Distal and non-distal ordered abelian groups

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Indiscernible sequences

Convention: $\mathbb{M}$ is a first order structure, possibly highly saturated.

Definition

A sequence $(a_i)_{i \in I}$ from $\mathbb{M}^n$ is A-**indiscernible** if for all $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_m$ from I we have $a_{i_1} \cdots a_{i_m} \equiv_A a_{j_1} \cdots a_{j_m}$

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- A sequence (a_n) of algebraically independent numbers in $(\mathbb{C}; 0, 1, +, -, \cdot)$, for example, $(\exp(\sqrt{p_n}))$, where $p_n = n$th prime.

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Definition (for monster model $\mathbb{M}$)

$\mathbb{M}$ is NIP iff for every formula $\varphi(x, y)$, every indiscernible sequence $(a_i)_{i \in I}$ from $\mathbb{M}^{|x|}$ and every $b \in \mathbb{M}^{|y|}$, there is $\epsilon \in \{0, 1\}$ such that eventually $\models \varphi(a_i, b)^\epsilon$.

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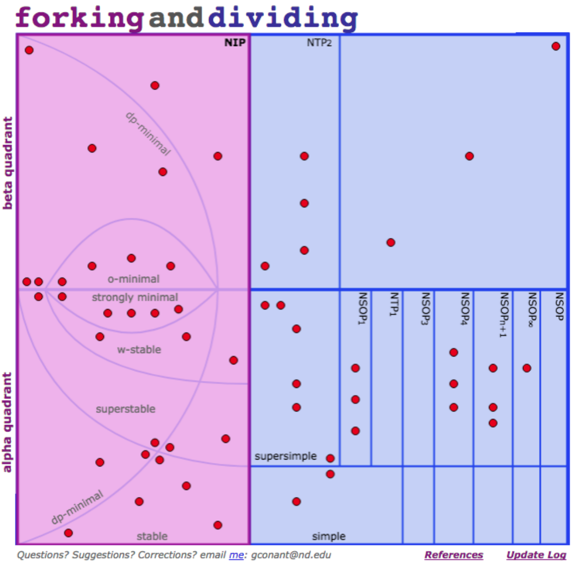
More examples of NIP theories



Map of the Universe

Nice Properties of Theories

ω-stable	superstable	stable (NOP)	
strongly minimal	o-minimal	dp-minimal	
NIP	supersimple	simple (NTP)	
$NSOP_1$	NTP_1	NTP_2	NSOP
$NSOP_3$	$NSOP_4$	$NSOP_{n+1}$	$NSOP_\infty$

Click a property above to highlight region and display details. Or click the map for specific region information.

Reset

NIP (dependent)

Examples

- $(\mathbb{Q}^n, <_1, \dots, <_n)$
- $(\mathbb{T}, +, \cdot, 0, 1, \partial, \le, \preccurlyeq)$

Contains:

- dp-minimal
- o-minimal
- strongly minimal
- stable
- superstable
- ω-stable

Definition

Features Displaying Poorly?

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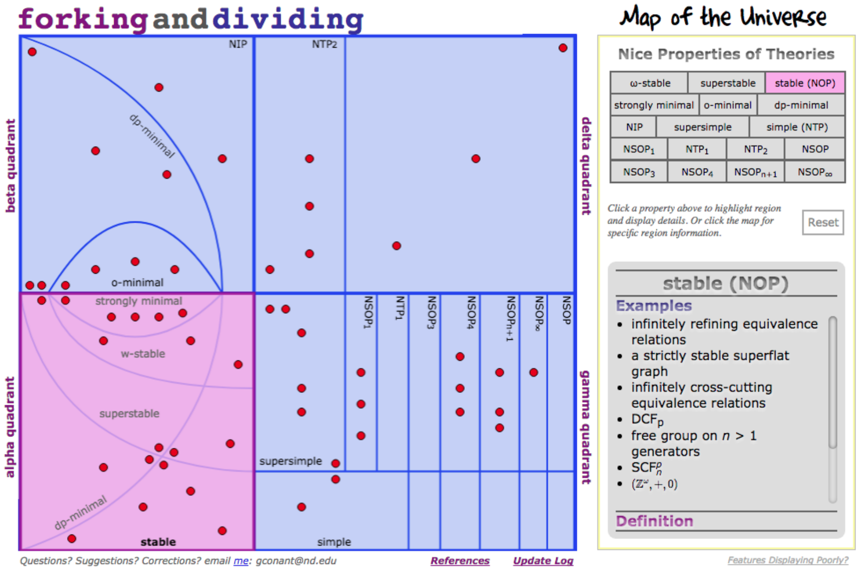
Examples of stable structures



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We say that $\mathbb{M}$ is distal if for every tuple d, for every indiscernible sequence $(a_i)_{i \in I}$ from $\mathbb{M}^p$ such that

1. $I = I_1 + (c) + I_2$, I_1 nonempty without greatest element, I_2 nonempty without least element,
2. $(a_i)_{i \in I_1 + I_2}$ is d-indiscernible,

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- $(\mathbb{R}, <, +, \cdot, 0, 1, 2^{\mathbb{Z}})$ is distal, whereas $(\mathbb{R}, <, +, \cdot, 0, 1, 2^{\mathbb{Q}})$ is **not** distal (Hieronymi, Nell, 2017)

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- Let G be an OAG. For each $n \geq 1$ we will define three new sorts $\mathcal{S}_n$, $\mathcal{T}_n$, and $\mathcal{T}_n^+$ which encode uniformly definable families of convex subgroups of G. The so-called *auxiliary sorts*.

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- The above shows that $\mathcal{S}_2, \mathcal{T}_2, \mathcal{T}_2^+$ are order-isomorphic to $(\mathbb{Q}, <)$.

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- Given $a = \sum_{q\in\mathbb{Q}} r_q\epsilon_q \in G$, let $q_0 = \min\{q : r_q \not\equiv_2 0\}$, then:

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- The above shows that $\mathcal{S}_2, \mathcal{T}_2, \mathcal{T}_2^+$ are order-isomorphic to $(\mathbb{Q}, <)$.
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Theorem (Cluckers, Halupczok)

In the theory of ordered abelian groups, each L_{qe}-formula $\psi(\bar{x}, \bar{\eta})$, where $\bar{x}$ are home sort variables, and $\bar{\eta}$ are auxiliary sort variables, is equivalent to an L_{qe}-formula $\phi(\bar{x}, \bar{\eta})$ in family union form:

$$\phi(\bar{x}, \bar{\eta}) = \bigvee_{i=1}^{k} \exists \bar{\theta}(\xi_i(\bar{\eta}, \bar{\theta}) \wedge \psi_i(\bar{x}, \bar{\theta})),$$

where $\bar{\theta}$ are auxiliary sort variables, the formulas $\xi_i(\bar{\eta}, \bar{\theta})$ live purely in the auxiliary sorts, each $\psi_i(\bar{x}, \bar{\theta})$ is a conjunction of literals (i.e., of atoms and negated atoms), and for any ordered abelian group G and any $\bar{\beta}$ in the auxiliary sort of G corresponding to $\bar{\eta}$, the $L_{qe}(G)$-formulas $\{\xi_i(\bar{\beta}, \bar{\alpha}) \wedge \psi_i(\bar{x}, \bar{\alpha}) : 1 \leq i \leq k, \bar{\alpha} \in \text{auxiliary sorts of } G\}$ are pairwise inconsistent.

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Distal and non-distal OAGs in an easy case

Theorem

Suppose G is an ordered abelian group such that $\mathcal{S}_p$ is finite for all primes p. Then the following are equivalent:

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