Towards a model theory of logarithmic transseries

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The (Ordered) Valued Field $\mathbb{T}_{\log}$

Definition (The valued field $\mathbb{T}_{\log}$ of logarithmic transseries)

$$\mathbb{T}_{\log} := \bigcup_n \mathbb{R}[[\mathfrak{L}_n]] \quad \text{union of spherically complete Hahn fields}$$

where $\mathfrak{L}_n$ is the *ordered group of logarithmic transmonomials*:

$$\mathfrak{L}_n := \ell_0^{\mathbb{R}} \cdots \ell_n^{\mathbb{R}} = \{\ell_0^{r_0} \cdots \ell_n^{r_n} : r_i \in \mathbb{R}\}, \quad \ell_0 = x, \ell_{m+1} = \log \ell_m$$

ordered such that $\ell_i \succ \ell_{i+1}^m \succ 1$ for all $m \geq 1$, $i = 0, \ldots, n-1$.

Typical elements of $\mathbb{T}_{\log}$ look like:

- $-2x^3 \log x + \sqrt{x} + 2 + \frac{1}{\log\log x} + \frac{1}{(\log\log x)^2} + \cdots$
- $\frac{1}{\log\log x} + \frac{1}{(\log\log x)^2} + \cdots + \frac{1}{\log x} + \frac{1}{(\log x)^2} + \cdots + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots$

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Note: $\mathbb{T}_{\log}$ is a real closed field and thus has a definable ordering. Also: Residue field is $\mathbb{R}$ and value group $\Gamma_{\log}$ is additive copy of $\cup_n \mathfrak{L}_n$ with reverse ordering.

The derivation on $\mathbb{T}_{\log}$

$\mathbb{T}_{\log}$ comes equipped with the usual termwise derivative and logarithmic derivative:

$$\begin{aligned} f &\mapsto f' \\ f &\mapsto f^\dagger := f'/f, \quad (f \neq 0) \end{aligned}$$

subject to the usual rules: $\ell_0' = 1, \ell_1' = \ell_0^{-1}$, etc.
For example:

- $(x^3 \log x + \sqrt{x} + 2 + \cdots)' = 3x^2 \log x + x^2 + \frac{1}{2x^{1/2}} + \cdots$
- $\ell_n^\dagger = \frac{1}{\ell_0 \ell_1 \cdots \ell_n}$
- $\left(\frac{1}{\log\log x} + \frac{1}{(\log\log x)^2} + \cdots\right)' = -\frac{1}{x \log x (\log\log x)^2} - \frac{2}{x \log x (\log\log x)^3} + \cdots$
- $(\ell_0^{r_0} \cdots \ell_n^{r_n})^\dagger = r_0 \ell_0^{-1} + r_1 \ell_0^{-1} \ell_1^{-1} + \cdots + r_n \ell_0^{-1} \cdots \ell_n^{-1}$

This derivative makes $\mathbb{T}_{\log}$ into a differential field with field of constants $\mathbb{R}$.

H-fields: ordered valued differential fields with asymptotics

Definition

K an ordered valued differential field. We call K an H-**field** if

- H1 for all $f \in K$, if $f > C$, then $f' > 0$;
- H2 $\mathcal{O} = C + \mathcal{o}$ where $\mathcal{O} = \{g \in K : |g| \leq c \text{ for some } c \in C\}$ is the (convex) valuation ring of K and $\mathcal{o}$ is the maximal ideal of $\mathcal{O}$

Example

$\mathbb{T}_{\log}$ is an H-field, also any Hardy field containing $\mathbb{R}$ is an H-field.

Example

$\mathbb{T}$, the differential field of *logarithmic-exponential transseries* is naturally an H-field, and contains $\mathbb{T}_{\log}$. It is closed under exp. Typical element:

$$-3e^{e^x} + e^{\frac{e^x}{\log x} + \frac{e^x}{\log^2 x} + \frac{e^x}{\log^3 x} + \cdots} - x^{11} + 7 + \frac{\pi}{x} + \frac{1}{x \log x} + \cdots + e^{-x} + 2e^{-x^2} + \cdots$$

The asymptotic couple (Γ, ψ) of an H-field K

Fact

For $f \in K^{\times}$ such that $v(f) \neq 0$, the values $v(f')$ and $v(f^{\dagger})$ depend only on $v(f)$.

$$\begin{array}{ccc} K & \xrightarrow{\ \prime\ } & K \\ \downarrow v & & \downarrow v \\ \Gamma & \overset{\prime}{\dashrightarrow} & \Gamma \end{array} \qquad \begin{array}{ccc} K & \xrightarrow{\ \dagger\ } & K \\ \downarrow v & & \downarrow v \\ \Gamma & \overset{\psi}{\dashrightarrow} & \Gamma \end{array}$$

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The pair (Γ, ψ) is the *asymptotic couple of K*.

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Theorem (G)

$\mathrm{Th}(\Gamma_{\log}, \psi)$, *the asymptotic couple of $\mathbb{T}_{\log}$, has quantifier elimination in a natural language and is model complete and has NIP.*

H-fields: two technical properties

Both $\mathbb{T}$ and $\mathbb{T}_{\log}$ enjoy two additional (first-order) properties:

- **ω-free**: this is a very strong and robust property which prevents certain deviant behavior
  $\forall f \neq 0\ \exists g \not\asymp 1[g' \asymp f]\ \ \&\ \ \forall f\ \exists g \succ 1[f + 2g^{\dagger\dagger'} + 2(g^{\dagger\dagger})^2 \succcurlyeq g^{\dagger}]$
- **newtonian**: this is a variant of "differential-henselian"; it essentially means that you can simulate being differential henselian arbitrarily well by sufficient coarsenings and compositional conjugations ($\partial \mapsto \phi\partial$).

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$\mathbb{T}_{\log}$ satisfies both of these properties because it has integration and is a union of spherically complete H-fields, each with a smallest "comparability class":

$$\mathbb{T}_{\log} := \bigcup_n \mathbb{R}[[\ell_0^{\mathbb{R}} \cdots \ell_n^{\mathbb{R}}]]$$

Another nice property:

Definition

We call a real closed H-field K **Liouville closed** if

$$K' = K \quad \text{and} \quad (K^\times)^\dagger = K$$

H-fields: integrals and exponential integrals

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$\mathbb{T}$ is Liouville closed, however...
$\mathbb{T}_{\log}$ is NOT Liouville closed:

$$(\mathbb{T}_{\log})' = \mathbb{T}_{\log} \quad \text{but} \quad (\mathbb{T}_{\log}^{\times})^{\dagger} \neq \mathbb{T}_{\log}$$

E.g., an element f such that $f^{\dagger} = 1$ would have to behave like e^x..

The field $\mathbb{T}$: a success story

Let $\mathcal{L} = \{0, 1, +, -, \cdot, \partial, \leq, \preccurlyeq\}$
The following result is the starting point for the model theory of $\mathbb{T}_{\log}$:

Theorem (Aschenbrenner, van den Dries, van der Hoeven, 2015)

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Recall: a structure M is *model complete* if every definable subset of M^n is existentially definable (for every n).

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Furthermore, $\mathbb{T}$ *is model complete as an* $\mathcal{L}$*-structure.*

Recall: a structure M is *model complete* if every definable subset of M^n is existentially definable (for every n). A starting point for model completeness of $\mathbb{T}_{\log}$ is to try to make both $(\mathbb{T}_{\log}^{\times})^{\dagger}$ and its complement existentially definable.

Investigating $(\mathbb{T}_{\log}^{\times})^{\dagger}$

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Given $f \in \mathbb{T}_{\log}^{\times}$, we can write it uniquely as

$$f = c\ell_0^{r_0} \cdots \ell_n^{r_n}(1+\epsilon) \quad \text{for some infinitesimal } \epsilon \prec 1 \text{ and some } c \in \mathbb{R}^{\times}$$

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Then we compute the logarithmic derivative:

$$(c\ell_0^{r_0} \cdots \ell_n^{r_n}(1+\epsilon))^{\dagger} = r_0\ell_0^{-1} + r_1\ell_0^{-1}\ell_1^{-1} + \cdots + r_n\ell_0^{-1} \cdots \ell_n^{-1} + \underbrace{\frac{\epsilon'}{1+\epsilon}}_{\text{“small”}}$$

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Fact

$f \notin (\mathbb{T}_{\log}^{\times})^{\dagger} \iff$ **there exists** $g \in \mathbb{T}_{\log}^{\times}$ *such that* $v(f - g^{\dagger}) \in \Psi^{\downarrow} \setminus \Psi$

Introducing LD-H-fields

From now on all H-fields will have asymptotic integration ($\Gamma = (\Gamma^{\neq})'$).
Let K be an H-field and $\mathrm{LD} \subseteq K$.
We call the pair (K, LD) an LD-H-**field** if:

LD1 LD is a C_K-vector subspace of K;

LD2 $(K^\times)^\dagger \subseteq \mathrm{LD}$;

LD3 $\mathrm{I}(K) := \{y \in K : y \preccurlyeq f' \text{ for some } f \in \mathcal{O}\} \subseteq \mathrm{LD}$; and

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- LD4 $v(\mathrm{LD}) \subseteq \Psi \cup (\Gamma^{>})' \cup \{\infty\}$.

We say an LD-H-field (K, LD) is Ψ-**closed** if:

- E1* For every $a \in K \setminus \mathrm{LD}$, there is $b \in \mathrm{LD}$ such that $v(a-b) \in \Psi^{\downarrow} \setminus \Psi$; and
- E2 $\mathrm{LD} = (K^{\times})^{\dagger}$.

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Example

$(\mathbb{T}_{\log}, (\mathbb{T}_{\log}^{\times})^{\dagger})$ and $(\mathbb{T}, \mathbb{T})$ are both Ψ-closed LD-H-fields.

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1. K is real closed, ω-free, and newtonian;
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3. $(\Gamma, \psi) \models \text{Th}(\Gamma_{\log}, \psi)$, where (Γ, ψ) is the asymptotic couple of K.

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Conjecture

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Embedding version of conjecture

Let (K, LD) and (L, LD_1) be models of $T_{\log}$ and suppose (E, LD_0) is an ω-free LD-H-subfield of (K, LD) with E1 such that $(\mathbb{Q}\Gamma_E, \psi) \models \mathrm{Th}(\Gamma_{\log}, \psi)$. Let $i : (E, \mathrm{LD}_0) \to (L, \mathrm{LD}_1)$ be an embedding of LD-H-fields. Assume (L, LD_1) is $|K|^+$ saturated. Then i extends to an embedding $(K, \mathrm{LD}) \to (L, \mathrm{LD}_1)$ of LD-H-fields.

Algebraic Extensions of LD-H-fields

Given LD-H-fields (K, LD) and (L, LD^*) such that $K \subseteq L$, we say that (L, LD^*) **is an extension of** (K, LD) (notation $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$) is $\mathrm{LD}^* \cap K = \mathrm{LD}$.

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Proposition

Suppose L is an algebraic extension of K, (K, LD) has E1, and $(\Gamma, \psi) \models \mathrm{Th}(\Gamma_{\log}, \psi)$. Then there is a **unique** LD-set $\mathrm{LD}^* \subseteq L$ such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$; equipped with this LD-set, (L, LD^*) **also has E1**. Important case: L is a real closure of K.

Constant Field Extensions of LD-H-fields

Suppose $K \subseteq L$ is an extension of H-fields such that $L = K(C_L)$, so L is a constant field extension of K.

Proposition

Suppose K is henselian, $(\Gamma, \psi) \models \mathrm{Th}(\Gamma_{\log}, \psi)$, and (K, LD) has E1. Then there is a **unique** LD-set $\mathrm{LD}^* \subseteq L$ such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$; equipped with this LD-set, (L, LD^*) **also has E1**.

Thus adding new constants will never be an issue!

The Ψ-closure of an LD-H-field

Definition

We say an LD-H-field extension $(K^{\Psi}, \mathrm{LD}^{\Psi})$ of (K, LD) is a **Ψ-closure of** (K, LD) if K^{Ψ} is real closed, $(K^{\Psi}, \mathrm{LD}^{\Psi})$ is Ψ-closed, and for any LD-H-field extension (L, LD^*) of (K, LD) such that L is real closed and (L, LD^*) is Ψ-closed, there is an embedding $(K^{\Psi}, \mathrm{LD}^{\Psi}) \to (L, \mathrm{LD}^*)$ of LD-H-fields over (K, LD).

Proposition

Suppose (K, LD) has E1, is λ-free, and $(\Gamma, \psi) \models \mathrm{Th}(\Gamma_{\log}, \psi)$. Then (K, LD) has a Ψ-closure. Furthermore, every Ψ-closure will be differentially-algebraic over K, and its asymptotic couple will model $\mathrm{Th}(\Gamma_{\log}, \psi)$.

Newtonization: a reduction to the linear case

Suppose K is ω-free, $(\Gamma, \psi) \models \mathrm{Th}(\Gamma_{\log}, \psi)$ and let K^{nt} be the *newtonization* of K (a newtonian extension of K with a universal property).

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What we would like to prove:

Suppose (K, LD) has E1. Then $\mathrm{LD}^{\mathrm{nt}} := \mathrm{LD} + \mathrm{I}(K^{\mathrm{nt}})$ is the unique LD-set on K^{nt} such that $(K, \mathrm{LD}) \subseteq (K^{\mathrm{nt}}, \mathrm{LD}^{\mathrm{nt}})$; equipped with this LD-set, $(K^{\mathrm{nt}}, \mathrm{LD}^{\mathrm{nt}})$ also satisfies E1.

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This can be reduced to the linear case:

Conjecture 1

There is a linearly newtonian H-field L such that $K \subseteq L \subseteq K^{\mathrm{nt}}$ and $\mathrm{LD}^* := \mathrm{LD} + \mathrm{I}(L)$ is the unique LD-set on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$; equipped with this LD-set, (L, LD^*) also satisfies E1.

Linearly newtonian is the fragment of *newtonian* that only involves degree 1 differential polynomials (differential operators).

Newtonization: a reduction to the linear case

Suppose K is ω-free, $(\Gamma, \psi) \models \mathrm{Th}(\Gamma_{\log}, \psi)$ and let K^{nt} be the *newtonization* of K (a newtonian extension of K with a universal property).

What we would like to prove:

Suppose (K, LD) has E1. Then $\mathrm{LD}^{\mathrm{nt}} := \mathrm{LD} + \mathrm{I}(K^{\mathrm{nt}})$ is the unique LD-set on K^{nt} such that $(K, \mathrm{LD}) \subseteq (K^{\mathrm{nt}}, \mathrm{LD}^{\mathrm{nt}})$; equipped with this LD-set, $(K^{\mathrm{nt}}, \mathrm{LD}^{\mathrm{nt}})$ also satisfies E1.

This can be reduced to the linear case:

Conjecture 1

There is a linearly newtonian H-field L such that $K \subseteq L \subseteq K^{\mathrm{nt}}$ and $\mathrm{LD}^* := \mathrm{LD} + \mathrm{I}(L)$ is the unique LD-set on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$; equipped with this LD-set, (L, LD^*) also satisfies E1.

Linearly newtonian is the fragment of *newtonian* that only involves degree 1 differential polynomials (differential operators).

Two more cases we need to handle

Conjecture 2 (Differentially-transcendental immediate extension case)

Suppose (L, LD^*) is an LD-H-field extension of (K, LD) such that $(K, \mathrm{LD}), (L, \mathrm{LD}^*) \models T_{\log}$, and suppose there is $y \in L \setminus K$ such that $K\langle y \rangle$ is an immediate extension of K (so y is necessarily differentially transcendental over K since K is asymptotically d-algebraically maximal). Then $\mathrm{LD}_y := \mathrm{LD} + \mathrm{I}(K\langle y \rangle)$ is the unique LD-set on $K\langle y \rangle$ such that $(K, \mathrm{LD}) \subseteq (K\langle y \rangle, \mathrm{LD}_y)$; equipped with this LD-set, $(K\langle y \rangle, \mathrm{LD}_y)$ also satisfies E1.

Conjecture 3 (Copy of $\mathbb{Z}$ case)

Similar statement, but for adjoining "copies of $\mathbb{Z}$" to the Ψ-set of K.

Model completeness follows from resolving Conjectures 1, 2, and 3.