small submodel of $\mathbb{M}$.

Variants of o-minimality. In contrast to the o-minimality of Ψ (Corollary 4.3.11), it is important to note that (Γ,ψ) is not even weakly o-minimal because the definable set $\Psi\subseteq\Gamma$ is infinite and discrete. In fact, (Γ,ψ) is not even *locally o-minimal* (in the sense of [**38**]) because the definable set $(\Psi-\Psi)^{>0}\subseteq\Gamma$ does not have the local o-minimality property at 0.

However, (Γ,ψ) is "o-minimal at infinity" in the following sense:

Lemma 4.6.1. *If $X\subseteq\Gamma$ is definable in (Γ,ψ), then there is $a\in\Gamma$ such that $(a,\infty)\subseteq X$ or $(a,\infty)\cap X=\emptyset$.*

This is immediate from the following claim:

Claim 4.6.2. *Let $F:\Gamma\to\Gamma_\infty$ be a definable function. Then there is $a\in\Gamma$ such that on the restriction (a,∞), F is either constant, or of the form $x\mapsto qx+\beta$ for $q\in\mathbb{Q}^{\neq}$ and $\beta\in\Gamma$.*

Proof. By quantifier elimination and universal axiomatization of $T_{AC,\log}$, it suffices to prove the claim just for L-terms $t(x)$. This we can do by induction on the complexity of $t(x)$.

The cases $t(x)=\beta$ for some $\beta\in\Gamma_\infty$ is clear since this is already a constant function. The cases $t(x)=t_1(x)+t_2(x)$, $t(x)=-t_1(x)$, $t(x)=\delta_n t_1(x)$ are also clear.

If $t(x)$ is constant on (b,∞) for some $b\in\Gamma$, then so are $\psi\big(t(x)\big), s\big(t(x)\big)$ and $p\big(t(x)\big)$. If $t(x)$ is $qx+\beta$ on (b,∞), then $t(x)$ is either strictly increasing and cofinal in Γ, or strictly decreasing and coinitial in Γ. Thus $\psi\big(t(x)\big)$ and $s\big(t(x)\big)$ will eventually be the constant value $s0$ and $p\big(t(x)\big)$ will eventually be the constant value ∞. $\square$

Strongly NIP, finite dp-rank, and dp-minimality. There are various strengthenings of NIP that would be natural to consider. Among these are dp-*minimality*, having *finite* dp-*rank*, and being *strongly NIP*; see [**10**] for a definition of these notions.

As it turns out, T_{AC} and $T_{AC,\log}$ do not have any of these properties:

Theorem 4.6.3. *Neither T_{AC} nor $T_{AC,\log}$ are strongly NIP. Therefore they also do not have finite* dp-*rank nor are they* dp-*minimal.*

It is sufficient to show that neither T_{AC} nor $T_{AC,\log}$ are *strong*, since if a theory is strongly NIP, then it is strong (see [**10**]). To do this, we will use the following criterion:

Proposition 4.6.4. [**10**, 2.14] *Suppose that $\boldsymbol{M}=(M;+,<,\dots)$ is an expansion of a densely-ordered abelian group. Let $\boldsymbol{N}$ be a saturated model of* $\mathrm{Th}(\boldsymbol{M})$*, and suppose that for every $\varepsilon>0$ in $\boldsymbol{N}$ there is an infinite definable discrete set $X\subseteq\boldsymbol{N}$ such that $X\subseteq(0,\varepsilon)$. Then* $\mathrm{Th}(\boldsymbol{M})$ *is not strong.*

Proof of Theorem 4.6.3. Let $\boldsymbol{N}$ be a saturated model of T_{AC} or $T_{AC,\log}$. The infinite definable set $\Psi_{\boldsymbol{N}}$ is discrete and has the property that for every $\alpha\in\Psi_{\boldsymbol{N}}$, the set $\Psi_{\boldsymbol{N}}^{>\alpha}$ is also infinite and discrete. Let $\varepsilon>0$ and take $\alpha\in\Psi_{\boldsymbol{N}}$ such that $\big(\alpha+2(s\alpha-\alpha)\big)-\alpha=-2\int\alpha<\varepsilon$. Note that then $\alpha+2(s\alpha-\alpha)>\Psi_{\boldsymbol{N}}$ by Lemma 2.4.3. The definable infinite discrete set $X:=\Psi_{\boldsymbol{N}}^{>\alpha}-\alpha$ has the desired property. $\square$

The Steinitz exchange property. Given an arbitrary theory T, a parameter set A and an element a in $\mathbb{M}$, we say that a is **algebraic over** A if a belongs to a *finite* A-definable subset of $\mathbb{M}$. Then we define the **algebraic closure of** A **in** $\mathbb{M}$ as the set

$$\operatorname{acl}(A) \;:=\; \{a \in \mathbb{M} : a \text{ is algebraic over } A\}.$$

Definition 4.6.5. A theory T is said to have the **Steinitz exchange property** if for all sets A and all elements $a, b \in \mathbb{M}$, if $a \notin \operatorname{acl}(A)$ and $b \notin \operatorname{acl}(A)$, then

$$a \in \operatorname{acl}\big(A \cup \{b\}\big) \iff b \in \operatorname{acl}\big(A \cup \{a\}\big).$$

If a theory T has the Steinitz exchange property, then the algebraic closure operator acl will be a so-called *pregeometry*. For more on the role of pregeometries in model theory, we refer the reader to [**29**, Chapter 8]. For our theory $T_{AC,\log}$, the algebraic closure operator will *not* be a pregeometry:

Proposition 4.6.6. *$T_{AC,\log}$ does not have the Steinitz exchange property.*

Proof. Since $T_{AC,\log}$ has a universal axiomatization and is model complete, we have that for all A, $\operatorname{acl}(A) = \langle A\rangle$. Let Γ be a small model and construct an elementary extension Γ_ρ of Γ for some $\rho : 2 \to \operatorname{sded}(\Psi)$ inside $\mathbb{M}$. Let $(\beta_{k,0})$ and $(\beta_{k,1})$ be the two copies of $\mathbb{Z}$ which were added to Γ in Γ_ρ. Let $a = \beta_{0,0}$ and $b = \beta_{0,0}+\beta_{0,1}$. By calculations done in Section 4.4, we have $\operatorname{acl}\big(\Gamma \cup \{b\}\big) = \Gamma\langle b\rangle = \Gamma_\rho$ whereas $\operatorname{acl}\big(\Gamma\cup\{a\}\big) = \Gamma\langle a\rangle = \Gamma_{\rho\restriction 1}$. ☐

4.7. Relation to precontraction groups

In this section (Γ,ψ) is an H-asymptotic couple with asymptotic integration, which we construe as an $\mathcal{L}_{AC}$-structure.

In general, if $(\Gamma,\widetilde{\psi})$ is a (B,ε)-shift of (Γ,ψ), then we do not expect these asymptotic couples, as $\mathcal{L}_{AC}$-structures, to be elementarily equivalent. Indeed, if $(\Gamma,\psi) \models T_{AC}$, then the $(\emptyset,-s0)$-shift $(\Gamma,\widetilde{\psi})$ will not be a model of T_0 because $\min\widetilde{\Psi} = 0$ in that case. However, we do have the following:

Proposition 4.7.1. *Suppose $(\Gamma,\psi) \models T_{AC}$ and $B \in \operatorname{sded}(\Psi)$ is such that $B \neq \emptyset$ and $\varepsilon \in \Gamma$ is such that $\psi(\varepsilon) > B$. Then the (B,ε)-shift $(\Gamma,\widetilde{\psi})$ is also a model of T_{AC}.*

Proof. $(\Gamma,\widetilde{\psi})$ is a divisible H-asymptotic couple with asymptotic integration such that $\chi + \widetilde{\psi}\circ\chi = \widetilde{\psi}$. Let $\widetilde{s}$ be the successor function of $(\Gamma,\widetilde{\psi})$. It is clear that $\widetilde{\Psi}$ is a successor set with least element $s0 = \widetilde{s} > 0$, since the order types of Ψ and $\widetilde{\Psi}$ are the same and these Ψ-sets have at least the first copy of $\mathbb{N}$ in common.

Claim 4.7.2. *Suppose α is such that $\psi(\alpha) > B$. Then $\widetilde{s}\big(\widetilde{\psi}(\alpha)\big) = s\psi(\alpha) + \varepsilon$.*

Proof of Claim. By the relation $s\psi = \psi\chi$, which holds in every H-asymptotic couple with asymptotic integration, and the fact that $\widetilde{\chi} = \chi$, we have

$$\widetilde{s}\big(\widetilde{\psi}(\alpha)\big) \;=\; \widetilde{\psi}\big(\widetilde{\chi}(\alpha)\big) \;=\; \widetilde{\psi}\big(\chi(\alpha)\big) \;=\; \psi\big(\chi(\alpha)\big) + \varepsilon \;=\; s\big(\psi(\alpha)\big) + \varepsilon. \qquad \square$$

By the claim it follows that each $\alpha \in \widetilde{\Psi}$ has immediate successor $\widetilde{s}(\alpha)$ and that $\gamma \mapsto \widetilde{s}\gamma : \widetilde{\Psi} \to \widetilde{\Psi}^{>s0}$ is a bijection. ☐

In the rest of this section we will remark on the relationship between our asymptotic couples and the precontraction groups of Kuhlmann. Precontraction groups arise as the value groups of certain ordered exponential fields, and in this way they are similar in spirit to asymptotic couples which arise as the value groups of certain valued differential fields. We refer the interested reader to **[21, 22]** for a treatment of the model theory of precontraction groups and to **[24]** for their connection to ordered exponential fields. For our purposes, it suffices to recall the definition:

Definition 4.7.3. A **precontraction group** is a pair (Γ, χ) where Γ is an ordered abelian group and $\chi : \Gamma \to \Gamma$ satisfies for all $\alpha, \beta \in \Gamma$:

(1) $\chi(\alpha) = 0 \iff \alpha = 0$;
(2) $\alpha \leqslant \beta \implies \chi(\alpha) \leqslant \chi(\beta)$;
(3) $\chi(-\alpha) = -\chi(\alpha)$;
(4) $[\alpha] = [\beta]$ and $\mathrm{sign}(\alpha) = \mathrm{sign}(\beta) \implies \chi(\alpha) = \chi(\beta)$.

If in addition, for all $\alpha \in \Gamma^{\neq}$:

(5) $|\alpha| > \big|\chi(\alpha)\big|$,

then (Γ, χ) is said to be a **centripetal** precontraction group. Finally, we say that a precontraction group (Γ, χ) is **divisible** if the underlying ordered abelian group Γ is divisible.

We let $\mathcal{L}_{PG} = \{0, +, -, <, \chi\}$ denote the natural first-order language of precontraction groups and construe all precontraction groups (Γ, χ) as $\mathcal{L}_{PG}$-structures in the obvious way.

If (Γ, ψ) is a divisible H-asymptotic couple with asymptotic integration, then we may associate to (Γ, ψ) a divisible centripetal precontraction group (Γ, χ_{PG}) by defining for all $\alpha \in \Gamma$,

$$\chi_{PG}(\alpha) \;=\; \begin{cases} \chi(\alpha) & \text{if } \alpha < 0, \\ 0 & \text{if } \alpha = 0, \\ -\chi(-\alpha) & \text{if } \alpha > 0, \end{cases}$$

where $\chi : \Gamma \to \Gamma^{\leqslant}$ is the contraction map of (Γ, ψ) as defined in Definition 2.3.1. Thus every divisible H-asymptotic couple with asymptotic integration yields a divisible centripetal precontraction group as a reduct. Conversely, it is worth considering whether this process is reversible, i.e., given a divisible centripetal precontraction group (Γ, χ_{PG}), can one define a ψ-map on Γ in the $\mathcal{L}_{PG}$-structure (Γ, χ_{PG}) such that (Γ, ψ) is a divisible H-asymptotic couple with asymptotic integration and such that the contraction map of (Γ, ψ) is $\chi_{PG}|\Gamma^{<}$. It turns out this is impossible for models of T_{AC}:

Proposition 4.7.4. *In no precontraction group (Γ, χ) can one define, even allowing parameters, a function $\psi : \Gamma^{\neq} \to \Gamma$ such that (Γ, ψ) is a model of T_{AC} and $\chi + \psi \circ \chi = \psi$ $\Gamma^{<}$.*

Proof. Suppose $(\Gamma, \psi) \models T_{AC}$ is such that we can define ψ in (Γ, χ). We may assume that (Γ, ψ) is $\aleph_0$-saturated. Take $B \in \mathrm{sded}(\Psi)$ large enough so that it is to the right of the Ψ-set of the definable closure of all the finitely-many parameters needed from Γ to define ψ in (Γ, χ). Consider any (B, ε)-shift $\widetilde{\psi}$ of ψ such that $\psi(\varepsilon) \in B$. Then $(\Gamma, \psi) \equiv (\Gamma, \widetilde{\psi})$ and $(\Gamma, \chi) = (\Gamma, \widetilde{\chi})$. By completeness of T_{AC}, the same formula that defines ψ in (Γ, χ) must define $\widetilde{\psi}$ in $(\Gamma, \widetilde{\chi})$ and so $\psi = \widetilde{\psi}$, a contradiction. $\square$

Our method of proof for Proposition 4.7.4 mirrors the proof given in [**2**, Prop 5.1] for the corresponding result about closed asymptotic couples. A **closed asymptotic couple** is a divisible H-asymptotic couple with asymptotic integration such that $(\Gamma^{<})' = \Psi$ (see [**3**]). There they use essentially the same trick with

(B, ε)-shifts, except they consider iterates of ψ instead of iterates of s. However, by Corollary 2.3.12, one can see that this is essentially the same notion for elements $\alpha \ll 0$.

Furthermore, it seems likely that this trick can be used for any theory $\mathrm{Th}(\Gamma, \psi)$ of interest, where (Γ, ψ) is a divisible H-asymptotic couple with asymptotic integration. Provided that the first order theory of (Γ, ψ) is preserved under sufficiently subtle (B, ε)-shifts, the same proof can be used. This leads us to the following:

Conjecture 4.7.5. *In no nontrivial precontraction group (Γ, χ) can one define, even allowing parameters, a function $\psi : \Gamma^{\neq} \to \Gamma$ such that (Γ, ψ) is an H-asymptotic couple and $\chi + \psi \circ \chi = \psi$ on $\Gamma^{<}$.*

CHAPTER 5

Valued fields, differential fields, and valued differential fields

In this chapter we recall the theory of valued fields, differential fields, and various types of valued differential fields which we will need for later chapters. We also include here various technical results which we could have postponed to later chapters when they are actually used, but instead naturally fit in with the current chapter due to the level of generality.

In Section 5.1, we recall many of the basic notions of *valued fields*. Much of this section is a review from [**6**], although we do prove a useful generalization of *Kaplansky's Lemma* (Lemma 5.1.4) to the setting of rational functions.

In Section 5.2 we define *differential fields* without any additional structure, and also recall some basic extension theory of differential fields.

Section 5.3 is essentially a crash course in different types of valued differential fields from [**6**] that we wish to consider: *valued differential fields*, *asymptotic fields*, *pre-differential-valued fields*, *differential-valued fields*. We also define some closely related ordered valued differential fields: the *pre-H-fields* and *H-fields*. Much of the later chapters will deal almost exclusively with differential-valued fields and H-fields. We also give precise definitions there of the concepts of ω-*free* and *newtonian*; something we neglected to do in Chapter 1.

In Section 5.4, we extract information from an asymptotic differential Kaplansky Lemma [**6**, 11.3.8] to study almost special elements in an asymptotic field extension.

Section 5.5 deals with extending the constant field of a differential-valued field.

Finally, in Section 5.6 we compute and study the eventual generic valuation v_P^e for the differential polynomial $P(Y) = Y' - sY$.

5.1. Valued fields

In this section K is a valued field. Let $\mathcal{O}_K$ denote its valuation ring, $\smallo_K$ the maximal ideal of $\mathcal{O}_K$, $v : K^\times \to \Gamma_K := v(K^\times)$ its valuation with value group Γ_K, and $\operatorname{res} : \mathcal{O}_K \to \boldsymbol{k}_K := \mathcal{O}_K/\smallo_K$ its residue map with residue field $\boldsymbol{k}_K$, which we may also denote as $\operatorname{res}(K)$. We will suppress the subscript K when the valued field K is clear from context. By convention we extend v to a map $v : K \to \Gamma_\infty$ by setting $v(0) := \infty$.

Given $f, g \in K$ we have the following relations:

$$f \preccurlyeq g \;:\Longleftrightarrow\; vf \geqslant vg \quad (f \text{ is } \textbf{dominated} \text{ by } g)$$
$$f \prec g \;:\Longleftrightarrow\; vf > vg \quad (f \text{ is } \textbf{strictly dominated} \text{ by } g)$$
$$f \asymp g \;:\Longleftrightarrow\; vf = vg \quad (f \text{ is } \textbf{asymptotic} \text{ to } g)$$

For $f, g \in K^\times$, we have the additional relation:

$$f \sim g \ :\Longleftrightarrow\ v(f-g) > vf \quad (f \text{ and } g \text{ are } \textbf{equivalent})$$

Both $\asymp$ and $\sim$ are equivalence relations on K and $K^\times$, respectively. We shall also use the following notation:

$$\begin{aligned} K^{\prec 1} &:= \{f \in K : f \prec 1\} = \smallO_K \\ K^{\preccurlyeq 1} &:= \{f \in K : f \preccurlyeq 1\} = \mathcal{O}_K \\ K^{\succ 1} &:= \{f \in K : f \succ 1\} = K \setminus \mathcal{O}_K \end{aligned}$$

Pseudocauchy sequences and a Kaplansky lemma. Let (a_ρ) be a well-indexed sequence in K and $a \in K$. Then (a_ρ) is said to **pseudoconverge to** a (written: $a_\rho \rightsquigarrow a$) if for some index ρ_0 we have $a - a_\sigma \prec a - a_\rho$ whenever $\sigma > \rho > \rho_0$. In this case we also say that a is a **pseudolimit of** (a_ρ). We say that (a_ρ) is a **pseudocauchy sequence in** K (or **pc-sequence in** K) if for some index ρ_0 we have

$$\tau > \sigma > \rho > \rho_0 \implies a_\tau - a_\sigma \prec a_\sigma - a_\rho.$$

If $a_\rho \rightsquigarrow a$, then (a_ρ) is necessarily a pc-sequence in K. A pc-sequence (a_ρ) is **divergent in** K if (a_ρ) does not have a pseudolimit in K.

Suppose that (a_ρ) is a pc-sequence in K and $a \in K$ is such that $a_\rho \rightsquigarrow a$. Also let $\gamma_\rho := v(a - a_\rho) \in \Gamma_\infty$, which is eventually in Γ and strictly increasing as a function of ρ. Recall *Kaplansky's Lemma*:

ADH 5.1.1. [**6**, Prop. 3.2.1] *Suppose $P \in K[X] \setminus K$. Then $P(a_\rho) \rightsquigarrow P(a)$. Furthermore, there are $\alpha \in \Gamma$ and $i \geqslant 1$ such that eventually $v\big(P(a_\rho) - P(a)\big) = \alpha + i\gamma_\rho$.*

Note that ADH 5.1.1 concerns *polynomials* $P \in K[X]$. Below we give a version for rational functions, but first a few remarks.

Roughly speaking, we think of the eventual nature of the sequence (γ_ρ) as a "rate of convergence" for the pseudoconvergence $a_\rho \rightsquigarrow a$. ADH 5.1.1 tells us that the rate of convergence for $P(a_\rho) \rightsquigarrow P(a)$ is very similar to that of $a_\rho \rightsquigarrow a$. Indeed, $(\alpha + i\gamma_\rho)$ is just an affine transform of (γ_ρ) in Γ. We want to show that applying rational functions to (a_ρ) will also have this property. Before we can do this, we need to recall a few more facts from valuation theory.

Suppose that (a_ρ) is a pc-sequence in K. A main consequence of ADH 5.1.1 is that (a_ρ) falls into one of two categories:

(1) (a_ρ) is of **algebraic type over** K if for *some* nonconstant $P \in K[X]$, $v\big(P(a_\rho)\big)$ is eventually strictly increasing (equivalently, $P(a_\rho) \rightsquigarrow 0$).
(2) (a_ρ) is of **transcendental type over** K if for *all* nonconstant $P \in K[X]$, $v\big(P(a_\rho)\big)$ is eventually constant (equivalently, $P(a_\rho) \not\rightsquigarrow 0$).

Suppose (a_ρ) is a pc-sequence of transcendental type over K. Then (a_ρ) is divergent in K. Moreover, if $a_\rho \rightsquigarrow b$ with b in a valued field extension of K, then b will necessarily be transcendental over K.

Now suppose that (a_ρ) is a pc-sequence in K. Take ρ_0 as in the definition of "pseudocauchy sequence" and define $\gamma_\rho := v(a_{\rho'} - a_\rho) \in \Gamma$ for $\rho' > \rho > \rho_0$; this depends only on ρ and the sequence $(\gamma_\rho)_{\rho > \rho_0}$ is strictly increasing. We define the **width** of (a_ρ) to be the following upward closed subset of Γ_∞:

$$\operatorname{width}(a_\rho) \;=\; \{\gamma \in \Gamma_\infty : \gamma > \gamma_\rho \text{ for all } \rho > \rho_0\}$$

The width of (a_ρ) is independent of the choice of ρ_0. The following follows from various results in [**6**, Chapters 2 and 3]:

ADH 5.1.2. *Let (a_ρ) be a divergent pc-sequence in K and let b be an element of a valued field extension of K such that $a_\rho \rightsquigarrow b$. Then for $\sigma_\rho := v(b-a_\rho) \in \Gamma_\infty$, eventually $\sigma_\rho = \gamma_\rho$ and*

$$\operatorname{width}(a_\rho) \ = \ \Gamma_\infty^{>v(b-K)} \quad \textit{and} \quad v(b-K) \ = \ \Gamma_\infty^{<\operatorname{width}(a_\rho)}$$

where $v(b-K) = \big\{v(b-a) : a \in K\big\} \subseteq \Gamma$.

Remark 5.1.3. Let b be an element of an immediate valued field extension of K. If $b \notin K$, then $v(b-K) \subseteq \Gamma$ is a nonempty downward closed subset of Γ without a greatest element. We think of $v(b-K)$ as encoding how well elements from K can approximate b. Below we will consider various qualitative properties of such a set $v(b-K)$ and consider what these properties say about the element b itself.

We say that pc-sequences (a_ρ) and (b_σ) in K are **equivalent** if they satisfy any of the following equivalent conditions:

(1) (a_ρ) and (b_σ) have the same pseudolimits in every valued field extension of K;
(2) (a_ρ) and (b_σ) have the same width, and have a common pseudolimit in some valued field extension of K;
(3) there are arbitrarily large ρ and σ such that for all $\rho' > \rho$ and $\sigma' > \sigma$ we have $a_{\rho'} - b_{\sigma'} \prec a_{\rho'} - a_\rho$, and there are arbitrarily large ρ and σ such that for all $\rho' > \rho$ and $\sigma' > \sigma$ we have $a_{\rho'} - b_{\sigma'} \prec b_{\sigma'} - b_\sigma$.

See [**6**, 2.2.17] for details of this equivalence.

Now we assume that L is an immediate extension of K, $a \in L \setminus K$, and (a_ρ) is a pc-sequence in K of transcendental type over K such that $a_\rho \rightsquigarrow a$.

Lemma 5.1.4. *Let $R(X) \in K(X) \setminus K$. Then there exists an index ρ_0 such that for $\rho > \rho_0$:*

(1) *$R(a_\rho) \in K$ (that is, $R(a_\rho) \neq \infty$);*
(2) *$R(a_\rho) \rightsquigarrow R(a)$;*
(3) *$v\big(R(a_\rho) - R(a)\big) = \alpha + i\gamma_\rho$, eventually, for some $\alpha \in \Gamma$ and $i \geqslant 1$;*
(4) *$(\alpha + i\gamma_\rho)$ is eventually cofinal in $v\big(R(a) - K\big)$, with α and i as in (3);*
(5) *$\big(R(a_\rho)\big)$ is a divergent pc-sequence in K; and*
(6) *$v\big(R(a) - K\big) = \big(\alpha + iv(a-K)\big)^{\downarrow}$, for some $\alpha \in \Gamma$ and $i \geqslant 1$.*

Proof. Let $R(X) = P(X)/Q(X)$ with $P, Q \in K[X]^{\neq}$. It is clear there exists ρ_0 such that $R(a_\rho) \in K$ for all $\rho > \rho_0$. Fix such a ρ_0 and assume $\rho > \rho_0$ for the rest of this proof.

We first consider the case that $R(X) = P(X) \in K[X] \setminus K$ is a polynomial. Then (2) and (3) follow from ADH 5.1.1. We will prove (5) and then (4) and (6) will follow. Assume towards a contradiction that there is $b \in K$ such that $R(a_\rho) \rightsquigarrow b$. Then $R(a_\rho) - b \rightsquigarrow 0$, so (a_ρ) is of algebraic type in view of $R(X) - b \in K[X] \setminus K$. This contradicts the assumption that (a_ρ) is a pc-sequence of transcendental type.

Next consider the case that $R(X) \in K(X) \setminus K[X]$. In particular, $Q(X) \in K[X] \setminus K$ and $Q \nmid P$. Then note that

$$\begin{aligned} v\left(\frac{P(a_\rho)}{Q(a_\rho)} - \frac{P(a)}{Q(a)}\right) \ &= \ v\left(\frac{P(a_\rho)Q(a) - P(a)Q(a_\rho)}{Q(a_\rho)Q(a)}\right) \\ &= \ v\big(P(a_\rho)Q(a) - P(a)Q(a_\rho)\big) - v\big(Q(a_\rho)\big) - v\big(Q(a)\big). \end{aligned}$$

The quantity $v\big(Q(a_\rho)\big)$ is eventually constant since (a_ρ) is of transcendental type. Next, set $S(X) := P(X)Q(a) - P(a)Q(X) \in K(a)[X]$. Note that eventually $S(a_\rho) \neq 0$ and thus $S \neq 0$ (otherwise, the polynomial $Q(X) - (Q/P)(a)P(X)$ would be identically zero since it would have infinitely many distinct zeros, which would imply $Q \mid P$). Furthermore, $S(a) = 0$, which shows that $S \in K(a)[X] \setminus K(a)$. By ADH 5.1.1, it follows that $S(a_\rho) \rightsquigarrow S(a) = 0$. In particular, $v\big(S(a_\rho)\big)$ is eventually strictly increasing and there are $\alpha \in \Gamma$ and $i \geqslant 1$ such that eventually $v\big(S(a_\rho)\big) = \alpha + i\gamma_\rho$. This shows (2) and (3).

Finally, we will prove (5), and then (4) and (6) will follow. Assume towards a contradiction that $R(a_\rho) \rightsquigarrow b$ with $b \in K$. Then

$$v\left(\frac{P(a_\rho)}{Q(a_\rho)} - b\right) \ = \ v\big(P(a_\rho) - bQ(a_\rho)\big) - v\big(Q(a_\rho)\big)$$

is eventually strictly increasing. Since $v\big(Q(a_\rho)\big)$ is eventually constant, $v\big(P(a_\rho) - bQ(a_\rho)\big)$ is eventually strictly increasing, so (a_ρ) is of algebraic type, a contradiction. □

For pc-sequences of algebraic type, we have the following result from Kuhlmann [**23**, Theorem 1]. It concerns "rates of convergences" of pc-sequences approximating elements in immediate algebraic extensions. Recall that K^h is the henselization of K. We say that a pc-sequence (x_ρ) from K is **special** if $\Gamma_\infty^{<\mathrm{width}(x_\rho)}$ is Δ-special for some nontrivial convex subgroup Δ of Γ.

ADH 5.1.5. [**6**, 3.4.24] *For each $x \in K^h \setminus K$ there is a divergent special pc-sequence (x_ρ) in K and some $a \in K^\times$ such that $x_\rho \rightsquigarrow x/a$. Moreover, for each $x \in K^h \setminus K$ there is $\alpha \in \Gamma$ and a nontrivial convex subgroup Δ of Γ such that $(\alpha + \Delta)^\downarrow = v(x - K)$.*

The Valuation Property. *In this subsection K is a henselian valued field of equicharacteristic zero.*

Proposition 5.1.6. *Suppose $L = K(x)$ is a valued field extension of K, where x is transcendental over K and $v(L) = \Gamma \oplus \mathbb{Z}v(x)$. Then for every $f \in L \setminus K$, there is $a \in K$ and $h \in L$ such that $f = a + h$ and $vh \notin \Gamma$.*

Proof. Let $f \in L \setminus K$. We want to prove there is $a \in K$ such that $v(f - a) \notin \Gamma$. Assume towards a contradiction that $v(f - K) \subseteq \Gamma$.

Claim. *$v(f - K) \subseteq \Gamma$ does not have a largest element.*

Proof of claim. Take $b \in K$ and consider $v(f - b) \in v(f - K)$. As $v(f - K) \subseteq \Gamma$, we can take $a \in K$ such that $f - b \asymp a$. Since $\mathrm{res}(K) = \mathrm{res}(L)$ by [**6**, 3.1.30], there is $c \in K^{\asymp 1}$ such that $f - b \sim ca$. Thus $f - b - ca \prec f - b$ for such c. □

Next, let (a_ρ) be a well-indexed sequence such that $v(f - a_\rho)$ is strictly increasing and cofinal in $v(f - K)$. Then $a_\rho \rightsquigarrow f$ and (a_ρ) is a divergent pc-sequence K. As K is henselian and of equicharacteristic zero, and $nv(x) \notin \Gamma$ for all $n \geqslant 1$, this contradicts [**6**, 3.3.24]. □

5.2. Differential fields

A **differential ring** is by definition a commutative ring K containing $\mathbb{Q}$, equipped with a derivation ∂ on K, i.e., an additive map $\partial : K \to K$ which satisfies the Leibniz identity: $\partial(ab) = \partial(a)b + a\partial(b)$ for all $a, b \in K$. For a differential ring K with derivation ∂, when ∂ is clear from the context, we set $a' := \partial(a)$, and similarly, $a^{(n)} = \partial^n(a)$, with ∂^n the nth iterate of ∂.

A **differential field** is a differential ring whose underlying ring is a field (of characteristic 0 since it contains $\mathbb{Q}$). Let K be a differential field. For $a \in K^\times$ we will denote the **logarithmic derivative** of a as $a^\dagger := a'/a = \partial(a)/a$. For $a, b \in K^\times$, note that $(ab)^\dagger = a^\dagger + b^\dagger$, in particular, $(a^k)^\dagger = ka^\dagger$ for $k \in \mathbb{Z}$. We denote the additive abelian group of logarithmic derivatives of K as:

$$K^\dagger \ := \ \{a^\dagger : a \in K^\times\} \ = \ (K^\times)^\dagger$$

The set $\{a \in K : a' = 0\} \subseteq K$ is a subfield of K and is called the **field of constants** of K, and denoted by C_K (or just C if K is clear from the context). If $c \in C$, then $(ca)' = ca'$ for $a \in K$. If $a, b \in K^\times$, then $a^\dagger = b^\dagger$ iff $a = bc$ for some $c \in C^\times$.

The following is routine:

Lemma 5.2.1. *Let K be a differential field. Suppose that $y_0, y_1, \ell \in K$ are such that $y_0, y_1 \notin C$ and $y_i'' = \ell y_i'$ for $i = 0, 1$. Then there are $c_0, c_1 \in C$ such that $c_0 \neq 0$ and $y_1 = c_0 y_0 + c_1$.*

We will often be concerned with algebraic extensions and simple transcendental extensions of differential fields. In these cases, the following are relevant:

ADH 5.2.2. [**6**, 1.9.2] *Suppose K is a differential field and L is an algebraic field extension of K. Then ∂ extends uniquely to a derivation on L.*

ADH 5.2.3. [**6**, 1.9.4] *Suppose K is a differential field with field extension $L = K(x)$ where $x = (x_i)_{i \in I}$ is a family in L that is algebraically independent over K. Then there is for each family $(y_i)_{i \in I}$ in L a unique extension of ∂ to a derivation on L with $\partial(x_i) = y_i$ for all $i \in I$.*

If K is a differential field and $s \in K \setminus \partial(K)$, then ADH 5.2.3 allows us to *adjoin an integral for s*: let $K(x)$ be a field extension of K such that x is transcendental over K. Then by ADH 5.2.3 there is a unique derivation on $K(x)$ extending ∂ such that $x' = s$. Likewise, if $s \in K \setminus K^\dagger$, then we can *adjoin an exponential integral for s*: take $K(x)$ as before and by ADH 5.2.3 there is a unique derivation on $K(x)$ extending ∂ such that $x' = sx$, and thus $x^\dagger = s$, i.e., "$x = \exp(\int s)$". Adjoining integrals and exponential integrals are basic examples of *Liouville extensions*:

A **Liouville extension** of K is a differential field extension L of K such that C_L is algebraic over C and for each $a \in L$ there are $t_1, \dots, t_n \in L$ with $a \in K(t_1, \dots, t_n)$ and for $i = 1, \dots, n$,

(1) t_i is algebraic over $K(t_1, \dots, t_{i-1})$, or
(2) $t_i' \in K(t_1, \dots, t_{i-1})$, or
(3) $t_i \neq 0$ and $t_i^\dagger \in K(t_1, \dots, t_{i-1})$.

Differential polynomials. *In the rest of this section K is a differential field with derivation ∂.* We shall briefly give some definitions concerning *differential polynomials* over K, and we refer the reader to [**6**, Chapter 4] for a more complete exposition. We let $K\{Y\}$ denote the **ring of differential polynomials in Y over K**. As a ring, $K\{Y\}$ is just the polynomial ring $K[Y, Y', Y'', \dots]$ in the distinct indeterminates $Y^{(n)}$ over K, where as usual we write Y, Y', Y'' instead of $Y^{(0)}, Y^{(1)}, Y^{(2)}$. We consider $K\{Y\}$ to be a differential ring extension of K, equipped with the unique derivation ∂ such that $\partial(Y^{(n)}) = Y^{(n+1)}$ for every n. Furthermore, $K\{Y\}$ comes equipped with an *evaluation rule*: for $P \in K\{Y\}$ and y an element of a differential ring extension of K, we let $P(y)$ be the element of that extension obtained by substituting $y, y', \dots$ for $Y, Y', \dots$ in P, respectively.

Given $P = P(Y) \in K\{Y\}$ and $f, g \in K$, define the following differential polynomials:

$$\begin{aligned} P_{+f} &= P_{+f}(Y) &:=& \; P(f+Y) \in K\{Y\} \quad \textbf{(additive conjugate of } P \textbf{ by } f\textbf{)}, \\ P_{\times g} &= P_{\times g}(Y) &:=& \; P(gY) \in K\{Y\} \qquad \textbf{(multiplicative conjugate of } P \textbf{ by } g\textbf{)}. \end{aligned}$$

Additive and multiplicative conjugation plays a role in finding zeros of a differential polynomial.

Compositional conjugation. *In this subsection we let ϕ range over $K^\times$.* In this subsection we will define the notion of *compositional conjugation*. We refer the reader to [**6**, §5.7] for more details.

We define K^ϕ to be the differential ring with the same underlying ring as K but with the derivation δ given by $\delta(a) = \phi^{-1} \cdot \partial(a)$ for $a \in K$, so $\delta = \phi^{-1}\partial$. We call K^ϕ the **compositional conjugate of K by ϕ**.

The compositional conjugate K^ϕ gives rise to the ring $K^\phi\{Y\}$ of differential indeterminates over K^ϕ. The underlying ring of $K^\phi\{Y\}$ is the same as the underlying ring of $K\{Y\}$; however, the derivation of $K^\phi\{Y\}$ extends the derivation $\delta = \phi^{-1}\partial$ of K^ϕ. In [**6**, §5.7] they define a certain K-algebra morphism

$$P(Y) \mapsto P^\phi(Y) : K\{Y\} \to K^\phi\{Y\}$$

where P^ϕ is called the **compositional conjugate of P by ϕ**. We will not give a definition for this morphism here, but we shall recall the following properties:

ADH 5.2.4. [**6**, §5.8] *Let $P \in K\{Y\}$ and $y \in K$. Then*

(1) $Y^\phi = Y$ *and* $(Y')^\phi = \phi Y'$,
(2) $P(y) = P^\phi(y)$,
(3) $(P^\phi)_{+y} = (P_{+y})^\phi$, *and*
(4) $(P^\phi)_{\times y} = (P_{\times y})^\phi$.

In view of the last two identities we let P^ϕ_{+y} denote both $(P^\phi)_{+y}$ and $(P_{+y})^\phi$, and let $P^\phi_{\times y}$ denote both $(P^\phi)_{\times y}$ and $(P_{\times y})^\phi$.

5.3. Valued differential fields

A **valued differential field** is a differential field K equipped with a valuation ring $\mathcal{O} \supseteq \mathbb{Q}$ of K. In particular, all valued differential fields have char $\boldsymbol{k} = 0$. We say that a valued differential field K has **small derivation** if $\partial\smallo \subseteq \smallo$.

An **asymptotic differential field**, or just **asymptotic field**, is a valued differential field K such that for all $f, g \in K^\times$ with $f, g \prec 1$,

(A) $f \prec g \iff f' \prec g'$.

If K is an asymptotic field, then $C \subseteq \mathcal{O}$ and thus $v(C^\times) = \{0\}$. The following consequence of Lemma 5.2.1 will be used in §6.7 to obtain the main result of Chapter 6:

Lemma 5.3.1. *Let K be an asymptotic field. Suppose that $y_0, y_1, \ell \in K$ are such that $y_0, y_1 \notin C$ and $y_i'' = \ell y_i'$ for $i = 0, 1$. Then $y_0 \succ 1$ iff $y_1 \succ 1$.*

The value group of an asymptotic field always has a natural asymptotic couple structure associated to it:

ADH 5.3.2. [**6**, 9.1.3] *Let K be a valued differential field. The following are equivalent:*

(1) *K is an asymptotic field;*

(2) *there is an asymptotic couple* (Γ, ψ) *with underlying ordered abelian group* $\Gamma = v(K^\times)$ *such that for all* $g \in K^\times$ *with* $g \not\asymp 1$ *we have* $\psi(vg) = v(g^\dagger)$.

If K is an asymptotic field, we call (Γ, ψ) as defined in ADH 5.3.2(2), the **asymptotic couple of** K.

Convention 5.3.3. *Let* L *be an expansion of an asymptotic field, and* P *a property that an asymptotic couple may or may not have. Then "*L *has property* P*", means "the asymptotic couple of* L *has property* P*". For instance, when we say* L *is "of* H***-type****", equivalently "is* H***-asymptotic****", we mean that the asymptotic couple* (Γ_L, ψ_L) *of* L *is* H*-type. Likewise for the properties "asymptotic integration", "grounded", etc.*

We say that an asymptotic field K is **pre-differential-valued**, or **pre-d-valued**, if the following holds:

(PDV) for all $f, g \in K^\times$, if $f \preccurlyeq 1$, $g \prec 1$, then $f' \prec g^\dagger$.

Every ungrounded asymptotic field is pre-d-valued by [**6**, 10.1.3].

Finally, we say that an asymptotic field field K is **differential-valued**, or **d-valued**, if it satisfies one of the following three equivalent conditions:

(1) $\mathcal{O} = C + \scriptstyle\mathcal{O}$;
(2) $\{\operatorname{res}(a) : a \in C\} = \boldsymbol{k}$;
(3) for all $f \asymp 1$ in K there exists $c \in C$ with $f \sim c$.

All differential-valued fields are necessarily pre-differential-valued, see [**6**, 9.1.3(iv)]. We shall often tacitly use the following:

ADH 5.3.4. [**6**, 9.1.2] *If* L *is an asymptotic extension of a* d*-valued field* K *with* $\operatorname{res}(K) = \operatorname{res}(L)$*, then* L *is* d*-valued, with* $C_L = C$.

Suppose K is a pre-d-valued field of H-type. Define the $\mathcal{O}$-submodule

$$\mathrm{I}(K) \ := \ \{y \in K : y \preccurlyeq f' \text{ for some } f \in \mathcal{O}\}$$

of K. We say that K has **integration** if $K = \partial K$, has **exponential integration** if $K = K^\dagger$, has **small integration** if $\mathrm{I}(K) = \partial\scriptstyle\mathcal{O}$, and has **small exponential integration** if $\mathrm{I}(K) = (1 + \scriptstyle\mathcal{O})^\dagger$.

Lemma 5.3.5. *Let* K *be a pre-*d*-valued field of* H*-type with small integration. Then* K *is* d*-valued.*

Proof. Take $f \in K$ such that $f \asymp 1$. Then $f' \in \mathrm{I}(K) = \partial\scriptstyle\mathcal{O}$, so we have $\varepsilon \in \scriptstyle\mathcal{O}$ such that $f' = \varepsilon'$. Hence $f - \varepsilon = c$ with $c \in C^\times$ and thus $f \sim c$. □

Ordered valued differential fields. A **pre-**H**-field** is an ordered pre-d-valued field K whose ordering, valuation, and derivation interact as follows:

(PH1) the valuation ring $\mathcal{O}$ is convex with respect to the ordering;
(PH2) for all $f \in K$, if $f > \mathcal{O}$, then $f' > 0$.

It follows from (PH1) and (PH2) that pre-H-fields are necessarily of H-type. Any ordered differential field with the trivial valuation is a pre-H-field.

An H**-field** is an ordered differential field K such that:

(H1) for all $f \in K$, if $f > C$, then f'>0;

(H2) $\mathcal{O} = C + o$ where $\mathcal{O} = \{g \in K : |g| \preccurlyeq c \text{ for some } c \in C\}$ and o is the maximal ideal of the convex subring $\mathcal{O}$ of K.

We construe an H-field K as an ordered *valued* differential field by taking the valuation given by the valuation ring $\mathcal{O}$ defined in (H2). An ordered valued differential field is an H-field iff it is a d-valued pre-H-field.

Example 5.3.6. Consider the field $L = \mathbb{R}(x)$ with x transcendental over $\mathbb{R}$, equipped with the unique derivation which has constant field $\mathbb{R}$ and $x' = 1$. Furthermore, equip L with the trivial valuation and the unique field ordering determined by requiring $x > \mathbb{R}$. It follows that L is a pre-H-field with residue field isomorphic to $\mathbb{R}(x)$. However, L is not an H-field. Indeed, the residue field is not even algebraic over the image of the constant field $\mathbb{R}$ under the residue map.

Example 5.3.7. Consider the Hardy field $\mathbb{Q}$. Using [**34**, Theorem 2] twice, we can extend to the Hardy field $\mathbb{Q}(x)$ where $x' = 1$, and further extend to the Hardy field $K = \mathbb{Q}\big(x, \arctan(x)\big)$ where $\big(\arctan(x)\big)' = 1/(1+x^2)$. Each of these three Hardy fields are pre-H-fields (see [**6**, §10.5]); however, $\mathbb{Q}$ and $\mathbb{Q}(x)$ are H-fields whereas K is *not* an H-field: the constant field of K is $\mathbb{Q}$ whereas the residue field of K is $\mathbb{Q}(\pi)$. Note that in this example the residue field $\mathbb{Q}(\pi)$ is also not algebraic over the image of the constant field $\mathbb{Q}$. For details of these Hardy field extensions and justification of the claims about K, see the table and discussion below:

Hardy field	Value group	Residue field	Constant field	H-field?
$\mathbb{Q}$	$\{0\}$	$\mathbb{Q}$	$\mathbb{Q}$	Yes
$\mathbb{Q}(x)$	$\mathbb{Z}v(x)$	$\mathbb{Q}$	$\mathbb{Q}$	Yes
$K = \mathbb{Q}\big(x, \arctan(x)\big)$	(1) $\mathbb{Z}v(x)$	(1) $\mathbb{Q}(\pi)$	(2) $\mathbb{Q}$	No

(1) Note that $\lim_{x\to\infty} \arctan(x) = \pi/2$, hence $\arctan(x) \preccurlyeq 1$ and the residue field $\operatorname{res}(K)$ of K contains $\mathbb{Q}(\pi)$. Recall that by the Lindemann-Weierstrass theorem [**26**], π is transcendental over $\mathbb{Q}$, so $\operatorname{res}\big(\arctan(x)\big) = \pi/2$ is transcendental over $\operatorname{res}\big(\mathbb{Q}(x)\big) = \mathbb{Q}$. It follows that $\arctan(x)$ is transcendental over $\mathbb{Q}(x)$ (otherwise $\operatorname{res}(K)$ would be algebraic over $\operatorname{res}\big(\mathbb{Q}(x)\big) = \mathbb{Q}$). By [**6**, 3.1.31], it follows that $\Gamma_K = \Gamma_{\mathbb{Q}(x)} = \mathbb{Z}v(x)$, and

$$\operatorname{res}(K) \;=\; \operatorname{res}\big(\mathbb{Q}(x)\big)\big(\operatorname{res}(\arctan(x))\big) \;=\; \mathbb{Q}(\pi/2) \;=\; \mathbb{Q}(\pi).$$

(2) As K is a pre-H-field, it follows that the constant field is necessarily a subfield of the residue field $\mathbb{Q}(\pi)$. A routine brute force verification shows that $1/(1+x^2) \notin \partial\big(\mathbb{Q}(x)\big)$. Thus the differential ring $\mathbb{Q}(x)\big[\arctan(x)\big]$ is simple by [**6**, 4.6.10] (see [**6**] for definitions of *differential ring* and *simple differential ring*). Furthermore, as $\mathbb{Q}(x)\big[\arctan(x)\big]$ is finitely generated as a $\mathbb{Q}(x)$-algebra, it follows that C_K is algebraic over $\mathbb{Q}$ by [**6**, 4.6.12]. However, $\mathbb{Q}$ is algebraically closed in $\mathbb{Q}(\pi)$ (because π is transcendental over $\mathbb{Q}$) and so $C_K = \mathbb{Q}$.

Algebraic extensions. *In this subsection K is an asymptotic field.* We fix an algebraic field extension L of K. By ADH 5.2.2 we equip L with the unique derivation extending the derivation ∂ of K. By *Chevalley's Extension Theorem* [**6**, 3.1.15] we equip L with a valuation extending the valuation of K. Thus L is a valued differential field extension of K. We record here several properties that are preserved in this algebraic extension:

ADH 5.3.8. *The valued differential field L is an asymptotic field* [**6**, 9.5.3]. *Also:*

(1) *If K is of H-type, then so is L.*
(2) *If K is pre-*d*-valued, then so is L* [**6**, 10.1.22].
(3) *K is grounded iff L is grounded.*

(1) and (3) of ADH 5.3.8 follow from the corresponding facts about the divisible hull of an asymptotic couple; see ADH 2.2.10.

Furthermore, assume that K is equipped with an ordering making it a pre-H-field, and $L|K$ is an algebraic extension of ordered differential fields.

ADH 5.3.9. *There is a unique convex valuation ring of L extending the valuation ring of K* [**6**, 3.5.18]*. Equipped with this valuation ring, L is a pre-H-field extension of K* [**6**, 10.5.4]*. Furthermore, if K is an H-field and $L = K^{rc}$, a real closure of K, then L is also an H-field* [**6**, 10.5.6].

ω-freeness. *In this subsection assume that K is an ungrounded H-asymptotic field with $\Gamma \neq \{0\}$.* We review here the important and robust property of ω-*freeness*.

Definition 5.3.10. A **logarithmic sequence (in K)** is a well-indexed sequence (ℓ_ρ) in $K^{\succ 1}$ such that
(1) $\ell'_{\rho+1} \asymp \ell^\dagger_\rho$, i.e., $v(\ell_{\rho+1}) = \chi(v\ell_\rho)$, for all ρ;
(2) $\ell_{\rho'} \prec \ell_\rho$ whenever $\rho' > \rho$;
(3) (ℓ_ρ) is coinitial in $K^{\succ 1}$: for each $f \in K^{\succ 1}$ there is an index ρ with $\ell_\rho \preccurlyeq f$.

Such sequences exist and can be constructed by transfinite recursion. Next, we define the function:

$$\omega : K \to K, \quad \omega(z) = -(2z' + z^2).$$

Definition 5.3.11. An **ω-sequence (in K)** is a sequence of the form $(\omega_\rho) = \big(\omega(-(\ell^{\dagger\dagger}_\rho))\big)$ where (ℓ_ρ) is a logarithmic sequence in K.

It follows from results in [**6**, §11.7] that all ω-sequences in K are pc-sequences, and are equivalent as pc-sequences. Furthermore, the divergence of an ω-sequence is actually a first order property:

ADH 5.3.12. [**6**, 11.7.8] *The following conditions on K are equivalent:*
(1) *there is a divergent ω-sequence in K;*
(2) *every ω-sequence in K is divergent;*
(3) *for every $f \in K$ there is $g \succ 1$ in K with $f - \omega(-g^{\dagger\dagger}) \succcurlyeq (g^\dagger)^2$.*

Definition 5.3.13. We say that an asymptotic field L is **ω-free** (or has **ω-freeness**) if L is ungrounded of H-type with $\Gamma_L \neq \{0\}$, and satisfies condition (3) in ADH 5.3.12 for L in the role of K

The following is immediate from the definition:

Lemma 5.3.14. *If K is a directed union of ω-free asymptotic subfields, then K is ω-free.*

Recall that $\mathbb{T}_{\log} = \cup_n \mathbb{R}[[\mathfrak{L}_n]]$ is a union of grounded H-asymptotic subfields. Thus in view of the following, $\mathbb{T}_{\log}$ is ω-free.

ADH 5.3.15. [**6**, 11.7.15] *If K is a union of grounded H-asymptotic subfields, then K is ω-free.*

We will not dive into the details of ω-freeness in this thesis. However, we will often include "ω-free" as a convenient hypothesis in results below. Here is one of the more compelling consequences of ω-freeness:

ADH 5.3.16. *Suppose K is* ω*-free. Then*

(1) *K has rational asymptotic integration (so K does not have a gap)* [**6**, 11.7.3 and 11.6.8]*; and*
(2) *if L is a pre-*d*-valued field extension of K of H-type which is* d*-algebraic over K, then L is* ω*-free* [**6**, 13.6.1].

ADH 5.3.16 above casts ω-freeness as a very robust *gap prevention property* in the sense that if K is ω-free and L is a d-algebraic pre-d-valued extension of K of H-type, then L has no gap. In Chapter 6 we consider a weaker gap prevention property: λ-*freeness*.

Newtonianity. *In this subsection K is a valued differential field.*

Definition 5.3.17. Let $P \in K\{Y\}^{\neq}$. We define the **dominant degree** of P to be the natural number

$$\operatorname{ddeg} P \ := \ \max\big\{d : v(P) = v(P_d)\big\}$$

and the **dominant multiplicity** of P to be the natural number

$$\operatorname{dmul} P \ := \ \min\big\{e : v(P) = v(P_e)\big\}.$$

For valued differential fields, we have the notion of *differential-henselian* which is a differential analogue of *henselian* for valued fields:

Definition 5.3.18. We say that K is **differential-henselian** if K has small derivation and every $P \in K\{Y\}^{\neq}$ such that $\operatorname{ddeg} P = 1$ has a zero in the valuation ring $\mathcal{O}$ of K.

Even though $\mathbb{T}_{\log}$ is a valued differential field with small derivation, it is *not* differential-henselian. Indeed, the differential polynomial $P(Y) = 1 - Y'$ has dominant degree 1, however it does not have a zero in $\mathcal{O}_{\mathbb{T}_{\log}}$ since $\ell_0 \succ 1$.

The recourse in this situation is to work with an "eventual" variant of differential-henselianity called *newtonianity.*

In the rest of this subsection K is an ungrounded H-asymptotic field with $\Gamma \neq \{0\}$. We say an element $\phi \in K^{\times}$ is **active** if $v\phi \in \Psi^{\downarrow}$.

ADH 5.3.19. [**6**, §11.1] *Let $P \in K\{Y\}^{\neq}$. Then there is an active $\phi_0 \in K^{\times}$ and $M, N \in \mathbb{N}$ such that for all active $\phi \preccurlyeq \phi_0$, $M = \operatorname{dmul}(P^{\phi})$ and $N = \operatorname{ddeg}(P^{\phi})$.*

The natural number M from ADH 5.3.19 above is called the **Newton multiplicity** of P (notation: $\operatorname{nmul} P$) and the natural number N is called the **Newton degree** of P (notation: $\operatorname{ndeg} P$). The values of M and N do not depend on the choice of ϕ_0.

We are now in a position to define the most important notion from [**6**]:

Definition 5.3.20. We define $P \in K\{Y\}$ to be **quasilinear** if $\operatorname{ndeg} P = 1$, and we define K to be **newtonian** if every quasilinear $P \in K\{Y\}$ has a zero in $\mathcal{O}$.

Newtonianity is an "eventual" variant of differential-henselianity in the sense that Newton degree is computed in ADH 5.3.19 as an eventual quantity for sufficiently large $v\phi$ for active $\phi \in K^{\times}$.

The following is very useful for showing that specific asymptotic fields of interest are newtonian. In particular, it shows that $\mathbb{T}_{\log}$ is newtonian:

ADH 5.3.21. [**6**, 15.0.1] *If K is* d*-valued with $\partial K = K$, and K is a directed union of spherically complete grounded* d*-valued subfields, then K is newtonian.*

We will also be interested in the following "fragments" of newtonianity:

Definition 5.3.22. We say that K is m**-linearly newtonian** if every $P \in K\{Y\}^{\neq}$ with $\deg P = 1$, $\operatorname{ndeg} P = 1$, and $\operatorname{order}(P) \leqslant m$, has a zero in $\mathcal{O}$. We say that K is **linearly newtonian** if K is m-linearly newtonian for every m.

Assuming asymptotic integration, 1-linearly newtonian implies small integration and small exponential integration:

ADH 5.3.23. [**6**, 14.2.5] *Assume K has asymptotic integration and is* 1*-linearly newtonian. Then K is* d*-valued and $\partial\smallo = \mathrm{I}(K) = (1+\smallo)^{\dagger}$.*

In fact, ADH 5.3.23 is perhaps (in retrospect) the original motivation for our fascination with small integration and small exponential integration in Chapters 6 and 7 below.

5.4. Almost special elements

In this section K is a valued differential field. We first recall two types of pc-sequences which can occur in a valued differential field:

Let (a_ρ) be a pc-sequence in K. We say that (a_ρ) is of **differential-algebraic type over** K (or d**-algebraic type over** K, for short) if $G(b_\lambda) \rightsquigarrow 0$ for some $G(Y) \in K\{Y\}$ and some pc-sequence (b_λ) in K equivalent to (a_ρ). A **minimal differential polynomial of** (a_ρ) over K is a differential polynomial $G(Y) \in K\{Y\}$ with the following properties:

(1) $G(b_\lambda) \rightsquigarrow 0$ for some pc-sequence (b_λ) in K equivalent to (a_ρ) (so $G \notin K$);
(2) $H(b_\lambda) \not\rightsquigarrow 0$ whenever $H \in K\{Y\}$ has lower complexity than G and the pc-sequence (b_λ) in K is equivalent to (a_ρ).

Thus (a_ρ) is of d-algebraic type over K iff (a_ρ) has a minimal differential polynomial over K.

We say that (a_ρ) is of **differential-transcendental type over** K (or d**-transcendental type over** K, for short) if it is not of d-algebraic type over K, that is, $G(b_\lambda) \not\rightsquigarrow 0$ for each $G \in K\{Y\}$ and each pc-sequence (b_λ) in K equivalent to (a_ρ).

We now further assume $K \subseteq L$ is an immediate asymptotic extension of H-asymptotic fields with rational asymptotic integration. First a consequence of [**6**, 11.3.8]:

Lemma 5.4.1. *Let $a \in L \setminus K$ and let $\mathcal{F} \subseteq L\{Y\} \setminus L$ be a finite set of differential polynomials. Then there is a divergent pc-sequence $(a_\rho)_{\rho<\lambda}$ in K such that $a_\rho \rightsquigarrow a$ and for every ρ, $\gamma_\rho := v(a_{s(\rho)} - a_\rho) \in \Gamma$ and $\rho \mapsto \gamma_\rho$ is strictly increasing, and for each $G \in \mathcal{F}$ there is $d_G \in \mathbb{N}^{\geqslant 1}$ such that*

$$v\big(G(a_{\rho'}) - G(a)\big) - v\big(G(a_\rho) - G(a)\big) \ = \ d_G(\gamma_{\rho'} - \gamma_\rho) + o(\gamma_{\rho'} - \gamma_\rho) \quad \text{for every } \rho < \rho'.$$

In particular, $G(a_\rho) \rightsquigarrow G(a)$ for each $G \in \mathcal{F}$.

Proof. This follows from the proof of [**6**, 11.3.8]. The ε from the last display of that proof can be absorbed into the $o(\gamma_{\rho'} - \gamma_\rho)$ by requiring the following in the course of that proof:

(1) $\delta_\rho = o(\gamma_{s(\rho)} - \gamma_\rho) = o(\gamma_{\rho'} - \gamma_\rho)$ whenever $\rho' > \rho$,

(2) $vx_\rho = o(\delta_\rho)$ for each ρ, and

(3) $\varepsilon_\rho = o(\delta_\rho)$ for each ρ. $\square$

Let Δ be a nontrivial convex subgroup of Γ. We say that $a \in L \setminus K$ is **Δ-special (over K)** (respectively, **almost Δ-special (over K)**) if the set $v(a-K)$ is Δ-special (respectively, almost Δ-special) as a subset of Γ, in the sense of Section 2.1. We say that $a \in L \setminus K$ is **special (over K)** (respectively, **almost special (over K)**) if it is Δ-special over K (respectively, almost Δ-special over K) for some nontrivial convex subgroup Δ of Γ. Finally, we say that an extension $L \supseteq K$ is **special** (respectively, **almost special**) if every $a \in L \setminus K$ is special over K (respectively, almost special over K).

Lemma 5.4.2. *Suppose $a \in L\setminus K$ is Δ-special over K and (a_ρ) is a divergent pc-sequence in K with minimal differential polynomial $G(Y)$ over K such that $a_\rho \rightsquigarrow a$. Let $P \in K\{Y\} \setminus K$ have lower complexity than G. Then $P(a)$ is almost Δ-special over K.*

Proof. By Lemma 5.4.1, we may assume the divergent pc-sequence (a_ρ) in K is such that $\gamma_\rho := v(a_{s(\rho)} - a_\rho) \in \Gamma$ for every $\rho < \lambda$, the map $\rho \mapsto \gamma_\rho$ is strictly increasing, $P(a_\rho) \rightsquigarrow P(a)$, and that there is $d \in \mathbb{N}^{\geqslant 1}$ such that

$$v\big(P(a_{\rho'}) - P(a)\big) - v\big(P(a_\rho) - P(a)\big) \;=\; d(\gamma_{\rho'} - \gamma_\rho) + o(\gamma_{\rho'} - \gamma_\rho) \quad \text{for every } \rho < \rho'.$$

In particular, $\big(P(a_\rho)\big)$ is a pc-sequence in K, and since P has lower complexity than G, $\big(P(a_\rho)\big)$ is a divergent pc-sequence in K (if there were $b \in K$ such that $P(a_\rho) \rightsquigarrow b$, then $P(a_\rho) - b \rightsquigarrow 0$, but $P - b \in K\{Y\} \setminus K$ also has lower complexity than G).

Next, as a is Δ-special, the sequence (γ_ρ) is cofinal in Δ. Let ρ_0 be some index such that $\gamma_{\rho_0} \in \Delta^{>}$. Then $0 < \gamma_{\rho_0} < \gamma_\rho$ for all $\rho_0 < \rho$ and so $o(\gamma_\rho - \gamma_{\rho_0}) = o(\gamma_\rho)$ for all $\rho_0 < \rho$. For this fixed ρ_0 we get for $\rho_0 < \rho$:

$$\begin{aligned} v\big(P(a_\rho) - P(a)\big) &= d\gamma_\rho + o(\gamma_\rho) - \underbrace{\big(d\gamma_{\rho_0} + v(P(a_{\rho_0}) - P(a))\big)}_{:=\alpha} \\ &= d\gamma_\rho + o(\gamma_\rho) - \alpha \end{aligned}$$

which increases cofinally through $\Delta - \alpha$. Together with the earlier observation that $\big(P(a_\rho)\big)$ is a divergent pc-sequence in K, we conclude that $P(a)$ is almost Δ-special over K. $\square$

Lemma 5.4.3. *Suppose $a \in L \setminus K$ is Δ-special over K. Furthermore, suppose there is a divergent pc-sequence (a_ρ) in K with minimal differential polynomial $G(Y)$ over K such that $a_\rho \rightsquigarrow a$. Further assume that the order of G is at least* 1*. Then, given $P, Q \in K\{Y\} \setminus K$ of lower complexity than G such that Q has lower order than G, we have $Q(a) \neq 0$ and for $R := P/Q$, if $R(a) \in L \setminus K$, then $R(a)$ is almost Δ-special over K.*

Proof. Without loss of generality, we may assume $Q \nmid P$. Furthermore, we may arrange that (a_ρ) satisfies the conclusion of Lemma 5.4.1 with $\mathcal{F} = \big\{P, Q, P(X)Q(a) - P(a)Q(X)\big\}$. Then as $\big(Q(a_\rho)\big)$ is a pc-sequence in K, we get that $Q(a_\rho) \neq 0$ eventually, and $Q(a_\rho) \rightsquigarrow Q(a)$, so $Q(a) \neq 0$ since Q has lower complexity than G. Thus we will pass to a cofinal subsequence and arrange that $Q(a_\rho) \neq 0$ for all ρ. Next note that

$$\begin{aligned} v\left(\frac{P(a_\rho)}{Q(a_\rho)} - \frac{P(a)}{Q(a)}\right) &= v\left(\frac{P(a_\rho)Q(a) - P(a)Q(a_\rho)}{Q(a_\rho)Q(a)}\right) \\ &= v\big(P(a_\rho)Q(a) - P(a)Q(a_\rho)\big) - v\big(Q(a_\rho)\big) - v\big(Q(a)\big). \end{aligned}$$

The quantity $v\big(Q(a_\rho)\big)$ eventually takes a constant value in Γ. By an argument as in Lemma 5.4.2, we get that $\big(R(a_\rho)\big)$ is a pc-sequence in K and that $v\big(R(a_\rho) - R(a)\big)$ eventually increases cofinally through $\Delta - \alpha$ for some $\alpha \in \Gamma$. Finally, we must show that $\big(R(a_\rho)\big)$ is a divergent pc-sequence in K. Assume towards a contradiction that there is $b \in K$ such that $R(a_\rho) \rightsquigarrow b$. Then

$$v\left(\frac{P(a_\rho)}{Q(a_\rho)} - b\right) \;=\; v\big(P(a_\rho) - bQ(a_\rho)\big) - v\big(Q(a_\rho)\big)$$

is eventually strictly increasing. Since $v\big(Q(a_\rho)\big)$ is eventually constant, $v\big(P(a_\rho) - bQ(a_\rho)\big)$ is eventually strictly increasing, i.e., $\big(P(a_\rho) - bQ(a_\rho)\big)$ is a pc-sequence and $P(a_\rho) - bQ(a_\rho) \rightsquigarrow 0$. This is a contradiction because $P(X) - bQ(X)$ has lower complexity than G. $\square$

Note that in Lemma 5.4.3 above we did not require that $G(a) = 0$, and so we could not necessarily conclude there that all elements of $K\langle a\rangle \setminus K$ were Δ-special over K (for instance, if a is d-transcendental over K, but is approximated by a divergent pc-sequence of d-algebraic type). If we do assume $G(a) = 0$, then we get the following:

Corollary 5.4.4. *Suppose that $a \in L \setminus K$ is special over K. Furthermore, suppose (a_ρ) is a divergent pc-sequence in K with minimal differential polynomial $G(Y)$ over K such that $a_\rho \rightsquigarrow a$ and $G(a) = 0$. Then every element in $K\langle a\rangle \setminus K$ is almost special over K.*

5.5. Constant field extensions

In this section K is a d*-valued field.* We will first review the state of affairs when it comes to extending the constant field of K.

First, if L is a differential field extension of K with constant field $D \supseteq C$ such that $L = K(D)$, then K and D are linearly disjoint over C by [**6**, 4.6.16].

Conversely, suppose L is a field extension of K with a subfield $D \supseteq C$ such that K and D are linearly disjoint over C and $L = K(D)$. Then by [**6**, 4.6.21] there is a unique derivation on L extending the derivation on K that is trivial on D; with this derivation, the constant field of L is D.

In the rest of this section L is a differential field extension of K with constant field $D \supseteq C$ such that $L = K(D)$. In this case we can naturally make $L \supseteq K$ into an extension of d-valued fields:

ADH 5.5.1. [**6**, 10.5.15] *There exists a unique valuation on the differential field L extending the valuation of K which is trivial on D. This valuation has the same value group as K. Equipped with this valuation, L is* d*-valued and thus L is a* d*-valued asymptotic extension of K.*

We now consider L to be equipped with the valuation of ADH 5.5.1 for the rest of the section. Furthermore, if K is an H-field and D comes equipped with an appropriate ordering on it, then there is a natural way to make $L \supseteq K$ into an extension of H-fields:

ADH 5.5.2. [**6**, 10.5.16] *Suppose K and D are equipped with orderings which make K an H-field and D an ordered field extension of C. Then there is a unique field ordering of L extending the orderings of K and D in which the valuation ring of L is convex. With this ordering L is an H-field and thus $L \supseteq K$ is an extension of H-fields.*

The following shows that we can essentially reduce to a gaussian valuation in the current situation. It will be useful when dealing with new constants, either algebraic or transcendental:

Lemma 5.5.3. *Suppose $c_0, \ldots, c_n \in C_L$ are linearly independent over C and $a_0, \ldots, a_n \in K$. Then*

$$v\left(\sum_i c_i a_i\right) \;=\; \min_i v(a_i).$$

Proof. We may assume that $v(a_0) \leqslant v(a_1), \ldots, v(a_n)$. It suffices to show that $1 \asymp \sum_i c_i a_i / a_0$. As K is d-valued, there are $d_i \in C$ and $\varepsilon_i \in \smallo$ such that $a_i/a_0 = d_i + \varepsilon_i$ for $i = 0, \ldots, n$. In particular, $d_0 = 1$. Thus

$$\sum_i c_i a_i / a_0 \;=\; \sum_i c_i d_i + \sum_i c_i \varepsilon_i.$$

As the c_i's are C-linearly independent and not every d_i is equal to zero, it follows that $\sum_i c_i d_i \in C_L^\times$ and $\sum_i c_i \varepsilon_i \prec 1$. In particular, $\sum_i c_i a_i / a_0 \asymp 1$. □

To motivate Proposition 5.5.4 below we present the following scenario. Suppose we have $f \in L \setminus K[D] = K(D) \setminus K[D]$ and we are interested in how well we can approximate f by elements from the *ring* $K[D]$. As a simple example, suppose $c \in D$ is transcendental over C, and $\eta \in K$ is such that $\eta \succ 1$. Consider the element:

$$f \;:=\; \frac{\eta}{\eta - c} \;=\; \frac{1}{1 - c\varepsilon}$$

for $\varepsilon := \eta^{-1}$. By the geometric series we can very naively expand f as follows:

$$f \;=\; 1 + c\varepsilon + c^2\varepsilon^2 + c^3\varepsilon^3 + \cdots$$

With any luck, we might hope the following things are true:

(1) The sequence of partial sums $(\sum_{i=0}^n c^i \varepsilon^i)$ pseudoconverges to f but has no pseudolimit in $K[D]$;
(2) f is Δ-*special over* $K[D]$ i.e., $v(f - K[D]) = (\alpha + \Delta)^{\downarrow} \subseteq \Gamma$ for some $\alpha \in \Gamma$, where Δ is the smallest convex subgroup of Γ which contains $v\varepsilon$.
(3) Something analogous to (1) and (2) above holds for *all* elements of $L \setminus K[D]$.

Proposition 5.5.4 below gives us something along these lines, but with "does not decelerate" in place of "Δ-special". The main point is, we show that pc-sequences from $K[D]$ which approximate elements from $K(D) \setminus K[D]$ cannot have very exotic rates of pseudoconvergence.

Proposition 5.5.4. *Suppose K is henselian and let $f \in L \setminus K[D]$. Then $v\big(f - K[D]\big) \subseteq \Gamma$ does not decelerate and has cofinality ω.*

Proof. We have $a_i, b_j \in K$ and $c_i, d_j \in D$ for $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$ such that

$$f \;=\; \frac{a_0 c_0 + a_1 c_1 + \cdots + a_m c_m}{b_0 d_0 + b_1 d_1 + \cdots + b_n d_n}, \quad \text{where } b_0 d_0 + b_1 d_1 + \cdots + b_n d_n \neq 0.$$

Next we define the subfield $E_1 \subseteq K$ to be the algebraic closure of

$$E_0 \;:=\; C(a_0, \ldots, a_m, b_0, \ldots, b_n)$$

inside K. In particular, E_1 is henselian, $\operatorname{res}(E_1) = \operatorname{res}(K) = \operatorname{res}(C)$, and $\Delta := v(E_1) \subseteq \Gamma$ is countable, and has finite rank because

$$\operatorname{trdeg}_C E_0 \;=\; \operatorname{trdeg}_C E_1 \;<\; \infty$$

and $v(C) = \{0\} \subseteq \Gamma$. In general, Δ will not be a convex subgroup of Γ; however, Δ is contained in the divisible hull of $\Delta_0 = v(E_0^\times)$. Next, let E be a maximal immediate extension of E_1 inside of K (which exists by Zorn). Note that E is henselian since K is henselian.

The setup now is the following:

$$\begin{tikzcd}
 & K \arrow[dl] \arrow[d, dash] \arrow[dr, "v"] & & \\
\mathrm{res}(K) \arrow[d, dash] & E \arrow[dl] \arrow[d, dash] \arrow[dr, "v"] & \Gamma \arrow[d, dash] & \\
\mathrm{res}(K) \arrow[d, dash] & E_1 \arrow[dl] \arrow[d, dash] \arrow[dr, "v"] & \Delta \arrow[d, dash] & \text{(finite rank)} \\
\mathrm{res}(K) \arrow[d, dash] & E_0 \arrow[dl] \arrow[d, dash] \arrow[dr, "v"] & \Delta \arrow[d, dash] & \\
\mathrm{res}(K) \arrow[d, dash] & C \arrow[dl] \arrow[dr, "v"] & \Delta_0 \arrow[d, dash] & \\
\mathrm{res}(K) & & \{0\} &
\end{tikzcd}$$

The idea now is that E is in some sense a best finite-rank subfield of K when it comes to approximating the element f with pc-sequences.

Claim. *For each $h \in K \setminus E$, the set $v(h - E) \subseteq \Gamma$ has a maximum element in $\Gamma \setminus \Delta$.*

Proof of claim. First assume towards a contradiction that $v(h-E) \subseteq \Gamma$ does not have a largest element. Take a well-indexed sequence (e_ρ) in E such that $v(h - e_\rho)$ is strictly increasing and cofinal in $v(h - E)$. Then (e_ρ) is a pc-sequence in E and $e_\rho \rightsquigarrow h$. By maximality of E inside K, there is $e \in E$ such that $e_\rho \rightsquigarrow e$. Then $v(h-e) > v(f-E)$, a contradiction. Thus there is $e \in E$ such that $v(h-e) = \max v(h-E)$. Suppose again towards a contradiction that $v(h-e) \in \Delta$. Then take $e_0 \in E$ such that $h - e \asymp e_0$. By the d-valued assumption on K, there is $c \in C \subseteq E$ such that $h - e \sim ce_0$, or rather, $v(h-e-ce_0) > v(h-e)$, a contradiction. $\square$

Next we will show that elements from $E[D]$ can approximate f just as well as elements from $K[D]$ can:

Claim. *The sets $v\big(f - E[D]\big)$ and $v\big(f - K[D]\big)$ are mutually cofinal as subsets of Γ, i.e., $v\big(f - E[D]\big)^{\downarrow} = v\big(f - K[D]\big)^{\downarrow}$.*

Proof of claim. The direction $v\big(f - E[D]\big)^{\downarrow} \subseteq v\big(f - K[D]\big)^{\downarrow}$ is immediate from $E \subseteq K$. Thus it suffices to prove that for every $h \in K[D]$, there is $g \in E[D]$ such that $v(f-g) \geqslant v(f-h)$. Let $h \in K[D]$ and assume that $h \notin E[D]$ to avoid the trivial case. In picture form:

$$\begin{tikzcd}
h \in K[D] \arrow[r, "\subseteq"] \arrow[d, dash] & K(D) \arrow[d, dash] \\
\exists ? g \in E[D] \arrow[r, "\subseteq"] & E(D) \ni f
\end{tikzcd}$$

We have $h = \sum_i k_i c_i$ where all $k_i \in K, c_i \in D$ and the c_is are linearly independent over C. By the above claim, for each k_i, we may pick $e_i \in E$ such that $v(k_i - e_i) \in \Gamma_\infty \setminus \Delta$. Then, setting $g := \sum_i e_i c_i$, by Lemma 5.5.3 we have that

$$v(h-g) \;=\; v\left(\sum_i c_i(k_i - e_i)\right) \;=\; \min_i v(k_i - e_i) \in \Gamma \setminus \Delta.$$

Note also that $v(f-g) \in \Delta$ since $f, g \in E(D)$ and $v(E(D)^\times) = \Delta$. Assume towards a contradiction that $v(f-g) < v(f-h)$. Then

$$v(f-g) \;=\; v(g-h),$$

but $v(f-g) \in \Delta$ whereas $v(g-h) \in \Gamma \setminus \Delta$, a contradiction. We conclude that $v(f-g) \geqslant v(f-h)$. □

To conclude our proof of the proposition, note that the subset $v\big(f - E[D]\big) \subseteq \Delta$ has cofinality ω since Δ is countable. Furthermore, $[\Delta^{\neq}]$ is finite and so $v\big(f - E[D]\big) \subseteq \Delta$ does not decelerate by Lemma 2.1.10. By Lemma 2.1.9, it follows that $v\big(f - K[D]\big)$ does not decelerate either. □

5.6. Eventual generic valuations of linear differential operators

In this section K is an $\upomega$-free d*-valued field, g ranges over $K^\times$, ϕ over the active elements of K, and $P(Y)$ over $K\{Y\}^{\neq}$.*

We first recall the definitions of other quantities and functions associated to a nonzero differential polynomial. For existence of these quantities, see [**6**, §4.2, 4.5, and 11.1].

Definition 5.6.1. Given the isobaric decomposition $P = \sum_w P_{[w]}$ of P, we define the **dominant weight** of P to be

$$\operatorname{dwt}(P) \;:=\; \max\big\{w : v(P_{[w]}) = v(P)\big\}.$$

We also define the function $v_P : \Gamma_\infty \to \Gamma_\infty$ by

$$v_P(\gamma) \;:=\; v(P_{\times g}) \text{ if } vg = \gamma, \text{ and } v_P(\infty) := \infty$$

which we call the **generic valuation** of P. We define the **Newton weight** of P to be

$$\operatorname{nwt}(P) \;:=\; \text{eventual value of } \operatorname{dwt}(P^\phi).$$

Related to Newton weight is the function $\operatorname{nwt}_P : \Gamma \to \mathbb{N}$ defined by

$$\operatorname{nwt}_P(\gamma) \;:=\; \operatorname{nwt}(P_{\times g}) \text{ if } vg = \gamma.$$

We define the **eventual generic valuation** of P to be the function $v^{\mathrm{e}}_P : \Gamma \to \Gamma$ given by

$$v^{\mathrm{e}}_P(\gamma) \;:=\; \text{eventual value of } v_{P^\phi}(\gamma) - \operatorname{nwt}_P(\gamma)v(\phi).$$

Finally, if P is homogeneous of degree 1, we define the set of **eventual exceptional values** of P to be

$$\mathscr{E}^{\mathrm{e}}(P) \;:=\; \big\{\gamma : \operatorname{nwt}_P(\gamma) \geqslant 1\big\} \subseteq \Gamma.$$

The $\upomega$-free assumption on K gives us the following:

ADH 5.6.2. [**6**, 14.2.7] *Suppose P is homogeneous of degree 1. Then the map $\gamma \mapsto v^{\mathrm{e}}_P(\gamma) : \Gamma \setminus \mathscr{E}^{\mathrm{e}}(P) \to \Gamma$ is strictly increasing and surjective.*

By convention, we consider $(v^e_P)^{-1}$ to be the inverse of the bijection $v^e_P : \Gamma \setminus \mathscr{E}^e(P) \to \Gamma$.

In the rest of this section we fix $s \in K$ and assume there is an $h \in K^\times$ such that

$$v(s - h^\dagger) \ = \ \max v(s - K^\dagger) \ \in \ (\Psi^{\downarrow} \setminus \Psi) \cup \{\infty\}.$$

Fix such an h. Then $h^\dagger$ is a best approximation to s among the elements of $K^\dagger$. In terms of this h, we get a more explicit version of [**6**, 9.7.2]:

Lemma 5.6.3. *If $g \not\asymp h$, then*

$$v(g^\dagger - s) \ = \ \min\big(\psi(vg - vh), v(h^\dagger - s)\big) \ \in \ \Psi^{\downarrow}$$

and if $g \asymp h$, then

$$v(g^\dagger - s) \ \in \ (\Gamma^>)' \cup \big\{v(h^\dagger - s)\big\}.$$

Proof. Note that $v(g^\dagger - s) \leqslant v(h^\dagger - s)$ gives $g^\dagger - h^\dagger \not\sim s - h^\dagger$. Assuming first $g \not\asymp h$,

$$g^\dagger - h^\dagger \ = \ (g/h)^\dagger$$

gives

$$v(g^\dagger - h^\dagger) \ = \ \psi(vg - vh).$$

Using

$$g^\dagger - s \ = \ (g^\dagger - h^\dagger) - (s - h^\dagger),$$

it follows that

$$v(g^\dagger - s) \ = \ \min\big(\underbrace{v(g^\dagger - h^\dagger)}_{\in\Psi}, \underbrace{v(s - h^\dagger)}_{\notin\Psi}\big) \ = \ \min\big(\psi(vg - vh), v(h^\dagger - s)\big).$$

Next, suppose that $g \asymp h$. Then $g/h \asymp 1$ and so

$$v(g^\dagger - h^\dagger) \ = \ v\big((g/h)^\dagger\big) \ \in \ (\Gamma^>)' \cup \{\infty\}.$$

Thus

$$v(g^\dagger - s) \ = \ \min\big(v(g^\dagger - h^\dagger), v(s - h^\dagger)\big) \ \in \ (\Gamma^>)' \cup \big\{v(s - h^\dagger)\big\}. \qquad \square$$

Note that in Lemma 5.6.3, if $g \not\asymp h$, then $v(g^\dagger - s)$ depends only on vg.

We now compute several of the above quantities for the differential polynomial $P(Y) := Y' - sY$:

Lemma 5.6.4. *The function $\operatorname{nwt}_P : \Gamma \to \mathbb{N}$ is given by:*

$$\operatorname{nwt}_P(\gamma) \ = \ \begin{cases} 0 & \text{if } \gamma \neq vh \text{ or } s \notin K^\dagger \\ 1 & \text{if } \gamma = vh \text{ and } s \in K^\dagger \end{cases}$$

Thus the set of eventual exceptional values of P is:

$$\mathscr{E}^e(P) \ = \ \begin{cases} \emptyset & \text{if } s \notin K^\dagger \\ \{vh\} & \text{if } s \in K^\dagger \end{cases}$$

Given $\gamma \in \Gamma$ we have, eventually:

$$v_{P^\phi}(\gamma) \;=\; \begin{cases} \gamma + \psi(\gamma - vh) & \text{if } \psi(\gamma - vh) < v(h^\dagger - s) \text{ (so } \gamma \neq vh\text{)} \\ \gamma + v(h^\dagger - s) & \text{if } \psi(\gamma - vh) > v(h^\dagger - s) \text{ (so } s \notin K^\dagger\text{)} \\ \gamma + v\phi & \text{if } \psi(\gamma - vh) = v(h^\dagger - s) \text{ (so } s \in K^\dagger \text{ and } \gamma = vh\text{)} \end{cases}$$

The function $v_P^e : \Gamma \to \Gamma$ is given by:

$$v_P^e(\gamma) \;=\; \begin{cases} \gamma + \psi(\gamma - vh) & \text{if } \psi(\gamma - vh) < v(h^\dagger - s) \\ \gamma + v(h^\dagger - s) & \text{if } \psi(\gamma - vh) > v(h^\dagger - s) \\ \gamma & \text{if } \psi(\gamma - vh) = v(h^\dagger - s) \end{cases}$$

Proof. Let $\gamma = vg$. We need to study the eventual behavior of

$$P^\phi_{\times g} \;=\; \big(gY' + (g' - gs)Y\big)^\phi \;=\; g\big(\phi Y' + (g^\dagger - s)Y\big).$$

By Lemma 5.6.3, if $g \not\asymp h$, then $v(g^\dagger - s) \in \Psi^\downarrow$, so $g^\dagger - s \succ \phi$, eventually. If $h \asymp g$ and $s \notin K^\dagger$, then $v(g^\dagger - s) \in \Psi^\downarrow \setminus \Psi$ by our assumption on h, and so $g^\dagger - s \succ \phi$, eventually. If $h \asymp g$ and $s \in K^\dagger$, then $v(g^\dagger - s) \in (\Gamma^>)' \cup \{\infty\}$, so $g^\dagger - s \prec \phi$, eventually. $\square$

In the next corollary we set $\Gamma_P := \Gamma \setminus \mathscr{E}^e(P)$:

Corollary 5.6.5. *Define the following sets:*

$$\begin{aligned} C_P^1 &:= \big\{\gamma < vh : \psi(\gamma - vh) < v(h^\dagger - s)\big\} \\ C_P^2 &:= \big\{\gamma \in \Gamma_P : \psi(\gamma - vh) > v(h^\dagger - s)\big\} \\ C_P^3 &:= \big\{\gamma > vh : \psi(\gamma - vh) < v(h^\dagger - s)\big\} \end{aligned}$$

Then $C_P^1 < C_P^2 < C_P^3$, $C_P^1 \cup C_P^2 \cup C_P^3 = \Gamma_P$, and for $\gamma \in \Gamma_P$,

$$\begin{aligned} \gamma \in C_P^1 \cup C_P^3 &\implies v_P^e(\gamma) = \gamma + \psi(\gamma - vh) \\ \gamma \in C_P^2 &\implies v_P^e(\gamma) = \gamma + v(h^\dagger - s). \end{aligned}$$

Furthermore for $i \in \{1,2,3\}$, define

$$I_P^i \;:=\; v_P^e(C_P^i).$$

Then $I_P^1 < I_P^2 < I_P^3$ and $I_P^1 \cup I_P^2 \cup I_P^3 = \Gamma$. Also, for $\gamma \in \Gamma$,

$$\begin{aligned} \gamma \in I_P^1 \cup I_P^3 &\implies (v_P^e)^{-1}(\gamma) = \textstyle\int(\gamma - vh) + vh \\ \gamma \in I_P^2 &\implies (v_P^e)^{-1}(\gamma) = \gamma - v(h^\dagger - s). \end{aligned}$$

Proof. $C_P^1 < C_P^2 < C_P^3$ follows from the observation that $\gamma \mapsto \psi(\gamma - vh) : \Gamma^{<vh} \to \Gamma$ is an increasing function and $\gamma \mapsto \psi(\gamma - vh) : \Gamma^{>vh} \to \Gamma$ is a decreasing function, both of which are consequences of (HC). The formula for $(v_P^e)^{-1}$ is easily verified, considering the cases $\gamma \in C_P^1 \cup C_P^3$ and $\gamma \in C_P^2$ separately. Everything else follows from the fact that $v_P^e : \Gamma_P \to \Gamma$ is a strictly increasing bijection. $\square$

Lemma 5.6.6 follows immediately from the observation that $(v_P^e)^{-1}$ is a strictly increasing function:

Lemma 5.6.6. *Let $S \subseteq \Gamma$ be nonempty. Then the following conditions on S are equivalent:*

(1) *For cofinally many $\gamma \in S$, there is an $\varepsilon \in \Gamma^{>}$ such that $(v_P^{\mathrm{e}})^{-1}(\gamma + \varepsilon') \in S^{\downarrow}$;*
(2) *for all $\gamma \in S^{\downarrow}$, there is an $\varepsilon \in \Gamma^{>}$ such that $(v_P^{\mathrm{e}})^{-1}(\gamma + \varepsilon') \in S^{\downarrow}$.*

To motivate the following definition, suppose $\mathscr{E}^{e}(P) = \emptyset$, $\gamma \in \Gamma$, and $\varepsilon \in \Gamma^{>}$. Then $v_P^{\mathrm{e}}(\gamma) < \gamma + \varepsilon'$ by Lemma 5.6.4. Applying $(v_P^{\mathrm{e}})^{-1}$ to both sides of this inequality yields $\gamma < (v_P^{\mathrm{e}})^{-1}(\gamma + \varepsilon')$. In other words, if a nonempty subset S of Γ satisfies one of the two equivalent conditions of Lemma 5.6.6, then if you have an element $\gamma \in S$, then you can increase upwards at least a distance of $(v_P^{\mathrm{e}})^{-1}(\gamma+\varepsilon') - \gamma > 0$, for some $\varepsilon \in \Gamma^{>}$, and still remain in $S^{\downarrow}$.

Definition 5.6.7. We say that $S \subseteq \Gamma$ has the **v_P^{e}-yardstick property** if it is nonempty and satisfies one of the two equivalent conditions of Lemma 5.6.6.

Proposition 5.6.8. *Suppose $S \subseteq \Gamma$ has the v_P^{e}-yardstick property. Then one of the following holds:*

(1) *$S^{\downarrow}$ is Δ-fluent for some nontrivial convex subgroup Δ;*
(2) *$S^{\downarrow} = \Gamma^{<vh}$.*

In particular, S does not have a largest element.

Proof. Without loss of generality, we may assume that $S = S^{\downarrow}$.

Case 1: *For cofinally many $\gamma \in S$, there is $\varepsilon \in \Gamma^{>}$ such that $\gamma + \varepsilon' \in I_P^1$ and $(v_P^{\mathrm{e}})^{-1}(\gamma+\varepsilon') \in S$.* Then this holds for all $\gamma \in S$. Let $\gamma \in S$, and take $\varepsilon \in \Gamma^{>}$ such that $\gamma + \varepsilon' \in I_P^1$ and $(v_P^{\mathrm{e}})^{-1}(\gamma + \varepsilon') \in S$. By the v_P^{e}-yardstick property we have

$$\textstyle\int \big((\gamma+\varepsilon') - vh\big) + vh \ \in\ S.$$

Then by Lemma 2.5.10

$$\gamma \ <\ \gamma - \chi(\gamma - vh) \ <\ \textstyle\int \big((\gamma+\varepsilon') - vh\big) + vh,$$

and so S has the vh-yardstick property. Therefore by Lemma 2.5.6, either $S = \Gamma^{<vh}$ (if S happens to be jammed), or else S is Δ-fluent for some nontrivial convex subgroup Δ of Γ.

Case 2: *For cofinally many $\gamma \in S$, there is $\varepsilon \in \Gamma^{>}$ such that $\gamma + \varepsilon' \in I_P^2$ and $(v_P^{\mathrm{e}})^{-1}(\gamma+\varepsilon') \in S$.* Then this holds for all $\gamma \in S$: given $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1 \leqslant \gamma_2$ and $\varepsilon_2 \in \Gamma^{>}$, we can take $\varepsilon_1 \geqslant \varepsilon_2$ in Γ such that $\gamma_1 + \varepsilon_1' = \gamma_2 + \varepsilon_2'$.

From $I_P^2 \neq \emptyset$ we get $C_P^2 \neq \emptyset$, so $v(h^\dagger - s) \in \Psi^{\downarrow}$. By the v_P^{e}-yardstick property, we obtain $\varepsilon > 0$ in Γ such that $\gamma + \varepsilon' - v(h^\dagger - s) \in S$. Since $s\big(v(h^\dagger - s)\big) < \varepsilon'$, we have

$$\gamma + s\big(v(h^\dagger - s)\big) - v(h^\dagger - s) \ \in\ S.$$

However, from $v(h^\dagger - s) \in \Psi^{\downarrow}$ we get $s\big(v(h^\dagger - s)\big) - v(h^\dagger - s) > 0$. Let Δ be a nontrivial convex subgroup of Γ such that $\Delta < s\big(v(h^\dagger - s)\big) - v(h^\dagger - s)$. Then S is Δ-fluent.

Case 3: *For cofinally many $\gamma \in S$, there is $\varepsilon \in \Gamma^{>}$ such that $\gamma + \varepsilon' \in I_P^3$ and $(v_P^{\mathrm{e}})^{-1}(\gamma+\varepsilon') \in S$.* Then this holds for all $\gamma \in S$, as in Case 2. Now the argument of Case 1 works. □

CHAPTER 6

λ-freeness and the number of Liouville closures of an H-field

Most of this chapter is directly from [**16**].

In Section 6.1 we give a survey of the property of λ-*freeness*, citing many definitions and results from [**6**, §11.5 and §11.6]. Many of these results we cite, and later use, involve situations where λ-freeness is preserved in certain valued differential field extensions. The main result of this section is Proposition 6.1.18 which shows that a rather general type of asymptotic field extension preserves λ-freeness. Proposition 6.1.18 is related to the yardstick property of Section 2.5.

In Sections 6.2, 6.3, and 6.4, we show that under various circumstances, if a pre-differential-valued field or a pre-H-field K is λ-free, and we adjoin an integral or an exponential integral to K for an element in K that does not already have an integral or exponential integral, then the resulting field extension will also be λ-free. The arguments in all three sections mirror one another and the main results, Propositions 6.2.2, 6.3.3, and 6.4.3 are all instances of Proposition 6.1.18.

In Sections 6.5, and 6.6 we give two minor applications of the results of Sections 6.2, 6.3, and 6.4. In Section 6.5, we show that λ-freeness is preserved when passing to the *differential-valued hull* of a λ-free pre-d-field K (Theorem 6.5.2). In Section 6.6, we show that for λ-free d-valued fields K, the minimal henselian, integration-closed extension $K(\int)$ of K is also λ-free (Theorem 6.6.2).

In Section 6.7 we prove the main result of this chapter, Theorem 6.7.1. Combining this with Section 6.5, we also give a generalization of Theorem 6.7.1 to the setting of pre-H-fields (Corollary 6.7.3). Finally, we provide proofs of claims made in [**4**] and [**5**] (Corollary 6.7.6 and Remark 6.7.7).

6.1. λ-freeness

In this section K is an ungrounded H-asymptotic field with $\Gamma \neq \{0\}$.

λ-sequences and λ-freeness.

Definition 6.1.1. A **λ-sequence (in K)** is a sequence of the form $(\lambda_\rho) = \big(-(\ell_\rho^{\dagger\dagger})\big)$ where (ℓ_ρ) is a logarithmic sequence in K.

ADH 6.1.2. [**6**, 11.5.2] *Every λ-sequence is a pc-sequence of width $\{\gamma \in \Gamma_\infty : \gamma > \Psi\}$.*

ADH 6.1.3. [**6**, 11.5.3] *All λ-sequences are equivalent as pc-sequences.*

In the rest of this section we fix in K a distinguished logarithmic sequence (ℓ_ρ) along with its corresponding λ-sequence (λ_ρ). Nothing that we will discuss depends on the choice of this λ-sequence.

ADH 6.1.4. [**6**, 11.6.1] *The following conditions on K are equivalent:*

(1) (λ_ρ) *has no pseudolimit in K;*

(2) *for all $s \in K$ there is $g \in K^{\succ 1}$ such that $s - g^{\dagger\dagger} \succcurlyeq g^{\dagger}$.*

Definition 6.1.5. An asymptotic field L is said to be **$\uplambda$-free** (or has **$\uplambda$-freeness**) if it is ungrounded of H-type with $\Gamma_L \neq \{0\}$ and satisfies condition (2) in ADH 6.1.4 for L in the role of K.

The following is immediate from the definition of $\uplambda$-freeness and is a remark made after [**6**, 11.6.4]:

ADH 6.1.6. *Suppose L is an H-asymptotic extension of K such that Ψ is cofinal in Ψ_L. If L is $\uplambda$-free, then so is K.*

ADH 6.1.7. [**6**, 11.6.4] *If K is a union of grounded asymptotic subfields, then K is $\uplambda$-free.*

Lemma 6.1.8. *If K is a directed union of $\uplambda$-free asymptotic subfields, then K is $\uplambda$-free.*

Proof. This follows easily from the ADH 6.1.4(2) characterization of $\uplambda$-freeness. ☐

Algebraic extensions. Ultimately, we will show that $\uplambda$-freeness is preserved under arbitrary Liouville extensions of H-fields. For the time being, we have the following results concerning $\uplambda$-freeness for algebraic extensions:

ADH 6.1.9. [**6**, 11.6.7] *If K is $\uplambda$-free, then so is its henselization K^{h}.*

ADH 6.1.10. [**6**, 11.6.8] *K is $\uplambda$-free iff the algebraic closure K^{a} of K is $\uplambda$-free.*

Lemma 6.1.11. *Suppose K is equipped with an ordering making it a pre-H-field. If K is $\uplambda$-free, then so is its real closure K^{rc}.*

Proof. This follows from ADH 6.1.10 and 6.1.6, using the fact that $\Psi_{K^{\mathrm{a}}} = \Psi_{K^{\mathrm{rc}}} = \Psi$. ☐

Big exponential integration. The "big" exponential integral extensions considered here complement the Liouville extensions considered in §6.2, §6.3, and §6.4 below. In particular, we fix an element $s \in K$ that does not have an exponential integral in K, i.e., $s \notin K^{\dagger}$, and we assume that s is *bounded away* from the logarithmic derivatives in K in the sense that

$$S \ := \ \big\{ v(s - a^{\dagger}) : a \in K^{\times} \big\} \subseteq \Psi^{\downarrow}.$$

Then under the following circumstances, $\uplambda$-freeness is preserved when adjoining an exponential integral for such an s:

ADH 6.1.12. [**6**, 11.6.12] *Suppose K is $\uplambda$-free and Γ is divisible, and let $f^{\dagger} = s$, where $f \neq 0$ lies in an H-asymptotic field extension of K. Suppose*

(1) *S does not have a largest element, or*

(2) *S has a largest element and $[\gamma + vf] \notin [\Gamma]$ for some $\gamma \in \Gamma$.*

Then $K(f)$ is $\uplambda$-free.

ADH 6.1.13. [**6**, 10.5.20 and 11.6.13] *Suppose K is equipped with an ordering making it a real closed H-field such that $s < 0$. Let $L = K(f)$ be a field extension of K such that f is transcendental over K, equipped with the unique derivation extending the derivation of K such that $f^{\dagger} = s$. Then there is a unique pair consisting of a valuation of $L = K(f)$ and a field ordering on L making it a pre-H-field extension of K with $f > 0$. With this valuation and ordering L is an H-field and Ψ is cofinal in Ψ_L. Furthermore, if K is $\uplambda$-free, then so is L.*

Gap creators. Let $s \in K$. We say that s **creates a gap over** K if vf is a gap in $K(f)$, for some element $f \neq 0$ in some H-asymptotic field extension of K with $f^\dagger = s$.

ADH 6.1.14. [**6**, 11.6.1 and 11.6.8] *If K is λ-free, then K has rational asymptotic integration, and no element of K creates a gap over K.*

Remark 6.1.15. ADH 6.1.14 suggests that one way to view λ-freeness is as a *gap prevention property.* How good is λ-freeness as a gap prevention property? Already the above results show that it is impossible to create a gap from algebraic extensions and certain exponential integral extensions of a λ-free field. However, we can do a little bit better than that: by our results Propositions 6.2.2, 6.3.3, and 6.4.3 below, it follows that λ-freeness is also safely preserved (and so gaps are prevented) when passing to much more general Liouville extensions of a λ-free field.

On the other hand, *not* being λ-free does not bode well for preventing a gap:

ADH 6.1.16. *Suppose K has asymptotic integration, Γ is divisible, and $\lambda_\rho \rightsquigarrow \lambda \in K$. Then $s = -\lambda$ creates a gap over K. Furthermore, for every H-asymptotic extension $K(f)$ of K such that $f^\dagger = s$, vf is a gap in $K(f)$.*

Proof. The first claim is [**6**, 11.5.14] and the second claim is a remark after [**6**, 11.5.14]. $\square$

The following will be our main method of producing gaps in Liouville extensions of H-fields in Section 6.7 below:

ADH 6.1.17. *Suppose that K is equipped with an ordering making it a real closed H-field with asymptotic integration, and $\lambda_\rho \rightsquigarrow \lambda \in K$. Let $L = K(f)$ be a field extension of K with f transcendental over K equipped with the unique derivation extending the derivation of K such that $f^\dagger = -\lambda$. Then there is a unique pair consisting of a valuation of L and a field ordering on L making it an H-field extension of K with $f > 0$. With this valuation and ordering, vf is a gap in L.*

Proof. By [**6**, 11.5.13] we can apply [**6**, 10.5.20] with either $-\lambda$ or λ playing the role of s, whichever one is negative. Either way, a positive exponential integral f of $-\lambda$ will be adjoined. By ADH 6.1.16, vf is a gap in L. $\square$

The yardstick argument. Assume that $L = K(y)$ is an immediate H-asymptotic extension of K where y is transcendental over K. In particular, $v(y - K)$ is a nonempty downward closed subset of Γ without a greatest element.

Proposition 6.1.18. *Assume K is henselian and λ-free, and $v(y-K) \subseteq \Gamma$ has the yardstick property. Then $L = K(y)$ is λ-free.*

Proof. Assume towards a contradiction that L is not λ-free. Take $\lambda \in L \setminus K$ such that $\lambda_\rho \rightsquigarrow \lambda$. By ADH 6.1.2, ADH 5.1.2, and Lemma 2.4.4, $v(\lambda - K) = \Psi^{\downarrow}$ is jammed. Furthermore, $v(\lambda - K)$ does not have a supremum in $\mathbb{Q}\Gamma$ because K is λ-free and hence has rational asymptotic integration. By the henselian assumption and Lemma 5.1.4, there are $\alpha \in \Gamma$ and $n \geqslant 1$ such that $v(\lambda - K) = \big(\alpha + nv(y-K)\big)^{\downarrow}$. Thus by Lemmas 2.1.6 and 2.1.5, $v(y - K)$ is jammed as well. Since $v(y - K)$ also has the yardstick property, by Lemma 2.5.6 we have $v(y - K) = \Gamma^{<}$. However, since $v(\lambda - K)$ does not have a supremum in $\mathbb{Q}\Gamma$, neither does $v(y - K)$ by Lemma 2.1.1, a contradiction. $\square$

6.2. Small exponential integration

*In this section K is a henselian pre-*d*-valued field of H-type and $s \in K \setminus K^\dagger$ is such that $v(s) \in (\Gamma^>)'$.* In particular, K *does not* have small exponential integration. Take a field extension $L = K(y)$ with y transcendental over K, equipped with the unique derivation extending the derivation of K such that $(1+y)^\dagger = y'/(1+y) = s$.

ADH 6.2.1. [**6**, 10.4.3 and 10.5.18] *There is a unique valuation of L that makes it an H-asymptotic extension of K with $y \not\asymp 1$. With this valuation L is pre-*d*-valued, and is an immediate extension of K with $y \prec 1$. Furthermore, if K is equipped with an ordering making it a pre-H-field, then there is a unique ordering on L making it a pre-H-field extension of K.*

In the rest of this section L is equipped with this valuation. Here is the main result of this section:

Proposition 6.2.2. *If K is λ-free, then so is $L = K(y)$.*

The proof of Proposition 6.2.2 is delayed until the end of the section. The following nonempty set will be of importance in our analysis:

$$S \;:=\; \left\{ v\left(s - \frac{\varepsilon'}{1+\varepsilon}\right) : \varepsilon \in K^{\prec 1} \right\} \;\subseteq\; (\Gamma^>)' \;\subseteq\; \Gamma.$$

ADH 6.2.3. *The set S does not have a largest element.*

Proof. This is Claim 1 in the proof of [**6**, 10.4.3]. $\square$

Lemma 6.2.4. *S is a downward closed subset of $(\Gamma^>)'$; in particular, S is convex.*

Proof. Let $\varepsilon_1 \prec 1$ in K and $\alpha, \beta \in (\Gamma^>)'$ be such that

$$\alpha \;<\; v\left(s - \frac{\varepsilon_1'}{1+\varepsilon_1}\right) \;=\; \beta.$$

Let $\delta \prec 1$ in K be such that $v(\delta') = \alpha$ and set $\varepsilon_0 := \delta + \varepsilon_1 + \delta\varepsilon_1$. Note that

$$\begin{aligned}
\frac{\varepsilon_1'}{1+\varepsilon_1} - \frac{\varepsilon_0'}{1+\varepsilon_0} \;&=\; \frac{\varepsilon_1'}{1+\varepsilon_1} - (1+\delta+\varepsilon_1+\delta\varepsilon_1)^\dagger \\
&=\; \frac{\varepsilon_1'}{1+\varepsilon_1} - \big((1+\delta)(1+\varepsilon_1)\big)^\dagger \\
&=\; \frac{\varepsilon_1'}{1+\varepsilon_1} - \frac{\delta'}{1+\delta} - \frac{\varepsilon_1'}{1+\varepsilon_1} \\
&=\; -\frac{\delta'}{1+\delta}
\end{aligned}$$

and thus

$$v\left(\frac{\varepsilon_1'}{1+\varepsilon_1} - \frac{\varepsilon_0'}{1+\varepsilon_0}\right) \;=\; v\left(\frac{\delta'}{1+\delta}\right) \;=\; \alpha.$$

Finally,

$$v\left(s - \frac{\varepsilon_0'}{1+\varepsilon_0}\right) \;=\; v\left(\left(s - \frac{\varepsilon_1'}{1+\varepsilon_1}\right) + \left(\frac{\varepsilon_1'}{1+\varepsilon_1} - \frac{\varepsilon_0'}{1+\varepsilon_0}\right)\right) \;=\; \min(\beta,\alpha) \;=\; \alpha \in S. \qquad \square$$

The next lemma shows that S is a transform of the positive portion of $v(y-K)$.

Lemma 6.2.5. $\big(v(y-K)^{>0}\big)' = S$.

Proof. ($\subseteq$) Let $\varepsilon \in K$ be such that $v(y-\varepsilon) > 0$. Then necessarily $\varepsilon \prec 1$ since $y \prec 1$ and so it suffices to prove that $\big(v(y-\varepsilon)\big)' = v(y'-\varepsilon') \in S$. By (PDV) it follows that $(y-\varepsilon)' \succ \varepsilon'(y-\varepsilon)$. Thus

$$\begin{aligned} s - \frac{\varepsilon'}{1+\varepsilon} \;&=\; \frac{y'}{1+y} - \frac{\varepsilon'}{1+\varepsilon} \;=\; \frac{y'(1+\varepsilon)-\varepsilon'(1+y)}{(1+y)(1+\varepsilon)} \;=\; \frac{(1+\varepsilon)(y-\varepsilon)'-\varepsilon'(y-\varepsilon)}{(1+y)(1+\varepsilon)} \\ &\asymp\; (1+\varepsilon)(y-\varepsilon)'-\varepsilon'(y-\varepsilon) \;\asymp\; y'-\varepsilon'. \end{aligned}$$

We conclude that $v(y'-\varepsilon') = \big(v(y-\varepsilon)\big)' \in S$.

For the ($\supseteq$) direction, suppose that $\alpha = v\big(s-\varepsilon'/(1+\varepsilon)\big) \in S$ where $\varepsilon \in K^{\prec 1}$. Then the calculation in reverse shows that $\alpha = v(y'-\varepsilon') = \big(v(y-\varepsilon)\big)' \in \big(v(y-K)^{>0}\big)'$. $\square$

The next lemma gives us a "definable yardstick" that we can use for going up the set S. If K has small integration, then we can obtain a longer yardstick in the sense of Lemma 2.5.2, however the shorter yardstick will be good enough for our purposes.

Lemma 6.2.6. *Suppose K has asymptotic integration. Then for all $\gamma \in S$ we have $\gamma < \gamma - \int s\gamma \in S$. If $\mathrm{I}(K) = \partial\smallfrown$, then for all $\gamma \in S$ we have $\gamma < \gamma + \int \gamma \in S$. Thus S has the derived yardstick property and so $v(y-K)^{>0}$ and $v(y-K)$ both have the yardstick property.*

Proof. Let $\gamma \in S$ and take $\varepsilon \prec 1$ in K such that $\gamma = v\big(s-\varepsilon'/(1+\varepsilon)\big)$. Next take $b \prec 1$ in K such that $v(b') = \big(v(b)\big)' = \gamma$ (and so $v(b) = \int \gamma$). Take $u \in K$ with $s - \varepsilon'/(1+\varepsilon) = ub'$, so $u \asymp 1$. Next let $\delta \prec 1$ be such that $(1+\varepsilon)(1+ub) = 1+\delta$. Now note that

$$\begin{aligned} s - \frac{\delta'}{1+\delta} \;&=\; s - \big((1+\varepsilon)(1+ub)\big)^{\dagger} \\ &=\; s - \frac{\varepsilon'}{1+\varepsilon} - \frac{(ub)'}{1+ub} \\ &=\; ub' - \frac{(ub)'}{1+ub} \\ &=\; \frac{u^2bb'-u'b}{1+ub}. \end{aligned}$$

However, since $\Psi \ni s^2\gamma < v(u') \in \Gamma^{>\Psi}$, we have

$$\begin{aligned} v(u'b) \;&=\; v\big(u'b'(b^{\dagger})^{-1}\big) \\ &=\; v(u') - \psi \textstyle\int \gamma + \gamma \\ &>\; s^2\gamma - s\gamma + \gamma \\ &=\; -\textstyle\int s\gamma + \gamma \quad \text{(by Lemma 2.3.3)}, \end{aligned}$$

and so by Lemma 2.5.2,

$$\begin{aligned} v\left(s - \frac{\delta'}{1+\delta}\right) \;&\geqslant\; \min\big(v(u^2bb'), v(u'b)\big) \\ &\geqslant\; \min(\gamma + \textstyle\int\gamma, -\int s\gamma + \gamma) \\ &=\; \gamma - \textstyle\int s\gamma \;>\; \gamma. \end{aligned}$$

Finally, by Lemma 6.2.4, it follows that $\gamma - \int s\gamma \in S$.

If $\mathrm{I}(K) = \partial\smallfrown$, then we can arrange $u = 1$ above and thus

$$s - \frac{\delta'}{1+\delta} \;=\; \frac{bb'}{1+b} \;\asymp\; bb'$$

and so $v(bb') = \gamma + \int \gamma$.

The claim about $v(y-K)^{>0}$ now follows from Lemma 6.2.5 and Proposition 2.5.8. $\square$

Proposition 6.2.2 now follows immediately from Lemma 6.2.6 and Proposition 6.1.18.

6.3. Small integration

In this section K is a henselian pre-d-valued field of H-type and $s \in K$ is such that $v(s) \in (\Gamma^{>})'$ and $s \notin \partial\mathcal{O}$. In particular, K *does not* have small integration. Define the following nonempty set:

$$S \ := \ \big\{v(s-\varepsilon') : \varepsilon \in K^{\prec 1}\big\} \ \subseteq \ (\Gamma^{>})' \ \subseteq \ \Gamma.$$

As K is pre-d-valued, we have the following, which elaborates on [**6**, 10.2.5(iii)]:

Lemma 6.3.1. *S has no largest element and is a downward closed subset of $(\Gamma^{>})'$; in particular, S is convex.*

Proof. First note that $v(s) \in S$. Next take $\gamma \in S$ with $\gamma \geqslant v(s)$; then $\gamma = v(s-\varepsilon')$ where $\varepsilon \prec 1$ in K. As $\gamma \in (\Gamma^{>})'$, we have $b \prec 1$ in K such that $v(b') = \gamma$. Thus for some $u \asymp 1$ in K we have $v(s-\varepsilon'-ub') > \gamma$. By (PDV), $v(u'b) > v(b') = \gamma$ and so $v\big(s-\varepsilon'-(ub)'\big) > \gamma$. This shows that S has no largest element. The claim that S is a downward closed subset of $(\Gamma^{>})'$ follows easily from $S \subseteq (\Gamma^{>})'$. $\square$

Take a field extension $L = K(y)$ with y transcendental over K, equipped with the unique derivation extending the derivation of K such that $y' = s$.

ADH 6.3.2. [**6**, 10.2.4 and 10.5.8] *There is a unique valuation of L that makes it an H-asymptotic extension of K with $y \not\asymp 1$. With this valuation L is an immediate extension of K with $y \prec 1$ and L is pre-d-valued. Furthermore, if K is equipped with an ordering making it a pre-H-field, then there is a unique ordering on L making it a pre-H-field extension of K.*

In the rest of this section L is equipped with this valuation. Here is the main result of this section:

Proposition 6.3.3. *If K is λ-free, then so is $L = K(y)$.*

We delay the proof of Proposition 6.3.3 until the end of the section.

In the rest of this section we assume that K has asymptotic integration.

Lemma 6.3.4. $\big(v(y-K)^{>0}\big)' = S$.

Proof. ($\subseteq$) Let $\varepsilon \in K$ be such that $y - \varepsilon \prec 1$. Then necessarily $\varepsilon \prec 1$ because $y \prec 1$. Let $\alpha = v(y-\varepsilon)$. We want to show that $\alpha' \in S$. From $y - \varepsilon \not\asymp 1$ we get

$$\alpha' \ = \ \big(v(y-\varepsilon)\big)' \ = \ v(y'-\varepsilon') \ = \ v(s-\varepsilon') \in S.$$

For the ($\supseteq$) direction, let $\alpha = v(s-\varepsilon')$ with $\varepsilon \prec 1$. By arguing as above, $v(y-\varepsilon) > 0$ and $(v(y-\varepsilon))' = \alpha$. $\square$

Lemma 6.3.5. *Suppose K has asymptotic integration. Then for all $\gamma \in S$ we have $\gamma < \gamma - \int s\gamma \in S$. If $\mathrm{I}(K) = (1+\mathcal{o})^{\dagger}$, then for all $\gamma \in S$ we have $\gamma < \gamma + \int \gamma \in S$. Thus S has the derived yardstick property and so $v(y-K)^{>0}$ and $v(y-K)$ both have the yardstick property.*

Proof. Suppose $\gamma \in S$ and take $\varepsilon \prec 1$ in K such that $\gamma = v(s-\varepsilon')$. As $\gamma \in (\Gamma^{>})'$, we may take $b \prec 1$ in K such that $b' \asymp s-\varepsilon'$. Thus there is $u \asymp 1$ in K such that $ub' = s-\varepsilon'$. By (PDV), it follows that $v(u') > \Psi$. Thus

$$\begin{aligned} v\big(s-(\varepsilon-ub)'\big) \ &= \ v(s-\varepsilon'-ub'-u'b) \\ &= \ v(u'b) \\ &= \ v\big(u'b'(b^\dagger)^{-1}\big) \\ &= \ v(u') - \psi \textstyle\int \gamma + \gamma \\ &> \ s^2\gamma - s\gamma + \gamma \\ &= \ -\textstyle\int s\gamma + \gamma. \end{aligned}$$

Next, assume that $(1+\mathcal{O})^\dagger = \mathrm{I}(K)$. Since $s-\varepsilon' \in \mathrm{I}(K)$, there is $\delta \prec 1$ such that $s-\varepsilon' = (1+\delta)^\dagger$, i.e.,

$$s-\varepsilon' \ = \ \frac{\delta'}{1+\delta}.$$

Now note that

$$s-(\varepsilon+\delta)' \ = \ s-\varepsilon'-\delta' \ = \ \frac{\delta'}{1+\delta} - \delta' \ = \ \frac{-\delta'\delta}{1+\delta} \ \asymp \ \delta'\delta$$

and so

$$S \ \ni \ v\big(s-(\varepsilon+\delta)'\big) \ = \ v(\delta'\delta) \ = \ \gamma + \textstyle\int \gamma.$$

The claim about $v(y-K)^{>0}$ now follows from Lemma 6.3.4 and Proposition 2.5.8. □

Proposition 6.3.3 now follows immediately from Lemma 6.3.5 and Proposition 6.1.18.

6.4. Big integration

*In this section K is a henselian pre-*d*-valued field of H-type and $s \in K$ is such that*

$$S \ := \ \big\{v(s-a') : a \in K\big\} \ \subseteq \ (\Gamma^{<})' \ \subseteq \ \Gamma.$$

Thus $s \notin \partial K$ and $v(s) \in (\Gamma^{<})'$.

Lemma 6.4.1. *S is downward closed and does not have a largest element.*

Proof. Let $\gamma = v(s-a') \in S$ with $a \in K$. Suppose $\delta < \gamma$ in Γ. Take $f \in K$ such that $v(f') = \delta$ and so $\delta = v\big(s-(a+f)'\big) \in S$. Next, using $S \subseteq (\Gamma^{<})'$, take $b \in K^{\succ 1}$ such that $b' \asymp s-a'$, and then take $u \asymp 1$ in K with $ub' = s-a'$. By (PDV), $u'b \prec b'$ and thus $\gamma < v\big(s-a'-(ub)'\big) \in S$. □

Take a field extension $L = K(y)$ with y transcendental over K, equipped with the unique derivation extending the derivation of K such that $y' = s$.

ADH 6.4.2. [**6**, 10.2.6 and 10.5.8] *There is a unique valuation of L making it an H-asymptotic extension of K. With this valuation L is an immediate extension of K with $y \succ 1$ and L is pre-*d*-valued. Furthermore, if K is equipped with an ordering making it a pre-H-field, then there is a unique ordering on L making it a pre-H-field extension of K.*

In the rest of this section L is equipped with this valuation. Here is the main result of this section:

Proposition 6.4.3. *If K is $\uplambda$-free, then so is $L = K(y)$.*

We delay the proof of Proposition 6.4.3 until the end of the section.

In the rest of this section we assume that K has asymptotic integration.

Lemma 6.4.4. $\big(v(y-K)\big)' = S$.

Proof. Let $\gamma = v(y-a)$ with $a \in K$. Then $v(y'-a') = v(s-a') \in S \subseteq (\Gamma^{<})'$ and so $y-a \succ 1$. Thus $\gamma' = \big(v(y-a)\big)' = v(y'-a') = v(s-a') \in S$. Conversely, for $a \in K$ we have $v(s-a') = v(y'-a') = \big(v(y-a)\big)'$. $\square$

Below we fix $g \in K^{\succ 1}$ such that $g' \sim s$; such g exists by Lemma 6.4.1 and because $v(s) \in S$.

Lemma 6.4.5. *$S^{>v(s)}$ is cofinal in S, and*

$$S^{>v(s)} \;=\; \big\{v((g(1+\varepsilon))'-s) : \varepsilon \in K^{\prec 1}\big\}.$$

Proof. $S^{>v(s)}$ is cofinal in S since $v(s) \in S$ and S does not have a largest element. Suppose $\varepsilon \in K^{\prec 1}$. Then by (PDV), $\big(g(1+\varepsilon)\big)' = g' + \varepsilon' g + \varepsilon g' \sim g' \sim s$ and so $\big(g(1+\varepsilon)\big)' - s \prec s$. Conversely, suppose $\gamma = v(x'-s) > vs$. Then $x' \sim s$ and so $x' \sim g'$, i.e., $x'-g' \prec g'$. As $g \succ 1$, we get $x - g \prec g$ and so $x = g(1+\varepsilon)$ for some $\varepsilon \in K^{\prec 1}$. $\square$

Lemma 6.4.6. *Suppose K has asymptotic integration. If $\gamma \in S^{>v(s)}$, then $\gamma < \gamma - \int s\gamma \in S$. Thus S has the derived yardstick property and so $v(y-K)$ has the yardstick property.*

Proof. Let $\gamma = v\big((g(1+\varepsilon))'-s\big)$ with $\varepsilon \in K^{\prec 1}$. Note that

$$\big(g(1+\varepsilon)\big)' - s \;=\; g' + g\varepsilon' + g'\varepsilon - s.$$

Next take $a \in K^{\succ 1}$ such that

$$a' \;\asymp\; g' + g\varepsilon' + g'\varepsilon - s,$$

so $v(a') = \gamma$, and take $u \asymp 1$ in K such that

$$ua' \;=\; g' + g\varepsilon' + g'\varepsilon - s.$$

Then $a' \prec g' \asymp s$ and so $a \prec g$, i.e., $a/g \prec 1$. Furthermore, $u^\dagger \prec a^\dagger$, so $u'a \prec ua'$. Now consider the following element of $S^{>v(s)}$:

$$\beta \;:=\; v\big((g(1+\varepsilon - ua/g))' - s\big).$$

Note that:

$$\begin{aligned}
\big(g(1+\varepsilon-ua/g)\big)' - s \quad &= \quad (g + g\varepsilon - ua)' - s \\
&= \quad g' + g\varepsilon' + g'\varepsilon - u'a - ua' - s \\
&= \quad (g' + g\varepsilon' + g'\varepsilon - s - ua') - u'a \\
&= \quad -u'a.
\end{aligned}$$

Thus we can use that $v(u') > \Psi$ and $\gamma = v(a) + v(a^\dagger)$ to get the yardstick:

$$\begin{aligned} v(-u'a) &= v\big(u'(a^\dagger)^{-1}a'\big) \\ &= v\big(u'(a^\dagger)^{-1}\big) + \gamma \\ &= v(u') - \psi \textstyle\int \gamma + \gamma \\ &= v(u') - s\gamma + \gamma \\ &> s^2\gamma - s\gamma + \gamma \\ &= -\textstyle\int s\gamma + \gamma \end{aligned}$$

The claim about $v(y - K)$ now follows from Lemma 6.4.4 and Proposition 2.5.8. $\square$

Proposition 6.4.3 follows immediately from Lemma 6.4.6 and Proposition 6.1.18.

6.5. The differential-valued hull and H-field hull

*In this section K is a pre-*d*-valued field of H-type.*

ADH 6.5.1. [**6**, 10.3.1] *K has a* d*-valued extension* $\mathrm{dv}(K)$ *of H-type such that any embedding of K into any* d*-valued field L of H-type extends uniquely to an embedding of* $\mathrm{dv}(K)$ *into L.*

The d-valued field $\mathrm{dv}(K)$ as in ADH 6.5.1 above is called the **differential-valued hull of** K.

Theorem 6.5.2. *If K is λ-free, then* $\mathrm{dv}(K)$ *is λ-free.*

Proof. Assume K is λ-free. By iterating applications of ADH 6.1.9, Proposition 6.3.3, and Lemma 6.1.8, we get an immediate henselian λ-free H-asymptotic extension L of K which has small integration. By Lemma 5.3.5, L will also be d-valued. Thus by ADH 6.5.1, $\mathrm{dv}(K)$ can be identified with a subfield of L which contains K. Finally, by Lemma 6.1.6 it follows that $\mathrm{dv}(K)$ is λ-free. $\square$

Definition 6.5.3. A gap β in K is said to be a **true gap** if no $b \asymp 1$ in K satisfies $v(b') = \beta$, and is said to be a **fake gap** otherwise (that is, there is $b \asymp 1$ in K such that $v(b') = \beta$).

Remark 6.5.4. Suppose K has a gap β. Then the asymptotic couple (Γ, ψ) "believes" it can make a choice about β, in the sense of Remark 3.1.3. However, if β is a fake gap, then this choice is completely predetermined by K itself. Indeed, if L is a d-valued extension of K of H-type and β is a fake gap, then there is $\varepsilon \in \mathcal{O}_L$ such that $v(\varepsilon') = \beta$. However, if β is a true gap, then both options of this choice are still available to K, see [**6**, 10.3.2(ii), 10.2.1, and 10.2.2].

Lemma 6.5.5. *If K is* d*-valued and has a gap β, then β is a true gap.*

Proof. Let K be a d-valued field and consider $\beta \in \Gamma$. Suppose that there is $b \asymp 1$ in K such that $v(b') = \beta$. Then there are $c \in C^\times$ and $\varepsilon \prec 1$ in $K^\times$ such that $b = c + \varepsilon$ and thus $v(b') = v(\varepsilon') = \beta \in (\Gamma^>)'$. In particular, β is not a gap. $\square$

Corollary 6.5.6. *The differential-valued hull of K has the following properties:*

(1) *If K is grounded, then* $\mathrm{dv}(K)$ *is grounded.*
(2) *If K has a fake gap, then* $\mathrm{dv}(K)$ *is grounded.*
(3) *If K has a true gap, then* $\mathrm{dv}(K)$ *has a true gap.*

(4) *If K has asymptotic integration and is not* λ*-free, then* $\operatorname{dv}(K)$ *has asymptotic integration and is not* λ*-free.*
(5) *If K is* λ*-free, then* $\operatorname{dv}(K)$ *is* λ*-free.*

PROOF. (1)-(4) are a restatement of [**6**, 10.3.2]. (5) is Theorem 6.5.2. $\square$

The H-field hull of a pre-H-field. *In this subsection we further assume that K is equipped with an ordering making it a pre-H-field.*

ADH 6.5.7. [**6**, 10.5.13] *A unique field ordering on* $\operatorname{dv}(K)$ *makes* $\operatorname{dv}(K)$ *a pre-H-field extension of K. Let $H(K)$ be* $\operatorname{dv}(K)$ *equipped with this ordering. Then $H(K)$ is an H-field and embeds uniquely over K into any H-field extension of K.*

The H-field $H(K)$ in ADH 6.5.7 above is called the **H-field hull of** K. We have the following H-field analogues of Theorem 6.5.2 and Corollary 6.5.6:

Corollary 6.5.8. *If K is* λ*-free, then $H(K)$ is* λ*-free.*

Corollary 6.5.9. *The H-field hull of K has the following properties:*
(1) *If K is grounded, then $H(K)$ is grounded.*
(2) *If K has a fake gap, then $H(K)$ is grounded.*
(3) *If K has a true gap, then $H(K)$ has a true gap.*
(4) *If K has asymptotic integration and is not* λ*-free, then $H(K)$ has asymptotic integration and is not* λ*-free.*
(5) *If K is* λ*-free, then $H(K)$ is* λ*-free.*

6.6. The integration closure

In this section K is a d*-valued field of H-type with asymptotic integration.*

ADH 6.6.1. [**6**, 10.2.7] *K has an immediate asymptotic extension $K(\int)$ such that:*
(1) *$K(\int)$ is henselian and has integration;*
(2) *$K(\int)$ embeds over K into any henselian* d*-valued H-asymptotic extension of K that has integration.*

Furthermore, given any such $K(\int)$ with the above properties, the only henselian asymptotic subfield of $K(\int)$ containing K and having integration is $K(\int)$.

Theorem 6.6.2. *If K is* λ*-free, then so is $K(\int)$.*

PROOF. Assume K is λ-free. By iterating Lemma 6.1.8, ADH 6.1.9, and Propositions 6.3.3 and 6.4.3, we obtain a λ-free d-valued immediate H-asymptotic extension L of K that is henselian and has integration. Using ADH 6.6.1 to identify $K(\int)$ with a subfield of L which contains K, $K(\int)$ is λ-free by ADH 6.1.6. $\square$

6.7. The number of Liouville closures

In this section K is a pre-H-field. K is said to be **Liouville closed** if it is a real closed H-field with integration and exponential integration. A **Liouville closure** of K is a Liouville closed H-field extension of K which is also a Liouville extension of K.

Theorem 6.7.1. *Suppose K is an H-field. Then K has at least one and at most two Liouville closures up to isomorphism over K. In particular,*

(1) *K has exactly one Liouville closure up to isomorphism over K iff*
   (a) *K is grounded, or*
   (b) *K is $\boldsymbol{\lambda}$-free.*
(2) *K has exactly two Liouville closures up to isomorphism over K iff*
   (c) *K has a gap, or*
   (d) *K has asymptotic integration and is not $\boldsymbol{\lambda}$-free.*

Theorem 6.7.1 will follow from the following Proposition, whose proof we delay until later in the section:

Proposition 6.7.2. *Suppose K is an H-field.*

(1) *If K is $\boldsymbol{\lambda}$-free, then K has exactly one Liouville closure up to isomorphism over K.*
(2) *If K has asymptotic integration and is not $\boldsymbol{\lambda}$-free, then K has at least two Liouville closures up to isomorphism over K.*

Proof of Theorem 6.7.1 assuming Proposition 6.7.2. It is clear that K will be in case (a), (b), (c) or (d), and all four cases are mutually exclusive. If K is in case (a), then K has exactly one Liouville closure up to isomorphism over K, by [**6**, 10.6.23]. If K is in case (c), then K has exactly two Liouville closures up to isomorphism over K, by [**6**, 10.6.25]. Cases (b) and (d) are taken care of by Proposition 6.7.2 and [**6**, 10.6.12]. □

In general, a pre-H-field which is not also an H-field might not have any Liouville closures at all. For instance, the pre-H-field L from Example 5.3.6 cannot have any Liouville closures: a Liouville closure of L would necessarily contain $H(L)$, but $H(L)$ cannot be contained inside any Liouville extension of L because $C_{H(L)}$ is not an algebraic extension of $C_L = \mathbb{R}$. In such a situation, the next best thing is to consider Liouville closures of the H-field hull:

Corollary 6.7.3. *$H(K)$ has at least one and at most two Liouville closures up to isomorphism over K. In particular,*

(1) *$H(K)$ has exactly one Liouville closure up to isomorphism over K iff*
   (a) *K is grounded, or*
   (b) *K has a fake gap, or*
   (c) *K is $\boldsymbol{\lambda}$-free.*
(2) *$H(K)$ has exactly two Liouville closures up to isomorphism over K iff*
   (d) *K has a true gap, or*
   (e) *K has asymptotic integration and is not $\boldsymbol{\lambda}$-free.*

Proof. If we replace in the statement of Corollary 6.7.3 all instances of "up to isomorphism over K" with "up to isomorphism over $H(K)$", then this would follow from Corollary 6.5.9 and Theorem 6.7.1. Now, to strengthen the statements to "up to isomorphism over K", use that $H(K)$ is determined up-to-unique-isomorphism in Proposition 6.5.7. □

Liouville towers. *In this subsection K is an H-field.* The primary method of constructing Liouville closures of an H-field is with a *Liouville tower.* A **Liouville tower on** K is a strictly increasing chain $(K_\lambda)_{\lambda\leqslant\mu}$ of H-fields, indexed by the ordinals less than or equal to some ordinal μ, such that

(1) $K_0 = K$;
(2) if λ is a limit ordinal, $0 < \lambda \leqslant \mu$, then $K_\lambda = \bigcup_{\iota<\lambda} K_\iota$;

(3) for $\lambda < \lambda + 1 \leqslant \mu$, *either*

   (a) K_λ is not real closed and $K_{\lambda+1}$ is a real closure of K_λ,

   or K_λ is real closed, $K_{\lambda+1} = K_\lambda(y_\lambda)$ with $y_\lambda \notin K_\lambda$ (so y_λ is transcendental over K_λ), and one of the following holds, with $(\Gamma_\lambda, \psi_\lambda)$ the asymptotic couple of K_λ and $\Psi_\lambda := \psi_\lambda(\Gamma_\lambda^{\neq})$:

   (b) $y'_\lambda = s_\lambda \in K_\lambda$ with $y_\lambda \prec 1$ and $v(s_\lambda)$ is a gap in K_λ,

   (c) $y'_\lambda = s_\lambda \in K_\lambda$ with $y_\lambda \succ 1$ and $v(s_\lambda)$ is a gap in K_λ,

   (d) $y'_\lambda = s_\lambda \in K_\lambda$ with $v(s_\lambda) = \max \Psi_\lambda$,

   (e) $y'_\lambda = s_\lambda \in K_\lambda$ with $y_\lambda \prec 1$, $v(s_\lambda) \in (\Gamma_\lambda^{>})'$, and $s_\lambda \neq \varepsilon'$ for all $\varepsilon \in K_\lambda^{\prec 1}$,

   (f) $y'_\lambda = s_\lambda \in K_\lambda$ such that $S_\lambda := \{v(s_\lambda - a') : a \in K_\lambda\} < (\Gamma_\lambda^{>})'$, and S_λ has no largest element,

   (g) $y_\lambda^\dagger = s_\lambda \in K_\lambda$ with $y_\lambda \sim 1$, $v(s_\lambda) \in (\Gamma_\lambda^{>})'$, and $s_\lambda \neq a^\dagger$ for all $a \in K_\lambda^\times$,

   (h) $y_\lambda^\dagger = s_\lambda \in K_\lambda^{<}$ with $y_\lambda > 0$, and $v(s_\lambda - a^\dagger) \in \Psi_\lambda^{\downarrow}$ for all $a \in K_\lambda^\times$.

The H-field K_μ is called the **top** of the tower $(K_\lambda)_{\lambda \leqslant \mu}$. We say that a Liouville tower $(K_\lambda)_{\lambda \leqslant \mu}$ is **maximal** if it cannot be extended to a Liouville tower $(K_\lambda)_{\lambda \leqslant \mu+1}$ on K. Given a Liouville tower $(K_\lambda)_{\lambda \leqslant \mu}$ on K, $0 \leqslant \lambda < \lambda + 1 \leqslant \mu$, we say $K_{\lambda+1}$ is an **extension of type** $(*)$ for $(*) \in \{(\mathrm{a}), (\mathrm{b}), \ldots, (\mathrm{h})\}$ if $K_{\lambda+1}$ and K_λ satisfy the properties of item $(*)$ as in the definition of Liouville tower.

ADH 6.7.4. (1) *Let $(K_\lambda)_{\lambda \leqslant \mu}$ be a Liouville tower on K. Then:*

   (a) *K_μ is a Liouville extension of K;*

   (b) *the constant field C_μ of K_μ is a real closure of C if $\mu > 0$;*

   (c) *$|K_\mu| = |K|$, hence $\mu < |K|^+$.*

(2) *There is a maximal Liouville tower on K.*

(3) *The top of a maximal Liouville tower on K is Liouville closed, and hence a Liouville closure of K.*

(4) *If $(K_\lambda)_{\lambda \leqslant \mu}$ is a Liouville tower on K such that no K_λ with $\lambda < \mu$ has a gap, and if K_μ is Liouville closed, then K_μ is the unique Liouville closure of K up to isomorphism over K.*

Proof. (1) is [**6**, 10.6.13], (2) follows from (1)(c), (3) is [**6**, 10.6.14], and (4) is [**6**, 10.6.17]. □

For a set $\Lambda \subseteq \{(\mathrm{a}), (\mathrm{b}), \ldots, (\mathrm{h})\}$ with $(\mathrm{a}) \in \Lambda$, the definition of a **Λ-tower on** K is identical to that of *Liouville tower on* K, except that in clause (3) of the above definition only the items from Λ occur. Thus every Λ-tower on K is also a Liouville tower on K. Note that by Zorn's Lemma and ADH 6.7.4(1)(c), maximal Λ-towers exist on K.

Proof of Proposition 6.7.2. (1) Assume K is $\boldsymbol{\lambda}$-free. By ADH 6.7.4(4), it suffices to find a Liouville tower $(K_\lambda)_{\lambda \leqslant \mu}$ on K such that K_μ is Liouville closed and no K_λ with $\lambda < \mu$ has a gap. Take a maximal $\{$(a),(e),(f),(g),(h)$\}$-tower $(K_\lambda)_{\lambda \leqslant \mu}$ on K. By Lemmas 6.1.8, 6.1.11, Propositions 6.2.2, 6.3.3, 6.4.3 and ADH 6.1.13, K_λ is $\boldsymbol{\lambda}$-free for every $\lambda \leqslant \mu$. In particular, no K_λ with $\lambda < \mu$ has a gap. Finally, by maximality, it follows that K_μ is Liouville closed.

(2) Assume that K has asymptotic integration and is not $\boldsymbol{\lambda}$-free. First consider the case that K does not have rational asymptotic integration. Then $K_1 = K^{\mathrm{rc}}$ has a gap. By [**6**, 10.6.25] K_1 has two Liouville closures which are not isomorphic over K_1. As K_1 is a real closure of K, they are not isomorphic over K either because the real closure is unique up-to-unique-isomorphism. Thus K has at least two Liouville closures which are not isomorphic over K.

Next, consider the case that K is real closed. In this case, if L is a Liouville closure of K, then $C_L = C$ since C is necessarily real closed. As K is not $\boldsymbol{\lambda}$-free, there is $\boldsymbol{\lambda} \in K$ such that $\boldsymbol{\lambda}_\rho \rightsquigarrow \boldsymbol{\lambda}$. Next, let $K_1 = K(f)$

be the H-field extension from ADH 6.1.17 such that $f^\dagger = -\lambda$ and $v(f)$ is a gap in K_1. Again by [**6**, 10.6.25], K_1 has two Liouville closures L_1 and L_2 which are not isomorphic over K_1. There is $\widetilde{y} \in L_1^{\prec 1}$ such that $\widetilde{y}' = f$ whereas every $y \in L_2$ such that $y' = f$ has the property that $y \succ 1$. Furthermore, as both L_1 and L_2 are Liouville closed, they both contain nonconstant elements y such that $y'' = -\lambda y'$.

Claim. *If $y \in L_1 \setminus C$ is such that $y'' = -\lambda y'$, then $y \preccurlyeq 1$. If $y \in L_2 \setminus C$ is such that $y'' = -\lambda y'$, then $y \succ 1$.*

Proof of Claim. Suppose $y \in L_1 \setminus C$ is such that $y'' = -\lambda y'$. Let $\widetilde{y} \in L_1^{\prec 1}$ be such that $\widetilde{y}' = f$. Then $\widetilde{y} \in L_1 \setminus C$ since $f \neq 0$. Furthermore $\widetilde{y}'' = -\lambda \widetilde{y}'$ so there are $c_0 \in C^\times$ and $c_1 \in C$ such that $y = c_0 \widetilde{y} + c_1$, by Lemma 5.3.1. It follows that $y \preccurlyeq 1$.

Next, let $y \in L_2 \setminus C$ and let $\widetilde{y} \in L_2$ be such that $\widetilde{y}' = f$. Then $\widetilde{y} \notin C$ because $\widetilde{y} \succ 1$ and $\widetilde{y}'' = -\lambda \widetilde{y}'$. As in the first case, it will follow from Lemma 5.3.1 that $y \succ 1$. □

It follows from the claim that L_1 and L_2 are not isomorphic over K.

Finally, consider the case that K is not real closed, and has rational asymptotic integration. By the above case, the real closure K^{rc} has two Liouville closures L_1 and L_2 which are not isomorphic over K^{rc}. These two Liouville closures will also not be isomorphic over K, as real closures are unique-up-to-unique-isomorphism. □

The next lemma concerns the appearances of gaps in arbitrary Liouville H-field extensions, not necessarily extensions occurring as the tops of Liouville towers.

Lemma 6.7.5. *Suppose K is grounded or is λ-free and L is a Liouville H-field extension of K. Then L does not have a gap.*

Proof. We first consider the case that K is λ-free. Let M be the Liouville closure of K which was constructed in the proof of Proposition 6.7.2. We claim that Ψ is cofinal in Ψ_M. This follows from the fact that M is constructed as the top of an $\big\{$(a),(e),(f),(g),(h)$\big\}$-tower on K: the Ψ-set remains unchanged when passing to extensions of type (a), (e), (f) or (g) and for extensions of type (h), the original Ψ-set is cofinal in the larger Ψ-set by ADH 6.1.13. Finally, as M is the unique Liouville closure of K up to isomorphism over K, we may identify L with a subfield of M which contains K. Thus Ψ_L is cofinal in Ψ_M. As M is λ-free, so is L by ADH 6.1.6. In particular, L has rational asymptotic integration and so it does not have a gap.

We next consider the case that K is grounded. Let M be the Liouville closure of K as constructed in the proof of [**6**, 10.6.24] and the remarks following it. In particular, using the notation from the remarks following the proof of [**6**, 10.6.24], $M = \bigcup_{n<\omega} \ell^n(K)$ where $\ell^0(K) = K$ and for each n, $\ell^{n+1}(K)$ is a grounded Liouville H-field extension of K such that $\max \Psi_{\ell^{n+1}(K)} = s(\max \Psi_{\ell^n(K)})$. It follows that the set $\big\{s^n(\max \Psi) : n < \omega\big\}$ is a cofinal subset of Ψ_M. We now identify L with a subfield of M that contains K and consider two cases:

Case 1: $\big\{s^n(\max \Psi) : n < \omega\big\} \not\subseteq \Psi_L$. In this case there is a least $N < \omega$ such that $s^N(\max \Psi) \in \Psi_L$ but $s\big(s^N(\max \Psi)\big) \in \Psi_M \setminus \Psi_L$. Thus the element $s^N(\max \Psi) \in \Psi_L$ cannot be asymptotically integrated in L. The only way this can happen is if $s^N(\max \Psi) = \max \Psi_L$. In particular, L is grounded and does not have a gap.

Case 2: $\big\{s^n(\max \Psi) : n < \omega\big\} \subseteq \Psi_L$. In this case Ψ_L is cofinal in Ψ_M and so L is λ-free by ADH 6.1.6. In particular, L has rational asymptotic integration and therefore does not have a gap. □

We also give a characterization of the dichotomy of Theorem 6.7.1 entirely in terms of gaps appearing in Liouville towers and arbitrary Liouville extensions:

Corollary 6.7.6. *The following are equivalent:*

(1) *K has exactly two Liouville closures up to isomorphism over K,*
(2) *there is a Liouville tower $(K_\lambda)_{\lambda \leqslant \mu}$ on K such that some K_λ has a gap,*
(3) *every maximal Liouville tower $(K_\lambda)_{\lambda \leqslant \mu}$ on K has some K_λ with a gap,*
(4) *there is a Liouville tower $(K_\lambda)_{\lambda \leqslant \mu}$ on K with $\mu \geqslant \omega$ such that either K_0, K_1 or K_2 has a gap,*
(5) *there is an H-field L which has a gap and is a Liouville extension of K.*

Proof. (4) $\Rightarrow$ (2) and (3) $\Rightarrow$ (2) are clear. (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (5) follow from ADH 6.7.4(4).

(1) $\Rightarrow$ (4): If K has exactly two Liouville closures up to isomorphism over K, then in particular K itself is not Liouville closed. A routine argument shows that every maximal Liouville tower $(K_\lambda)_{\lambda \leqslant \mu}$ has $\mu \geqslant \omega$. By Theorem 6.7.1 either K has a gap or K has asymptotic integration and is not $\boldsymbol{\lambda}$-free. If K has a gap, then for any maximal Liouville tower $(K_\lambda)_{\lambda \leqslant \mu}$, K_0 has a gap. Otherwise, the proof of Proposition 6.7.2 shows how we can arrange either K_1 or K_2 to have a gap.

(2) $\Rightarrow$ (1): We will prove the contrapositive. Suppose that K has exactly one Liouville closure up to isomorphism over K and let $(K_\lambda)_{\lambda \leqslant \mu}$ be a Liouville tower on K. We will prove by induction on λ that K_λ is either grounded or $\boldsymbol{\lambda}$-free, and thus no K_λ has a gap. The case $\lambda = 0$ is clear and the limit ordinal case is taken care of by ADH 6.1.7 and Lemma 6.1.8. Suppose $\lambda = \nu + 1$ for some ordinal $0 \leqslant \nu < \mu$. If K_λ is a real closure of K_ν, then K_λ will be grounded if K_ν is and K_λ will be $\boldsymbol{\lambda}$-free if K_ν is by Lemma 6.1.11. By the inductive hypothesis, K_λ will never be an extension of type (b) or (c). If K_λ is an extension of type (d), then K_λ will also be grounded by [**6**, 10.2.3]. Extensions of type (e), (f) and (g) are necessarily immediate extensions, so if K_ν is grounded, then so is K_λ and if K_ν is $\boldsymbol{\lambda}$-free, then so is K_λ by Propositions 6.2.2, 6.3.3, and 6.4.3. Finally, if K_λ is an extension of type (h) and if K_ν is grounded, then so is K_λ by [**6**, 10.5.20], and if K_ν is $\boldsymbol{\lambda}$-free, then so is K_λ by ADH 6.1.13.

(5) $\Rightarrow$ (1): Suppose K has a Liouville H-field extension with a gap. Then by Lemma 6.7.5, K has a gap or K has asymptotic integration and is not $\boldsymbol{\lambda}$-free. By Theorem 6.7.1, it follows that K has exactly two Liouville closures up to isomorphism over K. $\square$

Remark 6.7.7. The implication (2) $\Rightarrow$ (1) of our Corollary 6.7.6 above occurs without proof in [**4**] (see item (II) before [**4**, 6.11]). Also, (1) $\Leftrightarrow$ (5) of our Corollary 6.7.6 is stated without proof in [**5**] (see the paragraph after [**5**, 4.3]).

CHAPTER 7

LD-fields

Recall from Chapter 1 that the main difference between $\mathbb{T}_{\log}$ and $\mathbb{T}$ is that $\mathbb{T}_{\log}$ only has partial exponential integration, whereas $\mathbb{T}$ has full exponential integration. In this chapter we turn our attention to studying the set of logarithmic derivatives $\mathbb{T}_{\log}^{\dagger}$ of $\mathbb{T}_{\log}$. More generally, we introduce the framework of LD-*fields* for studying certain d-valued fields K with a distinguished set $\mathrm{LD} \subseteq K$. First, some motivation:

Our eventual goal is model completeness for $\mathbb{T}_{\log}$ (in some natural language). In particular, all definable sets need to be existentially definable. Clearly the set of logarithmic derivatives $\mathbb{T}_{\log}^{\dagger}$ is existentially definable:

$$f \in \mathbb{T}_{\log}^{\dagger} \quad \Longleftrightarrow \quad \text{there exists } g \in \mathbb{T}_{\log}^{\times} \text{ such that } g^{\dagger} = f$$

What about the complement, $\mathbb{T}_{\log} \setminus \mathbb{T}_{\log}^{\dagger}$? In other words, can we find an existential answer to the following question:

Question. When is $f \in \mathbb{T}_{\log}$ *not* a logarithmic derivative?

To answer the above question, we first give an explicit description of $\mathbb{T}_{\log}^{\dagger}$:

Lemma 7.0.1. *Given a logarithmic transseries $f \in \mathbb{T}_{\log}$,*

$$f \in \mathbb{T}_{\log}^{\dagger} \quad \Longleftrightarrow \quad v\big(\operatorname{supp}(f)\big) \subseteq \Psi_{\log} \cup (\Gamma_{\log}^{>})'.$$

Proof. ($\Rightarrow$) Let $g \in \mathbb{T}_{\log}^{\times}$ be arbitrary. By factoring out the dominant monomial $\mathfrak{d}(g)$, we see that

$$g \;=\; c\ell_0^{r_0} \cdots \ell_n^{r_n}(1+\varepsilon)$$

for some $c \in \mathbb{R}^{\times}$, some n, some $r_0, \ldots, r_n \in \mathbb{R}$ and $\varepsilon \in \mathbb{T}_{\log}^{\prec 1}$. Thus

$$g^{\dagger} \;=\; r_0\ell_0^{-1} + \cdots + r_n\ell_0^{-1}\cdots\ell_n^{-1} + \frac{\varepsilon'}{1+\varepsilon},$$

and so $v\big(\operatorname{supp}(g^{\dagger})\big) \subseteq \Psi_{\log} \cup (\Gamma_{\log}^{>})'$.

($\Leftarrow$) Conversely, first suppose $v\big(\operatorname{supp}(f)\big) \subseteq \Psi_{\log}$. Since there is an upper bound on n such that ℓ_n which can occur in the support of f (because $\mathbb{T}_{\log}$ is a certain direct union), we have

$$f \;=\; r_0\ell_0^{-1} + \cdots + r_n\ell_0^{-1}\cdots\ell_n^{-1}$$

for suitable n and $r_0, \ldots, r_n \in \mathbb{R}$. Thus

$$f \;=\; (\ell_0^{r_0}\cdots\ell_n^{r_n})^{\dagger} \in \mathbb{T}_{\log}^{\dagger}.$$

Next, suppose that $v\big(\operatorname{supp}(f)\big) \subseteq (\Gamma_{\log}^{>})'$. Then $f \in \mathrm{I}(\mathbb{T}_{\log}) = (1+\smallo_{\log})^{\dagger} \subseteq \mathbb{T}_{\log}^{\dagger}$ by ADH 5.3.23 since $\mathbb{T}_{\log}$ is newtonian.

Finally, for arbitrary $f \in \mathbb{T}_{\log}$ such that $v\big(\operatorname{supp}(f)\big) \subseteq \Psi_{\log} \cup (\Gamma_{\log}^{>})'$, there are $g, h \in \mathbb{T}_{\log}$ such that $f = g + h$ and $v\big(\operatorname{supp}(g)\big) \subseteq \Psi_{\log}$ and $v\big(\operatorname{supp}(h)\big) \subseteq (\Gamma_{\log}^{>})'$. As $\mathbb{T}_{\log}^{\dagger}$ is closed under addition, it follows that $f \in \mathbb{T}_{\log}^{\dagger}$. $\square$

Now, suppose that $f \notin \mathbb{T}_{\log}^{\dagger}$. As a series, we write

$$f \;=\; \sum_{\mathfrak{m}\in\Gamma_{\log}} f_{\mathfrak{m}}\mathfrak{m}.$$

Next, we extract the following subseries from f:

$$f_{\dagger} \;:=\; \sum_{\mathfrak{m}\in\Psi_{\log}\cup(\Gamma_{\log}^{>})'} f_{\mathfrak{m}}\mathfrak{m}$$

By Lemma 7.0.1, it follows that $f_{\dagger} \in \mathbb{T}_{\log}^{\dagger}$. Furthermore, $f - f_{\dagger} \neq 0$ and

$$v\big(\operatorname{supp}(f - f_{\dagger})\big) \;\subseteq\; \Psi_{\log}^{\downarrow} \setminus \Psi_{\log}.$$

In particular,

$$v(f - f_{\dagger}) \;\in\; \Psi_{\log}^{\downarrow} \setminus \Psi_{\log}.$$

We summarize the above calculation in the following "existential" answer:

Answer. Given a logarithmic transseries $f \in \mathbb{T}_{\log}$,

$$f \notin \mathbb{T}_{\log}^{\dagger} \quad\Longleftrightarrow\quad \text{there exists } g \in \mathbb{T}_{\log}^{\times} \text{ such that } v(f - g^{\dagger}) \in \Psi_{\log}^{\downarrow} \setminus \Psi_{\log}.$$

In other words, if a logarithmic transseries f is not a logarithmic derivative, then there is a logarithmic derivative g which witnesses $f \notin \mathbb{T}_{\log}^{\dagger}$. In particular, this witnessing only relies on the valuation v and a set definable in the asymptotic couple $(\Gamma_{\log}, \psi)$. Furthermore, this witnessing does *not* refer to the non-first order notion of *support*.

In the parlance of this chapter (see Example 7.1.12 below), we will summarize this phenomenon as follows:

The H-field $\mathbb{T}_{\log}$ is Ψ-closed.

In Section 7.1, we introduce LD-fields. An LD-field is a pair (K, LD) where K is a certain type of d-valued field and $\mathrm{LD} \subseteq K$ is a distinguished subset which captures the essence of "being a logarithmic derivative in an appropriate extension". This is analogous to the theory of formally real fields, where one considers pairs (F, P) where F is a formally real field and $P \subseteq F$ is a so-called *positive cone* which captures the essence of "being a square in an appropriate real closed extension of F" (e.g., see **[32]**).

Intuitively, an LD-field is a pair (K, LD) which satisfies many of the same universal properties as the pair $(\mathbb{T}_{\log}, \mathbb{T}_{\log}^{\dagger})$. We pay close attention to when the pair (K, LD) satisfies the same witnessing property given in the Answer above. Indeed, we define (K, LD) to be *Ψ-closed* in this case. We also consider a useful generalization of being Ψ-closed, a property we call *being full*.

The rest of this chapter after Section 7.1 is a study in the extension theory of LD-fields. In particular, we are interested in showing that if (K, LD) is a full LD-field, and L is a d-valued extension of K with certain properties, then there is a unique subset $\mathrm{LD}^* \subseteq L$ such that (L, LD^*) is a full LD-field such that $\mathrm{LD} = \mathrm{LD}^* \cap K$. We accomplish this in many cases of interest.

7.1. LD-fields

In this section K is a d*-valued field of H-type with asymptotic integration.*

Recall that $\mathrm{I}(K) := \{y \in K : y \preccurlyeq f' \text{ for some } f \in \mathcal{O}\}$. It follows that $\mathrm{I}(K) = \{h \in K : vh > \Psi\}$.

Definition 7.1.1. An LD**-set (on** K**)** is a subset $\mathrm{LD} \subseteq K$ satisfying the following conditions:

(LD1) LD is a C_K-vector subspace of K;
(LD2) $K^\dagger \subseteq \mathrm{LD}$;
(LD3) $\mathrm{I}(K) \subseteq \mathrm{LD}$; and
(LD4) $v(\mathrm{LD}) \subseteq \Psi \cup (\Gamma^>)' \cup \{\infty\}$.

If $\mathrm{LD} \subseteq K$ is an LD-set on K, then we call the pair (K, LD) an LD**-field**; if in addition K is equipped with an ordering making it an H-field, then we call the pair (K, LD) an LD**-*H*-field**.

In general, if we refer to a pair (L, LD) as an LD-field (respectively, an LD-H-field) it is implied that L is a d-valued field of H-type with asymptotic integration (respectively, an H-field with asymptotic integration) and LD is an LD-set on L.

Here are some basic properties of LD-fields:

Lemma 7.1.2. *Suppose* (K, LD) *is an* LD*-field and let* $a, b \in K$. *Then:*

(1) $v(\mathrm{LD}) = \Psi \cup (\Gamma^>)' \cup \{\infty\}$;
(2) *if* $v(a - \mathrm{LD}) \not< (\Gamma^>)'$, *then* $a \in \mathrm{LD}$;
(3) *if* $b \in \mathrm{LD}$ *and* $v(a-b) \in \Psi^\downarrow \setminus \Psi$, *then* $a \notin \mathrm{LD}$ *and* $v(a-b) = \max v(a - \mathrm{LD})$; *and*
(4) *if* $v(a-b) > \Psi$, *then* $a \in \mathrm{LD}$ *iff* $b \in \mathrm{LD}$.

Proof. (1) follows from (LD2), (LD3), and (LD4). For (2), if $a \in K$ and $b \in \mathrm{LD}$ are such that $v(a-b) > \Psi$, then $a - b \in \mathrm{LD}$ by (LD3) and thus $a \in \mathrm{LD}$ by (LD1). For (3), if $b \in \mathrm{LD}$ and $v(a-b) \in \Psi^\downarrow \setminus \Psi$, then $a - b \notin \mathrm{LD}$ by (LD4) and so $a \notin \mathrm{LD}$ by (LD1). For (4), if $v(a-b) > \Psi$, then $a - b \in \mathrm{LD}$ and so $a \in \mathrm{LD}$ iff $b \in \mathrm{LD}$ by (LD1). $\square$

Minimal and maximal LD**-sets.** It is not clear that K has any LD-sets on it at all. The following is a natural candidate for a smallest LD-set on K:

Definition 7.1.3. $\mathrm{LD}(K) := CK^\dagger + \mathrm{I}(K)$. Note that $\mathrm{LD}(K)$ automatically satisfies (LD1), (LD2) and (LD3).

One condition that guarantees that $\mathrm{LD}(K)$ is an LD-set on K is being closed under powers:

Definition 7.1.4. We say that K is **closed under powers** if for every $c \in C$ and $f \in K^\times$ there is $y \in K^\times$ such that $y^\dagger = cf^\dagger$.

Lemma 7.1.5. *Suppose K is closed under powers. Then* $\mathrm{LD}(K)$ *is an* LD*-set on* K. *Furthermore,* $\mathrm{LD}(K)$ *is smallest among all* LD*-sets on* K.

Proof. It remains to check (LD4). Let $f = \sum_{i=1}^n c_i f_i^\dagger + g$ be an arbitrary element of $\mathrm{LD}(K)$ where $c_i \in C$, $f_i \in K^\times$ for $i = 1, \dots, n$ and $g \in \mathrm{I}(K)$. By the assumption that K is closed under powers, there is $y \in K^\times$ such that $y^\dagger = \sum_{i=1}^n c_i f_i^\dagger$. It follows that $v(y^\dagger + g) \in \Psi \cup (\Gamma^>)' \cup \{\infty\}$. The minimality property is clear by (LD1), (LD2) and (LD3). $\square$

The following is clear:

Lemma 7.1.6. *The union of a nonempty chain of* LD*-sets on* K *is an* LD*-set on* K.

Lemma 7.1.7. *Suppose K has an* LD*-set. Then K has a maximal* LD*-set and* $\mathrm{LD}(K)$ *is the smallest* LD*-set on K.*

Proof. That K has a maximal LD-set is clear from Zorn's Lemma. Let $\mathrm{LD} \subseteq K$ be any LD-set on K. Then $\mathrm{LD}(K) \subseteq \mathrm{LD}$. It suffices to check that that $\mathrm{LD}(K)$ satisfies (LD4), but this follows from $\mathrm{LD}(K) \subseteq \mathrm{LD}$. $\square$

Lemma 7.1.8. *Let (K, LD) be an* LD*-field. The following conditions on (K, LD) are equivalent:*

(1) LD *is a maximal* LD*-set on K;*
(2) *for every $a \in K$ there is $b \in \mathrm{LD}$ such that $v(a-b) \notin \Psi$;*
(3) *for every $a \in K \setminus \mathrm{LD}$ there is $b \in \mathrm{LD}$ such that $v(a-b) \in \Psi^{\downarrow} \setminus \Psi$.*

Proof. $(1 \Rightarrow 2)$ To prove the contrapositive, assume (2) is false. Take $a \in K$ such that $v(a - \mathrm{LD}) \subseteq \Psi$, so $a \in K \setminus \mathrm{LD}$. It suffices to prove the following:

Claim. *$Ca + \mathrm{LD}$ is an* LD*-set on K properly containing* LD*.*

Proof of claim. (LD1), (LD2), and (LD3) are clear. For (LD4), take $b \in \mathrm{LD}$ and $c \in C^{\times}$. Then $v(ca+b) = v(a + c^{-1}b) \in \Psi$ since $v(a - \mathrm{LD}) \subseteq \Psi$. $\square$

$(2 \Rightarrow 3)$ Suppose $a \in K \setminus \mathrm{LD}$ and take $b \in \mathrm{LD}$ such that $v(a-b) \notin \Psi$. If $v(a-b) > \Psi$, then $a - b \in \mathrm{I}(K) \subseteq \mathrm{LD}$, so $a \in \mathrm{LD}$ by (LD1). Thus $v(a-b) \in \Psi^{\downarrow} \setminus \Psi$.

$(3 \Rightarrow 1)$ Suppose $a \in K \setminus \mathrm{LD}$ and LD^* is an LD-set on K with $\mathrm{LD} \subseteq \mathrm{LD}^*$. Then (3) gives $b \in \mathrm{LD} \subseteq \mathrm{LD}^*$ such that $v(a-b) \in \Psi^{\downarrow} \setminus \Psi$, and so $a - b \notin \mathrm{LD}^*$ by (LD4). By (LD1) we conclude that $a \notin \mathrm{LD}^*$ since $b \in \mathrm{LD}^*$. $\square$

Definition 7.1.9. We say that an LD-field (K, LD) is **full** if (K, LD) satisfies any of the equivalent conditions in Lemma 7.1.8.

Definition 7.1.10. We say that an LD-field (K, LD) is **Ψ-closed** if it is full and $\mathrm{LD} = K^{\dagger}$. We say that the d-valued field K is **Ψ-closed** if

(1) $K^{\dagger}$ is an LD-set on K, and
(2) $(K, K^{\dagger})$ is Ψ-closed as an LD-field.

Note that if an LD-field is Ψ-closed, then it is automatically closed under powers.

Example 7.1.11. Suppose K is equipped with an ordering making it a Liouville closed H-field. Then $K = K^{\dagger}$ and $\Gamma_{\infty} = \Psi \cup (\Gamma^{>})' \cup \{\infty\}$. It follows that $(K, K) = (K, K^{\dagger})$ is a Ψ-closed LD-H-field and that K is a Ψ-closed H-field. Conversely, given an LD-H-field (K, LD), if $\partial K = K$, (K, LD) is Ψ-closed, and (Γ, ψ) is closed, then K is Liouville closed.

Example 7.1.12. The pair $(\mathbb{T}_{\log}, \mathbb{T}_{\log}^{\dagger})$ is a Ψ-closed LD-H-field, and thus $\mathbb{T}_{\log}$ is a Ψ-closed H-field. Property (LD1) follows from the fact that $\mathbb{T}_{\log}$ is closed under powers, a consequence of newtonianity and ADH 5.3.23. Everything else should be clear from the discussion at the beginning of this chapter.

A convenient consequence of fullness is that LD is "closed under pseudolimits" in the following sense:

Lemma 7.1.13. *Suppose (K, LD) is full and let (a_{ρ}) be a pc-sequence in K such that $a_{\rho} \in \mathrm{LD}$ for all ρ. Further suppose $b \in K$ is such that $a_{\rho} \rightsquigarrow b$. Then there is $a \in \mathrm{LD}$ such that $a_{\rho} \rightsquigarrow a$.*

PROOF. If $b \in \mathrm{LD}$ we are done, so assume $b \notin \mathrm{LD}$. Take $a \in \mathrm{LD}$ such that $v(b-a) \in \Psi^{\downarrow} \setminus \Psi$. For ρ big enough and $\sigma > \rho$ we have $v(b - a_\rho) = v(a_\sigma - a_\rho) \notin \Psi^{\downarrow} \setminus \Psi$ by (LD4), so $v(b - a_\rho) \neq v(b - a)$ eventually. Assume towards a contradiction that $v(b - a_\rho) > v(b - a)$ eventually. Then eventually $v(a_\rho - a) = v(b - a) \in \Psi^{\downarrow} \setminus \Psi$ contradicting (LD4). Thus $v(b - a) > v(b - a_\rho)$ eventually and so $a_\rho \rightsquigarrow a$. $\square$

Suppose L is a d-valued H-asymptotic extension of K with asymptotic integration, LD_0 is an LD-set on K, and LD_1 is an LD-set on L. Then we say that (L, LD_1) is an **extension** of (K, LD_0) (notation: $(K, \mathrm{LD}_0) \subseteq (L, \mathrm{LD}_1)$), if $\mathrm{LD}_1 \cap K = \mathrm{LD}_0$. A similar definition applies to **extension of LD-H-fields**.

Suppose the LD-fields (L, LD_1) and (M, LD_2) are extensions of (K, LD_0) and $i : L \to M$ is an embedding over K of valued differential fields. Then we say that $i : (L, \mathrm{LD}_1) \to (M, \mathrm{LD}_2)$ is an **embedding over K of LD-fields** if $i^{-1}(\mathrm{LD}_2) = \mathrm{LD}_1$. A similar definition applies to **embedding of LD-H-fields**.

The following is routine:

Lemma 7.1.14. *Let I be a nonempty ordered set and for all $i \in I$, let (K_i, LD_i) be an LD-field such that $(K_i, \mathrm{LD}_i) \subseteq (K_j, \mathrm{LD}_j)$ for all indices $i < j$. Then $\mathrm{LD} := \cup_{i \in I} \mathrm{LD}_i$ is the unique LD-set LD^* on $L := \cup_{i \in I} K_i$ such that $(K_i, \mathrm{LD}_i) \subseteq (L, \mathrm{LD}^*)$ for all $i \in I$. Furthermore, (L, LD) is full if (K_i, LD_i) is full and $\Psi_L \cap \Gamma_{K_i} = \Psi_{K_i}$ for all $i \in I$.*

7.2. Immediate extensions

In this section (K, LD) is an LD-field.

Lemma 7.2.1. *Suppose L is an immediate asymptotic extension of K. Then $\mathrm{LD}^* := \mathrm{LD} + \mathrm{I}(L)$ is an LD-set on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$. Furthermore, LD^* is the smallest LD-set LD_1 on L with the property that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}_1)$.*

PROOF. We first show that LD^* is an LD-set on L. (LD1) follows from $C_L = C_K$. (LD3) and (LD4) are clear. For (LD2), let $f \in L^\times$ be arbitrary. Then $f = gu$ with $g \in K$ and $u \in L^{\asymp 1}$ since $\Gamma_L = \Gamma_K$. Then $f^\dagger = g^\dagger + u^\dagger \in K^\dagger + \mathrm{I}(L) \subseteq \mathrm{LD} + \mathrm{I}(L) = \mathrm{LD}^*$.

Next, we must show that (L, LD^*) is an LD-field extension of (K, LD). It is clear that $\mathrm{LD}^* \cap K \supseteq \mathrm{LD}$. Suppose towards a contradiction that there is $h \in (\mathrm{LD}^* \cap K) \setminus \mathrm{LD}$. Then $h = f + g$ for some $f \in \mathrm{LD}$ and $g \in \mathrm{I}(L)$. Then $g = h - f \in K \cap \mathrm{I}(L)$, so $vg > \Psi$ and thus $g \in \mathrm{I}(K) \subseteq \mathrm{LD}$. By (LD1) it follows that $h \in \mathrm{LD}$, a contradiction.

The minimality of LD^* is clear since every LD-set LD_1 on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}_1)$ must necessarily contain both LD and $\mathrm{I}(L)$. $\square$

The next proposition gives a sufficient condition for fullness to be preserved by an immediate extension.

Proposition 7.2.2. *Suppose (K, LD) is full and L is an immediate asymptotic extension of K. Set $\mathrm{LD}^* := \mathrm{LD} + \mathrm{I}(L)$. If for all $f \in L \setminus K$,*

(1) *$v(f - K) < \Psi$, or*
(2) *the set $v(f - K) \cap \Psi$ has a maximum, or*
(3) *$v(f - g) > \Psi$ for some $g \in K$,*

then (L, LD^*) is full.

Proof. We will prove the contrapositive; assume $f \in L \setminus K$ is such that $v(f - \mathrm{LD}^*) \subseteq \Psi$. We will show that f cannot satisfy (1), (2), or (3). Note that $\emptyset \neq v(f - \mathrm{LD}) \subseteq v(f - \mathrm{LD}^*) \subseteq \Psi$. As $v(f - \mathrm{LD}) \subseteq v(f - K)$, f cannot satisfy (1).

Claim. $v(f - \mathrm{LD}) = v(f - \mathrm{LD}) \cap \Psi$ *does not have a largest element.*

Proof of claim. Let $a \in \mathrm{LD}$. Then $v(f - a) \in \Psi$ and so we have $g \in K^{\not\asymp 1}$ such that $f - a \asymp g^\dagger$. As K is d-valued and $L \supseteq K$ is immediate, we get $c \in C_K = C_L$ such that $f - a \sim cg^\dagger$. Then $a + cg^\dagger \in \mathrm{LD}$ and $v(f - a) < v(f - a - cg^\dagger) \in \Psi$. $\square$

Claim. $v(f - \mathrm{LD}) \cap \Psi$ *is downward closed as a subset of* Ψ*.*

Proof of claim. Let $g \prec 1$ and $a \in \mathrm{LD}$ be such that $v(g^\dagger) < v(f - a)$. Then $a + g^\dagger \in \mathrm{LD}$ by (LD1) and (LD2), and so $v(f - a - g^\dagger) = v(g^\dagger) \in v(f - \mathrm{LD}) \cap \Psi$. $\square$

Claim. $v(f - \mathrm{LD}) \cap \Psi = v(f - K) \cap \Psi$*.*

Proof of claim. Assume towards a contradiction that $g \in K$ is such that $v(f - g) \in \Psi$ and $v(f - g) > v(f - \mathrm{LD}) \cap \Psi$. By the first claim we can take a well-indexed sequence (a_ρ) in LD such that $v(f - a_\rho)$ is strictly increasing and cofinal in $v(f - \mathrm{LD})$. Then $a_\rho \rightsquigarrow f$ and $a_\rho \rightsquigarrow g$ since $v(f - g)$ is in the width of (a_ρ). By Lemma 7.1.13, there is $a \in \mathrm{LD}$ such that $a_\rho \rightsquigarrow a$. Thus $v(f - a) > v(f - \mathrm{LD})$, a contradiction. $\square$

As $v(f - K) \cap \Psi = v(f - \mathrm{LD}) \cap \Psi$ does not have a maximum, f cannot satisfy (2). Finally, assume towards a contradiction that f satisfies (3); we have $g \in K$ such that $v(f - g) > \Psi$. Since (K, LD) is full, we get $a \in \mathrm{LD}$ such that $v(g - a) \notin \Psi$. Then $v(f - a) \notin \Psi$ and $a \in \mathrm{LD}^*$, contradicting the assumption that $v(f - \mathrm{LD}^*) \subseteq \Psi$. Thus f cannot satisfy (3). $\square$

We do not know if Proposition 7.2.2 gives a *necessary* condition for preserving fullness when passing to an immediate extension, even in the special case $(\Gamma, \psi) \models T_{AC}$, where T_{AC} is as defined in Chapter 4.

7.3. Almost Special Immediate Extensions

In this section (K, LD) *is a full* LD*-field.* We consider here a general type of immediate extensions: the *almost special* extensions; see §5.4.

Definition 7.3.1. We say that an H-asymptotic couple (Γ^*, ψ^*) has the **successor property** if it has asymptotic integration and for every $\alpha, \beta \in \Psi^*$, if $\alpha < \beta$, then $s\alpha \leqslant \beta$. Furthermore, we say that such (Γ^*, ψ^*) has the **predecessor property** if it has asymptotic integration and for every $\alpha \in \mathrm{conv}(\Psi^*)$, there is $\beta \in (\Psi^*)^{\leqslant \alpha}$ such that $s\beta > \alpha$.

Note that if $(\mathbb{Q}\Gamma^*, \psi^*) \models T_{AC}$, then (Γ^*, ψ^*) has the successor and predecessor properties.

Lemma 7.3.2. *Suppose* (Γ, ψ) *has the successor and predecessor properties, and* L *is an almost special immediate asymptotic extension of* K*. Then there is a unique* LD*-set* LD^* *on* L *such that* $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$*, namely* $\mathrm{LD}^* := \mathrm{LD} + \mathrm{I}(L)$*; and with this* LD*-set,* (L, LD^*) *is full.*

Proof. By Lemma 7.2.1, $\mathrm{LD}^* := \mathrm{LD} + \mathrm{I}(L)$ is an LD-set on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$. For uniqueness, it is sufficient to establish that (L, LD^*) is full. To do this we will use Proposition 7.2.2. Suppose that $f \in L \setminus K$. By assumption, we have $\alpha \in \Gamma$ and a nontrivial convex subgroup Δ of Γ such that

$v(f-K)=(\alpha+\Delta)^{\downarrow}$. If $\alpha+\Delta<\Psi$ or $\alpha+\Delta>\Psi$, then we are done, so for the rest of the proof we assume that $(\alpha+\Delta)\cap\operatorname{conv}(\Psi)\neq\emptyset$.

Assume first that $(\alpha+\Delta)\cap\Psi=\emptyset$. Then $\alpha\in\operatorname{conv}(\Psi)$, so the predecessor property gives $\beta\in\Psi$ such that $\beta<\alpha$ and $s\beta>\alpha$. Then the successor property yields $\beta=\max(\alpha+\Delta)^{\downarrow}\cap\Psi$ and we are done.

Finally, assume that $(\alpha+\Delta)\cap\Psi\neq\emptyset$. If $(\alpha+\Delta)\cap\Psi=\{\beta\}$, then β is the maximum of $v(f-K)\cap\Psi$, and we are done. Otherwise, we have distinct $\beta,\delta\in(\alpha+\Delta)\cap\Psi$. We may assume that $\beta<\delta$ and by the successor property and convexity of Δ, we may also assume that $\delta=s\beta$. Thus $n(s\beta-\beta)\in\Delta$ for every n and so $\beta+2(s\beta-\beta)\in\alpha+\Delta$. However, by Lemma 2.4.3, $\beta+2(s\beta-\beta)\in(\Gamma^{>})'$. Therefore $\alpha+\Delta\not\subseteq\Psi^{\downarrow}$ and we are done. □

Proposition 7.3.3. *Suppose (Γ,ψ) has the successor and predecessor properties, L is an asymptotic extension of K, and $a\in L\setminus K$ is special over K. Furthermore, suppose (a_ρ) is a divergent pc-sequence from K of* d*-algebraic type with minimal differential polynomial $G(Y)$ over K such that $a_\rho\rightsquigarrow a$ and $G(a)=0$. Then there is a unique* LD*-set* LD^* *on $K\langle a\rangle$ such that $(K,\mathrm{LD})\subseteq\big(K\langle a\rangle,\mathrm{LD}^*\big)$, namely $\mathrm{LD}^*:=\mathrm{LD}+\mathrm{I}\big(K\langle a\rangle\big)$, and $\big(K\langle a\rangle,\mathrm{LD}^*\big)$ is full.*

Proof. By Corollary 5.4.4, $K\langle a\rangle\supseteq K$ is an almost special immediate extension. The rest follows from Lemma 7.3.2. □

7.4. Algebraic extensions

In this section (K,LD) is an LD*-field and $L\supseteq K$ is an algebraic asymptotic extension.* Since $(\Gamma_L,\psi_L)\subseteq(\mathbb{Q}\Gamma,\psi)$, L is necessarily of H-type and $\Psi_L=\Psi$. We will show that if L is d-valued with asymptotic integration, then there is always an LD-set LD^* on L such that $(K,\mathrm{LD})\subseteq(L,\mathrm{LD}^*)$. Under some additional technical assumptions, we will show that if (K,LD) is full, then so is (L,LD^*). We do this by considering three types of algebraic extensions.

Lemma 7.4.1 (Unramified extensions)**.** *Suppose that $[L:K]=[C_L:C]<\infty$. Then L is* d*-valued with asymptotic integration, and $\mathrm{LD}^*:=C_L\,\mathrm{LD}$ is an* LD*-set on L such that $(K,\mathrm{LD})\subseteq(L,\mathrm{LD}^*)$. Furthermore, if (K,LD) is full, then (L,LD^*) is full, and there is no* LD*-set $\mathrm{LD}_1\neq\mathrm{LD}^*$ on L such that $(K,\mathrm{LD})\subseteq(L,\mathrm{LD}_1)$.*

Proof. First we will show that L is d-valued. Since K is d-valued, $\operatorname{res}(C)=\operatorname{res}(K)$ and we have an extension of fields $\operatorname{res}(C)\subseteq\operatorname{res}(C_L)\subseteq\operatorname{res}(L)$. By [**6**, 3.1.8],

$$[L:K]\;=\;[C_L:C]\;=\;\big[\operatorname{res}(C_L):\operatorname{res}(C)\big]\;\leqslant\;\big[\operatorname{res}(L):\operatorname{res}(C)\big]\;=\;\big[\operatorname{res}(L):\operatorname{res}(K)\big]\;\leqslant\;[L:K].$$

Thus $\operatorname{res}(C_L)=\operatorname{res}(L)$. In particular, L is d-valued. It also follows from [**6**, 3.1.8] that $\Gamma_L=\Gamma$ and so L has asymptotic integration.

Next we will show that $\mathrm{LD}^*:=C_L\,\mathrm{LD}$ is an LD-set on L. (LD1) is clear. For (LD3), suppose that $f=\sum_i c_ia_i\in\mathrm{I}(L)$, where the $c_i\in C_L$ are C-linearly independent, and $a_i\in K$ for all i. Then $vf=\min_i va_i$ by Lemma 5.5.3. Since $vf>\Psi$, we get $va_i>\Psi$ for all i and so $a_i\in\mathrm{I}(K)\subseteq\mathrm{LD}$ for all i. It follows that $f\in\mathrm{LD}^*$. For (LD2), take $f\in L^\times$. Since $\Gamma_L=\Gamma$ we have $f=ua$ with $u\asymp1$ and $a\in K^\times$, so $f^\dagger=a^\dagger+u^\dagger\in K^\dagger+\mathrm{I}(L)\subseteq\mathrm{LD}^*$. For (LD4), take an arbitrary $f=\sum_i c_ia_i\in\mathrm{LD}^*$ with $c_i\in C_L$, and $a_i\in\mathrm{LD}$. We may again assume that the $c_i\in C_L^\times$ are C-linearly independent. Therefore $vf=\min_i va_i\in\Psi\cup(\Gamma_L^{>})'\cup\{\infty\}$.

To show that (L,LD^*) is an LD-field extension of (K,LD), take a basis $1,c_1,\dots,c_n$ of C_L as a vector space over C. Then $1,c_1,\dots,c_n$ is also a basis of L as a vector space over K because K and C_L are linearly disjoint over C and $[L:K]=[C_L:C]$. In particular, if $f=a_0+c_1a_1+\cdots+c_na_n\in \mathrm{LD}^*\cap K$ where each $a_i\in \mathrm{LD}$, then necessarily $f=a_0\in \mathrm{LD}$.

Finally, suppose (K,LD) is full. It remains to show that (L,LD^*) is full, which also shows uniqueness by maximality. Let $f\in L\setminus \mathrm{LD}^*$ be arbitrary. Then $f=\sum_i c_ia_i$ with C-linearly independent $c_i\in C_L^\times$ and $a_i\in K$. Since (K,LD) is full, we can take $b_i\in \mathrm{LD}$ with $v(a_i-b_i)\notin\Psi$ for each i. Then for $b:=\sum_i c_ib_i$ we have $f-b=\sum_i c_i(a_i-b_i)$, so $v(f-b)=\min_i v(a_i-b_i)\notin\Psi$, by Lemma 5.5.3. $\square$

Lemma 7.4.2 (Purely ramified extensions)**.** *Suppose K has rational asymptotic integration and $L=K(w^{1/p})$ is an extension of K with prime p and $v(w)\notin p\Gamma$ (and thus $[L:K]=[\Gamma_L:\Gamma]=p$). Then L is* d*-valued with asymptotic integration, and* $\mathrm{LD}^*:=\mathrm{LD}+\mathrm{I}(L)$ *is an* LD*-set on L such that $(K,\mathrm{LD})\subseteq(L,\mathrm{LD}^*)$. Furthermore, if (K,LD) is full, then (L,LD^*) is full, and there is no* LD*-set* $\mathrm{LD}_1\neq\mathrm{LD}^*$ *on L such that $(K,\mathrm{LD})\subseteq(L,\mathrm{LD}_1)$.*

Proof. The rational asymptotic integration assumption of K ensures that L has asymptotic integration. By [**6**, 3.1.8], $\big[\operatorname{res}(L):\operatorname{res}(K)\big]=1$, and so $\operatorname{res}(C_L)=\operatorname{res}(L)$. In particular, L is d-valued with $C_L=C$.

Next we will show that $\mathrm{LD}^*:=\mathrm{LD}+\mathrm{I}(L)$ is an LD-set on L. (LD1) and (LD3) are clear. For (LD2), let $f\in L^\times$ be arbitrary. Then $f=w^{i/p}ug$ with $i\in\{0,\dots,p-1\}, g\in K^\times$ and $u\asymp 1$. Then

$$f^\dagger=\frac{i}{p}w^\dagger+g^\dagger+u^\dagger\ \in\ \mathrm{LD}+\mathrm{I}(L).$$

(LD4) follows from the fact that (K,LD) satisfies (LD4) and $v\big(\mathrm{I}(L)\big)>\Psi$.

To show that (L,LD^*) is an LD-field extension of (K,LD), take an element

$$f=g+a_0+a_1w^{1/p}+\cdots+a_{p-1}w^{(p-1)/p}\ \in\ \mathrm{LD}^*\cap K$$

such that $g\in\mathrm{LD}$ and

$$a_0+a_1w^{1/p}+\cdots+a_{p-1}w^{(p-1)/p}\ \in\ \mathrm{I}(L)$$

with every $a_i\in K$. Then necessarily $a_i=0$ for $i\geqslant 1$. Thus $a_0\in\mathrm{I}(L)$, which gives $va_0>\Psi$ and so $a_0\in\mathrm{I}(K)$. We conclude that $f=g+a_0\in\mathrm{LD}$.

Finally, suppose (K,LD) is full. It remains to show that (L,LD^*) is full, which also shows uniqueness by maximality. Let $f\in L\setminus K$ be arbitrary. We want to find $b\in\mathrm{LD}^*$ such that $v(f-b)\notin\Psi$. We have $f=a+h$ where $a\in K$ and $vh\notin\Gamma\supseteq\Psi=\Psi_L$. If $vf\notin\Psi$ we can set $b:=0$ and we are done. Otherwise, since (K,LD) is full we can take $b\in\mathrm{LD}$ such that $v(a-b)\in\Gamma_\infty\setminus\Psi$. Then $v(f-b)=\min\big(v(a-b),vh\big)\notin\Psi$. $\square$

Lemma 7.4.3 (Immediate algebraic extensions)**.** *Suppose L is an immediate extension of K. Then L is* d*-valued with asymptotic integration, and* $\mathrm{LD}^*:=\mathrm{LD}+\mathrm{I}(L)$ *is an* LD*-set on L such that $(K,\mathrm{LD})\subseteq(L,\mathrm{LD}^*)$. Furthermore, if (K,LD) is full and (Γ,ψ) has the successor and predecessor properties, then (L,LD^*) is full, and there is no* LD*-set* $\mathrm{LD}_1\neq\mathrm{LD}^*$ *on L such that $(K,\mathrm{LD})\subseteq(L,\mathrm{LD}_1)$.*

Proof. The first claim is a special case of Lemma 7.2.1. By ADH 5.1.5, L is almost special over K. The second claim now follows from Lemma 7.3.2. $\square$

Lemma 7.4.4 (Algebraic closure)**.** *Suppose K has rational asymptotic integration and $L=K^a$ is an algebraic closure of K. Then L is* d*-valued with asymptotic integration, and there is an* LD*-set* LD^* *on L such that $(K,\mathrm{LD})\subseteq(L,\mathrm{LD}^*)$. Furthermore, if (K,LD) is full and (Γ,ψ) has the successor and predecessor properties, then there is a unique* LD*-set* LD^* *on L such that $(K,\mathrm{LD})\subseteq(L,\mathrm{LD}^*)$, and (L,LD^*) is full for this* LD^**.*

PROOF. We will prove both claims simultaneously (the first claim is mere existence, the second claim is uniqueness and fullness with stronger assumptions on (K, LD) and (Γ, ψ)). Note that the assumptions on the asymptotic couple in the second claim are inherited by every algebraic extension of K. We go from (K, LD) to L in three steps.

First, Lemma 7.4.3, gives us an LD-set LD_1 on the henselization K^{h} of K such that $(K, \mathrm{LD}) \subseteq (K^{\mathrm{h}}, \mathrm{LD}_1)$. Under the assumptions of the second claim $(K^{\mathrm{h}}, \mathrm{LD}_1)$ is full.

Next, Lemmas 7.4.1 and 7.1.14 gives us an LD-set $\mathrm{LD}_{\mathrm{unr}}$ on the maximal unramified extension $(K^{\mathrm{h}})^{\mathrm{unr}}$ of K^{h} such that $(K^{\mathrm{h}}, \mathrm{LD}_1) \subseteq \big((K^{\mathrm{h}})^{\mathrm{unr}}, \mathrm{LD}_{\mathrm{unr}}\big)$. Under the assumptions of the second claim $\big((K^{\mathrm{h}})^{\mathrm{unr}}, \mathrm{LD}_{\mathrm{unr}}\big)$ is full. Note that in this step, $(K^{\mathrm{h}})^{\mathrm{unr}}$ is reached from K^{h} by a direct union which we are suppressing, which is why we need Lemma 7.1.14.

Finally, we obtain L as a purely ramified extension of $(K^{\mathrm{h}})^{\mathrm{unr}}$ by [**6**, 3.3.48], so Lemmas 7.4.2 and 7.1.14 yield an LD-set LD^* on L such that $\big((K^{\mathrm{h}})^{\mathrm{unr}}, \mathrm{LD}_{\mathrm{unr}}\big) \subseteq (L, \mathrm{LD}^*)$. Under the assumptions of the second claim, (L, LD^*) is full. In this step, L is likewise reached from $(K^{\mathrm{h}})^{\mathrm{unr}}$ by a direct union which we are suppressing.

The uniqueness part of the second claim follows in a routine way from the uniqueness parts of Lemmas 7.4.1, 7.4.2, and 7.4.3. □

Proposition 7.4.5 (Arbitrary algebraic extensions)**.** *Suppose K has rational asymptotic integration and L is a* d*-valued extension of K. Then L has asymptotic integration, and there is an* LD*-set* LD^* *on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$. Furthermore, if (K, LD) is full and (Γ, ψ) has the successor and predecessor properties, then there is a unique* LD*-set* LD^* *on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$, and (L, LD^*) is full for this* LD^*.

PROOF. We may view L as a subfield of $K^{\mathrm{a}} = L^{\mathrm{a}}$. By Lemma 7.4.4, there is an LD-set LD_1 on K^{a} such that $(K, \mathrm{LD}) \subseteq (K^{\mathrm{a}}, \mathrm{LD}_1)$. Define $\mathrm{LD}^* := \mathrm{LD}_1 \cap L$. It is clear that LD^* is an LD-set on L. Furthermore, $\mathrm{LD}^* \cap K = (\mathrm{LD}_1 \cap L) \cap K = \mathrm{LD}_1 \cap K = \mathrm{LD}$ and so $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$.

For the second claim, we will now show that LD^* above is the unique LD-set on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$. Suppose that LD^{**} is an LD-set on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^{**})$. Then by the *existence* part of Lemma 7.4.4, there is an LD-set LD_2 on K^{a} such that $(L, \mathrm{LD}^{**}) \subseteq (K^{\mathrm{a}}, \mathrm{LD}_2)$. Thus $(K, \mathrm{LD}) \subseteq (K^{\mathrm{a}}, \mathrm{LD}_2)$, and so $\mathrm{LD}_1 = \mathrm{LD}_2$ by the uniqueness part of Lemma 7.4.4 for (K, LD). It follows that $\mathrm{LD}^{**} = \mathrm{LD}_2 \cap L = \mathrm{LD}_1 \cap L = \mathrm{LD}^*$.

To show that (L, LD^*) is full, assume that $\mathrm{LD}^{**} \supseteq \mathrm{LD}^*$ is a maximal LD-set on L. Then $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^{**})$ since (K, LD) is full. Thus $\mathrm{LD}^{**} = \mathrm{LD}^*$ by the uniqueness shown above. □

In the case where K is an H-field and L is the H-field real closure of K, then Proposition 7.4.5 gives the following:

Corollary 7.4.6 (Real closure)**.** *Suppose K has rational asymptotic integration and is equipped with an ordering making it an H-field, and L is a real closure of K equipped with the unique convex valuation ring extending the valuation ring of K, hence making L an H-field extension of K. Then there is an* LD*-set* $\mathrm{LD}^{\mathrm{rc}}$ *on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^{\mathrm{rc}})$. Furthermore, if (K, LD) is full and (Γ, ψ) has the successor and predecessor properties, then there is a unique* LD*-set* $\mathrm{LD}^{\mathrm{rc}}$ *on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^{\mathrm{rc}})$, and $(L, \mathrm{LD}^{\mathrm{rc}})$ is full for this* $\mathrm{LD}^{\mathrm{rc}}$.

7.5. Transcendental constant extension

In this section (K, LD) *is an* LD*-field.* Let $L = K(D)$ be a differential field extension of K with constant field $D = C_L$. Furthermore, assume that L is equipped with the valuation from ADH 5.5.1 making L a d-valued asymptotic extension of K. Note that L will have the same value group as K.

Lemma 7.5.1. $\mathrm{LD}^* := D\,\mathrm{LD} + \mathrm{I}(L)$ *is an* LD*-set on* L *such that* $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$*. Furthermore, if* (K, LD) *is full,* K *is henselian, and* (Γ, ψ) *has the successor and predecessor properties, then* (L, LD^*) *is full, and so there is no* LD*-set* $\mathrm{LD}_1 \neq \mathrm{LD}^*$ *on* L *such that* $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}_1)$.

Proof. We first show that LD^* is an LD-set on L. (LD1) and (LD3) are clear. (LD2) follows from writing arbitrary $f \in L^\times$ as $f = gu$ where $g \in K^\times$ and $u \in L^{\asymp 1}$. Then $f^\dagger = g^\dagger + u^\dagger \in K^\dagger + \mathrm{I}(L) \subseteq \mathrm{LD}^*$. For (LD4), write $f \in \mathrm{LD}^*$ as $f = \sum_{i=1}^n c_i a_i + z$ where all $a_i \in \mathrm{LD}$, the $c_i \in D$ are C-linearly independent and $z \in \mathrm{I}(L)$. Then $vf \in \{va_1, \dots, va_n\} \cup (\Gamma^>)' \cup \{\infty\}$ by Lemma 5.5.3.

We will now show that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$. Let $f \in \mathrm{LD}^* \cap K$. We have $f = a_0 + \sum_{i=1}^n c_i a_i + z$ where $a_0, a_1, \dots, a_n \in \mathrm{LD}$, $va_1, \dots, va_n < (\Gamma^>)'$, $1, c_1, \dots, c_n \in D$ are C-linearly independent, and $z \in \mathrm{I}(L)$. If $n = 0$, then $f - a_0 = z \in K \cap \mathrm{I}(L) = \mathrm{I}(K)$, so $f \in \mathrm{LD}$. The case $n \geqslant 1$ does not occur: in that case we would have

$$v(f - a_0) = \min_{1 \leqslant i \leqslant n} va_i < (\Gamma^>)',$$

and taking residues gives

$$1 \;=\; \sum_{i=1}^n \operatorname{res}\big(c_i a_i/(f - a_0)\big) + \underbrace{\operatorname{res}\big(z/(f - a_0)\big)}_{=0} \;=\; \sum_{i=1}^n \underbrace{\operatorname{res}\big(a_i/(f - a_0)\big)}_{\in \operatorname{res}(K) \cong C} c_i,$$

contradicting that $1, c_1, \dots, c_n$ are C-linearly independent.

Next we assume that K is henselian, (Γ, ψ) has the successor and predecessor properties, and (K, LD) is full. It remains to show that (L, LD^*) is full. We assume towards a contradiction that $f \in L \setminus K$ is such that $v(f - \mathrm{LD}^*) \subseteq \Psi$. In particular, $v(f - D\,\mathrm{LD}) \subseteq \Psi$.

Note that $v(f - D\,\mathrm{LD}) \subseteq \Psi$ does not have a largest element: given the element $v(f - \sum_i c_i b_i)$ where all $c_i \in D$ and $b_i \in \mathrm{LD}$, take $g \in K^\times$ such that $v(g^\dagger) = v(f - \sum_i c_i b_i)$. Since L is d-valued, there is $c \in D$ such $cg^\dagger \sim f - \sum_i c_i b_i$, and so $f - \sum_i c_i b_i - cg^\dagger \prec f - \sum_i c_i b_i$

Hence $v(f - D\,\mathrm{LD})$ decelerates as a subset of Γ by the successor property and Corollary 2.4.5.

Claim. $v\big(f - K[D]\big) < (\Gamma^>)'$.

Proof of claim. Assume towards a contradiction that $g \in K[D]$ is such that $v(f - g) > \Psi$. We have $g = \sum_{i=1}^n c_i a_i$, $n \geqslant 1$, where $c_1, \dots, c_n \in D$ are C-linearly independent and $a_1, \dots, a_n \in K$. Since (K, LD) is full we can take $b_1, \dots, b_n \in \mathrm{LD}$ with $v(a_i - b_i) \notin \Psi$ for all i. Then $h := \sum_i c_i b_i \in D\,\mathrm{LD}$ gives $g - h = \sum_i c_i (a_i - b_i)$, so $v(g - h) = \min_i v(a_i - b_i) \notin \Psi$. In view of $f - h = (f - g) + (g - h)$ we get $v(f - h) \notin \Psi$, a contradiction. □

Claim. *The sets* $v(f - D\,\mathrm{LD})$ *and* $v\big(f - K[D]\big)$ *are mutually cofinal as subsets of* Γ*, i.e.,* $v(f - D\,\mathrm{LD})^{\downarrow} = v\big(f - K[D]\big)^{\downarrow}$.

Proof of claim. The following is obvious:

$$v(f - D\,\mathrm{LD}) \;\subseteq\; v\big(f - K[D]\big) \;<\; (\Gamma^>)'$$

Next suppose that $\alpha = v(f - \sum_i c_i a_i) \in v\big(f - K[D]\big) \cap \Psi$, with $c_i \in D$ and $a_i \in K$. We may assume that the tuple (c_i) is linearly independent over C. Then we can choose $b_i \in \mathrm{LD}$ such that $v(a_i - b_i) \notin \Psi$, since (K, LD) is full. Thus $v\big(\sum_i c_i(a_i - b_i)\big) \notin \Psi$. Then $v(f - \sum_i c_i b_i) \in \Psi$ gives

$$v(f - \textstyle\sum_i c_i b_i) \;=\; v\big((f - \sum_i c_i a_i) - \sum_i c_i(a_i - b_i)\big) \;=\; \alpha$$

and so

$$v\big(f - K[D]\big) \cap \Psi \;=\; v(f - D\,\mathrm{LD}).$$

Since $v(f - D\,\mathrm{LD})$ does not have a largest element, this shows $v\big(f - K[D]\big)$ and $v(f - D\,\mathrm{LD})$ are mutually cofinal using the predecessor property. □

We now arrive at our overall contradiction. By Proposition 5.5.4, $v\big(f - K[D]\big)$ does not decelerate in Γ, whereas $v(f - D\,\mathrm{LD})$ does. However, by the last claim these sets are mutually cofinal in Γ, a contradiction by Lemma 2.1.9. □

7.6. Big exponential integration

In this section (K, LD) is an LD*-field with divisible value group and $s \in \mathrm{LD} \setminus K^\dagger$. We further assume that $L = K(f)$ is a* d*-valued extension of K of H-type with asymptotic integration such that f is transcendental over K, $f^\dagger = s$, $vf \notin \Gamma$, and $\Psi = \Psi_L$.* In particular, $C_L = C_K$.

Lemma 7.6.1. *Suppose K is algebraically closed. Then $\mathrm{LD}^* := \mathrm{LD} + \mathrm{I}(L)$ is an* LD*-set on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$. Furthermore, if (K, LD) is full, then (L, LD^*) is full, and there is no* LD*-set $\mathrm{LD}_1 \neq \mathrm{LD}^*$ on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}_1)$.*

Proof. (LD1), (LD3), and (LD4) are clear. It is also clear that $\mathrm{LD}^* \cap K = \mathrm{LD}$. To prove (LD2), since K is algebraically closed, it suffices to show that $(f - a)^\dagger \in \mathrm{LD}^*$ for arbitrary $a \in K$. If $f \prec a$, then

$$(f - a)^\dagger \;=\; \big((f/a) - 1\big)^\dagger + a^\dagger \;=\; \frac{(f/a)'}{(f/a) - 1} + a^\dagger,$$

which is in LD^* since $a^\dagger \in K^\dagger \subseteq \mathrm{LD} \subseteq \mathrm{LD}^*$ and $(f/a)'/\big((f/a) - 1\big) \in \mathrm{I}(L) \subseteq \mathrm{LD}^*$. The case $f \succ a$ procedes as before, using

$$(f - a)^\dagger \;=\; s + \big(1 - (a/f)\big)^\dagger \;=\; s - \frac{(a/f)'}{1 - (a/f)}.$$

Now suppose that (K, LD) is full. It is sufficient to prove that (L, LD^*) is full. Let $g \in L \setminus K$. Proposition 5.1.6 gives $a \in K$ and $h \in L$ such that $g = a + h$ and $vh \notin \Gamma \supseteq \Psi_L$. Since (K, LD) is full, we can take $b \in \mathrm{LD}$ such that $v(a - b) \notin \Psi = \Psi_L$. Then $v(g - b) = v\big((a - b) + h\big) \notin \Psi = \Psi_L$ since $vh \notin \Gamma$. □

In the rest of this section K is equipped with an ordering making (K, LD) a real closed LD*-H-field.*

Lemma 7.6.2. *Suppose L is equipped with an ordering making it an H-field extension of K. Then there is an* LD*-set LD^* on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$. Furthermore, if (K, LD) is full, then there is a unique* LD*-set LD^* on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$, and (L, LD^*) is full for this unique LD^*.*

Proof. Consider the d-valued extension $K(f)[i] = K[i](f)$ of $L = K(f)$. This will contain the algebraically closed d-valued field $K[i]$, an algebraic closure of K. By Lemma 7.4.1, there is an LD-set LD_0 on $K[i]$ such that $(K, \mathrm{LD}) \subseteq \big(K[i], \mathrm{LD}_0\big)$.

Next, $s \notin K[i]^\dagger$, so we can apply Lemma 7.6.1 to the extension $K[i] \subseteq K[i](f)$ to get an LD-set LD_1 on the d-valued field $K[i](f)$ such that $\big(K[i], \mathrm{LD}_0\big) \subseteq \big(K[i](f), \mathrm{LD}_1\big)$.

Set $\mathrm{LD}^* := \mathrm{LD}_1 \cap K(f) \subseteq K(f)$.

$$\begin{tikzcd}[column sep=small]
 & \big(K[i](f), \mathrm{LD}_1\big) \arrow[dl, dash] \arrow[dr, dash] & \\
\big(K[i], \mathrm{LD}_0\big) \arrow[dr, dash] & & \big(K(f), \mathrm{LD}_1 \cap K(f)\big) = (L, \mathrm{LD}^*) \arrow[dl, dash] \\
 & (K, \mathrm{LD}) &
\end{tikzcd}$$

It is clear that LD^* is an LD-set on $K(f)$. Furthermore, note that

$$\mathrm{LD}^* \cap K \;=\; \big(\mathrm{LD}_1 \cap K(f)\big) \cap K \;=\; \mathrm{LD}_1 \cap K \;=\; \mathrm{LD}$$

since $(K, \mathrm{LD}) \subseteq \big(K[i](f), \mathrm{LD}_1\big)$, as witnessed by the left side of the extension diagram. Thus $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$.

For the second claim assume (K, LD) is full. We will now show that there is no LD-set $\mathrm{LD}^{**} \neq \mathrm{LD}^*$ on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^{**})$. Suppose that LD^{**} is an LD-set on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^{**})$. Then by the *existence* part of Proposition 7.4.5, there is an LD-set LD_1^* on $K[i](f)$ such that $(L, \mathrm{LD}^{**}) \subseteq \big(K[i](f), \mathrm{LD}_1^*\big)$. In particular, $(K, \mathrm{LD}) \subseteq \big(K[i](f), \mathrm{LD}_1^*\big)$. However, by the *uniqueness* parts of Lemma 7.4.1 and 7.6.1, and fullness for (K, LD), we have $\mathrm{LD}_1 = \mathrm{LD}_1^*$ and so $\mathrm{LD}^* = \mathrm{LD}^{**}$.

Finally, to show (L, LD^*) is full, assume $\mathrm{LD}^{**} \supseteq \mathrm{LD}^*$ is a maximal LD-set on L. Then $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^{**})$ since (K, LD) is full. Therefore $\mathrm{LD}^{**} = \mathrm{LD}^*$ by the uniqueness shown above. $\square$

For later use, we summarize the results in this section as follows:

Corollary 7.6.3. *Assume K is λ-free and $v(s - K^\dagger) \subseteq \Psi^{\downarrow}$. Take a field extension $K(y)$ of K with y transcendental over K equipped with the unique derivation extending the derivation of K such that $y^\dagger = s$. Then there is a unique pair consisting of a valuation of $K(y)$ and a field ordering on $K(y)$ making it a pre-H-field extension of K with $y > 0$. With this valuation and ordering we have:*

(1) *$K(y)$ is an H-field, hence* d*-valued of H-type;*
(2) *$vy \notin \Gamma$;*
(3) *$\Psi = \Psi_{K(y)}$;*
(4) *$K(y)$ is λ-free.*

Furthermore, there is an LD*-set LD^* on $K(y)$ such that $(K, \mathrm{LD}) \subseteq \big(K(y), \mathrm{LD}^*\big)$. If (K, LD) is full, then there is a unique* LD*-set LD^* on $K(y)$ such that $(K, \mathrm{LD}) \subseteq \big(K(y), \mathrm{LD}^*\big)$, and $\big(K(y), \mathrm{LD}^*\big)$ is full for this unique LD^*.*

Proof. [**6**, 10.5.20] gives the unique pair consisting of a valuation of $K(y)$ and a field ordering on $K(y)$ making it a pre-H-field extension of K with $y > 0$. With that valuation and ordering, [**6**, 10.5.20] also gives (1) and (2). To show $\Psi = \Psi_{K(y)}$, the proof of [**6**, 10.5.20] gives

$$\Psi_{K(y)} \;=\; \Psi \cup \{v(a^\dagger + js) : a \in K^\times \text{ and } j \in \mathbb{Z}^{\neq}\} \;<\; (\Gamma^{>})'.$$

However, for $a \in K^\times$ and $j \in \mathbb{Z}^{\neq}$, we have $a^\dagger + js \in \mathrm{LD}$ and so necessarily $v(a^\dagger + js) \in \Psi$. Items (3) and (5) follow from ADH 6.1.13. The statement about LD-sets follows from Lemma 7.6.2. $\square$

7.7. Small exponential integration

We begin with a lemma for asymptotic couples. Lemma 7.7.1 will also be used in Section 7.8 below.

Lemma 7.7.1. *Assume $(\Gamma,\psi) \models T_{AC}$. Suppose $S \subseteq \Gamma$ is nonempty, does not have a greatest element, has the yardstick property, and $0 \in S^{\downarrow}$. Let $\alpha \in \Gamma$ and $n \geqslant 1$. Then exactly one of the following holds:*

(1) $(\alpha+nS)^{\downarrow} < \Psi$;
(2) *the set* $(\alpha+nS)^{\downarrow} \cap \Psi$ *has a maximum;*
(3) $(\alpha+nS)^{\downarrow} \not\subseteq \Psi^{\downarrow}$.

Proof. It is clear that conditions (1), (2) and (3) are mutually exclusive. We will assume that neither (1) nor (2) holds. If $\alpha > \Psi$, then we are done since $S^{>0} \neq \emptyset$. Thus we assume that $\alpha \in \Psi^{\downarrow}$. In particular, $(\alpha+nS)^{\downarrow} \cap \Psi^{>\alpha} \neq \emptyset$ by T_{AC}, the assumption that neither (1) nor (2) holds, and the observation that $\alpha \in (\alpha+nS)^{\downarrow}$. Here and below we let δ range over $\Psi \cap (\alpha+nS)^{\downarrow}$. Take $\gamma = \gamma(\delta) \in S^{\downarrow}$ such that $\alpha + n\gamma = \delta$, i.e., $\gamma = (\delta-\alpha)/n$. By increasing δ and γ, we may arrange $\delta > \alpha$ and thus $\gamma > 0$. By increasing δ and γ further, we arrange that $\delta > s^2\alpha > s\alpha > \alpha$. Then the yardstick property gives

$$\gamma \;<\; \gamma + s\gamma^{\dagger} - \gamma^{\dagger} \;=\; \gamma + s\psi(\delta-\alpha) - \psi(\delta-\alpha) \in S^{\downarrow}.$$

Applying the appropriate affine transformation to the above, and Lemma 2.3.5 gives

$$\delta \;<\; \alpha + n\gamma + ns\psi(\delta-\alpha) - n\psi(\delta-\alpha) \;=\; \delta + ns^2\alpha - ns\alpha \in (\alpha+nS)^{\downarrow}.$$

Now note that

$$\begin{aligned}
\textstyle\int(\delta + ns^2\alpha - ns\alpha) &= \delta + ns^2\alpha - ns\alpha - s(\delta + ns^2\alpha - ns\alpha) \quad \text{(by Lemma 2.3.3)}\\
&= \delta + ns^2\alpha - ns\alpha - s^2\alpha \quad \text{(by Lemma 4.3.5)}\\
&= \delta + (n-1)s^2\alpha - ns\alpha\\
&= (\delta - s\alpha) + (n-1)(s^2\alpha - s\alpha)\\
&> 0.
\end{aligned}$$

We conclude that $\delta + ns^2\alpha - ns\alpha \in (\Gamma^{>})'$ and thus $(\alpha+nS)^{\downarrow} \not\subseteq \Psi$. □

In the rest of this section (K,LD) is a henselian LD*-field and $s \in \mathrm{I}(K) \setminus K^{\dagger}$.* Let $L = K(f)$ be a field extension of f with f transcendental over K, equipped with the unique derivation extending the derivation of K such that $f^{\dagger} = s$. Using ADH 6.2.1 we equip L with the unique valuation that makes it an H-asymptotic extension of K with $f - 1 \not\asymp 1$. With this valuation, L is d-valued, and an immediate extension of K with $f \sim 1$.

Proposition 7.7.2. $\mathrm{LD}^* := \mathrm{LD} + \mathrm{I}(L)$ *is an* LD*-set on L such that $(K,\mathrm{LD}) \subseteq (L,\mathrm{LD}^*)$. Furthermore, if (K,LD) is full and $(\Gamma,\psi) \models T_{AC}$, then (L,LD^*) is full, and there is no* LD*-set $\mathrm{LD}_1 \neq \mathrm{LD}^*$ on L such that $(K,\mathrm{LD}) \subseteq (L,\mathrm{LD}_1)$.*

Proof. By Lemma 7.2.1, LD^* is an LD-set on L such that $(K,\mathrm{LD}) \subseteq (L,\mathrm{LD}^*)$. For the second claim assume that (K,LD) is full and $(\Gamma,\psi) \models T_{AC}$. It suffices to establish that (L,LD^*) is full. To do this, we will use Proposition 7.2.2. Let $g \in L \setminus K$ be arbitrary. Then by Lemma 5.1.4 there are $\alpha \in \Gamma$ and $n \geqslant 1$ such that $v(g-K) = (\alpha+nS)^{\downarrow}$ where $S := v(f-K)$. From $f \sim 1$ we get $0 \in S$, and so S has the yardstick property by Lemma 6.2.6. The conclusion now follows from Lemma 7.7.1 □

Let K be equipped with an ordering making it an H-field. Then we have the following LD-H-field version of the above, which for later use we combine with [**6**, 10.4.3, 10.5.8 and 10.5.18]:

Corollary 7.7.3. *Take a differential field extension $K(y)$ of K with y transcendental over K equipped with the unique derivation extending the derivation of K such that $y^\dagger = s$. Then there is a unique pair consisting of a valuation of $K(y)$ and a field ordering on $K(y)$ making it a pre-H-field extension of K with $y \sim 1$. Equipped with this valuation and ordering we have:*

(1) *$K(y)$ is an H-field, hence* d*-valued of H-type;*
(2) *$K(y)$ is an immediate extension of K and hence has asymptotic integration;*
(3) *if K is $\boldsymbol{\lambda}$-free, then so is $K(y)$ by Proposition 6.2.2.*

Furthermore, $\mathrm{LD}^* := \mathrm{LD} + \mathrm{I}(L)$ *is an* LD*-set on L such that* $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$*, and if* (K, LD) *is full and* $(\Gamma, \psi) \models T_{AC}$*, then* (L, LD^*) *is full, and there is no* LD*-set* $\mathrm{LD}_1 \neq \mathrm{LD}^*$ *on L such that* $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}_1)$.

7.8. Small integration

In this section (K, LD) *is a henselian* LD*-field and* $s \in \mathrm{I}(K) \setminus \partial K$. Let $L = K(f)$ be a field extension of K with f transcendental over K, equipped with the unique derivation extending the derivation of K such that $f' = s$. Then by ADH 6.3.2 we can equip L with the unique valuation that makes it an H-asymptotic extension of K with $f \not\asymp 1$. With this valuation L is an immediate extension of K with $f \prec 1$ and L is d-valued.

Proposition 7.8.1. $\mathrm{LD}^* := \mathrm{LD} + \mathrm{I}(L)$ *is an* LD*-set on L such that* $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$*. Furthermore, if* (K, LD) *is full and* $(\Gamma, \psi) \models T_{AC}$*, then* (L, LD^*) *is full, and there is no* LD*-set* $\mathrm{LD}_1 \neq \mathrm{LD}^*$ *on L such that* $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}_1)$.

Proof. The proof is identical to that of Proposition 7.7.2 except that it uses Lemmas 6.3.4 and 6.3.5. □

The statement of ADH 6.3.2 also gives us an H-field version:

Corollary 7.8.2. *Suppose K is equipped with an ordering making it an H-field and that L is equipped with the unique ordering making it a pre-H-field extension of K. Then* $\mathrm{LD}^* := \mathrm{LD} + \mathrm{I}(L)$ *is an* LD*-set on L such that* $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$*. Furthermore, if* (K, LD) *is full and* $(\Gamma, \psi) \models T_{AC}$*, then* (L, LD^*) *is full, and there is no* LD*-set* $\mathrm{LD}_1 \neq \mathrm{LD}^*$ *on L such that* $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}_1)$.

7.9. The Ψ-closure

In this section (K, LD) *is an* LD*-H-field.*

Definition 7.9.1. A **Ψ-closure of** (K, LD) is an LD-H-field extension $(K^\Psi, \mathrm{LD}^\Psi)$ of (K, LD) such that K^Ψ is real closed, $(K^\Psi, \mathrm{LD}^\Psi)$ is Ψ-closed and $(K^\Psi, \mathrm{LD}^\Psi)$ embeds over K into any LD-H-field extension of (K, LD) that is real closed and Ψ-closed.

The main result of this section is the following:

Theorem 7.9.2. *Suppose* (K, LD) *is full, K is $\boldsymbol{\lambda}$-free, and* $(\Gamma, \psi) \models T_{AC}$*. Then* (K, LD) *has up to isomorphism over K a unique Ψ-closure* $(K^\Psi, \mathrm{LD}^\Psi)$.

Theorem 7.9.2 will be proven below in Lemmas 7.9.6, 7.9.7, and 7.9.8 using the notion of LD-*H-towers.*

LD-H**-towers.** *In this subsection further suppose* $(\Gamma,\psi)\models T_{AC}$*,* K *is* λ*-free, and* (K,LD) *is full.* An **LD-H-tower on** (K,LD) is a strictly increasing chain $\big((K_\lambda,\mathrm{LD}_\lambda)\big)_{\lambda\leqslant\mu}$ of LD-H-fields, indexed by the ordinals less than or equal to some ordinal μ, such that

(1) $(K_0,\mathrm{LD}_0)=(K,\mathrm{LD})$;
(2) if λ is a limit ordinal, $0<\lambda\leqslant\mu$, then $K_\lambda=\bigcup_{\iota<\lambda}K_\iota$ and $\mathrm{LD}_\lambda=\bigcup_{\iota<\lambda}\mathrm{LD}_\iota$;
(3) for $\lambda<\lambda+1\leqslant\mu$, *either*
 (a) K_λ is not real closed and $K_{\lambda+1}$ is a real closure of K_λ,
 or, K_λ is real closed, $K_{\lambda+1}=K_\lambda(y_\lambda)$ with $y_\lambda\notin K_\lambda$ (so y_λ is transcendental over K_λ), and one of the following holds,
 (b) $y_\lambda^\dagger=s_\lambda\in \mathrm{I}(K_\lambda)\setminus K_\lambda^\dagger$ with $y_\lambda\sim 1$,
 (c) $y_\lambda^\dagger=s_\lambda\in \mathrm{LD}_\lambda^{<0}\setminus K_\lambda^\dagger$ with $y_\lambda>0$ and $v(s_\lambda-K_\lambda^\dagger)\subseteq\Psi_\lambda^{\downarrow}$.

The LD-H-field (K_μ,LD_μ) is called the **top** of the tower $\big((K_\lambda,\mathrm{LD}_\lambda)\big)_{\lambda\leqslant\mu}$.

Lemma 7.9.3. *If* $\big((K_\lambda,\mathrm{LD}_\lambda)\big)_{\lambda\leqslant\mu}$ *is an LD-H-tower on* (K,LD)*, then for all* $\lambda\leqslant\mu$

(1) K_λ *is* λ*-free, hence has rational asymptotic integration;*
(2) $(K_\lambda,\mathrm{LD}_\lambda)$ *is full;*
(3) $\Psi_\lambda=\Psi_0$*;*
(4) *if* K_λ *is real closed, then* $(\Gamma_\lambda,\psi_\lambda)\models T_{AC}$*.*

Proof. These follow from transfinite induction using Lemmas 6.1.8, 6.1.11, 7.1.14, and 7.6.3, Proposition 6.2.2, Corollaries 7.4.6, and 7.7.3, and ADH 6.1.13. □

Note that if $\big((K_\lambda,\mathrm{LD}_\lambda)\big)_{\lambda\leqslant\mu}$ is an LD-H-tower, then the "reduct" $(K_\lambda)_{\lambda\leqslant\mu}$ is a Liouville tower. Thus:

Lemma 7.9.4. *If* $\big((K_\lambda,\mathrm{LD}_\lambda)\big)_{\lambda\leqslant\mu}$ *is an LD-H-tower on* (K,LD)*, then*

(1) *the constant field* C_μ *of* K_μ *is a real closure of* C *if* $\mu>0$*; and*
(2) $|K_\mu|=|K|$*, hence* $\mu<|K|^+$*.*

We say that an LD-H-tower $\big((K_\lambda,\mathrm{LD}_\lambda)\big)_{\lambda\leqslant\mu}$ on K is **maximal** if it cannot be extended to an LD-H-tower $\big((K_\lambda,\mathrm{LD}_\lambda)\big)_{\lambda\leqslant\mu+1}$ on K.

The following is a consequence of Lemma 7.9.4(2):

Lemma 7.9.5. *Maximal* LD-H*-towers on* K *exist.*

We now fix a maximal LD-H-tower $\big((K_\lambda,\mathrm{LD}_\lambda)\big)_{\lambda\leqslant\mu}$ on K. We claim that the top (K_μ,LD_μ) is a Ψ-closure of (K,LD).

Lemma 7.9.6. (K_μ,LD_μ) *is real closed and* Ψ*-closed.*

Proof. By maximality and Corollary 7.4.6, it follows that (K_μ,LD_μ) is real closed. Likewise, by maximality, Corollaries 7.6.3 and 7.7.3, it follows that (K_μ,LD_μ) is Ψ-closed. □

Lemma 7.9.7. (K_μ,LD_μ) *embeds over* K *into any* Ψ*-closed real closed* LD-H*-field extension of* (K,LD)*.*

Proof. Let (L,LD^*) be a Ψ-closed real closed extension of (K,LD). The embedding of (K_μ,LD_μ) over K into (L,LD^*) can be constructed by transfinite recursion going up the tower $\big((K_\lambda,\mathrm{LD}_\lambda)\big)_{\lambda\leqslant\mu}$. At limit stages use Lemma 7.1.14. At successor stages, use the uniqueness of the LD-set extension given in Corollaries 7.4.6, 7.6.3, and 7.7.3. □

Lemma 7.9.8. *Suppose (L, LD^*) is a Ψ-closure of (K, LD). Then (K_μ, LD_μ) is isomorphic to (L, LD^*) over K. In particular, (L, LD^*) is* d*-algebraic over K and its asymptotic couple models T_{AC}.*

Proof. By the semiuniversal property of a Ψ-closure, we embed (L, LD^*) into (K_μ, LD_μ) over K, thus identifying (L, LD^*) with a subfield of (K_μ, LD_μ) which contains (K, LD). It suffices to show that $L = K_\mu$. This follows from arguing by transfinite induction that $K_\lambda \subseteq L$ for all $\lambda \leqslant \mu$. $\square$

7.10. Full newtonianity from linear newtonianity

In this section K is a d*-valued field of H-type with asymptotic integration.* We give here a method of extending LD-sets on K nicely to a certain newtonian extension of K (Corollary 7.10.8), *provided* that we have a way of handling the case where K is not linearly newtonian (Conjecture 7.10.4).

Recall that a **newtonization** of K is a newtonian extension of K that embeds over K into every newtonian extension of K. We also say that K is **asymptotically** d**-algebraically maximal** if it has no proper immediate d-algebraic asymptotic extension. We begin with some facts about newtonizations when K is ω-free with divisible value group:

ADH 7.10.1. *Suppose K is ω-free with divisible value group.*

(1) *K has a newtonization* [**6**, 14.5.2 and 14.5.4].
(2) *Any two newtonizations of K are isomorphic over K; this permits us to speak of* the *newtonization K^{nt} of K* [**6**, §14.5 and 14.3.12].
(3) *K^{nt} is an immediate* d*-algebraic extension of K, and no proper differential subfield of K^{nt} containing K is newtonian* [**6**, §14.5 and 14.3.12].
(4) *K^{nt} is asymptotically* d*-algebraically maximal* [**6**, 14.5.2].

Linear newtonianity allows us to reduce the problem to considering special pc-sequences, which are often easier to work with:

Lemma 7.10.2. *Suppose that K is λ-free, linearly newtonian, and every special pc-sequence in K of* d*-algebraic type over K has a pseudolimit in K. Then K is newtonian.*

Proof. Let $\phi \in K$ be active. Then the valuation ring of $(K^\phi, v^\flat_\phi)$ is linearly surjective by [**6**, 14.2.1]. Note that every special pc-sequence in $(K^\phi, v^\flat_\phi)$ of d-algebraic type over $(K^\phi, v^\flat_\phi)$ is also a special pc-sequence of K of d-algebraic type over K and hence has a pseudolimit in K by assumption. Thus by [**6**, 2.2.21 and 7.2.11] we conclude that $(K^\phi, v^\flat_\phi)$ is d-henselian, and so K is newtonian by [**6**, 14.1.4]. $\square$

Lemma 7.10.3. *Let K be ω-free with divisible value group. Let (a_ρ) be a divergent pc-sequence in K with minimal differential polynomial $G(Y)$ over K. Then there is $a \in K^{\mathrm{nt}} \setminus K$ such that $a_\rho \rightsquigarrow a$ and $G(a) = 0$.*

Proof. By compactness, we can take an element ℓ in an elementary extension of K such that $a_\rho \rightsquigarrow \ell$. By [**6**, 11.4.13], G is an element of $Z(K, \ell)$ of minimal complexity. Then by [**6**, 11.4.8], K has an immediate d-algebraic asymptotic extension $K\langle f\rangle$ with $G(f) = 0$ such that $a_\rho \rightsquigarrow f$.

Next we can consider the newtonizations of both K and $K\langle f\rangle$. By the defining property of the newtonization of K, there is an embedding $i : K^{\mathrm{nt}} \to K\langle f\rangle^{\mathrm{nt}}$ over K.

$$\begin{tikzcd}
 & K\langle f\rangle^{\mathrm{nt}} \\
K^{\mathrm{nt}} \arrow[ur, hook, "i"] \arrow[d, dash] & K\langle f\rangle \arrow[u, dash] \\
K \arrow[ur, dash] &
\end{tikzcd}$$

Then the extension $K\langle f\rangle^{\mathrm{nt}} \supseteq i(K^{\mathrm{nt}})$ is an immediate d-algebraic extension. As $i(K^{\mathrm{nt}})$ is asymptotically d-algebraically maximal, we have $i(K^{\mathrm{nt}}) = K\langle f\rangle^{\mathrm{nt}}$. Thus $f \in i(K^{\mathrm{nt}})$. Then $a := i^{-1}(f) \in K^{\mathrm{nt}} \setminus K$ has the desired properties. $\square$

In the rest of this section (K, LD) *ranges over* ω*-free full* LD-H*-fields with* $(\Gamma, \psi) \models T_{AC}$. Thus if L is an immediate d-algebraic asymptotic extension of K, then L is d-valued, $\mathrm{LD} + \mathrm{I}(L)$ is an LD-set on L with $(K, \mathrm{LD}) \subseteq \big(L, \mathrm{LD} + \mathrm{I}(L)\big)$, and L has a unique ordering making it an H-field extension of K. Here is our key conjecture about non-linearly newtonian K:

Conjecture 7.10.4 (Linear Newtonian Conjecture)**.** *Every* Ψ*-closed but non-linearly newtonian* (K, LD) *has a proper* d*-algebraic immediate asymptotic extension* L *such that* $\big(L, \mathrm{LD} + \mathrm{I}(L)\big)$ *is full.*

This conjecture allows us (Corollary 7.10.8) to handle the entire non-newtonian case ...and then some. First, we must recall some facts about the so-called *Newton-Liouville closure* from [**6**]:

Definition 7.10.5. Let E be an ω-free H-field. A **Newton-Liouville closure** of E is a newtonian Liouville closed H-field extension of E which embeds over E into every newtonian Liouville closed H-field extension of E.

ADH 7.10.6. *Suppose E is an ω-free H-field. Then:*

(1) *E has a Newton-Liouville closure* [**6**, 14.5.10]*;*
(2) *Any such Newton-Liouville closure of E is* d*-algebraic over E, thus ω-free, and its constant field is a real closure of C_E* [**6**, 14.5.10]*;*
(3) *Any two Newton-Liouville closures of E are isomorphic over E; this permits us to speak of* the *Newton-Liouville closure E^{nl} of E* [**6**, 16.2.2]*.*

The next definition is an adaptation of the definition of *Newton-Liouville closure* to the non-Liouville closed setting of (K, LD) and T_{AC}:

Definition 7.10.7. A **Newton-Ψ-closure of** (K, LD) is an LD-H-field extension $(K^{\Psi,\mathrm{nt}}, \mathrm{LD}^{\Psi,\mathrm{nt}})$ of (K, LD) with the following properties:

(1) $K^{\Psi,\mathrm{nt}}$ is real closed, ω-free, and newtonian;
(2) $(K^{\Psi,\mathrm{nt}}, \mathrm{LD}^{\Psi,\mathrm{nt}})$ is Ψ-closed and the asymptotic couple of $(K^{\Psi,\mathrm{nt}}, \mathrm{LD}^{\Psi,\mathrm{nt}})$ is a model of T_{AC};
(3) $(K^{\Psi,\mathrm{nt}}, \mathrm{LD}^{\Psi,\mathrm{nt}})$ embeds over K into any real closed, Ψ-closed, newtonian extension (K^*, LD^*) of (K, LD).

Of course, K has a Newton-*Liouville* closure, since it is an ω-free H-field by ADH 7.10.6(1) with K in the role of E; but taking a Newton-Liouville closure of K completely changes the model theory of the asymptotic couple of K in an irreversible way. However, a Newton-Liouville closure of K does serve as a convenient bound on certain d-algebraic extensions of K, and it is in this sense that we will use it in Corollary 7.10.8:

Corollary 7.10.8. *Suppose Conjecture 7.10.4 holds. Then a Newton-Ψ-closure* $(K^{\Psi,\mathrm{nt}}, \mathrm{LD}^{\Psi,\mathrm{nt}})$ *of* (K, LD) *exists. Any such* $K^{\Psi,\mathrm{nt}}$ *is* d*-algebraic over* K*, thus* ω*-free, and its constant field is a real closure of* C*.*

Proof. We are given a full ω-free LD-H-field (K, LD) with $(\Gamma, \psi) \models T_{AC}$. Consider a strictly increasing chain $\big((K_\lambda, \mathrm{LD}_\lambda)\big)_{\lambda<\nu}$ of full LD-H-fields indexed by the ordinals less than some ordinal $\nu > 0$, such that:

(a) $(K_0, \mathrm{LD}_0) = (K, \mathrm{LD})$;

(b) whenever $\lambda < \nu$ is an infinite limit ordinal, then $(K_\lambda, \mathrm{LD}_\lambda) = \bigcup_{\iota<\lambda}(K_\iota, \mathrm{LD}_\iota)$;

(c) whenever $\lambda < \lambda + 1 < \nu$, then either

  (i) $(K_\lambda, \mathrm{LD}_\lambda)$ is not Ψ-closed and $(K_{\lambda+1}, \mathrm{LD}_{\lambda+1})$ is a Ψ-closure of $(K_\lambda, \mathrm{LD}_\lambda)$, or

  (ii) $(K_\lambda, \mathrm{LD}_\lambda)$ is Ψ-closed and $K_{\lambda+1}$ is an immediate d-algebraic extension of K_λ with $\mathrm{LD}_{\lambda+1} = \mathrm{LD}_\lambda + \mathrm{I}(K_{\lambda+1})$

An easy induction shows that then for all $\lambda < \nu$: K_λ is d-algebraic over K (hence ω-free), its asymptotic couple is a model of T_{AC}, and if $\lambda > 0$, its constant field C_λ is a real closure of C. Moreover, if (K^*, LD^*) is any real closed, Ψ-closed, newtonian extension of (K, LD), then Theorem 7.9.2 and Lemma 7.10.2 yield by transfinite recursion embeddings $i_\lambda : (K_\lambda, \mathrm{LD}_\lambda) \to (K^*, LD^*)$ for $\lambda < \nu$ such that that i_0 is the natural inclusion and i_{λ_1} extends i_{λ_0} whenever $\lambda_0 < \lambda_1 < \nu$. In particular, this holds when $K^* = K^{\mathrm{nl}}$, the Newton-Liouville closure of K, with $\mathrm{LD}^* = (K^*)^\dagger$. This last fact shows that among the chains considered above there is a maximal one, i.e., there is a strictly increasing chain $\big((K_\lambda, \mathrm{LD}_\lambda)\big)_{\lambda<\nu}$ of full LD-H-fields with properties (a)-(c) that cannot be extended to a strictly increasing chain $\big((K_\lambda, \mathrm{LD}_\lambda)\big)_{\lambda<\nu+1}$ of full LD-H-fields with properties (a)-(c).

Assume now that our chain is maximal. Then $\nu = \mu+1$ for some ordinal μ, and so we have a last member (K_μ, LD_μ) in our chain, which is necessarily Ψ-closed. So far we did not use the Linear Newtonian Conjecture. Assuming this conjecture now, it is clear that (K_μ, LD_μ) is linearly newtonian. By Proposition 7.3.3 and Lemma 7.10.2 it is even newtonian, and so it is a Newton-Ψ-closure of (K, LD). $\square$

In Section 8.2 we will generalize the notion of *Newton-Ψ-closure* to arbitrary LD-H-fields, in a way that covers both the Newton-Ψ-closure of (K, LD) as well as the Newton-Liouville closure of E, when the asymptotic couple of E is closed. We shall also generalize ADH 7.10.6(3) to this setting as well.

7.11. The linear newtonian conjecture in a very special case

In this section K is a ω-free d*-valued field with divisible value group. We let a, b, s, t range over K, g range over $K^\times$, and ϕ range over the active elements of K.*

We begin with a more explicit version of [**6**, 13.6.10]:

Lemma 7.11.1. *Let $P \in K\{Y\}^{\neq}$ be homogeneous of degree* 1*. Suppose $a \neq 0$ and choose g such that $vg = (v_P^{\mathrm{e}})^{-1}(va)$. Then*

$$P^\phi_{\times g} \asymp a, \quad \textit{eventually}.$$

Proof. By convention, $vg = (v_P^{\mathrm{e}})^{-1}(va)$ gives $v_P^{\mathrm{e}}(vg) = va$ and $vg \notin \mathscr{E}^e(P)$, i.e., $\mathrm{nwt}_P(vg) = 0$. In particular, eventually $v(P^\phi_{\times g}) = v_{P^\phi}(vg) = v_P^{\mathrm{e}}(vg) + \mathrm{nwt}_P(vg)v\phi = v_P^{\mathrm{e}}(vg) = va$. $\square$

Definition 7.11.2. Let $P \in K\{Y\}^{\neq}$. We say that P is **in newton position at** a if $\operatorname{nmul} P_{+a} = 1$. Suppose P is in newton position at a. By the discussion in the beginning of [**6**, §14.3], if $P(a) \neq 0$, then there is a g such that eventually $P(a) = (P_{+a})^{\phi}_{1,\times g}$, and as vg does not depend on the choice of such g, we set $v^e(P,a) := vg$. If $P(a) = 0$, then we set $v^e(P,a) = \infty \in \Gamma_\infty$. Intuitively, we think of $v^e(P,a)$ as the distance from a where we would expect to find a zero of P in an appropriate immediate extension. Note that if P is degree 1 and $P(a) \neq 0$, then $v^e(P,a) = (v^e_{P_1})^{-1}(vP(a))$ by Lemma 7.11.1.

In the rest of this section let $P(Y) = Y' - sY - t$ and set $A := P_1 = Y' - sY$.

Lemma 7.11.3. *Suppose that P is in newton position at a. Then $vs < v\big(P(a)\big)$ or $\Psi < v\big(P(a)\big)$.*

Proof. By definition of newton position, $\operatorname{nmul} P_{+a} = 1$. This means that

$$P^{\phi}_{+a} \;=\; \big(Y' - sY + P(a)\big)^{\phi} \;=\; \phi Y' - sY + P(a)$$

has dominant multiplicity 1, eventually. Thus either $vs < v\big(P(a)\big)$ or $\Psi < v\big(P(a)\big)$. $\square$

Lemma 7.11.4. *Suppose P is in newton position at a and $P(a) \neq 0$. Then there exists b such that:*

(1) *P is in newton position at b, $v(a-b) = v^e(P,a)$, and $P(b) \prec P(a)$;*
(2) *for all $b^* \in K$ with $v(a-b^*) \geqslant v^e(P,a)$: $P(b^*) \prec P(a) \Leftrightarrow a - b \sim a - b^*$;*
(3) *for all $b^* \in K$, if $a - b \sim a - b^*$, then P is in newton position at b^* and $v^e(P,b^*) > v^e(P,a)$;*
(4) *(Yardstick) we have*

$$v\big(P(b)\big) \;>\; v^e(P,a) + \Psi,$$

and if $P(b) \neq 0$, then there is $\varepsilon \in \Gamma^{>}$ such that

$$v\big(P(b)\big) \;=\; v^e(P,a) + \varepsilon'.$$

Proof. We follow the proof of [**6**, 14.3.2]. Since $P(a) \neq 0$, by Lemma 7.11.1 we have g such that $vg = (v^e_A)^{-1}\big(vP(a)\big)$ and $P(a) \asymp A^{\phi}_{\times g}$, eventually; fix such a g. Let $Q = P_{+a}$ and $\gamma = vg$. Thus $Q_1 = A$, so

$$P(a) \;\asymp\; Q^{\phi}_{1,\times g}, \quad \text{eventually.}$$

In particular, we have

$$Q^{\phi}_{\times g} \;=\; P^{\phi}_{+a,\times g} \;=\; \underbrace{g\phi Y' + A(g)Y}_{(Q^{\phi}_{\times g})_1 = A^{\phi}_{\times g}} + P(a), \quad \text{eventually,}$$

and so $A(g) \asymp P(a)$ and $g\phi Y' \prec P(a)$, eventually. Taking $y = -P(a)/A(g)$ gives $y \asymp 1$ and

$$Q(gy) \;=\; Q^{\phi}(gy) \;=\; Q^{\phi}_{\times g}(y) \;=\; g\phi\delta(y) \;\preccurlyeq\; g\phi \;\prec\; P(a), \quad \text{eventually,}$$

where $\delta := \phi^{-1}\partial$ is the (necessarily small) derivation of K^{ϕ}. Note that since $P(a) \prec A^{\phi}$, eventually, and $P(a) \asymp A^{\phi}_{\times g}$, eventually, we have $g \prec 1$ (see the discussion at the beginning of [**6**, §14.3]). With $b := a + gy$, this satisfies automatically properties (1), (2) and (3) by the proof [**6**, 14.3.2]. We just need to show (4).

First we compute:

$$\begin{aligned}\frac{P(b)}{g} &= \frac{A(gy)+P(a)}{g}\\ &= (g^{-1}A_{\times g})(y)+g^{-1}P(a)\\ &= y'+g^{\dagger}y-sy+g^{-1}P(a)\\ &= y'+\frac{P(a)}{g}-\frac{(g^{\dagger}-s)P(a)}{A(g)}\\ &= y'+\frac{A(g)P(a)}{gA(g)}-\frac{(g'-sg)P(a)}{gA(g)}\\ &= y'\end{aligned}$$

where we used the identity $g^{-1}A_{\times g}=Y'+(g^{\dagger}-s)Y$ for the third equality. Finally, we note that $v(y')\in(\Gamma^{>})'\cup\{\infty\}$ because $-P(a)/A(g)=y\asymp 1$. □

This next lemma now follows from [**6**, 14.3.3] and its proof:

Lemma 7.11.5. *Suppose P is in newton position at a and there is no b with $P(b)=0$ and $v(a-b)=v^{e}(P,a)$. Then there exists a divergent pc-sequence (a_ρ) in K such that $P(a_\rho)\rightsquigarrow 0$, and (a_ρ) has the following properties:*

1. *P is in newton position at a_ρ, for all ρ,*
2. *$v(a_\sigma-a_\rho)=v^{e}(P,a_\rho)$ whenever $\rho<\sigma$,*
3. *$P(a_\sigma)\prec P(a_\rho)$ and $v^{e}(P,a_\sigma)>v^{e}(P,a_\rho)$ whenever $\rho<\sigma$,*
4. *finally, whenever $\rho<\sigma$, there is $\varepsilon_{\rho,\sigma}\in\Gamma^{>}$ such that*

$$v\big(P(a_\sigma)\big) \;=\; v^{e}(P,a_\rho)+\varepsilon'_{\rho,\sigma},$$

and thus

$$v\big(P(a_\sigma)\big) \;>\; v^{e}(P,a_\rho)+\Psi.$$

In the rest of the section we assume that K is henselian and Ψ-closed. Let P, a, and (a_ρ) be as in Lemma 7.11.5. Since K is henselian, (a_ρ) is not of algebraic type over K, and so has P as a minimal differential polynomial over K. Choose a pseudolimit $f\in K^{\mathrm{nt}}$ of (a_ρ) such that $P(f)=0$ (see Lemma 7.10.3). Furthermore, define $\gamma_\rho:=v(a_\sigma-a_\rho)=v^{e}(P,a_\rho)=(v^{e}_{A})^{-1}\big(vP(a_\rho)\big)\in\Gamma$ for $\sigma>\rho$, so γ_ρ is strictly increasing as a function of ρ by Lemma 7.11.5(3). Applying $(v^{e}_{A})^{-1}$ to the equality in Lemma 7.11.5(4) gives

$$(v^{e}_{A})^{-1}\big(vP(a_\sigma)\big) \;=\; \gamma_\sigma \;=\; (v^{e}_{A})^{-1}(\gamma_\rho+\varepsilon'_{\rho,\sigma}),\quad \text{for } \rho<\sigma,$$

which shows that $v(f-K)$ has the v^{e}_{A}-yardstick property. This allows us to apply Proposition 5.6.8 in the proof of the result below.

Proposition 7.11.6. *Suppose $\Psi=\{s^{n}0:n\geqslant 1\}$. Then there is a unique* LD*-set* LD^{*} *on $K(f)$ such that $(K,K^{\dagger})\subseteq\big(K(f),\mathrm{LD}^{*}\big)$, namely $\mathrm{LD}^{*}:=K^{\dagger}+\mathrm{I}\big(K(f)\big)$, and $\big(K(f),\mathrm{LD}^{*}\big)$ is full.*

Proof. This reduces to showing that $\big(K(f),\mathrm{LD}^{*}\big)$ is full. By Proposition 7.2.2, it suffices to establish the following: □

Claim 7.11.7. *Let $h\in K(f)\setminus K=K\langle f\rangle\setminus K$. Then exactly one of the following holds:*

1. *$v(h-K)<\Psi$;*

(2) *the set* $v(h-K)\cap\Psi$ *has a maximum;*
(3) $v(h-g)>\Psi$ *for some* $g\in K$.

Proof of Claim. By Lemma 7.11.5, $v(f-K)$ has the v_A^e-yardstick property. By Proposition 5.6.8, $v(f-K)$ is either Δ-fluent for some nontrivial convex subgroup Δ of Γ or is the form $\Gamma^{<\alpha}$ for some $\alpha\in\Gamma$. By Lemma 2.1.13, the same is true for all affine translates of $v(f-K)$. In view of Lemma 5.1.4 the desired result now follows from Lemma 2.5.11. $\square$

CHAPTER 8

Towards a model theory of logarithmic transseries

In this chapter we mimic parts of [**6**, Chapter 16] to obtain some results for $\mathbb{T}_{\log}$ that are analogous to results for $\mathbb{T}$ in [**6**]. We also state some conjectures that we believe to be true, and derive some of their consequences.

In Section 8.1 we generalize many of the results of [**6**, §16.1] to an appropriate "Ψ-closed" setting. This gives further evidence that the notion of an H-field being Ψ-closed is of independent interest. In particular, Theorem 8.1.6 can be used to show that Newton-Ψ-closures are unique, whenever they exist.

In Section 8.2 we give a more general definition of *Newton-Ψ-closure* for LD-H-fields, and show that they are unique up to isomorphism whenever they exist.

In Section 8.3 we first give an axiomatization for a theory $T_{\log}$ in a certain language $\mathcal{L}_{\mathrm{LD}}$, such that $\mathbb{T}_{\log} \models T_{\log}$ and $T_{\log}$ is the conjectured complete theory of $\mathbb{T}_{\log}$. Then we state some additional conjectures, beyond the Linear Newtonian Conjecture, which we need for model completeness of $\mathbb{T}_{\log}$. Finally, we give a proof of model completeness for $T_{\log}$ from these conjectures.

In the final section, Section 8.4, we conjecture an Ax-Kochen-Ersov theorem for H-fields which involves the Ψ-closed property, Conjecture 8.4.1.

8.1. Consequences of Ψ-closed

We believe that the condition of being Ψ-closed is of independent interest, apart from its role in our current strategy to obtain a model completeness result for $\mathbb{T}_{\log}$. As evidence in this direction, we present here Ψ-closed generalizations of results from [**6**, §16.1]. There the authors assume that the K below is Liouville closed. Here we replace that assumption with "Ψ-closed" and an additional assumption about the extension of asymptotic couples, and everything still goes through.

In this section K is an ω-free, newtonian, Ψ-closed d*-valued field with divisible value group. Furthermore, L is a* d*-valued field extension of K of H-type, and the asymptotic couple of K is existentially closed in the asymptotic couple (Γ_L, ψ_L) of L (as $\mathcal{L}_{AC}$-structures).* In particular, K is asymptotically d-algebraically maximal.

Note that we do not equip here either K or L with an LD-set. It follows from Definition 7.1.10 that a d-valued field E of H-type with asymptotic integration is Ψ-closed iff

(1) $E^{\dagger}$ is a C_E-vector subspace of E,
(2) $\mathrm{I}(E) \subseteq E^{\dagger}$, and
(3) for every $a \in E \setminus E^{\dagger}$, there is $b \in E^{\dagger}$ such that $v(a-b) \in \Psi_E^{\downarrow} \setminus \Psi_E$.

The existentially closed assumption gets us the famous *Property (B)* from [**3**, §4] almost for free (using [**6**, Lemma 9.9.3]):

Lemma 8.1.1. *Suppose $n \geqslant 1$, $\alpha_1, \dots, \alpha_n \in \Gamma$. Define functions $(\psi_L)_{\alpha_1,\dots,\alpha_i} : \Gamma_{L,\infty} \to \Gamma_{L,\infty}$ for $1 \leqslant i \leqslant n$ by recursion on i:*

$$(\psi_L)_{\alpha_1}(\gamma) := \psi_L(\gamma - \alpha_1), \quad (\psi_L)_{\alpha_1,\dots,\alpha_i}(\gamma) := \psi_L\big((\psi_L)_{\alpha_1,\dots,\alpha_{i-1}}(\gamma) - \alpha_i\big) \text{ for } i \geqslant 2.$$

Suppose $q_1, \dots, q_n \in \mathbb{Q}$ and $\gamma \in \Gamma_L$ are such that

$$(\psi_L)_{\alpha_1,\dots,\alpha_n}(\gamma) \neq \infty \quad (\text{so } (\psi_L)_{\alpha_1,\dots,\alpha_i}(\gamma) \neq \infty \text{ for } i = 1, \dots, n), \quad \text{and}$$

$$\gamma + q_1(\psi_L)_{\alpha_1}(\gamma) + \cdots + q_n(\psi_L)_{\alpha_1,\dots,\alpha_n}(\gamma) \in \Gamma \quad (\text{computed in } \mathbb{Q}\Gamma_L).$$

Then $\gamma \in \Gamma$.

In Lemmas 8.1.2, 8.1.3, and 8.1.4 below, we derive more consequences from the assumption of being existentially closed.

Lemma 8.1.2. $\Psi_L \cap \Gamma = \Psi$ *and* $(\Gamma_L^>)' \cap \Gamma = (\Gamma^>)'$.

Proof. Apply the definition of *existentially closed* to the appropriate $\mathcal{L}_{AC,\Gamma}$-sentences. □

Lemma 8.1.3. $L^\dagger \cap K = K^\dagger$.

Proof. Assume towards a contradiction that $f \in (L^\dagger \cap K) \setminus K^\dagger$. Since K is Ψ-closed, we may take $y \in K^\times$ such that $v(f - y^\dagger) \in \Psi^\downarrow \setminus \Psi$. We may also take $z \in L^\times$ such that $z^\dagger = f$. Then $v(f - y^\dagger) = v(z^\dagger - y^\dagger) = v\big((z/y)^\dagger\big) \in \big(\Psi_L \cup (\Gamma_L^>)'\big) \cap \Gamma = (\Psi_L \cap \Gamma) \cup \big((\Gamma_L^>)' \cap \Gamma\big) = \Psi \cup (\Gamma^>)'$ by Lemma 8.1.2, contradicting that $v(f - y^\dagger) \in \Psi^\downarrow \setminus \Psi$. □

In the rest of this section we further assume $C_L = C$.

Lemma 8.1.4. *Suppose $y \in L^\times$ and $y^\dagger \in K$. Then $y \in K^\times$.*

Proof. By Lemma 8.1.3, we can take $z \in K^\times$ such that $z^\dagger = y^\dagger$. Then $0 = y^\dagger - z^\dagger = (y/z)^\dagger$. In particular, $y/z \in C_L = C \subseteq K$ and so $y \in K$. □

The following Lemma 8.1.5 generalizes [**6**, 16.1.1]:

Lemma 8.1.5. *Suppose there is no $y \in L \setminus K$ for which $K\langle y \rangle$ is an immediate extension of K and let $f \in L \setminus K$. Then the $\mathbb{Q}$-vector space $\mathbb{Q}\Gamma_{K\langle f \rangle}/\Gamma$ is infinite-dimensional, so f is* d*-transcendental over K.*

Proof. We claim there is no divergent pc-sequence in K with a pseudolimit in L. To see this, suppose (y_ρ) is a divergent pc-sequence in K. It cannot be of d-algebraic type, since K is asymptotically d-algebraically maximal (see [**6**, 11.4.8 and 11.4.13]). So it is of d-transcendental type, and if it had a pseudolimit $y \in L$, then $K\langle y \rangle$ would be an immediate extension of K (see [**6**, 11.4.7 and 11.4.13]). This proves our claim.

Thus for each $y \in L \setminus K$ the set $v(y - K) \subseteq \Gamma_L$ has a largest element. Given $y \in L \setminus K$, a *best approximation in K to y* is by definition an element $y_0 \in K$ such that $v(y - y_0) = \max v(y - K)$.

Claim. *Suppose $y \in L \setminus K$ and y_0 is a best approximation in K to y. Then $v(y - y_0) \notin \Gamma$.*

Proof of Claim. Assume towards a contradiction that $v(y - y_0) \in \Gamma$. Then we have $a \in K^\times$ such that $y - y_0 \asymp a$. Since $C_L = C \subseteq K$ and L is d-valued, we get $c \in C_L = C$ such that $y - y_0 \sim ca$, and so $y - y_0 - ca \prec y - y_0$, contradicting $v(y - y_0) = \max v(y - K)$. □

Claim. *Suppose $y \in L^\dagger \setminus K$. Then there is $b \in K^\dagger$ such that b is a best approximation in K to y.*

Proof of Claim. Let b^* be a best approximation in K to y. If $b^* \in K^\dagger$, we can set $b := b^*$ and we are done. Otherwise, since K is Ψ-closed, we can find $a \in K^\times$ such that $v(b^* - a^\dagger) \in \Psi^\downarrow \setminus \Psi$. Since b^* is a best approximation in K to y, we have

$$v(y - b^*) \ \geqslant \ v(y - a^\dagger).$$

Assume towards a contradiction that $v(y - b^*) > v(y - a^\dagger)$. Then

$$v\big((y - b^*) - (y - a^\dagger)\big) \ = \ v(b^* - a^\dagger) \ = \ v(y - a^\dagger) \in \Psi^\downarrow \setminus \Psi.$$

However, $y = z^\dagger$, with $z \in L^\times$, so

$$v(y - a^\dagger) \ = \ v(z^\dagger - a^\dagger) \ = \ v\big((z/a)^\dagger\big) \in \Psi_L \cup (\Gamma_L^>)',$$

a contradiction since $\big(\Psi_L \cup (\Gamma_L^>)'\big) \cap (\Psi^\downarrow \setminus \Psi) = \emptyset$ by Lemma 8.1.2. Thus $v(y - b^*) = v(y - a^\dagger)$ and we may take $b := a^\dagger \in K^\dagger$ to be a best approximation in K to y. □

Pick a best approximation b_0 in K to $f_0 := f$, and set $f_1 := (f_0 - b_0)^\dagger \in K\langle f\rangle$. Then $f_1 \in K\langle f\rangle^\dagger \setminus K$ by Lemma 8.1.4.

By the above claim, we can take an element $a_1 \in K^\times$ such that $b_1 := a_1^\dagger$ is a best approximation in K to f_1. Continuing this way, we obtain a sequence (f_n) in $K\langle f\rangle \setminus K$ and sequences $(a_n)_{n\geqslant 1}$ in $K^\times$ and $(b_n)_{n\geqslant 0}$ in K such that:

(1) b_n is a best approximation in K to f_n for all n,
(2) $f_{n+1} = (f_n - b_n)^\dagger \in K\langle f\rangle^\dagger \setminus K$ for all n,
(3) $b_n = a_n^\dagger \in K^\dagger$ for all $n \geqslant 1$,
(4) $v(f_n - b_n) \notin \Gamma$ for all n.

Claim. $v(f_0 - b_0), v(f_1 - b_1), v(f_2 - b_2), \ldots$ *are* $\mathbb{Q}$*-linearly independent over* Γ.

Proof of claim. Note that

$$f_n - b_n \ = \ (f_{n-1} - b_{n-1})^\dagger - a_n^\dagger \ = \ \left(\underbrace{\frac{f_{n-1} - b_{n-1}}{a_n}}_{\not\asymp 1}\right)^\dagger$$

for $n \geqslant 1$. Then with $\alpha_n := va_n \in \Gamma$ for $n \geqslant 1$ we get

$$v(f_n - b_n) \ = \ \psi_L\big(v(f_{n-1} - b_{n-1}) - \alpha_n\big),$$

so by an easy induction on n,

$$v(f_n - b_n) \ = \ (\psi_L)_{\alpha_1,\ldots,\alpha_n}\big(v(f_0 - b_0)\big)$$

for $n \geqslant 1$. Suppose towards a contradiction that $v(f_0 - b_0), \ldots, v(f_n - b_n)$ are $\mathbb{Q}$-linearly dependent over Γ. Then we have $m < n$ and $q_{m+1}, \ldots, q_n \in \mathbb{Q}$ such that

$$v(f_m - b_m) + q_{m+1}v(f_{m+1} - b_{m+1}) + \cdots + q_n v(f_n - b_n) \ \in \ \Gamma.$$

For $\gamma := v(f_m - b_m) \in \Gamma_L \setminus \Gamma$ this gives

$$\gamma + q_{m+1}(\psi_L)_{\alpha_{m+1}}(\gamma) + \cdots + q_n(\psi_L)_{\alpha_{m+1},\ldots,\alpha_n}(\gamma) \ \in \ \Gamma,$$

which contradicts Lemma 8.1.1. □

We conclude that $\mathbb{Q}\Gamma_{K\langle f\rangle}/\Gamma$ is an infinite-dimensional $\mathbb{Q}$-vector space. In view of [**6**, 3.1.11], it follows that f is d-transcendental over K. $\square$

Here is a generalization of [**6**, 16.0.3]:

Theorem 8.1.6. *K has no proper* d-*algebraic* d-*valued extension F with the same constant field such that the asymptotic couple of K is existentially closed in the asymptotic couple of F.*

Proof. Assume F is a proper d-valued extension of K with $C_F = C$ and the asymptotic couple of K is existentially closed in the asymptotic couple of F. Take $f \in F \setminus K$. Then by Lemma 8.1.5 applied to $L = F$, f is necessarily d-transcendental over K. In particular, F cannot be a d-algebraic extension of K. $\square$

In the rest of this section we let K, L and f be as in Lemma 8.1.5 above. Elaborating on the proof of this lemma we shall obtain a complete description of $K\langle f\rangle$ as a d-valued extension of K generated by f. For this we use the notations in that proof, and set $\beta_n := v(f_n - b_n) - \alpha_{n+1} \in \Gamma_{K\langle f\rangle}$. Thus $\beta_0, \beta_1, \beta_2, \ldots$ are $\mathbb{Q}$-linearly independent over Γ (because the last claim of the proof of Lemma 8.1.5 showed that the sequence $(\beta_n + \alpha_{n+1})_{n\geqslant 0}$ is $\mathbb{Q}$-linearly independent over Γ, but the α_{n+1}'s are in Γ).

Here is a generalization of [**6**, Lemma 16.1.2]:

Lemma 8.1.7. *The asymptotic couple of $K\langle f\rangle$ has the following properties:*

(1) $\Gamma_{K\langle f\rangle} = \Gamma \oplus \bigoplus_n \mathbb{Z}\beta_n$ *(internal direct sum);*
(2) $\beta_n^\dagger \notin \Gamma$ *for all n, and $\beta_m^\dagger \neq \beta_n^\dagger$ for all $m \neq n$;*
(3) $\psi(\Gamma_{K\langle f\rangle}^{\neq}) = \Psi \cup \{\beta_n^\dagger : n = 0, 1, 2, \ldots\}$*;*
(4) $[\Gamma_{K\langle f\rangle}] = [\Gamma] \cup \big\{[\beta_n] : n = 0, 1, 2, \ldots\big\}$*;*
(5) $\Gamma^<$ *is cofinal in $\Gamma_{K\langle f\rangle}^<$, and $\beta_0^\dagger < \beta_1^\dagger < \beta_2^\dagger < \cdots$.*

Proof. Consider the "monomials" $\mathfrak{m}_n := (f_n - b_n)/a_{n+1}$ with $v(\mathfrak{m}_n) = \beta_n$. Then

$$\mathfrak{m}_{n+1} \;=\; \frac{f_{n+1} - b_{n+1}}{a_{n+2}} \;=\; \frac{(f_n - b_n)^\dagger - b_{n+1}}{a_{n+2}} \;=\; \frac{(a_{n+1}\mathfrak{m}_n)^\dagger - b_{n+1}}{a_{n+2}} \;=\; \frac{a_{n+1}^\dagger + \mathfrak{m}_n^\dagger - b_{n+1}}{a_{n+2}} \;=\; \frac{\mathfrak{m}_n^\dagger}{a_{n+2}},$$

and so $\mathfrak{m}_n' = a_{n+2}\mathfrak{m}_n\mathfrak{m}_{n+1}$. Thus $f = b_0 + a_1\mathfrak{m}_0$ gives $f' = b_0' + a_1'\mathfrak{m}_0 + a_1a_2\mathfrak{m}_0\mathfrak{m}_1$, and continuing by induction on n gives

$$f^{(n)} = F_n(\mathfrak{m}_0, \ldots, \mathfrak{m}_n), \quad F_n(Y_0, \ldots, Y_n) \in K[Y_0, \ldots, Y_n], \quad \deg F_n \leqslant n+1.$$

As f is d-transcendental over K, for $P \in K\{Y\}^{\neq}$ of order $\leqslant r$ we have

$$P(f) \;=\; \sum_{\boldsymbol{i}\in I} a_{\boldsymbol{i}}\mathfrak{m}_0^{i_0}\cdots\mathfrak{m}_r^{i_r}$$

where the sum is over a finite nonempty set I of tuples $\boldsymbol{i} = (i_0, \ldots, i_r) \in \mathbb{N}^{1+r}$, and $a_{\boldsymbol{i}} \in K^\times$ for all $\boldsymbol{i} \in I$. Since $v(\mathfrak{m}_0) = \beta_0, v(\mathfrak{m}_1) = \beta_1, \ldots$ are $\mathbb{Q}$-linearly independent over Γ, we obtain $v\big(P(f)\big) \in \Gamma + \sum_n \mathbb{N}\beta_n$, which proves (1).

We have $\beta_n^\dagger \notin \Gamma$ because by the proof of Lemma 8.1.5,

$$\beta_n^\dagger \;=\; \psi\big(v(f_n - b_n) - \alpha_{n+1}\big) \;=\; v(f_{n+1} - b_{n+1}) \;=\; \beta_{n+1} + \alpha_{n+2} \notin \Gamma.$$

Since $\beta_1, \beta_2, \beta_3, \ldots$ are $\mathbb{Q}$-linearly independent, so are $\beta_0^\dagger, \beta_1^\dagger, \beta_2^\dagger, \ldots$ by these equalities. This proves (2), which in view of (1) and ψ being a valuation yields (3). From (2) and (HC) we get $[\beta_n] \notin [\Gamma]$ for all n, and $[\beta_m] \neq [\beta_n]$ for all $m \neq n$. Again in view of (1), this gives (4).

To get (5), assume towards a contradiction that $\Gamma^{<}$ is not cofinal in $\Gamma^{<}_{K\langle f\rangle}$. Then by (4) we get n with $[\beta_n] < [\alpha]$ for all $\alpha \in \Gamma^{\neq}$, hence $\Psi < \beta_n^{\dagger} < (\Gamma^{>})'$. Then $[\beta_n^{\dagger} - \alpha] \in [\Gamma]$ for all $\alpha \in \Gamma$, by [**6**, 9.8.6]. For $\alpha := \alpha_{n+2}$ this means $[\beta_n^{\dagger} - \alpha_{n+2}] = [\beta_{n+1}] \in [\Gamma]$, contradicting (2). Thus $\Gamma^{<}$ is indeed cofinal in $\Gamma^{<}_{K\langle f\rangle}$. For any n we can therefore take $\alpha \in \Gamma^{\neq}$ with $[\alpha] < [\beta_n]$. Also $[\beta_{n+1}] \notin [\Gamma]$ and $\beta_n^{\dagger} - \alpha^{\dagger} \in (\Gamma + \mathbb{Z}\beta_{n+1}) \setminus \Gamma$, and by [**6**, 2.4.4 and 6.5.4(ii)],

$$[\beta_{n+1}] \leqslant [\beta_n^{\dagger} - \alpha^{\dagger}] < [\beta_n - \alpha] = [\beta_n].$$

So we have a strictly decreasing sequence $[\beta_0] > [\beta_1] > [\beta_2] > \cdots$ in $[\Gamma_{K\langle f\rangle}]$, and therefore a strictly increasing sequence $\beta_0^{\dagger} < \beta_1^{\dagger} < \beta_2^{\dagger} < \cdots$ in view of (2). □

Corollary 8.1.8. *$K\langle f\rangle$ is ω-free.*

PROOF. $K\langle f\rangle$ is of H-type since L is, and Lemma 8.1.7(5) implies that $K\langle f\rangle$ is ungrounded since K is. As K is ω-free, the pc-sequence (ω_ρ) in K is divergent. If $K\langle f\rangle$ was not ω-free, then there would be an $\omega \in K\langle f\rangle$ such that $\omega_\rho \rightsquigarrow \omega$, and so necessarily $\omega \in L \setminus K$. This contradicts the beginning of the proof of Lemma 8.1.5. □

In the rest of this section we further assume that K and L are equipped with orderings making K an H-field and L an H-field extension of K. Furthermore, assume that M is an H-field extension of K such that the asymptotic couple of K is also existentially closed in the asymptotic couple of M.

In this H-field setting, we have the following partial generalization of [**6**, 16.1.4]:

Lemma 8.1.9. *Suppose g in M realizes the same cut in the ordered set K as f does. Then $v(g - b_0) = \max v(g - K) \notin \Gamma$, and $g_1 := (g - b_0)^{\dagger}$ realizes the same cut in the ordered set $K^{\dagger}$ as $f_1 = (f - b_0)^{\dagger}$.*

PROOF. Establishing that $v(g - b_0) = \max v(g - K) \notin \Gamma$ proceeds exactly as it does in the proof of [**6**, 16.1.4].

Following that same proof, to get that $(g - b_0)^{\dagger} \notin K$, assume towards a contradiction that $(g-b_0)^{\dagger} \in K$. By Lemma 8.1.3 with M in the role of L, we have $(g - b_0)^{\dagger} \in K^{\dagger}$. This means $(g - b_0)^{\dagger} = a^{\dagger}$ with $a \in K^{\times}$ and so $g - b_0 = ca$ for some $c \in C_M^{\times}$, and thus $v(g - b_0) = va \in \Gamma$, a contradiction.

Finally, to get that g_1 and f_1 realize the same cut in the ordered set $K^{\dagger}$, we assume towards a contradiction that there is $\phi \in K^{>}$ such that $(f - b_0)^{\dagger} < \phi^{\dagger}$ in L and $\phi^{\dagger} < (g - b_0)^{\dagger}$ in M, and proceed as in the proof of [**6**, 16.1.4]. Similarly for the other case. □

We believe that in Lemma 8.1.9 above g_1 and f_1 realize the same cut in the ordered set K in general, as a consequence of the existentially closed extension. Up until this point, the only consequences of "existentially closed" that we have used are Lemmas 8.1.1 and 8.1.2. For the rest of the section we restrict our attention to the main case of interest, $(\Gamma, \psi) \models T_{AC}$, where we will use additional instances of the assumption of being existentially closed:

Lemma 8.1.10. *Suppose $(\Gamma, \psi) \models T_{AC}$. Then f_1 and g_1 realize the same cut in the ordered set K.*

PROOF. By Lemma 8.1.9, f_1 and g_1 realize the same cut in the ordered abelian group $K^{\dagger}$. Thus $f^* := f_1 - a_1^{\dagger}$ and $g^* := g_1 - a_1^{\dagger}$ also realize the same cut in $K^{\dagger}$. It suffices to show that f^* and g^* realize the same cut in K. Note that $vf^* = \beta_0^{\dagger} \in \Psi_{K\langle f\rangle} \setminus \Gamma$ and so $vg^* \in \Psi_M \setminus \Gamma$ as well. Furthermore, vf^* and vg^* realize the same cut in the set Ψ. Since $(\Gamma, \psi) \models T_{AC}$, there are four cases of cuts in Ψ to consider, although only one of them can actually correspond to the cut of vf^* in Ψ:

Case 1: $vf^* < s0$. Then

$$(\Gamma_{K\langle f\rangle}, \psi_L) \models \exists x\big[\psi(x) < s0\big]$$

and so the same existential sentence must be true in (Γ, ψ), contradicting $(\Gamma, \psi) \models T_{AC}$.

Case 2: *There is $\alpha \in \Psi$ such that $\alpha < vf^* < s\alpha$ in $\Gamma_{K\langle f\rangle}$.* Then

$$(\Gamma_{K\langle f\rangle}, \psi_L) \models \exists x\big[\alpha < \psi(x) \wedge \psi(x) < s\alpha\big]$$

and so the same existential sentence must be true in (Γ, ψ), contradicting $(\Gamma, \psi) \models T_{AC}$.

Case 3: *There is a nonempty $B \in \operatorname{sded}(\Psi)$ such that $B \neq \Psi$, and $B < vf^* < \Psi \setminus B$.* In this case, B is mutually cofinal in Γ with $\Gamma^{<\Gamma^{>B}}$ and $\Psi \setminus B$ is mutually coinitial in Γ with $\Gamma^{>B}$. Thus vf^* and vg^* realize the same cut in Γ. As $vf^*, vg^* \notin \Gamma$, and f^* and g^* have the same sign since $0 \in K^\dagger$, it follows that f^* and g^* realize the same cut in K.

Case 4: $vf^* > \Psi$. Then $\Gamma^<$ is not cofinal in $\Gamma^<_{K\langle f\rangle}$, which contradicts Lemma 8.1.7(5). □

The proof of Corollary 8.1.11 proceeds exactly as in [**6**, 16.1.5]:

Corollary 8.1.11. *Suppose $(\Gamma, \psi) \models T_{AC}$ and g in M realizes the same cut in the ordered set K as f does. Then there is an embedding $K\langle f\rangle \to M$ of H-fields over K sending f to g.*

8.2. Uniqueness of Newton-Ψ-closure

In this section (K, LD) is an LD-H-field and all asymptotic couples are construed as $\mathcal{L}_{AC}$-structures in the natural way. As an application of the results of the previous section, we generalize [**6**, 16.2.1 and 16.2.2] to the Ψ-closed setting. We first give a more general definition of a *Newton-Ψ-closure* of (K, LD).

Definition 8.2.1. A **Newton-Ψ-closure** of (K, LD) is a d-algebraic LD-H-field extension $(K^{\Psi,\mathrm{nt}}, \mathrm{LD}^{\Psi,\mathrm{nt}})$ of (K, LD) such that:

(1) $K^{\Psi,\mathrm{nt}}$ is real closed and newtonian;
(2) $(K^{\Psi,\mathrm{nt}}, \mathrm{LD}^{\Psi,\mathrm{nt}})$ is Ψ-closed;
(3) the asymptotic couple of $K^{\Psi,\mathrm{nt}}$ is an elementary extension of the asymptotic couple of K; and
(4) given any LD-H-field extension (L, LD^*) of (K, LD) with real closed, newtonian L, and Ψ-closed (L, LD^*), such that the asymptotic couple of L is an elementary extension of the asymptotic couple of K, there is an embedding $i : (K^{\Psi,\mathrm{nt}}, \mathrm{LD}^{\Psi,\mathrm{nt}}) \to (L, \mathrm{LD}^*)$ of LD-H-fields over K such that the asymptotic couple of $K^{\Psi,\mathrm{nt}}$ is existentially closed in the asymptotic couple of L.

At this point, this is just a definition and a Newton-Ψ-closure of (K, LD) need not exist. We saw in Section 7.10 that the Linear Newtonian Conjecture implies the existence of a Newton-Ψ-closure of (K, LD) in the case where K is ω-free, $(\Gamma, \psi) \models T_{AC}$ and (K, LD) is full. The *Newton-Liouville closure* gives another example of a Newton-Ψ-closure provided K is ω-free and (Γ, ψ) is closed.

Proposition 8.2.2. *Suppose K is ω-free, (K, LD) is full, and (Γ, ψ) is a closed asymptotic couple (so $\mathrm{LD} = K$). Let L be an H-field extension of K. Then the following are equivalent:*

(1) L is a Newton-Liouville closure of K;
(2) (L, L) is a Newton-Ψ-closure of (K, LD).

Proof. Suppose that L is a Newton-Liouville closure of K. Then L is newtonian, and since it is Liouville closed, it is also real closed and (L, L) is a Ψ-closed LD-H-field by Example 7.1.11. In particular, (L, L) is an

LD-H-field extension of (K, LD). The asymptotic couple of L is also closed, so it is an elementary extension of the asymptotic couple of K, by model completeness for closed asymptotic couples (see [**3**]). Finally, given an LD-H-field extension (L^*, LD^*) of (K, LD) with real closed, newtonian L^*, and Ψ-closed (L^*, LD^*) such that the asymptotic couple of L^* is an elementary extension of the asymptotic couple of K, it follows that L^* is in fact a newtonian Liouville closed H-field extension of K such that $\mathrm{LD}^* = L^*$ by Example 7.1.11. By the semiuniversal property of the Newton-Liouville closure, there is an embedding $L \to L^*$ over K. This will give rise to an embedding of LD-H-fields $i : (L, L) \to (L^*, \mathrm{LD}^*)$ over K. Since all asymptotic couples here are closed asymptotic couples, we have that the asymptotic couple of $i(L)$ is existentially closed in the asymptotic couple of L^*.

Conversely suppose that (L, L) is a Newton-Ψ-closure of (K, LD). Then L is a newtonian Liouville closed d-algebraic H-field extension of K by Example 7.1.11. This makes L a Newton-Liouville closure of K by [**6**, 16.2.1]. $\square$

We now generalize [**6**, 16.2.1 and 16.2.2]. In particular, we show that a Newton-Ψ-closure (if one exists) is unique up to isomorphism.

Lemma 8.2.3. *Let (E, LD_0) be a Newton-Ψ-closure of (K, LD) and $i : (E, \mathrm{LD}_0) \to (L, \mathrm{LD}_1)$ an embedding of LD-H-fields over K into an LD-H-field (L, LD_1) such that $C_L \subseteq i(E)$ and $(\Gamma_{i(E)}, \psi_L)$ is existentially closed in (Γ_L, ψ_L). Then*

$$i(E) \;=\; \big\{f \in L : f \text{ is d-algebraic over } i(K)\big\}.$$

Proof. By definition of Newton-Ψ-closures, E is a d-algebraic over K. Thus every element of $i(E)$ is d-algebraic over $i(K)$. Furthermore, $i(E)$ is a newtonian, Ψ-closed H-subfield of L with the same constants as L. Since $(\Gamma_{i(E)}, \psi_L)$ is existentially closed in (Γ_L, ψ_L), every $f \in L$ that is d-algebraic over $i(E)$ lies in $i(E)$ by Theorem 8.1.6. $\square$

Corollary 8.2.4. *Any two Newton-Ψ-closures of (K, LD) are isomorphic over (K, LD). If (E, LD_0) is a Newton-Ψ-closure of (K, LD), then (E, LD_0) does not have any proper real closed newtonian Ψ-closed LD-H-subfield containing (K, LD).*

Proof. Let (E, LD_0) and (L, LD_1) be Newton-Ψ-closures of (K, LD). Then there exists an embedding $i : (E, \mathrm{LD}_0) \to (L, \mathrm{LD}_1)$ over (K, LD) such that the asymptotic couple of $i(E)$ is existentially closed in the asymptotic couple of L, and any such embedding is necessarily surjective by Lemma 8.2.3, as L is a d-algebraic extension of K. The minimality property of (E, LD_0) also follows from Lemma 8.2.3 by considering embeddings $(E, \mathrm{LD}_0) \to (E, \mathrm{LD}_0)$ over (K, LD). $\square$

8.3. Model completeness for $\mathbb{T}_{\log}$ modulo some conjectures

In this section we give our proof of model completeness of $\mathbb{T}_{\log}$, modulo some conjectures which still need to be established. First, let

$$\mathcal{L} \;:=\; \{0, 1, +, -, \cdot, \partial, \leqslant, \preccurlyeq\}$$

be the language of ordered differential fields. We augment $\mathcal{L}$ by adding a unary predicate symbol LD, to obtain the language $\mathcal{L}_{\mathrm{LD}} := \mathcal{L} \cup \{\mathrm{LD}\}$. We will construe LD-$H$-fields as $\mathcal{L}_{\mathrm{LD}}$-structures in the obvious way.

Next, let $T_{\log}$ be the $\mathcal{L}_{\mathrm{LD}}$-theory whose models are precisely the LD-H-fields (K, LD) such that:

(1) K is real closed, ω-free, and newtonian;

(2) (K, LD) is Ψ-closed; and
(3) $(\Gamma, \psi) \models T_{AC}$, where (Γ, ψ) is the asymptotic couple of K.

The main result of this section is the following:

Theorem 8.3.1. *If Conjectures 7.10.4, 8.3.6, and 8.3.9 hold, then the theory $T_{\log}$ is model complete.*

This theorem refers to the Linear Newtonian Conjecture as well as Conjectures 8.3.6 and 8.3.9 which we introduce below. We will now formulate a more precise form of our conjecture:

Let T_{LD} be the $\mathcal{L}_{\mathrm{LD}}$-theory whose models are precisely the LD-H-fields (K, LD) such that:
(1) K is ω-free;
(2) (K, LD) is full; and
(3) $(\mathbb{Q}\Gamma, \psi) \models T_{AC}$, where (Γ, ψ) is the asymptotic couple of K.

By Lemmas 5.3.14, 7.1.14, and Corollary 4.2.3, T_{LD} is an inductive theory. Thus by ADH 4.1.13 we will actually show the following:

Theorem 8.3.2. *Suppose Conjectures 7.10.4, 8.3.6, and 8.3.9 hold. Then $T_{\log}$ is the model companion of the theory T_{LD} and thus the models of $T_{\log}$ are exactly the existentially closed models of T_{LD}.*

Without relying on any conjectures, we already have the easy direction of Theorem 8.3.2:

Proposition 8.3.3. *Suppose (K, LD) is an existentially closed model of T_{LD}. Then $(K, \mathrm{LD}) \models T_{\log}$.*

Proof. The extension $(K^{\mathrm{rc}}, \mathrm{LD}^{\mathrm{rc}})$ of (K, LD) given by Corollary 7.4.6 is a real closed model of T_{LD}. Thus (K, LD) is real closed and Γ is divisible, hence $(\Gamma, \psi) \models T_{AC}$. Next, let K^{nt} be the newtonization of K, equipped with the unique ordering which makes it an H-field extension of K. Furthermore, let LD^* be a maximal LD-set on K^{nt} which contains $\mathrm{LD} + \mathrm{I}(K^{\mathrm{nt}})$. Then $(K^{\mathrm{nt}}, \mathrm{LD}^*)$ is a newtonian model of T_{LD} which extends (K, LD). Thus K is newtonian. $\square$

Remark 8.3.4. Usually for model completeness, one aims for a language where every symbol has a natural role. For instance, in [**6**, Chapter 16] it is shown that $\mathbb{T}$ as an ordered *valued* differential field is model complete, whereas $\mathbb{T}$ as just an ordered differential field is *not* model complete; [**6**, 16.2.6].

We have seen that the symbol LD has a natural role in specifying "good" substructures of models of $T_{\log}$, but it could be dropped for the sake of model completeness: Suppose $(K, \mathrm{LD}) \models T_{\log}$. Then the set $\mathrm{LD} = K^{\dagger}$ is clearly defined in the $\mathcal{L}$-structure K by an *existential* $\mathcal{L}$-formula that does not depend on (K, LD). It is also defined in K by a *universal* $\mathcal{L}$-formula independent of (K, LD): this reduces to showing that the condition "$v(x) \in \Psi^{\downarrow} \setminus \Psi$" can be expressed by an existential $\mathcal{L}$-formula independent of (K, LD), by the "Answer" given in the introduction to Chapter 7. However, the set $\Psi^{\downarrow} \setminus \Psi$ is existentially definable in the asymptotic couple since T_{AC} is model complete (e.g. see Example 4.2.5), and such an existential definition can be lifted to K to existentially define "$v(x) \in \Psi^{\downarrow} \setminus \Psi$".

In spite of all this, the language $\mathcal{L}_{\mathrm{LD}}$ is still more convenient for us since it allows us to state all of the embedding lemmas we use from Chapter 7. Furthermore, the predicate LD will likely be useful for QE.

Remark 8.3.5. Another approach to model completeness of $\mathbb{T}_{\log}$ would be to work in a certain 3-sorted language. Let $\mathcal{L}_{\log}$ be a 3-sorted language with sorts f (the LD-H-field sort), r (the differential residue field sort), and v (the asymptotic couple sort). This language consists of the one-sorted language $\mathcal{L}_{\mathrm{LD}}$ for the sort f, the one-sorted language $\mathcal{L}_{\mathrm{dr}} = \{0, 1, -, +, \cdot, \partial\}$ of differential rings for the sort r, and the one-sorted language $\mathcal{L}_{AC,\log} = \mathcal{L}_{AC} \cup \{s, p, \delta_1, \delta_2, \ldots\}$ for the sort v, together with a function symbol v of sort fv (for the valuation) and a function symbol res of sort fr (for the residue map).

Given an LD-H-field (K, LD) such that $(\Gamma, \psi) \models T_{AC}$, we may naturally view it as an $\mathcal{L}_{\log}$-structure $(K, C_K, \Gamma; \ldots)$ and then we can consider the $\mathcal{L}_{\log}$-theory $T_{\log,\mathrm{frv}}$ whose models are precisely the $\mathcal{L}_{\log}$-structures that arise from models of $T_{\log}$. A variant of Proposition 8.3.11 and its proof below will also go through in this 3-sorted setting, showing that if $T_{\log}$ is model complete, then so is $T_{\log,\mathrm{frv}}$.

This 3-sorted setting seems to be the natural starting point for doing QE. In a proof of QE for $\mathbb{T}_{\log}$, it is not clear how to mimic the useful role that the functions s and p play in QE for $T_{AC,\log}$ (Theorem 4.2.2) *without* having a separate sort for the asymptotic couple. Furthermore, the predicates Λ and Ω which are used to get QE for $\mathbb{T}$ ([**6**, 16.0.1]) will also need to be taken into account.

Conjectures we still need. *In this subsection* (K, LD) *is a model of* $T_{\log}$. We already stated and studied the consequences of the Linear Newtonian Conjecture in Section 7.10. To round things out, we will state and discuss two more conjectures that we need for model completeness of $T_{\log}$. The first conjecture involves differentially-transcendental immediate extensions:

Conjecture 8.3.6 (Differentially-Transcendental Immediate Extension Conjecture)**.** *Suppose* (L, LD^*) *is an* LD-*H-field extension of* (K, LD) *such that* $(L, \mathrm{LD}^*) \models T_{\log}$*, and suppose* $y \in L \setminus K$ *is such that* $K\langle y\rangle$ *is an immediate extension of* K *(so* y *is necessarily differentially transcendental over* K *since* K *is asymptotically* d*-algebraically maximal). Then* $\big(K\langle y\rangle, \mathrm{LD}_y\big)$ *is full, where* $\mathrm{LD}_y := \mathrm{LD} + \mathrm{I}\big(K\langle y\rangle\big)$.

Remark 8.3.7. Conjecture 8.3.6 gives a best case scenario. A weaker version would actually be good enough for our purpose: assume the hypothesis of Conjecture 8.3.6, and note that then Zorn's Lemma gives an LD-set LD^* on $K\langle y\rangle$ such that $(K\langle y\rangle, \mathrm{LD}^*)$ is full; the weaker version just says that then any two such LD^* are conjugate by some K-automorphism of $K\langle y\rangle$. Since Conjecture 8.3.6 gets used only in Case 5 of the proof of Proposition 8.3.11, we may also assume in this conjecture (and in its weaker version) that both K and $K\langle y\rangle$ are ω-free.

The next conjecture involves adding copies of $\mathbb{Z}$ to the Ψ-set of (Γ, ψ). For the purpose of stating the conjecture, we first give a "non-divisible" version of Lemma 3.3.1:

Lemma 8.3.8. *Suppose* (Γ_0, ψ_0) *is a divisible H-asymptotic couple with asymptotic integration, and let* $B \in \mathrm{sded}(\Psi_0)$ *be such that* $B \neq \emptyset$*. Then there is an H-asymptotic couple* $(\Gamma_B, \psi_B) \supseteq (\Gamma_0, \psi_0)$ *with a family* $(\beta_k)_{k\in\mathbb{Z}}$ *in* Γ_B *satisfying the following conditions:*

(1) (Γ_B, ψ_B) *has rational asymptotic integration;*
(2) $B < \beta_k < \Gamma_0^{>B}$*, and* $s_B(\beta_k) = \beta_{k+1}$ *for all* k*;*
(3) $\Psi_B = \Psi_0 \cup \{\beta_k : k \in \mathbb{Z}\}$*;*
(4) *for any embedding* $i : (\Gamma_0, \psi_0) \to (\Gamma^*, \psi^*)$ *into an H-asymptotic couple with rational asymptotic integration and any family* $(\beta_k^*)_{k\in\mathbb{Z}}$ *in* Ψ^* *such that* $i(B) < \beta_k^* < i(\Gamma_0^{>B})$ *and* $s^*(\beta_k^*) = \beta_{k+1}^*$ *for all* k*, there is a unique extension of* i *to an embedding* $(\Gamma_B, \psi_B) \to (\Gamma^*, \psi^*)$ *sending* β_k *to* β_k^* *for all* k*.*

Conjecture 8.3.9 (Copy of $\mathbb{Z}$ Conjecture)**.** *Suppose* $B \in \mathrm{sded}(\Psi)$ *is nonempty. Then there is an ω-free full* LD-*H-field extension* (K_B, LD_B) *of* (K, LD) *with the following properties:*

(1) *the asymptotic couple of* K_B *is* (Γ_B, ψ_B) *from Lemma 8.3.8 with family* $(\beta_k)_{k\in\mathbb{Z}}$*;*

(2) *for any* LD-H*-field extension* (L, LD^*) *of* (K, LD) *such that* $(L, \mathrm{LD}^*) \models T_{\log}$ *and family* $(\beta_k^*)_{k\in\mathbb{Z}}$ *in* Ψ_L *with* $B < \beta_k^* < \Gamma^{>B}$ *and* $s(\beta_k^*) = \beta_{k+1}^*$ *for all* k*, there is an embedding* $(K_B, \mathrm{LD}_B) \to (L, \mathrm{LD}^*)$ *over* K *inducing the asymptotic couple embedding* $(\Gamma_B, \psi_B) \to (\Gamma_L, \psi_L)$ *over* Γ *that sends* β_k *to* β_k^* *for all* k*.*

Remark 8.3.10. We actually view Conjecture 8.3.9 as two qualitatively distinct conjectures:

(1) the case where $B = \Psi$, i.e., where we need to adjoin a copy of $\mathbb{Z}$ to the end of the Ψ-set; and
(2) the case where $B \neq \Psi$, i.e., where we need to insert a copy of $\mathbb{Z}$ in the middle of the Ψ-set.

The asymptotic couple extensions involved in (1) and (2) above have seemingly distinct proofs (Lemma 3.2.4 versus Lemma 3.2.5). This suggests that (1) and (2) will need to be handled separately. For case (1), it seems that results in [**6**, §13.4] concerning adjoining a gap, as well as the construction of F_ω in [**6**, §11.7] may come in handy. For case (2), note that Corollary 8.1.11 already gives a construction of adjoining a copy of $\mathbb{N}$ with a certain semiuniversal property. In fact, we do not think the construction of K_B or the LD-set LD_B will be too difficult, the harder part will be establishing the desired semiuniversal property.

Model completeness of $\mathbb{T}_{\log}$ modulo some conjectures. The following embedding property immediately establishes Theorems 8.3.1 and 8.3.2 in view of ADH 4.1.6.

Proposition 8.3.11. *Assume Conjectures 7.10.4, 8.3.6, and 8.3.9 hold. Let* (K, LD) *and* (L, LD_1) *be models of* $T_{\log}$ *and suppose* (E, LD_0) *is a full* ω*-free* LD-H*-subfield of* (K, LD) *such that* $(\mathbb{Q}\Gamma_E, \psi) \models T_{AC}$*. Let* $i : (E, \mathrm{LD}_0) \to (L, \mathrm{LD}_1)$ *be an embedding of* LD-H*-fields. Assume* (L, LD_1) *is* $|K|^+$*-saturated. Then* i *extends to an embedding* $(K, \mathrm{LD}) \to (L, \mathrm{LD}_1)$ *of* LD-H*-fields.*

Proof. Assume $E \neq K$. It suffices to show that i can be extended to an embedding $(F, \mathrm{LD}^*) \to (L, \mathrm{LD}_1)$ for some ω-free LD-H-subfield (F, LD^*) of (K, LD) properly extending (E, LD_0) such that (F, LD^*) is full and $(\mathbb{Q}\Gamma_F, \psi) \models T_{AC}$. The picture to keep in mind is the following:

$$\begin{tikzcd}
 & (L,\mathrm{LD}_1) \arrow[dd, dash] \\
(K,\mathrm{LD}) \arrow[d, dash] & \\
(F,\mathrm{LD}^*) \arrow[r, dotted, "\exists ?", "\sim"'] \arrow[d, dash] & \big(i(F), i(\mathrm{LD}^*)\big) \arrow[d, dash] \\
(E,\mathrm{LD}_0) \arrow[r, "i", "\sim"'] & \big(i(E), i(\mathrm{LD}_0)\big)
\end{tikzcd}$$

We consider several cases:

Case 1: *E is not real closed or E is not Ψ-closed.* We set $(F, \mathrm{LD}^*) := (E^\Psi, \mathrm{LD}_0^\Psi)$, the Ψ-closure of (E, LD_0) inside (K, LD), which exists by Theorem 7.9.2. By the semiuniversal property of the Ψ-closure, i extends to an embedding $(F, \mathrm{LD}^*) \to (L, \mathrm{LD}_1)$ of LD-H-fields. Since F is d-algebraic over E, it is ω-free. Finally, (F, LD^*) is full and $(\Gamma_F, \psi) \models T_{AC}$ also by Theorem 7.9.2.

Case 2: *E is henselian and $C_E \neq C$.* The real closed constant field C_L is $|C|^+$-saturated, so the ordered field embedding $i|C_E : C_E \to C_L$ extends to an ordered field embedding $j : C \to C_L$. Then ADH 5.5.1 and 5.5.2 gives an extension of the underlying ordered valued differential field embedding of i to an embedding of H-fields $F := E(C) \to L$ that agrees with j on C. Since F is d-algebraic over E, it is ω-free. Furthermore,

$\Gamma_F = \Gamma_E$, so $(\mathbb{Q}\Gamma_F, \psi) \models T_{AC}$. Finally, by Lemma 7.5.1 and the assumption that E is henselian, there is a unique LD-set LD^* on F such that $(E, \mathrm{LD}_0) \subseteq (F, \mathrm{LD}^*)$. With this LD-set, (F, LD^*) is full and j is an embedding of LD-H-fields.

Note: Cases 1 and 2 did not rely on any conjectures. In the cases below, we assume that $C_E = C$, *and* E *is both real closed and* Ψ*-closed.*

Case 3: *E is not newtonian.* We set $(F, \mathrm{LD}^*) := (E^{\Psi,\mathrm{nt}}, \mathrm{LD}^{\Psi,\mathrm{nt}})$, the Newton-$\Psi$-closure of (E, LD_0) inside of (K, LD). This uses Corollary 7.10.8 which depends on Conjecture 7.10.4. It is clear that F is ω-free and that $(\Gamma_F, \psi) \models T_{AC}$. By the seminuniversal property of the Newton-Ψ-closure, there is an embedding $j : (F, \mathrm{LD}^*) \to (L, , \mathrm{LD}_1)$ which extends i.

Case 4: $(E, \mathrm{LD}_0) \models T_{\log}$ *and there is no $y \in K \setminus E$ such that $K\langle y \rangle$ is an immediate extension of K.* Take $y \in K \setminus E$. By Lemma 8.1.5 and 8.1.7, it follows that $\Psi_{E\langle y \rangle} \neq \Psi_E$. In particular, there is an entire copy of $\mathbb{Z}$ in $\Psi_{E\langle y \rangle}$ which is not present in Ψ_E. We will add this entire copy of $\mathbb{Z}$. Let $B \in \mathrm{sded}(\Psi_E)$ be the s-cut determined by this copy of $\mathbb{Z}$. Let $F := E_B$ be the H-field extension of K given by Conjecture 8.3.9, which we can take inside of K by the semiuniversal property of Conjecture 8.3.9. Furthermore, by saturation of L we can extend the underlying ordered valued differential field embedding of $i : E \to L$ to an embedding $j : E_B \to L$ over E. Finally, by Conjecture 8.3.9, we equip E_B with the unique LD-set LD_B such that $(E, \mathrm{LD}_0) \subseteq (E_B, \mathrm{LD}_B)$; this conjecture also says that (E_B, LD_B) is full. The uniqueness implies that $j : (E_B, \mathrm{LD}_B) \to (L, \mathrm{LD}^*)$ is an embedding of LD-H-fields.

Note that by the proof of Case 4, if we are not in Cases 1-4, then necessarily $\Psi_E = \Psi$ and so we are in Case 5 below.

Case 5: $(E, \mathrm{LD}_0) \models T_{\log}$, *$\Psi_E = \Psi$, and there is $y \in K \setminus E$ such that $K\langle y \rangle$ is an immediate extension of K.* Given such a y, we set $F := E\langle y \rangle$, which is ω-free since K is ω-free and $\Psi_E = \Psi$. We take a divergent pc-sequence (a_ρ) in E such that $a_\rho \rightsquigarrow y$. Since E is asymptotically d-algebraically maximal, (a_ρ) is of d-transcendental type over E. The saturation assumption of L gives $z \in L$ such that $i(a_\rho) \rightsquigarrow z$. Then [**6**, Lemma 11.4.7] gives a valued differential field embedding $j : F \to L$ that extends the underlying valued differential field embedding of i and sends y to z. By [**6**, 10.5.8] and the assumption that $K\langle y \rangle$ is an immediate extension of K, this embedding $F \to L$ is an embedding of *ordered* valued differential fields. Finally, by Conjecture 8.3.6 there is a unique LD-set LD_y on F such that $(E, \mathrm{LD}) \subseteq (F, \mathrm{LD}_y)$. Conjecture 8.3.6 also says that (F, LD_y) is full. The uniqueness of this LD-set implies that j is an embedding of LD-H-fields. $\square$

8.4. An Ax-Kochen-Ersov conjecture for H-fields

We conclude this thesis on a positive, optimistic note: a conjectured Ax-Kochen-Ersov (AKE) Theorem for H-fields. An AKE theorem is a theorem that gives conditions under which the first-order theory of a valued field-like object is completely determined by the theory of its residue field and value group. This was done for various types of valued fields by Ax-Kochen [**7**] and Ersov [**12**]. Also of interest here is the AKE theorem by Scanlon [**35**] for the *differential-henselian valued differential fields with many constants*, a theorem later generalized by Hakobyan [**17**].

There are not yet any known AKE theorems for the category of H-fields. We conjecture the following:

Conjecture 8.4.1. *Suppose K and L are real closed, ω-free, newtonian, Ψ-closed H-fields. Then $K \equiv L$ as ordered valued differential fields iff $(\Gamma_K, \psi) \equiv (\Gamma_L, \psi)$ as asymptotic couples (i.e., as $\mathcal{L}_{AC}$-structures).*

Of course, one direction of Conjecture 8.4.1 is trivial: if $K \equiv L$ as ordered valued differential fields, then $(\Gamma_K, \psi) \equiv (\Gamma_L, \psi)$ as $\mathcal{L}_{AC}$-structures, since the asymptotic couple of an asymptotic field is interpretable in that asymptotic field.

Furthermore, if K and L are real closed H-fields, then the residue fields $\operatorname{res}(K)$ and $\operatorname{res}(L)$ will necessarily be real closed since the valuation ring of an H-field is convex with respect to the ordering (see [**6**, 3.5.16]). Since all real closed fields are already elementarily equivalent, it is unnecessary to reference the residue fields of K and L in the statement of Conjecture 8.4.1.

The main result from [**6**] already provides us with two pieces of evidence in support of Conjecture 8.4.1:

Evidence 8.4.2. *Suppose K and L are real closed, ω-free, newtonian, Ψ-closed H-fields. If (Γ_K, ψ) and (Γ_L, ψ) are closed asymptotic couples such that $s0 > 0$, then $K \equiv L$ as ordered valued differential fields.*

Proof. Let K and L be as in the statement. Newtonianity implies that both K and L are closed under integration. The Ψ-closed condition, together with the assumption that the asymptotic couples are closed asymptotic couples imply that K and L are both closed under exponential integration. Thus both K and L are Liouville closed. In particular, K and L are both models of the (incomplete) $\mathcal{L}$-theory T^{nl} of ω-free newtonian Liouville closed H-fields, where $\mathcal{L} = \{0, 1, +, -, \cdot, \partial, \leqslant, \preccurlyeq\}$ is the language of ordered valued differential fields. That $s0 > 0$ in the asymptotic couple implies that both K and L have small derivation. Thus they are both models of the same complete $\mathcal{L}$-theory $T^{\mathrm{nl}}_{\mathrm{small}}$. In particular, $K \equiv L$ as ordered valued differential fields. $\square$

Similarly, we also have the "large" version:

Evidence 8.4.3. *Suppose K and L are real closed, ω-free, newtonian, Ψ-closed H-fields. If (Γ_K, ψ) and (Γ_L, ψ) are closed asymptotic couples such that $s0 < 0$, then $K \equiv L$ as ordered valued differential fields.*

Furthermore, model completeness of $\mathbb{T}_{\log}$ (Theorem 8.3.1) would also contribute to the evidence for Conjecture 8.4.1, once the relevant conjectures are resolved.

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