A Tale of Two Liouville Closures...

Allen Gehret

University of Illinois at Urbana-Champaign

agehret2@illinois.edu

Kolchin Research and Training Workshop III
May 5 and 7, 2017

Overview

1. Hardy fields

2. H-fields

Hardy fields

Hardy fields are the natural domain of asymptotic analysis, where all rules hold, without qualifying conditions
- Maxwell Rosenlicht

Hardy fields

Hardy fields are the natural domain of asymptotic analysis, where all rules hold, without qualifying conditions
- Maxwell Rosenlicht

- Let $\mathcal{G}$ be the ring of germs of real valued functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$ for some $a \in \mathbb{R}$

Hardy fields

Hardy fields are the natural domain of asymptotic analysis, where all rules hold, without qualifying conditions
- Maxwell Rosenlicht

- Let $\mathcal{G}$ be the ring of germs of real valued functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$ for some $a \in \mathbb{R}$
- A germ is said to be **differentiable**, **continuous**, etc. if it is the germ of a function that is differentiable, continuous, etc.

Hardy fields

Hardy fields are the natural domain of asymptotic analysis, where all rules hold, without qualifying conditions
- Maxwell Rosenlicht

- Let $\mathcal{G}$ be the ring of germs of real valued functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$ for some $a \in \mathbb{R}$
- A germ is said to be **differentiable**, **continuous**, etc. if it is the germ of a function that is differentiable, continuous, etc.
- Given differentiable $g \in \mathcal{G}$, $g' \in \mathcal{G}$ is defined to be the germ of the derivative of a differentiable representative of g

Hardy fields

Hardy fields are the natural domain of asymptotic analysis, where all rules hold, without qualifying conditions
- Maxwell Rosenlicht

- Let $\mathcal{G}$ be the ring of germs of real valued functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$ for some $a \in \mathbb{R}$
- A germ is said to be **differentiable**, **continuous**, etc. if it is the germ of a function that is differentiable, continuous, etc.
- Given differentiable $g \in \mathcal{G}$, $g' \in \mathcal{G}$ is defined to be the germ of the derivative of a differentiable representative of g
- A **Hardy field** is a subring K of $\mathcal{G}$ such that K is a field, all $g \in K$ are differentiable, and $g' \in K$ for all $g \in K$

Hardy fields

Hardy fields are the natural domain of asymptotic analysis, where all rules hold, without qualifying conditions
- Maxwell Rosenlicht

- Let $\mathcal{G}$ be the ring of germs of real valued functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$ for some $a \in \mathbb{R}$
- A germ is said to be **differentiable**, **continuous**, etc. if it is the germ of a function that is differentiable, continuous, etc.
- Given differentiable $g \in \mathcal{G}$, $g' \in \mathcal{G}$ is defined to be the germ of the derivative of a differentiable representative of g
- A **Hardy field** is a subring K of $\mathcal{G}$ such that K is a field, all $g \in K$ are differentiable, and $g' \in K$ for all $g \in K$
- Examples:
  $\mathbb{Q}, \mathbb{Q}(\sqrt{2}), \mathbb{R},$

Hardy fields

Hardy fields are the natural domain of asymptotic analysis, where all rules hold, without qualifying conditions
- Maxwell Rosenlicht

- Let $\mathcal{G}$ be the ring of germs of real valued functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$ for some $a \in \mathbb{R}$
- A germ is said to be **differentiable**, **continuous**, etc. if it is the germ of a function that is differentiable, continuous, etc.
- Given differentiable $g \in \mathcal{G}$, $g' \in \mathcal{G}$ is defined to be the germ of the derivative of a differentiable representative of g
- A **Hardy field** is a subring K of $\mathcal{G}$ such that K is a field, all $g \in K$ are differentiable, and $g' \in K$ for all $g \in K$
- Examples:
  $\mathbb{Q}, \mathbb{Q}(\sqrt{2}), \mathbb{R}, \mathbb{Q}(x), \mathbb{R}(x),$

Hardy fields

Hardy fields are the natural domain of asymptotic analysis, where all rules hold, without qualifying conditions
- Maxwell Rosenlicht

- Let $\mathcal{G}$ be the ring of germs of real valued functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$ for some $a \in \mathbb{R}$
- A germ is said to be **differentiable**, **continuous**, etc. if it is the germ of a function that is differentiable, continuous, etc.
- Given differentiable $g \in \mathcal{G}$, $g' \in \mathcal{G}$ is defined to be the germ of the derivative of a differentiable representative of g
- A **Hardy field** is a subring K of $\mathcal{G}$ such that K is a field, all $g \in K$ are differentiable, and $g' \in K$ for all $g \in K$
- Examples: $\mathbb{Q}, \mathbb{Q}(\sqrt{2}), \mathbb{R}, \mathbb{Q}(x), \mathbb{R}(x), \mathbb{R}(x, \sqrt{\log x}, e^x, \exp(x\sqrt{\log x} + e^x)), \ldots$

Hardy fields

Hardy fields are the natural domain of asymptotic analysis, where all rules hold, without qualifying conditions
- Maxwell Rosenlicht

- Let $\mathcal{G}$ be the ring of germs of real valued functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$ for some $a \in \mathbb{R}$
- A germ is said to be **differentiable**, **continuous**, etc. if it is the germ of a function that is differentiable, continuous, etc.
- Given differentiable $g \in \mathcal{G}$, $g' \in \mathcal{G}$ is defined to be the germ of the derivative of a differentiable representative of g
- A **Hardy field** is a subring K of $\mathcal{G}$ such that K is a field, all $g \in K$ are differentiable, and $g' \in K$ for all $g \in K$
- Examples: $\mathbb{Q}, \mathbb{Q}(\sqrt{2}), \mathbb{R}, \mathbb{Q}(x), \mathbb{R}(x), \mathbb{R}(x, \sqrt{\log x}, e^x, \exp(x\sqrt{\log x} + e^x)), \ldots$

Hardy fields

Hardy fields are the natural domain of asymptotic analysis, where all rules hold, without qualifying conditions
- Maxwell Rosenlicht

- Let $\mathcal{G}$ be the ring of germs of real valued functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$ for some $a \in \mathbb{R}$
- A germ is said to be **differentiable**, **continuous**, etc. if it is the germ of a function that is differentiable, continuous, etc.
- Given differentiable $g \in \mathcal{G}$, $g' \in \mathcal{G}$ is defined to be the germ of the derivative of a differentiable representative of g
- A **Hardy field** is a subring K of $\mathcal{G}$ such that K is a field, all $g \in K$ are differentiable, and $g' \in K$ for all $g \in K$
- Examples: $\mathbb{Q}, \mathbb{Q}(\sqrt{2}), \mathbb{R}, \mathbb{Q}(x), \mathbb{R}(x), \mathbb{R}(x, \sqrt{\log x}, e^x, \exp(x\sqrt{\log x} + e^x)), \ldots$

Some nice properties of Hardy fields

Let K be a Hardy field and $f \in K$

- If $f \neq 0$, then f does not have arbitrarily large zeros.

Some nice properties of Hardy fields

Let K be a Hardy field and $f \in K$

- If $f \neq 0$, then f does not have arbitrarily large zeros.
- All derivatives of f are eventually monotone

Some nice properties of Hardy fields

Let K be a Hardy field and $f \in K$

- If $f \neq 0$, then f does not have arbitrarily large zeros.
- All derivatives of f are eventually monotone
- $\lim_{x \to +\infty} f(x) \in \mathbb{R} \cup \{\pm\infty\}$ always exists!

Some nice properties of Hardy fields

Let K be a Hardy field and $f \in K$

- If $f \neq 0$, then f does not have arbitrarily large zeros.
- All derivatives of f are eventually monotone
- $\lim_{x \to +\infty} f(x) \in \mathbb{R} \cup \{\pm\infty\}$ always exists!
- $f > 0$ eventually, or $f = 0$ eventually, or $f < 0$ eventually. Thus every Hardy field is an *ordered field*

Some nice properties of Hardy fields

Let K be a Hardy field and $f \in K$

- If $f \neq 0$, then f does not have arbitrarily large zeros.
- All derivatives of f are eventually monotone
- $\lim_{x \to +\infty} f(x) \in \mathbb{R} \cup \{\pm\infty\}$ always exists!
- $f > 0$ eventually, or $f = 0$ eventually, or $f < 0$ eventually. Thus every Hardy field is an *ordered field*
- $\cos(x), \sin(x)$ and most other common oscillating functions can't belong to any Hardy field

Some nice properties of Hardy fields

Let K be a Hardy field and $f \in K$

- If $f \neq 0$, then f does not have arbitrarily large zeros.
- All derivatives of f are eventually monotone
- $\lim_{x \to +\infty} f(x) \in \mathbb{R} \cup \{\pm\infty\}$ always exists!
- $f > 0$ eventually, or $f = 0$ eventually, or $f < 0$ eventually. Thus every Hardy field is an *ordered field*
- $\cos(x), \sin(x)$ and most other common oscillating functions can't belong to any Hardy field
- However... Hardy fields can contain "crypto-oscillation", i.e., $\mathbb{R}\langle \Gamma'(x)/\Gamma(x) + x + \lambda e^{-x} \sin x \rangle$ is a Hardy field. This shows there is no maxi**mum** Hardy field.

Some nice properties of Hardy fields

Let K be a Hardy field and $f \in K$

- If $f \neq 0$, then f does not have arbitrarily large zeros.
- All derivatives of f are eventually monotone
- $\lim_{x \to +\infty} f(x) \in \mathbb{R} \cup \{\pm\infty\}$ always exists!
- $f > 0$ eventually, or $f = 0$ eventually, or $f < 0$ eventually. Thus every Hardy field is an *ordered field*
- $\cos(x), \sin(x)$ and most other common oscillating functions can't belong to any Hardy field
- However... Hardy fields can contain "crypto-oscillation", i.e., $\mathbb{R}\langle \Gamma'(x)/\Gamma(x) + x + \lambda e^{-x} \sin x \rangle$ is a Hardy field. This shows there is no maxi**mum** Hardy field.

Some nice properties of Hardy fields

Let K be a Hardy field and $f \in K$

- If $f \neq 0$, then f does not have arbitrarily large zeros.
- All derivatives of f are eventually monotone
- $\lim_{x \to +\infty} f(x) \in \mathbb{R} \cup \{\pm\infty\}$ always exists!
- $f > 0$ eventually, or $f = 0$ eventually, or $f < 0$ eventually. Thus every Hardy field is an *ordered field*
- $\cos(x), \sin(x)$ and most other common oscillating functions can't belong to any Hardy field
- However... Hardy fields can contain "crypto-oscillation", i.e., $\mathbb{R}\langle \Gamma'(x)/\Gamma(x) + x + \lambda e^{-x} \sin x \rangle$ is a Hardy field. This shows there is no maxi**mum** Hardy field.

Liouville extensions and closures

- A **differential field** is a characteristic zero field K equipped with a derivation $\partial : K \to K$ (additive map satisfying Leibniz identity: $\partial(ab) = \partial(a)b + a\partial(b)$). Also define $C_K = C = \ker \partial$, the **constant field of** K.

Liouville extensions and closures

- A **differential field** is a characteristic zero field K equipped with a derivation $\partial : K \to K$ (additive map satisfying Leibniz identity: $\partial(ab) = \partial(a)b + a\partial(b)$). Also define $C_K = C = \ker \partial$, the **constant field of** K.
- A **Liouville extension** of a differential field K is a differential field extension L of K such that C_L is algebraic over C_K and for each $a \in L$ there are $t_1, \ldots, t_n \in L$ with $a \in K(t_1, \ldots, t_n)$ and for $i = 1, \ldots, n$,
  - t_i is algebraic over $K(t_1, \ldots, t_{i-1})$, or
  - $\partial(t_i) \in K(t_1, \ldots, t_{i-1})$, or
  - $t_i \neq 0$ and $\partial(t_i)/t_i \in K(t_1, \ldots, t_{i-1})$.

Liouville extensions and closures

- A **differential field** is a characteristic zero field K equipped with a derivation $\partial : K \to K$ (additive map satisfying Leibniz identity: $\partial(ab) = \partial(a)b + a\partial(b)$). Also define $C_K = C = \ker \partial$, the **constant field of** K.
- A **Liouville extension** of a differential field K is a differential field extension L of K such that C_L is algebraic over C_K and for each $a \in L$ there are $t_1, \ldots, t_n \in L$ with $a \in K(t_1, \ldots, t_n)$ and for $i = 1, \ldots, n$,
  - t_i is algebraic over $K(t_1, \ldots, t_{i-1})$, or
  - $\partial(t_i) \in K(t_1, \ldots, t_{i-1})$, or
  - $t_i \neq 0$ and $\partial(t_i)/t_i \in K(t_1, \ldots, t_{i-1})$.
- A differential field K is **Liouville closed** if it is real closed and for every $f, g \in K$ there is $y \in K^\times$ such that $y' + fy = g$.

Liouville extensions and closures

- A **differential field** is a characteristic zero field K equipped with a derivation $\partial : K \to K$ (additive map satisfying Leibniz identity: $\partial(ab) = \partial(a)b + a\partial(b)$). Also define $C_K = C = \ker \partial$, the **constant field of** K.
- A **Liouville extension** of a differential field K is a differential field extension L of K such that C_L is algebraic over C_K and for each $a \in L$ there are $t_1, \ldots, t_n \in L$ with $a \in K(t_1, \ldots, t_n)$ and for $i = 1, \ldots, n$,
  - t_i is algebraic over $K(t_1, \ldots, t_{i-1})$, or
  - $\partial(t_i) \in K(t_1, \ldots, t_{i-1})$, or
  - $t_i \neq 0$ and $\partial(t_i)/t_i \in K(t_1, \ldots, t_{i-1})$.
- A differential field K is **Liouville closed** if it is real closed and for every $f, g \in K$ there is $y \in K^\times$ such that $y' + fy = g$.
- A **Liouville closure** of K is a Liouville closed differential field extension L of K that is a Liouville extension. (provisional definition)

Liouville extensions and closures

- A **differential field** is a characteristic zero field K equipped with a derivation $\partial : K \to K$ (additive map satisfying Leibniz identity: $\partial(ab) = \partial(a)b + a\partial(b)$). Also define $C_K = C = \ker \partial$, the **constant field of** K.
- A **Liouville extension** of a differential field K is a differential field extension L of K such that C_L is algebraic over C_K and for each $a \in L$ there are $t_1, \ldots, t_n \in L$ with $a \in K(t_1, \ldots, t_n)$ and for $i = 1, \ldots, n$,
  - t_i is algebraic over $K(t_1, \ldots, t_{i-1})$, or
  - $\partial(t_i) \in K(t_1, \ldots, t_{i-1})$, or
  - $t_i \neq 0$ and $\partial(t_i)/t_i \in K(t_1, \ldots, t_{i-1})$.
- A differential field K is **Liouville closed** if it is real closed and for every $f, g \in K$ there is $y \in K^\times$ such that $y' + fy = g$.
- A **Liouville closure** of K is a Liouville closed differential field extension L of K that is a Liouville extension. (provisional definition)

Liouville extensions and closures

- A **differential field** is a characteristic zero field K equipped with a derivation $\partial : K \to K$ (additive map satisfying Leibniz identity: $\partial(ab) = \partial(a)b + a\partial(b)$). Also define $C_K = C = \ker \partial$, the **constant field of** K.
- A **Liouville extension** of a differential field K is a differential field extension L of K such that C_L is algebraic over C_K and for each $a \in L$ there are $t_1, \ldots, t_n \in L$ with $a \in K(t_1, \ldots, t_n)$ and for $i = 1, \ldots, n$,
  - t_i is algebraic over $K(t_1, \ldots, t_{i-1})$, or
  - $\partial(t_i) \in K(t_1, \ldots, t_{i-1})$, or
  - $t_i \neq 0$ and $\partial(t_i)/t_i \in K(t_1, \ldots, t_{i-1})$.
- A differential field K is **Liouville closed** if it is real closed and for every $f, g \in K$ there is $y \in K^\times$ such that $y' + fy = g$.
- A **Liouville closure** of K is a Liouville closed differential field extension L of K that is a Liouville extension. (provisional definition)

Liouville Extensions of Hardy fields

Let K be a Hardy field

Theorem (Robinson, 1972)

Define $K^{rc} = \{g \in \mathcal{G} : g \text{ is continuous and algebraic over } K\} \subseteq \mathcal{G}$. Then K^{rc} is a Hardy field and a real closure of K. In the category of Hardy fields, it is **THE** *real closure of K.*

Let K be a Hardy field

Theorem (Robinson, 1972)

Define $K^{rc} = \{g \in \mathcal{G} : g \text{ is continuous and algebraic over } K\} \subseteq \mathcal{G}$*. Then* K^{rc} *is a Hardy field and a real closure of* K*. In the category of Hardy fields, it is* **THE** *real closure of* K*.*

Theorem (Hardy 1912, Marić 1972, Rosenlicht-Singer 1983)

If $P(Y) \in K(Y)$ *and* $g \in \mathcal{G}$ *is differentiable such that satisfies* $g' = P(g)$*, then* $K(g)$ *is a Hardy field.*

Liouville Extensions of Hardy fields

Let K be a Hardy field

Theorem (Robinson, 1972)

Define $K^{rc} = \{g \in \mathcal{G} : g$ is continuous and algebraic over $K\} \subseteq \mathcal{G}$. Then K^{rc} is a Hardy field and a real closure of K. In the category of Hardy fields, it is **THE** *real closure of K.*

Theorem (Hardy 1912, Marić 1972, Rosenlicht-Singer 1983)

If $P(Y) \in K(Y)$ and $g \in \mathcal{G}$ is differentiable such that satisfies $g' = P(g)$, then $K(g)$ is a Hardy field.

Corollary

If $f \in K$, then $K(\mathbb{R})$, $K(\int f)$, $K(e^f)$, $K(\log(|f|))$, $K(\exp(\int f))$ are all Hardy fields.

Liouville Extensions of Hardy fields

Let K be a Hardy field

Theorem (Robinson, 1972)

Define $K^{rc} = \{g \in \mathcal{G} : g \text{ is continuous and algebraic over } K\} \subseteq \mathcal{G}$. Then K^{rc} is a Hardy field and a real closure of K. In the category of Hardy fields, it is **THE** *real closure of K.*

Theorem (Hardy 1912, Marić 1972, Rosenlicht-Singer 1983)

If $P(Y) \in K(Y)$ and $g \in \mathcal{G}$ is differentiable such that satisfies $g' = P(g)$, then $K(g)$ is a Hardy field.

Corollary

If $f \in K$, then $K(\mathbb{R})$, $K(\int f)$, $K(e^f)$, $K(\log(|f|))$, $K(\exp(\int f))$ are all Hardy fields.

Liouville Extensions of Hardy fields

Let K be a Hardy field

Theorem (Robinson, 1972)

Define $K^{rc} = \{g \in \mathcal{G} : g$ is continuous and algebraic over $K\} \subseteq \mathcal{G}$. Then K^{rc} is a Hardy field and a real closure of K. In the category of Hardy fields, it is **THE** *real closure of K.*

Theorem (Hardy 1912, Marić 1972, Rosenlicht-Singer 1983)

If $P(Y) \in K(Y)$ and $g \in \mathcal{G}$ is differentiable such that satisfies $g' = P(g)$, then $K(g)$ is a Hardy field.

Corollary

If $f \in K$, then $K(\mathbb{R})$, $K(\int f)$, $K(e^f)$, $K(\log(|f|))$, $K(\exp(\int f))$ are all Hardy fields.

Liouville closure of Hardy fields

- Let K be a Hardy field containing $\mathbb{R}$

Liouville closure of Hardy fields

- Let K be a Hardy field containing $\mathbb{R}$
- Define $\mathrm{Li}(K) :=$

$$\{g \in \mathcal{G} : g \text{ lies in some Hardy field Liouville extension of } K\} \subseteq \mathcal{G}$$

Liouville closure of Hardy fields

- Let K be a Hardy field containing $\mathbb{R}$
- Define $\mathrm{Li}(K) :=$

$$\{g \in \mathcal{G} : g \text{ lies in some Hardy field Liouville extension of } K\} \subseteq \mathcal{G}$$

- In other words, $\mathrm{Li}(K)$ is obtained from K by closing off under integration, exponential integration and real-closing

Liouville closure of Hardy fields

- Let K be a Hardy field containing $\mathbb{R}$
- Define $\mathrm{Li}(K) :=$

  $$\{g \in \mathcal{G} : g \text{ lies in some Hardy field Liouville extension of } K\} \subseteq \mathcal{G}$$

- In other words, $\mathrm{Li}(K)$ is obtained from K by closing off under integration, exponential integration and real-closing
- $\mathrm{Li}(K)$ is itself a Hardy field and is a Liouville closure of K

Liouville closure of Hardy fields

- Let K be a Hardy field containing $\mathbb{R}$
- Define $\mathrm{Li}(K) :=$

$$\{g \in \mathcal{G} : g \text{ lies in some Hardy field Liouville extension of } K\} \subseteq \mathcal{G}$$

- In other words, $\mathrm{Li}(K)$ is obtained from K by closing off under integration, exponential integration and real-closing
- $\mathrm{Li}(K)$ is itself a Hardy field and is a Liouville closure of K
- In fact, in the category of Hardy fields, it is **THE** Liouville closure of K

Liouville closure of Hardy fields

- Let K be a Hardy field containing $\mathbb{R}$
- Define $\mathrm{Li}(K) :=$

$$\{g \in \mathcal{G} : g \text{ lies in some Hardy field Liouville extension of } K\} \subseteq \mathcal{G}$$

- In other words, $\mathrm{Li}(K)$ is obtained from K by closing off under integration, exponential integration and real-closing
- $\mathrm{Li}(K)$ is itself a Hardy field and is a Liouville closure of K
- In fact, in the category of Hardy fields, it is **THE** Liouville closure of K

Liouville closure of Hardy fields

- Let K be a Hardy field containing $\mathbb{R}$
- Define $\mathrm{Li}(K) :=$

$$\{g \in \mathcal{G} : g \text{ lies in some Hardy field Liouville extension of } K\} \subseteq \mathcal{G}$$

- In other words, $\mathrm{Li}(K)$ is obtained from K by closing off under integration, exponential integration and real-closing
- $\mathrm{Li}(K)$ is itself a Hardy field and is a Liouville closure of K
- In fact, in the category of Hardy fields, it is **THE** Liouville closure of K

H-fields

- An H-**field** is an ordered differential field K such that:

(H1) for all $f \in K$, if $f > C_K$, then $\partial(f) > 0$;

(H2) $\mathcal{O} = C_K + \mathcal{o}$ where

$$\mathcal{O} = \{g \in K : |g| \leq c \text{ for some } c \in C_K\}$$

and $\mathcal{o}$ is the maximal ideal of the convex subring $\mathcal{O}$ of K.

H-fields

- An H-**field** is an ordered differential field K such that:

(H1) for all $f \in K$, if $f > C_K$, then $\partial(f) > 0$;
(H2) $\mathcal{O} = C_K + \mathcal{o}$ where

$$\mathcal{O} = \{g \in K : |g| \leq c \text{ for some } c \in C_K\}$$

and $\mathcal{o}$ is the maximal ideal of the convex subring $\mathcal{O}$ of K.

- Example: A Hardy field $K \supseteq \mathbb{R}$ is an H-field, with
  - $\mathcal{O} = \{f \in K : \lim_{x\to+\infty} f \in \mathbb{R}\}$, the *bounded* elements, and
  - $\mathcal{o} = \{f \in K : \lim_{x\to+\infty} f = 0\}$, the *infinitesimal* elements

H-fields

- An H-**field** is an ordered differential field K such that:

(H1) for all $f \in K$, if $f > C_K$, then $\partial(f) > 0$;

(H2) $\mathcal{O} = C_K + \mathcal{o}$ where

$$\mathcal{O} = \{g \in K : |g| \leq c \text{ for some } c \in C_K\}$$

and $\mathcal{o}$ is the maximal ideal of the convex subring $\mathcal{O}$ of K.

- Example: A Hardy field $K \supseteq \mathbb{R}$ is an H-field, with
  - $\mathcal{O} = \{f \in K : \lim_{x \to +\infty} f \in \mathbb{R}\}$, the *bounded* elements, and
  - $\mathcal{o} = \{f \in K : \lim_{x \to +\infty} f = 0\}$, the *infinitesimal* elements
- Other examples: various fields of transseries such as $\mathbb{T}$ and $\mathbb{T}_{\log}$

H-fields

- An H-**field** is an ordered differential field K such that:

(H1) for all $f \in K$, if $f > C_K$, then $\partial(f) > 0$;
(H2) $\mathcal{O} = C_K + \scriptstyle\mathcal{O}$ where

$$\mathcal{O} = \{g \in K : |g| \leq c \text{ for some } c \in C_K\}$$

and $\scriptstyle\mathcal{O}$ is the maximal ideal of the convex subring $\mathcal{O}$ of K.

- Example: A Hardy field $K \supseteq \mathbb{R}$ is an H-field, with
  - $\mathcal{O} = \{f \in K : \lim_{x \to +\infty} f \in \mathbb{R}\}$, the *bounded* elements, and
  - $\scriptstyle\mathcal{O}$ $= \{f \in K : \lim_{x \to +\infty} f = 0\}$, the *infinitesimal* elements
- Other examples: various fields of transseries such as $\mathbb{T}$ and $\mathbb{T}_{\log}$
- Conway's **No**, the ordered field of surreal numbers is a proper class-sized H-field when equipped with the Berarducci-Mantova derivation ∂_{BM}

H-fields

- An H-**field** is an ordered differential field K such that:

(H1) for all $f \in K$, if $f > C_K$, then $\partial(f) > 0$;

(H2) $\mathcal{O} = C_K + \scriptstyle{\mathcal{O}}$ where

$$\mathcal{O} = \{g \in K : |g| \leq c \text{ for some } c \in C_K\}$$

and $\scriptstyle{\mathcal{O}}$ is the maximal ideal of the convex subring $\mathcal{O}$ of K.

- Example: A Hardy field $K \supseteq \mathbb{R}$ is an H-field, with
  - $\mathcal{O} = \{f \in K : \lim_{x \to +\infty} f \in \mathbb{R}\}$, the *bounded* elements, and
  - $\scriptstyle{\mathcal{O}}$ $= \{f \in K : \lim_{x \to +\infty} f = 0\}$, the *infinitesimal* elements
- Other examples: various fields of transseries such as $\mathbb{T}$ and $\mathbb{T}_{\log}$
- Conway's **No**, the ordered field of surreal numbers is a proper class-sized H-field when equipped with the Berarducci-Mantova derivation ∂_{BM}

H-fields

- An H-**field** is an ordered differential field K such that:

  (H1) for all $f \in K$, if $f > C_K$, then $\partial(f) > 0$;

  (H2) $\mathcal{O} = C_K + \scriptstyle{\mathcal{O}}$ where

$$\mathcal{O} = \{g \in K : |g| \leq c \text{ for some } c \in C_K\}$$

  and $\scriptstyle{\mathcal{O}}$ is the maximal ideal of the convex subring $\mathcal{O}$ of K.

- Example: A Hardy field $K \supseteq \mathbb{R}$ is an H-field, with
  - $\mathcal{O} = \{f \in K : \lim_{x \to +\infty} f \in \mathbb{R}\}$, the *bounded* elements, and
  - $\scriptstyle{\mathcal{O}}$ $= \{f \in K : \lim_{x \to +\infty} f = 0\}$, the *infinitesimal* elements
- Other examples: various fields of transseries such as $\mathbb{T}$ and $\mathbb{T}_{\log}$
- Conway's **No**, the ordered field of surreal numbers is a proper class-sized H-field when equipped with the Berarducci-Mantova derivation ∂_{BM}

Liouville closures of H-fields

If K is an H-field, by **Liouville closure of** K we now mean "H-field extension of K that is also a Liouville closure of K in the previous sense"

Liouville closures of H-fields

If K is an H-field, by **Liouville closure of** K we now mean "H-field extension of K that is also a Liouville closure of K in the previous sense"

Theorem (Aschenbrenner, van den Dries, 2002)

Let K be an H-field. Then one of the following occurs:

(I) *K has exactly one Liouville closure up to isomorphism over K,*

(II) *K has exactly two Liouville closures up to isomorphism over K.*

Liouville closures of H-fields

If K is an H-field, by **Liouville closure of** K we now mean "H-field extension of K that is also a Liouville closure of K in the previous sense"

Theorem (Aschenbrenner, van den Dries, 2002)

Let K be an H-field. Then one of the following occurs:

(I) *K has exactly one Liouville closure up to isomorphism over K,*

(II) *K has exactly two Liouville closures up to isomorphism over K.*

What causes one or two Liouville closures?

Liouville closures of H-fields

If K is an H-field, by **Liouville closure of** K we now mean "H-field extension of K that is also a Liouville closure of K in the previous sense"

Theorem (Aschenbrenner, van den Dries, 2002)

Let K be an H-field. Then one of the following occurs:

(I) *K has exactly one Liouville closure up to isomorphism over K,*

(II) *K has exactly two Liouville closures up to isomorphism over K.*

What causes one or two Liouville closures?

Liouville closures of H-fields

If K is an H-field, by **Liouville closure of K** we now mean "H-field extension of K that is also a Liouville closure of K in the previous sense"

Theorem (Aschenbrenner, van den Dries, 2002)

Let K be an H-field. Then one of the following occurs:

(I) *K has exactly one Liouville closure up to isomorphism over K,*

(II) *K has exactly two Liouville closures up to isomorphism over K.*

What causes one or two Liouville closures?

The valuation of an H-field

Let K be an H-field and let $f, g \in K$

- Define $f \preccurlyeq g :\Longleftrightarrow \exists c \in C_K^{>0} : |f| \leq c|g|$

The valuation of an H-field

Let K be an H-field and let $f, g \in K$

- Define $f \preccurlyeq g :\Longleftrightarrow \exists c \in C_K^{>0} : |f| \leq c|g|$
- Also define equivalence relation $\asymp$ on $K^\times$:
  $f \asymp g :\Longleftrightarrow f \preccurlyeq g$ and $g \preccurlyeq f$

The valuation of an H-field

Let K be an H-field and let $f, g \in K$

- Define $f \preccurlyeq g :\Longleftrightarrow \exists c \in C_K^{>0} : |f| \leq c|g|$
- Also define equivalence relation $\asymp$ on $K^\times$:
  $f \asymp g :\Longleftrightarrow f \preccurlyeq g$ and $g \preccurlyeq f$
- The equivalence classes vf are elements of an ordered abelian group $\Gamma_K := v(K^\times)$:

$$vf + vg = v(fg), \quad vf \geq vg \Longleftrightarrow f \preccurlyeq g.$$

The valuation of an H-field

Let K be an H-field and let $f, g \in K$

- Define $f \preccurlyeq g :\Longleftrightarrow \exists c \in C_K^{>0} : |f| \leq c|g|$
- Also define equivalence relation $\asymp$ on $K^\times$:
  $f \asymp g :\Longleftrightarrow f \preccurlyeq g$ and $g \preccurlyeq f$
- The equivalence classes vf are elements of an ordered abelian group $\Gamma_K := v(K^\times)$:

$$vf + vg = v(fg), \quad vf \geq vg \Longleftrightarrow f \preccurlyeq g.$$

- The map $f \mapsto vf : K^\times \to \Gamma$ is a valuation.

The valuation of an H-field

Let K be an H-field and let $f, g \in K$

- Define $f \preccurlyeq g :\Longleftrightarrow \exists c \in C_K^{>0} : |f| \leq c|g|$
- Also define equivalence relation $\asymp$ on $K^\times$:
  $f \asymp g :\Longleftrightarrow f \preccurlyeq g$ and $g \preccurlyeq f$
- The equivalence classes vf are elements of an ordered abelian group $\Gamma_K := v(K^\times)$:

$$vf + vg = v(fg), \quad vf \geq vg \Longleftrightarrow f \preccurlyeq g.$$

- The map $f \mapsto vf : K^\times \to \Gamma$ is a valuation.
- Example: in $K = \mathbb{R}(x, \arctan(x))$, $x \preccurlyeq x^2$ and $\arctan(x) \asymp \pi$

The valuation of an H-field

Let K be an H-field and let $f, g \in K$

- Define $f \preccurlyeq g :\Longleftrightarrow \exists c \in C_K^{>0} : |f| \leq c|g|$
- Also define equivalence relation $\asymp$ on $K^\times$:
  $f \asymp g :\Longleftrightarrow f \preccurlyeq g$ and $g \preccurlyeq f$
- The equivalence classes vf are elements of an ordered abelian group $\Gamma_K := v(K^\times)$:

$$vf + vg = v(fg), \quad vf \geq vg \Longleftrightarrow f \preccurlyeq g.$$

- The map $f \mapsto vf : K^\times \to \Gamma$ is a valuation.
- Example: in $K = \mathbb{R}(x, \arctan(x))$, $x \preccurlyeq x^2$ and $\arctan(x) \asymp \pi$

The valuation of an H-field

Let K be an H-field and let $f, g \in K$

- Define $f \preccurlyeq g :\Longleftrightarrow \exists c \in C_K^{>0} : |f| \leq c|g|$
- Also define equivalence relation $\asymp$ on $K^\times$:
  $f \asymp g :\Longleftrightarrow f \preccurlyeq g$ and $g \preccurlyeq f$
- The equivalence classes vf are elements of an ordered abelian group $\Gamma_K := v(K^\times)$:

$$vf + vg = v(fg), \quad vf \geq vg \Longleftrightarrow f \preccurlyeq g.$$

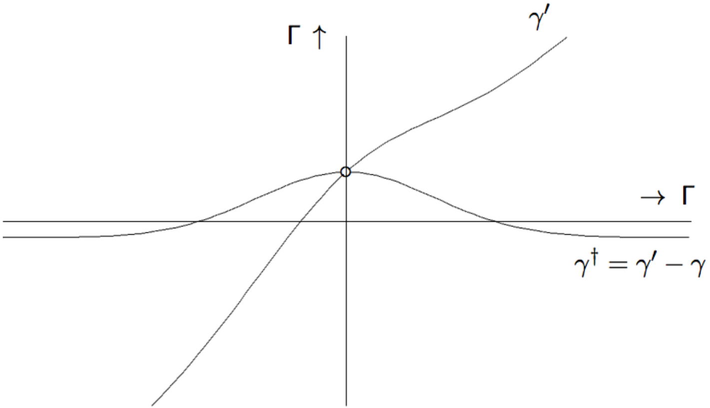
- The map $f \mapsto vf : K^\times \to \Gamma$ is a valuation.
- Example: in $K = \mathbb{R}(x, \arctan(x))$, $x \preccurlyeq x^2$ and $\arctan(x) \asymp \pi$

The Asymptotic Couple of an H-field

The derivation ∂ induces a map

$$\gamma = vf \mapsto \gamma' = v(f') : \quad \Gamma^{\neq} := \Gamma \setminus \{0\} \to \Gamma.$$

We set $\Psi := \{\gamma' - \gamma : \gamma \in \Gamma^{\neq}\}$. Then $\Psi < (\Gamma^{>0})'$.



Trichotomy for *H*-fields

Exactly one of the following statements holds:

(1) $\Psi < \beta < (\Gamma^{>0})'$ for a (necessarily unique) β. We call such a β a **gap** in K. Example: $K = \mathbb{R}$.

Trichotomy for H-fields

Exactly one of the following statements holds:

(1) $\Psi < \beta < (\Gamma^{>0})'$ for a (necessarily unique) β. We call such a β a **gap** in K. Example: $K = \mathbb{R}$.

(2) Ψ has a largest element. In this case we say that K is **grounded**. Example: $K = \mathbb{R}(x, \log x)$.

Trichotomy for H-fields

Exactly one of the following statements holds:

(1) $\Psi < \beta < (\Gamma^{>0})'$ for a (necessarily unique) β. We call such a β a **gap** in K. Example: $K = \mathbb{R}$.

(2) Ψ has a largest element. In this case we say that K is **grounded**. Example: $K = \mathbb{R}(x, \log x)$.

(3) $\sup \Psi$ does not exists; equivalently: $\Gamma = (\Gamma^{\neq})'$. In this case we say that K has **asymptotic integration**. Example: $K = \mathbb{R}(x, \log(x), \log\log(x), \log\log\log(x), \ldots)$.

Trichotomy for H-fields

Exactly one of the following statements holds:

(1) $\Psi < \beta < (\Gamma^{>0})'$ for a (necessarily unique) β. We call such a β a **gap** in K. Example: $K = \mathbb{R}$.

(2) Ψ has a largest element. In this case we say that K is **grounded**. Example: $K = \mathbb{R}(x, \log x)$.

(3) $\sup \Psi$ does not exists; equivalently: $\Gamma = (\Gamma^{\neq})'$. In this case we say that K has **asymptotic integration**. Example: $K = \mathbb{R}(x, \log(x), \log\log(x), \log\log\log(x), \ldots)$.

- In case (1) there are two Liouville closures. Why? You can integrate β in such a way that its antiderivative is either infinite or infinitesimal, your choice!

Trichotomy for H-fields

Exactly one of the following statements holds:

(1) $\Psi < \beta < (\Gamma^{>0})'$ for a (necessarily unique) β. We call such a β a **gap** in K. Example: $K = \mathbb{R}$.

(2) Ψ has a largest element. In this case we say that K is **grounded**. Example: $K = \mathbb{R}(x, \log x)$.

(3) $\sup \Psi$ does not exists; equivalently: $\Gamma = (\Gamma^{\neq})'$. In this case we say that K has **asymptotic integration**. Example: $K = \mathbb{R}(x, \log(x), \log\log(x), \log\log\log(x), \ldots)$.

- In case (1) there are two Liouville closures. Why? You can integrate β in such a way that its antiderivative is either infinite or infinitesimal, your choice!
- In case (2) there is one Liouville closure.

Trichotomy for H-fields

Exactly one of the following statements holds:

(1) $\Psi < \beta < (\Gamma^{>0})'$ for a (necessarily unique) β. We call such a β a **gap** in K. Example: $K = \mathbb{R}$.

(2) Ψ has a largest element. In this case we say that K is **grounded**. Example: $K = \mathbb{R}(x, \log x)$.

(3) $\sup \Psi$ does not exists; equivalently: $\Gamma = (\Gamma^{\neq})'$. In this case we say that K has **asymptotic integration**. Example: $K = \mathbb{R}(x, \log(x), \log\log(x), \log\log\log(x), \ldots)$.

- In case (1) there are two Liouville closures. Why? You can integrate β in such a way that its antiderivative is either infinite or infinitesimal, your choice!
- In case (2) there is one Liouville closure.
- What about case (3)?

Trichotomy for H-fields

Exactly one of the following statements holds:

(1) $\Psi < \beta < (\Gamma^{>0})'$ for a (necessarily unique) β. We call such a β a **gap** in K. Example: $K = \mathbb{R}$.

(2) Ψ has a largest element. In this case we say that K is **grounded**. Example: $K = \mathbb{R}(x, \log x)$.

(3) $\sup \Psi$ does not exists; equivalently: $\Gamma = (\Gamma^{\neq})'$. In this case we say that K has **asymptotic integration**. Example: $K = \mathbb{R}(x, \log(x), \log\log(x), \log\log\log(x), \ldots)$.

- In case (1) there are two Liouville closures. Why? You can integrate β in such a way that its antiderivative is either infinite or infinitesimal, your choice!
- In case (2) there is one Liouville closure.
- What about case (3)?

Trichotomy for H-fields

Exactly one of the following statements holds:

(1) $\Psi < \beta < (\Gamma^{>0})'$ for a (necessarily unique) β. We call such a β a **gap** in K. Example: $K = \mathbb{R}$.

(2) Ψ has a largest element. In this case we say that K is **grounded**. Example: $K = \mathbb{R}(x, \log x)$.

(3) $\sup \Psi$ does not exists; equivalently: $\Gamma = (\Gamma^{\neq})'$. In this case we say that K has **asymptotic integration**. Example: $K = \mathbb{R}(x, \log(x), \log\log(x), \log\log\log(x), \ldots)$.

- In case (1) there are two Liouville closures. Why? You can integrate β in such a way that its antiderivative is either infinite or infinitesimal, your choice!
- In case (2) there is one Liouville closure.
- What about case (3)?

How do gaps occur in Liouville extensions?

Let K be a real closed H-field

- If K is Liouville closed, then K does not have a gap.

How do gaps occur in Liouville extensions?

Let K be a real closed H-field

- If K is Liouville closed, then K does not have a gap.
- If $L = K(y)$ with $y' = f \in K$ ($y = \int f$), then L has a gap if and only if K has a gap.

How do gaps occur in Liouville extensions?

Let K be a real closed H-field

- If K is Liouville closed, then K does not have a gap.
- If $L = K(y)$ with $y' = f \in K$ $(y = \int f)$, then L has a gap if and only if K has a gap.
- If $L = K(z)$ with $z \neq 0$, $z'/z = g \in K$ $(z = \exp \int g)$, then L **may** have a gap even if K does not have a gap.

How do gaps occur in Liouville extensions?

Let K be a real closed H-field

- If K is Liouville closed, then K does not have a gap.
- If $L = K(y)$ with $y' = f \in K$ ($y = \int f$), then L has a gap if and only if K has a gap.
- If $L = K(z)$ with $z \neq 0$, $z'/z = g \in K$ ($z = \exp \int g$), then L **may** have a gap even if K does not have a gap.

How do gaps occur in Liouville extensions?

Let K be a real closed H-field

- If K is Liouville closed, then K does not have a gap.
- If $L = K(y)$ with $y' = f \in K$ $(y = \int f)$, then L has a gap if and only if K has a gap.
- If $L = K(z)$ with $z \neq 0$, $z'/z = g \in K$ $(z = \exp \int g)$, then L **may** have a gap even if K does not have a gap.

In fact, one can detect in K already whether some $g \in K$ **creates a gap over** K, i.e., $z = \exp \int g$ is a gap in $K(z)$...

Liouville closures of H-fields

Proposition

The following are equivalent, for a real closed H-field K:

(1) $\forall f \exists g [|g| > C_K$ and $f - g^{\dagger\dagger} \succcurlyeq g^{\dagger}]$, where $g^{\dagger} := \partial(g)/g$.

(2) K has asymptotic integration, and no element of K creates a gap.

Liouville closures of H-fields

Proposition

The following are equivalent, for a real closed H-field K:

(1) $\forall f \exists g [|g| > C_K$ and $f - g^{\dagger\dagger} \succcurlyeq g^{\dagger}]$, where $g^{\dagger} := \partial(g)/g$.

(2) K has asymptotic integration, and no element of K creates a gap.

We say that K is λ-free if it satisfies condition (1) in the proposition.

Liouville closures of H-fields

Proposition

The following are equivalent, for a real closed H-field K:

(1) $\forall f \exists g [|g| > C_K$ and $f - g^{\dagger\dagger} \succcurlyeq g^{\dagger}]$, where $g^{\dagger} := \partial(g)/g$.

(2) K has asymptotic integration, and no element of K creates a gap.

We say that K is λ-free if it satisfies condition (1) in the proposition.

Theorem (G.)

Let K be an H-field. Then

(1) *K has exactly one Liouville closure up to isomorphism over K iff*
- (a) *K is grounded, or*
- (b) *K has asymptotic integration and is λ-free*

(2) *K has exactly two Liouville closures up to isomorphism over K iff*
- (c) *K has asymptotic integration and is not λ-free, or*
- (d) *K has a gap.*

Liouville closures of H-fields

Proposition

The following are equivalent, for a real closed H-field K:

(1) $\forall f \exists g [|g| > C_K \text{ and } f - g^{\dagger\dagger} \succcurlyeq g^{\dagger}]$, where $g^{\dagger} := \partial(g)/g$.

(2) K has asymptotic integration, and no element of K creates a gap.

We say that K is λ-free if it satisfies condition (1) in the proposition.

Theorem (G.)

Let K be an H-field. Then

(1) *K has exactly one Liouville closure up to isomorphism over K iff*
   - (a) *K is grounded, or*
   - (b) *K has asymptotic integration and is λ-free*

(2) *K has exactly two Liouville closures up to isomorphism over K iff*
   - (c) *K has asymptotic integration and is not λ-free, or*
   - (d) *K has a gap.*

Liouville closures of H-fields

Proposition

The following are equivalent, for a real closed H-field K:

(1) $\forall f \exists g [|g| > C_K \text{ and } f - g^{\dagger\dagger} \succcurlyeq g^{\dagger}]$, where $g^{\dagger} := \partial(g)/g$.

(2) K has asymptotic integration, and no element of K creates a gap.

We say that K is λ-free if it satisfies condition (1) in the proposition.

Theorem (G.)

Let K be an H-field. Then

(1) *K has exactly one Liouville closure up to isomorphism over K iff*
- (a) *K is grounded, or*
- (b) *K has asymptotic integration and is λ-free*

(2) *K has exactly two Liouville closures up to isomorphism over K iff*
- (c) *K has asymptotic integration and is not λ-free, or*
- (d) *K has a gap.*

Proof sketch

- Assume K has asymptotic integration and is λ-free. Want to show that K has one Liouville closure up to isomorphism over K.

Proof sketch

- Assume K has asymptotic integration and is λ-free. Want to show that K has one Liouville closure up to isomorphism over K.
- Proof by procrastination!

Proof sketch

- Assume K has asymptotic integration and is λ-free. Want to show that K has one Liouville closure up to isomorphism over K.
- Proof by procrastination!
- The primary source of Two Liouville Closures is the occurrence of gaps.

Proof sketch

- Assume K has asymptotic integration and is λ-free. Want to show that K has one Liouville closure up to isomorphism over K.
- Proof by procrastination!
- The primary source of Two Liouville Closures is the occurrence of gaps.
- λ-freeness is a **gap prevention property** in the sense that you can't create a gap in "the next step"

Proof sketch

- Assume K has asymptotic integration and is λ-free. Want to show that K has one Liouville closure up to isomorphism over K.
- Proof by procrastination!
- The primary source of Two Liouville Closures is the occurrence of gaps.
- λ-freeness is a **gap prevention property** in the sense that you can't create a gap in "the next step"
- I proved that λ-freeness is preserved under adjoining integrals and adjoining exponential integrals

Proof sketch

- Assume K has asymptotic integration and is λ-free. Want to show that K has one Liouville closure up to isomorphism over K.
- Proof by procrastination!
- The primary source of Two Liouville Closures is the occurrence of gaps.
- λ-freeness is a **gap prevention property** in the sense that you can't create a gap in "the next step"
- I proved that λ-freeness is preserved under adjoining integrals and adjoining exponential integrals
- Thus we "kick the can down the road": in constructing a Liouville closure, we are forever λ-free, so we are always at least two steps away from creating a gap, so a gap never gets created!

Proof sketch

- Assume K has asymptotic integration and is λ-free. Want to show that K has one Liouville closure up to isomorphism over K.
- Proof by procrastination!
- The primary source of Two Liouville Closures is the occurrence of gaps.
- λ-freeness is a **gap prevention property** in the sense that you can't create a gap in "the next step"
- I proved that λ-freeness is preserved under adjoining integrals and adjoining exponential integrals
- Thus we "kick the can down the road": in constructing a Liouville closure, we are forever λ-free, so we are always at least two steps away from creating a gap, so a gap never gets created!

Proof sketch

- Assume K has asymptotic integration and is λ-free. Want to show that K has one Liouville closure up to isomorphism over K.
- Proof by procrastination!
- The primary source of Two Liouville Closures is the occurrence of gaps.
- λ-freeness is a **gap prevention property** in the sense that you can't create a gap in "the next step"
- I proved that λ-freeness is preserved under adjoining integrals and adjoining exponential integrals
- Thus we "kick the can down the road": in constructing a Liouville closure, we are forever λ-free, so we are always at least two steps away from creating a gap, so a gap never gets created!