Towards a model theory of logarithmic transseries

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Logarithmic transseries

The (Ordered) Valued Field \mathbb{T}_{log}

Definition (The valued field \mathbb{T}_{log} of logarithmic transseries)

 $\mathbb{T}_{\log} := \bigcup_n \mathbb{R}[[\mathfrak{L}_n]]$ union of spherically complete Hahn fields

where \mathfrak{L}_n is the ordered group of logarithmic transmonomials:

$$\mathfrak{L}_n := \ell_0^{\mathbb{R}} \cdots \ell_n^{\mathbb{R}} = \{\ell_0^{r_0} \cdots \ell_n^{r_n} : r_i \in \mathbb{R}\}, \quad \ell_0 = x, \ell_{m+1} = \log \ell_m$$

ordered such that $\ell_i \succ \ell_{i+1}^m \succ 1$ for all $m \ge 1$, $i = 0, \ldots, n-1$.

Typical elements of \mathbb{T}_{log} look like:

•
$$-2x^3 \log x + \sqrt{x} + 2 + \frac{1}{\log \log x} + \frac{1}{(\log \log x)^2} + \cdots$$

• $\frac{1}{\log \log x} + \frac{1}{(\log \log x)^2} + \cdots + \frac{1}{\log x} + \frac{1}{(\log x)^2} + \cdots + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots$

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Note: \mathbb{T}_{\log} is a real closed field and thus has a definable ordering. Also:
Residue field is \mathbb{R} and value group Γ_{\log} is additive copy of $\bigcup_n \mathfrak{L}_n$ with
reverse ordering.

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 $\mathbb{T}_{\mathsf{log}}$ comes equipped with the usual termwise derivative and logarithmic derivative:

$$egin{array}{ccc} f &\mapsto f' \ f &\mapsto f^{\dagger} := f'/f, \ (f
eq 0) \end{array}$$

subject to the usual rules: $\ell_0' = 1, \ell_1' = \ell_0^{-1}$, etc. For example:

•
$$(x^3 \log x + \sqrt{x} + 2 + \cdots)' = 3x^2 \log x + x^2 + \frac{1}{2x^{1/2}} + \cdots$$

• $\ell_n^{\dagger} = \frac{1}{\ell_0 \ell_1 \cdots \ell_n}$
• $(\frac{1}{\log \log x} + \frac{1}{(\log \log x)^2} + \cdots)' = -\frac{1}{x \log x (\log \log x)^2} - \frac{2}{x \log x (\log \log x)^3} + \cdots$
• $(\ell_0^{r_0} \cdots \ell_n^{r_n})^{\dagger} = r_0 \ell_0^{-1} + r_1 \ell_0^{-1} \ell_1^{-1} + \cdots + r_n \ell_0^{-1} \cdots \ell_n^{-1}$

This derivative makes \mathbb{T}_{log} into a differential field with field of constants \mathbb{R} .

H-fields: ordered valued differential fields with asymptotics

Definition

K an ordered valued differential field. We call K an H-field if

H1 for all
$$f \in K$$
, if $f > C$, then $f' > 0$;

H2 $\mathcal{O} = C + \sigma$ where $\mathcal{O} = \{g \in K : |g| \le c \text{ for some } c \in C\}$ is the (convex) valuation ring of K and σ is the maximal ideal of \mathcal{O}

Example

 \mathbb{T}_{log} is an *H*-field, also any Hardy field containing \mathbb{R} is an *H*-field.

Example

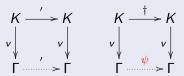
 \mathbb{T} , the differential field of *logarithmic-exponential transseries* is naturally an *H*-field, and contains \mathbb{T}_{log} . It is closed under exp. Typical element:

$$-3e^{e^{x}} + e^{\frac{e^{x}}{\log x} + \frac{e^{x}}{\log^{2}x} + \frac{e^{x}}{\log^{3}x} + \dots} - x^{11} + 7 + \frac{\pi}{x} + \frac{1}{x\log x} + \dots + e^{-x} + 2e^{-x^{2}} + \dots$$

The asymptotic couple (Γ, ψ) of an *H*-field *K*

Fact

For $f \in K^{\times}$ such that $v(f) \neq 0$, the values v(f') and $v(f^{\dagger})$ depend only on v(f).

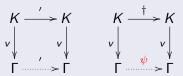


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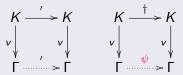
Definition (Rosenlicht)

The pair (Γ, ψ) is the *asymptotic couple of K*.

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Theorem (G)

Th(Γ_{log}, ψ), the asymptotic couple of \mathbb{T}_{log} , has quantifier elimination in a natural language and is model complete and has NIP.

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Logarithmic transseries

H-fields: two technical properties

Both ${\mathbb T}$ and ${\mathbb T}_{\mathsf{log}}$ enjoy two additional (first-order) properties:

• ω -free: this is a very strong and robust property which prevents certain deviant behavior

 $\forall f \neq 0 \; \exists g \not\asymp 1[g' \asymp f] \; \& \; \forall f \; \exists g \succ 1[f + 2g^{\dagger \dagger'} + 2(g^{\dagger \dagger})^2 \succcurlyeq g^{\dagger}]$

 newtonian: this is a variant of "differential-henselian"; it essentially means that you can simulate being differential henselian arbitrarily well by sufficient coarsenings and compositional conjugations (∂ → φ∂).

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• **newtonian**: this is a variant of "differential-henselian"; it essentially means that you can simulate being differential henselian arbitrarily well by sufficient coarsenings and compositional conjugations $(\partial \mapsto \phi \partial)$.

 \mathbb{T}_{log} satisfies both of these properties because it has integration and is a union of spherically complete *H*-fields, each with a smallest "comparability class":

$$\mathbb{T}_{\log} := \bigcup_{n} \mathbb{R}[[\ell_0^{\mathbb{R}} \cdots \ell_n^{\mathbb{R}}]]$$

Another nice property:

Definition

We call a real closed H-field K Liouville closed if

$${\sf K}'={\sf K}$$
 and $({\sf K}^ imes)^\dagger={\sf K}$

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 \mathbb{T} is Liouville closed, however... \mathbb{T}_{log} is NOT Liouville closed:

$$(\mathbb{T}_{\mathsf{log}})' = \mathbb{T}_{\mathsf{log}} \quad \mathsf{but} \quad (\mathbb{T}_{\mathsf{log}}^{ imes})^{\dagger}
eq \mathbb{T}_{\mathsf{log}}$$

E.g., an element f such that $f^{\dagger} = 1$ would have to behave like e^{x} ...

Let $\mathcal{L} = \{0, 1, +, -, \cdot, \partial, \leq, \preccurlyeq\}$

The following result is the starting point for the model theory of $\mathbb{T}_{\mathsf{log}}$:

Theorem (Aschenbrenner, van den Dries, van der Hoeven, 2015)

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 - real closed, ω -free, newtonian, H-field such that $\forall \epsilon \prec 1, \partial(\epsilon) \prec 1$;
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 - *K*′ = *K*(*K*[×])[†] = *K*

Furthermore, $\mathbb T$ is model complete as an $\mathcal L\text{-structure}.$

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Furthermore, $\mathbb T$ is model complete as an $\mathcal L\text{-structure}.$

Recall: a structure M is model complete if every definable subset of M^n is existentially definable (for every n). A starting point for model completeness of \mathbb{T}_{\log} is to try to make both $(\mathbb{T}_{\log}^{\times})^{\dagger}$ and its complement existentially definable.

Investigating $(\mathbb{T}^{ imes}_{ ext{log}})^{\dagger}$

$f\in (\mathbb{T}_{\mathsf{log}}^{ imes})^{\dagger}$ \Longleftrightarrow there exists $g\in \mathbb{T}_{\mathsf{log}}^{ imes}$ such that $g^{\dagger}=f$

 $f \in (\mathbb{T}_{\log}^{\times})^{\dagger} \iff$ there exists $g \in \mathbb{T}_{\log}^{\times}$ such that $g^{\dagger} = f$ Given $f \in \mathbb{T}_{\log}^{\times}$, we can write it uniquely as

 $f = c\ell_0^{r_0}\cdots\ell_n^{r_n}(1+\epsilon)$ for some infinitesimal $\epsilon\prec 1$ and some $c\in\mathbb{R}^{ imes}$

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Then we compute the logarithmic derivative:

$$(c\ell_0^{r_0}\cdots\ell_n^{r_n}(1+\epsilon))^{\dagger} = r_0\ell_0^{-1} + r_1\ell_0^{-1}\ell_1^{-1} + \cdots + r_n\ell_0^{-1}\cdots\ell_n^{-1} + \underbrace{\frac{\epsilon'}{1+\epsilon}}_{\text{"small"}}$$

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Note: $v(\ell_0^{-1}\cdots \ell_n^{-1}) \in \Psi := \psi(\Gamma_{\log}^{\neq}) \text{ and } v(\epsilon'/(1+\epsilon)) > \Psi.$

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Fact

 $f \not\in (\mathbb{T}_{\mathsf{log}}^{\times})^{\dagger} \Longleftrightarrow \mathsf{there} \ \mathsf{exists} \ g \in \mathbb{T}_{\mathsf{log}}^{\times} \ \mathsf{such} \ \mathsf{that} \ \mathsf{v}(f-g^{\dagger}) \in \Psi^{\downarrow} \setminus \Psi$

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Introducing LD-H-fields

From now on all *H*-fields will have asymptotic integration $(\Gamma = (\Gamma^{\neq})')$. Let *K* be an *H*-field and LD $\subseteq K$.

We call the pair (K, LD) an LD-*H*-field if:

LD1 LD is a C_{K} -vector subspace of K;

LD2 $(K^{\times})^{\dagger} \subseteq$ LD;

LD3 I(\mathcal{K}) := { $y \in \mathcal{K} : y \preccurlyeq f'$ for some $f \in \mathcal{O}$ } \subseteq LD; and LD4 $v(LD) \subseteq \Psi \cup (\Gamma^{>})' \cup \{\infty\}$.

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We say an LD-*H*-field (K, LD) is Ψ -closed if:

E1^{*} For every $a \in K \setminus LD$, there is $b \in LD$ such that $v(a - b) \in \Psi^{\downarrow} \setminus \Psi$; and

E2 LD = $(K^{\times})^{\dagger}$.

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 - E2 LD = $(K^{\times})^{\dagger}$.

Example

 $(\mathbb{T}_{\log}, (\mathbb{T}_{\log}^{\times})^{\dagger})$ and (\mathbb{T}, \mathbb{T}) are both Ψ -closed LD-*H*-fields.

Let $\mathcal{L}_{\mathsf{LD}}:=\mathcal{L}\cup\{\mathsf{LD}\}$ where LD is a unary relation symbol.

Let $\mathcal{L}_{LD} := \mathcal{L} \cup \{LD\}$ where LD is a unary relation symbol. Let \mathcal{T}_{log} be the \mathcal{L}_{LD} -theory whose models are precisely the LD-*H*-fields (*K*, LD) such that:

- K is real closed, ω -free, and newtonian;
- **2** (K, LD) is Ψ -closed; and
- $(\Gamma, \psi) \models \mathsf{Th}(\Gamma_{\mathsf{log}}, \psi)$, where (Γ, ψ) is the asymptotic couple of K.

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Embedding version of conjecture

Let (K, LD) and (L, LD_1) be models of T_{log} and suppose (E, LD_0) is an ω -free LD-*H*-subfield of (K, LD) with E1 such that $(\mathbb{Q}\Gamma_E, \psi) \models \text{Th}(\Gamma_{log}, \psi)$. Let $i : (E, LD_0) \rightarrow (L, LD_1)$ be an embedding of LD-*H*-fields. Assume (L, LD_1) is $|K|^+$ saturated. Then i extends to an embedding $(K, LD) \rightarrow (L, LD_1)$ of LD-*H*-fields.

Given LD-*H*-fields (*K*, LD) and (*L*, LD^{*}) such that $K \subseteq L$, we say that (*L*, LD^{*}) is an extension of (*K*, LD) (notation (*K*, LD) \subseteq (*L*, LD^{*})) is LD^{*} $\cap K =$ LD.

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Proposition

Suppose *L* is an algebraic extension of *K*, (*K*, LD) has E1, and $(\Gamma, \psi) \models \text{Th}(\Gamma_{\log}, \psi)$. Then there is a **unique** LD-set LD^{*} \subseteq *L* such that $(K, \text{LD}) \subseteq (L, \text{LD}^*)$; equipped with this LD-set, (L, LD^*) also has E1. Important case: *L* is a real closure of *K*.

Suppose $K \subseteq L$ is an extension of *H*-fields such that $L = K(C_L)$, so *L* is a constant field extension of *K*.

Proposition

Suppose K is henselian, $(\Gamma, \psi) \models \text{Th}(\Gamma_{\log}, \psi)$, and (K, LD) has E1. Then there is a **unique** LD-set $\text{LD}^* \subseteq L$ such that $(K, \text{LD}) \subseteq (L, \text{LD}^*)$; equipped with this LD-set, (L, LD^*) also has E1.

Thus adding new constants will never be an issue!

Definition

We say an LD-*H*-field extension (K^{Ψ}, LD^{Ψ}) of (K, LD) is a Ψ -closure of (K, LD) if K^{Ψ} is real closed, (K^{Ψ}, LD^{Ψ}) is Ψ -closed, and for any LD-*H*-field extension (L, LD^*) of (K, LD) such that *L* is real closed and (L, LD^*) is Ψ -closed, there is an embedding $(K^{\Psi}, LD^{\Psi}) \rightarrow (L, LD^*)$ of LD-*H*-fields over (K, LD).

Proposition

Suppose (K, LD) has E1, is λ -free, and $(\Gamma, \psi) \models Th(\Gamma_{\log}, \psi)$. Then (K, LD) has a Ψ -closure. Furthermore, every Ψ -closure will be differentially-algebraic over K, and its asymptotic couple will model $Th(\Gamma_{\log}, \psi)$.

Suppose K is ω -free, $(\Gamma, \psi) \models \text{Th}(\Gamma_{\log}, \psi)$ and let K^{nt} be the *newtonization* of K (a newtonian extension of K with a universal property).

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What we would like to prove:

Suppose (K, LD) has E1. Then $LD^{nt} := LD + I(K^{nt})$ is the unique LD-set on K^{nt} such that $(K, LD) \subseteq (K^{nt}, LD^{nt})$; equipped with this LD-set, (K^{nt}, LD^{nt}) also satisfies E1.

This can be reduced to the linear case:

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Conjecture 1

There is a linearly newtonian *H*-field *L* such that $K \subseteq L \subseteq K^{nt}$ and $LD^* := LD + I(L)$ is the unique LD-set on *L* such that $(K, LD) \subseteq (L, LD^*)$; equipped with this LD-set, (L, LD^*) also satisfies E1.

Linearly newtonian is the fragment of newtonian that only involves degree 1 differential polynomials (differential operators).

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Conjecture 2 (Differentially-transcendental immediate extension case)

Suppose (L, LD^*) is an LD-*H*-field extension of (K, LD) such that $(K, LD), (L, LD^*) \models T_{log}$, and suppose there is $y \in L \setminus K$ such that $K\langle y \rangle$ is an immediate extension of K (so y is necessarily differentially transcendental over K since K is asymptotically d-algebraically maximal). Then $LD_y := LD + I(K\langle y \rangle)$ is the unique LD-set on $K\langle y \rangle$ such that $(K, LD) \subseteq (K\langle y \rangle, LD_y)$; equipped with this LD-set, $(K\langle y \rangle, LD_y)$ also satisfies E1.

Conjecture 3 (Copy of \mathbb{Z} case)

Similar statement, but for adjoining "copies of \mathbb{Z} " to the Ψ -set of K.

Model completeness follows from resolving Conjectures 1, 2, and 3.

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