On degree theory for quasilinear elliptic equations with natural growth conditions

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Dedicated to Patrizia Pucci

Abstract. We show how the degree for maps of class $(S)_+$ can be used to define, by a suitable approximation technique, a degree for quasilinear elliptic equations with natural growth conditions.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$. Quasilinear problems of the form

$$\begin{cases} -\mathrm{div}\,[a(x,u,\nabla u)] + b(x,u,\nabla u) = 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

have been first of all studied under the so-called *controllable growth conditions* in the sense of [**13**], which ensure that the nonlinear operator

$$\{u \mapsto -\mathrm{div}\,[a(x,u,\nabla u)] + b(x,u,\nabla u)\}$$

is well defined and continuous from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ for some $p \in]1,\infty[$. Under suitable monotonicity and coercivity assumptions, a degree theory for this class of problems can be defined in the framework of operators of class $(S)_+$ (see e.g. [**6, 15**]).

On the other hand, it is well known that controllable growth conditions do not allow to include the Euler-Lagrange equation associated with functionals $f : W_0^{1,p}(\Omega) \to \mathbb{R}$ of the form

$$f(u) = \frac{1}{p}\int_\Omega \alpha(u)|\nabla u|^p\,dx + \int_\Omega G(u)\,dx\,,$$

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unless $p > n$ or α is constant. For this reason the *natural growth conditions* in the sense of [**13**] have been introduced. A feature is that now the operator

$$\{u \mapsto -\text{div}\,[a(x,u,\nabla u)] + b(x,u,\nabla u)\}$$

is well defined from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega) + L^1(\Omega)$. Consequently, test functions in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ have to be considered in the weak formulation of the equation.

While problems with natural growth conditions have been studied since the '60, from the point of view of regularity theory (see e.g. [**14**]), the existence and multiplicity of solutions has been treated more recently. In the case of Euler-Lagrange equations, several results are now available, starting from [**17, 7, 1**]. Let us mention, in particular, the monograph [**16**] and references therein. On the contrary, the existence of solutions in the general case has been treated in few papers (see [**3, 4, 2**]). In particular, to our knowledge, a degree theory for this class of problems has not been developed so far.

Our purpose is to show how the degree theory for maps of class $(S)_+$ can be used, by a suitable approximation technique, to define the degree in the presence of natural growth conditions. We plan to apply this tool in a subsequent paper.

We also consider the case with controllable growth conditions. As in [**8**] for the p-Laplace operator, we show in Theorem 3.4 that also the case with critical growth gives rise locally to an operator of class $(S)_+$.

2. Topological degree in reflexive Banach spaces

Let X be a finite dimensional normed space, U a bounded open subset of X and $F : \overline{U} \to X$ a continuous map. For every $w \in X \setminus F(\partial U)$, one can define the topological degree $\deg(F, U, w) \in \mathbb{Z}$ (see e.g. [**11, 12, 18**]).

Assume now that $F : \overline{U} \to X'$ is a continuous map, let $(\cdot|\cdot)$ be any scalar product in X and let $R : X \to X'$ be the homeomorphism defined as

$$\langle R(u), v\rangle = (v|u) \qquad \forall u, v \in X\,.$$

For every $\varphi \in X' \setminus F(\partial U)$, the integer $\deg(R^{-1} \circ F, U, R^{-1}\varphi)$ turns out to be independent of the scalar product. This is, by definition, the degree $\deg(F, U, \varphi)$.

Finally, according to [**6, 15**], let X be a reflexive real Banach space.

Definition 2.1. A map $F : D \to X'$, with $D \subseteq X$, is said to be *of class* $(S)_+$ if, for every sequence (u_k) in D weakly convergent to some u in X with

$$\limsup_k \langle F(u_k), u_k - u\rangle \leq 0\,,$$

it holds $\|u_k - u\| \to 0$.

More generally, if M is a metrizable topological space, a map $H : D \to X'$, with $D \subseteq X \times M$, is said to be *of class* $(S)_+$ if, for every sequence (u_k, μ_k) in D with (u_k) weakly convergent to u in X, (μ_k) convergent to μ in M and

$$\limsup_k \langle H_{\mu_k}(u_k), u_k - u\rangle \leq 0\,,$$

it holds $\|u_k - u\| \to 0$ (we write $H_\mu(u)$ instead of $H(u,\mu)$).

In the following of the section, U will denote an open and bounded subset of X, $F : \overline{U} \to X'$ a continuous map of class $(S)_+$ and φ an element of X'.

Given a linear subspace Y of X, we denote by $i_Y : Y \to X$ the inclusion map and by $i'_Y : X' \to Y'$ the dual map.

If $\varphi \notin F(\partial U)$, then there exists a finite dimensional linear subspace Y_0 of X such that:

(a) $i'_Y\varphi \notin (i'_Y \circ F \circ i_Y)(\partial_Y(U \cap Y))$ for every finite dimensional linear subspace Y of X with $Y_0 \subseteq Y$;

(b) for $Y_0 \subseteq Y$, the integer

$$\deg(i'_Y \circ F \circ i_Y, U \cap Y, i'_Y\varphi)$$

is independent of Y.

This is, by definition, the degree

$$\deg(F, U, \varphi)\,.$$

Let us recall from [**6, 15**] some basic properties.

PROPOSITION 2.2. *If* $\varphi \notin F(\partial U)$, *then* $\deg(F, U, \varphi) = \deg(F - \varphi, U, 0)$.

THEOREM 2.3. *If* $\varphi \notin F(\overline{U})$, *then* $\deg(F, U, \varphi) = 0$.

THEOREM 2.4. *If* $0 \in U$ *and*

$$\langle F(u), u\rangle > 0 \qquad \textit{for any } u \in \partial U\,,$$

then $\deg(F, U, 0) = 1$.

THEOREM 2.5. *If* $\varphi \notin F(\partial U)$ *and* $U = U_1 \cup U_2$, *where* U_1, U_2 *are two disjoint open subsets of* X, *then*

$$\deg(F, U, \varphi) = \deg(F, U_1, \varphi) + \deg(F, U_2, \varphi)\,.$$

THEOREM 2.6. *Let* V *be another open subset of* X *with* $V \subseteq U$ *and let* $\varphi \notin F(\overline{U} \setminus V)$. *Then* $\deg(F, U, \varphi) = \deg(F, V, \varphi)$.

THEOREM 2.7. *Let* $H : \overline{U} \times [0, 1] \to X'$ *be a continuous map of class* $(S)_+$ *and let* $\varphi \notin H(\partial U \times [0, 1])$.

Then $\deg(H_t, U, \varphi)$ *is independent of* $t \in [0, 1]$.

3. Quasilinear elliptic equations with controllable growth conditions

Let Ω be a bounded open subset of $\mathbb{R}^n$, let M be a metrizable topological space and let

$$a : \Omega \times (\mathbb{R} \times \mathbb{R}^n \times M) \to \mathbb{R}^n\,,$$

$$b : \Omega \times (\mathbb{R} \times \mathbb{R}^n \times M) \to \mathbb{R}$$

be two Carathéodory functions. We will denote by $\|\ \|_p$ the usual norm in L^p and write $a_\mu(x, s, \xi)$, $b_\mu(x, s, \xi)$ instead of $a(x, (s, \xi, \mu))$, $b(x, (s, \xi, \mu))$.

Assume that:

$(UC)_1$ *there exist* $p \in]1, n[$, $\alpha_0 \in L^{(p^*)'}(\Omega)$, $\alpha_1 \in L^{p'}(\Omega)$ *and* $\beta \geq 0$ *such that*

$$|a_\mu(x, s, \xi)| \leq \alpha_1(x) + \beta\,|s|^{\frac{p^*}{p'}} + \beta\,|\xi|^{p-1}\,,$$

$$|b_\mu(x, s, \xi)| \leq \alpha_0(x) + \beta\,|s|^{p^*-1} + \beta\,|\xi|^{\frac{p}{(p^*)'}}\,,$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, $\mu \in M$, *where* $p^* = \dfrac{np}{n-p}$.

It follows

$$\begin{cases} a_\mu(x,u,\nabla u) \in L^{p'}(\Omega) \\ b_\mu(x,u,\nabla u) \in L^{(p^*)'}(\Omega) \subseteq W^{-1,p'}(\Omega) \end{cases} \qquad \text{for any } \mu \in M \text{ and } u \in W_0^{1,p}(\Omega)$$

and one can define a continuous map $H : W_0^{1,p}(\Omega) \times M \to W^{-1,p'}(\Omega)$ by

$$H_\mu(u) = -\mathrm{div}\,[a_\mu(x,u,\nabla u)] + b_\mu(x,u,\nabla u)\,.$$

Assume also the monotonicity condition:

$(UC)_2$ *we have*

$$[a_\mu(x,s,\xi) - a_\mu(x,s,\eta)] \cdot (\xi - \eta) \geq 0$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^n$, $\mu \in M$.

Finally, fix $\vartheta \in C^1(\mathbb{R})$ such that

$$\begin{aligned} &\vartheta(s) = 1 && \text{for } s \leq 1\,, \\ &\vartheta(s) = 0 && \text{for } s \geq 2\,, \\ &0 \leq \vartheta(s) \leq 1 && \text{for any } s \in \mathbb{R}\,, \\ &-2 \leq \vartheta'(s) \leq 0 && \text{for any } s \in \mathbb{R}\,, \end{aligned}$$

and set, for any $h \in \mathbb{N}$ with $h \geq 1$ and $s \in \mathbb{R}$,

$$\mathcal{T}_h(s) = \vartheta\left(\frac{|s|}{h}\right) s\,, \qquad \mathcal{R}_h(s) = s - \mathcal{T}_h(s)\,.$$

It is easily seen that

$$|\mathcal{T}_h'(s)| \leq 5\,, \qquad |\mathcal{R}_h'(s)| \leq 5 \qquad \text{for any } h \text{ and } s\,. \tag{3.1}$$

Lemma 3.1. *Assume* $(UC)_1$ *and* $(UC)_2$. *Let* (u_k) *be a sequence weakly convergent to* u *in* $W_0^{1,p}(\Omega)$ *and* (μ_k) *a sequence convergent to* μ *in* M *such that*

$$\limsup_{k\to\infty} \langle H_{\mu_k}(u_k), u_k - u\rangle \leq 0\,.$$

Then

$$\liminf_{h\to\infty}\left(\liminf_{k\to\infty} \langle H_{\mu_k}(\mathcal{R}_h(u_k)), \mathcal{R}_h(u_k)\rangle\right) \leq 0\,.$$

Proof. By $(UC)_2$ and (3.1), we have

$$\begin{aligned} &a_{\mu_k}(x,u_k,\nabla u_k) \cdot \nabla(u_k - u) \\ &\quad = a_{\mu_k}(x,\mathcal{R}_h(u_k),\nabla\mathcal{R}_h(u_k)) \cdot \nabla(\mathcal{R}_h(u_k) - \mathcal{R}_h(u)) \\ &\qquad + [a_{\mu_k}(x,u_k,\nabla u_k) - a_{\mu_k}(x,\mathcal{R}_h(u_k),\nabla\mathcal{R}_h(u_k))] \cdot \nabla(\mathcal{R}_h(u_k) - \mathcal{R}_h(u)) \\ &\qquad + a_{\mu_k}(x,\mathcal{T}_h(u_k),\nabla\mathcal{T}_h(u_k)) \cdot \nabla(\mathcal{T}_h(u_k) - \mathcal{T}_h(u)) \\ &\qquad + [a_{\mu_k}(x,u_k,\nabla u_k) - a_{\mu_k}(x,\mathcal{T}_h(u_k),\nabla\mathcal{T}_h(u_k))] \cdot \nabla(\mathcal{T}_h(u_k) - \mathcal{T}_h(u)) \\ &\quad \geq a_{\mu_k}(x,\mathcal{R}_h(u_k),\nabla\mathcal{R}_h(u_k)) \cdot \nabla(\mathcal{R}_h(u_k) - \mathcal{R}_h(u)) \\ &\qquad - 5\,[\,|a_{\mu_k}(x,u_k,\nabla u_k)| + |a_{\mu_k}(x,\mathcal{R}_h(u_k),\nabla\mathcal{R}_h(u_k))|\,]\ \chi_{\{h<|u_k|<2h\}}\,|\nabla u_k| \\ &\qquad - [\,|a_{\mu_k}(x,u_k,\nabla u_k)| + |a_{\mu_k}(x,\mathcal{R}_h(u_k),\nabla\mathcal{R}_h(u_k))|\,]\ |\nabla\mathcal{R}_h(u)| \\ &\qquad + a_{\mu_k}(x,\mathcal{T}_h(u_k),\nabla\mathcal{T}_h(u)) \cdot \nabla(\mathcal{T}_h(u_k) - \mathcal{T}_h(u)) \\ &\qquad - 5\,[\,|a_{\mu_k}(x,u_k,\nabla u_k)| + |a_{\mu_k}(x,\mathcal{T}_h(u_k),\nabla\mathcal{T}_h(u_k))|\,]\ \chi_{\{h<|u_k|<2h\}}\,|\nabla u_k| \\ &\qquad - [\,|a_{\mu_k}(x,u_k,\nabla u_k)| + |a_{\mu_k}(x,\mathcal{T}_h(u_k),\nabla\mathcal{T}_h(u_k))|\,]\ \chi_{\{|u_k|>h\}}|\nabla\mathcal{T}_h(u)| \end{aligned}$$

and also

$$
\begin{aligned}
&b_{\mu_k}(x,u_k,\nabla u_k)\,(u_k-u)\\
&\qquad = b_{\mu_k}(x,\mathcal{R}_h(u_k),\nabla\mathcal{R}_h(u_k))\,(\mathcal{R}_h(u_k)-\mathcal{R}_h(u))\\
&\qquad\quad + [b_{\mu_k}(x,u_k,\nabla u_k)-b_{\mu_k}(x,\mathcal{R}_h(u_k),\nabla\mathcal{R}_h(u_k))]\,(\mathcal{R}_h(u_k)-\mathcal{R}_h(u))\\
&\qquad\quad + b_{\mu_k}(x,u_k,\nabla u_k)\,(\mathcal{T}_h(u_k)-\mathcal{T}_h(u))\\
&\qquad \geq b_{\mu_k}(x,\mathcal{R}_h(u_k),\nabla\mathcal{R}_h(u_k))\,(\mathcal{R}_h(u_k)-\mathcal{R}_h(u))\\
&\qquad\quad - [\,|b_{\mu_k}(x,u_k,\nabla u_k)|+|b_{\mu_k}(x,\mathcal{R}_h(u_k),\nabla\mathcal{R}_h(u_k))|\,]\;\chi_{\{h<|u_k|<2h\}}|u_k|\\
&\qquad\quad - [\,|b_{\mu_k}(x,u_k,\nabla u_k)|+|b_{\mu_k}(x,\mathcal{R}_h(u_k),\nabla\mathcal{R}_h(u_k))|\,]\;|\mathcal{R}_h(u)|\\
&\qquad\quad + b_{\mu_k}(x,u_k,\nabla u_k)\,(\mathcal{T}_h(u_k)-\mathcal{T}_h(u))\,.
\end{aligned}
$$

Since (u_k) is bounded in $W_0^{1,p}(\Omega)$, it holds

$$
\liminf_{h\to\infty}\left(\liminf_{k\to\infty}\int_{\{h<|u_k|<2h\}}\left(|\nabla u_k|^p+|u_k|^{p^*}\right)dx\right)=0
$$

(see e.g. [**10**, Lemma 2.6]). By $(UC)_1$ and (3.1) it follows

$$
\begin{aligned}
\liminf_{h\to\infty}\Bigg(\liminf_{k\to\infty}\int_{\{h<|u_k|<2h\}}&\Big\{5\big[\,2|a_{\mu_k}(x,u_k,\nabla u_k)|+|a_{\mu_k}(x,\mathcal{R}_h(u_k),\nabla\mathcal{R}_h(u_k))|\\
&\qquad + |a_{\mu_k}(x,\mathcal{T}_h(u_k),\nabla\mathcal{T}_h(u_k))|\,\big]\,|\nabla u_k|\\
&+\big[\,|b_{\mu_k}(x,u_k,\nabla u_k)|+|b_{\mu_k}(x,\mathcal{R}_h(u_k),\nabla\mathcal{R}_h(u_k))|\,\big]\,|u_k|\Big\}\;dx\Bigg)=0\,.
\end{aligned}
$$

It is also clear that

$$
\begin{aligned}
&\lim_{h\to\infty}\Bigg(\limsup_{k\to\infty}\int_\Omega\big[\,|a_{\mu_k}(x,u_k,\nabla u_k)|\\
&\qquad\qquad +|a_{\mu_k}(x,\mathcal{R}_h(u_k),\nabla\mathcal{R}_h(u_k))|\,\big]\,|\nabla\mathcal{R}_h(u)|\,dx\Bigg)=0\,,\\
&\lim_{h\to\infty}\Bigg(\limsup_{k\to\infty}\int_\Omega\big[\,|b_{\mu_k}(x,u_k,\nabla u_k)|\\
&\qquad\qquad +|b_{\mu_k}(x,\mathcal{R}_h(u_k),\nabla\mathcal{R}_h(u_k))|\,\big]\,|\mathcal{R}_h(u))|\,dx\Bigg)=0\,,\\
&\lim_{h\to\infty}\Bigg(\limsup_{k\to\infty}\int_{\{|u_k|>h\}}\big[\,|a_{\mu_k}(x,u_k,\nabla u_k)|\\
&\qquad\qquad +|a_{\mu_k}(x,\mathcal{T}_h(u_k),\nabla\mathcal{T}_h(u_k))|\,\big]\,|\nabla\mathcal{T}_h(u)|\,dx\Bigg)=0\,,
\end{aligned}
$$

since by Fatou's lemma

$$
\limsup_{k\to\infty}\int_\Omega\chi_{\{|u_k|\geq h\}}\,|\nabla\mathcal{T}_h(u)|^p\,dx\leq\int_\Omega\chi_{\{|u|\geq h\}}\,|\nabla\mathcal{T}_h(u)|^p\,dx\,.
$$

Finally, we have

$$
\begin{aligned}
&\lim_{k\to\infty}\ \|\mathcal{T}_h(u_k)-\mathcal{T}_h(u)\|_{p^*}=0\,,\\
&\lim_{k\to\infty}\ \|a_{\mu_k}(x,\mathcal{T}_h(u_k),\nabla\mathcal{T}_h(u))-a_\mu(x,\mathcal{T}_h(u),\nabla\mathcal{T}_h(u))\|_{p'}=0\,,\\
&\lim_{k\to\infty}\ \nabla\mathcal{T}_h(u_k)=\nabla\mathcal{T}_h(u)\qquad\text{weakly in } L^p(\Omega)\,,
\end{aligned}
$$

whence

$$\lim_{k\to\infty} \int_\Omega a_{\mu_k}(x, \mathcal{T}_h(u_k), \nabla \mathcal{T}_h(u)) \cdot \nabla(\mathcal{T}_h(u_k) - \mathcal{T}_h(u))\, dx = 0\,,$$

$$\lim_{k\to\infty} \int_\Omega b_{\mu_k}(x, u_k, \nabla u_k)\, (\mathcal{T}_h(u_k) - \mathcal{T}_h(u))\, dx = 0\,.$$

It follows

$$\liminf_{h\to\infty} \left(\liminf_{k\to\infty} \langle H_{\mu_k}(\mathcal{R}_h(u_k)), \mathcal{R}_h(u_k) - \mathcal{R}_h(u)\rangle \right) \leq 0\,.$$

On the other hand, we also have

$$\limsup_{h\to\infty} \left(\limsup_{k\to\infty} \langle H_{\mu_k}(\mathcal{R}_h(u_k)), \mathcal{R}_h(u)\rangle \right) \leq 0$$

and the assertion follows. □

We now consider two possible coercivity assumptions:

$(UC)_3$ (the critical case) *there exist* $\nu > 0$, $\gamma \in L^1(\Omega)$ *and* $\eta \geq 0$ *such that*

$$a_\mu(x, s, \xi) \cdot \xi + b_\mu(x, s, \xi)\, s \geq \nu |\xi|^p - \gamma(x) - \eta\, |s|^{p^*}$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, $\mu \in M$*;*

$(UC)_4$ (the subcritical case) *there exist* $\nu > 0$ *and, for every* $\varepsilon > 0$, $\gamma_\varepsilon \in L^1(\Omega)$ *such that*

$$a_\mu(x, s, \xi) \cdot \xi + b_\mu(x, s, \xi)\, s \geq \nu |\xi|^p - \gamma_\varepsilon(x) - \varepsilon\, |s|^{p^*}$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, $\mu \in M$*.*

Let us point out that assumption $(UC)_3$ allows to consider the critical case

$$a(x, s, \xi) = |\xi|^{p-2}\xi\,, \qquad b(x, s, \xi) = -|s|^{p^*-2}s\,.$$

Lemma 3.2. *Assume* $(UC)_1$–$(UC)_3$*. Then there exists* $r = r(n, p, \eta/\nu) > 0$ *such that, for every* $z \in L^{p^*}(\Omega)$*, every sequence* (u_k) *in*

$$\left\{ v \in W_0^{1,p}(\Omega) : \ \|v - z\|_{p^*} \leq r \right\}$$

weakly convergent to u *in* $W_0^{1,p}(\Omega)$ *and every sequence* (μ_k) *convergent to* μ *in* M *such that*

$$\limsup_{k\to\infty} \langle H_{\mu_k}(u_k), u_k - u\rangle \leq 0\,,$$

we have

$$\liminf_{h\to\infty} \left(\liminf_{k\to\infty} \|\nabla \mathcal{R}_h(u_k)\|_p \right) = 0\,, \qquad \lim_{k\to\infty} \|u_k - u\|_{p^*} = 0\,,$$

$$\lim_{k\to\infty} \|[a_{\mu_k}(x, u_k, \nabla u_k) - a_{\mu_k}(x, u_k, \nabla u)] \cdot \nabla(u_k - u)\|_1 = 0\,.$$

Proof. By $(UC)_3$ and (3.1), we have

$$\begin{aligned}
&\langle H_{\mu_k}(\mathcal{R}_h(u_k)), \mathcal{R}_h(u_k)\rangle \\
&\quad\geq \nu \int_\Omega |\nabla \mathcal{R}_h(u_k)|^p\,dx - \int_{\{|u_k|>h\}} \gamma\,dx - \eta \int_\Omega |\mathcal{R}_h(u_k)|^{p^*}\,dx \\
&\quad\geq \nu \|\nabla \mathcal{R}_h(u_k)\|_p^p - \int_{\{|u_k|>h\}} \gamma\,dx \\
&\qquad\qquad - 2^{p^*-1}\,\eta \|\mathcal{R}_h(u_k) - \mathcal{R}_h(z)\|_{p^*}^{p^*} - 2^{p^*-1}\,\eta \|\mathcal{R}_h(z)\|_{p^*}^{p^*} \\
&\quad\geq \nu \|\nabla \mathcal{R}_h(u_k)\|_p^p - \int_{\{|u_k|>h\}} \gamma\,dx - 2^{p^*-1}\,\eta \|\mathcal{R}_h(z)\|_{p^*}^{p^*} \\
&\qquad\qquad - 2^{p^*-1}\,5^{p^*-p}\,\eta \|u_k - z\|_{p^*}^{p^*-p}\,\|\mathcal{R}_h(u_k) - \mathcal{R}_h(z)\|_{p^*}^p \\
&\quad\geq \nu \|\nabla \mathcal{R}_h(u_k)\|_p^p - \int_{\{|u_k|>h\}} \gamma\,dx - 2^{p^*-1}\,\eta \|\mathcal{R}_h(z)\|_{p^*}^{p^*} \\
&\qquad - 2^{p^*+p-2}\,5^{p^*-p}\,\eta\, r^{p^*-p}\,\|\mathcal{R}_h(u_k)\|_{p^*}^p - 2^{p^*+p-2}\,5^{p^*-p}\,\eta\, r^{p^*-p}\,\|\mathcal{R}_h(z)\|_{p^*}^p\,.
\end{aligned}$$

If $S(n,p) > 0$ satisfies

$$\|\nabla v\|_p^p \geq S(n,p)\,\|v\|_{p^*}^p \qquad \text{for any } v \in W_0^{1,p}(\Omega)\,,$$

it follows

$$\begin{aligned}
&\left(\nu - \frac{2^{p^*+p-2}\,5^{p^*-p}}{S(n,p)}\,\eta\, r^{p^*-p}\right) \|\nabla \mathcal{R}_h(u_k)\|_p^p \leq \langle H_{\mu_k}(\mathcal{R}_h(u_k)), \mathcal{R}_h(u_k)\rangle \\
&\qquad + \int_{\{|u_k|>h\}} \gamma\,dx + 2^{p^*-1}\,\eta \|\mathcal{R}_h(z)\|_{p^*}^{p^*} + 2^{p^*+p-2}\,5^{p^*-p}\,\eta\, r^{p^*-p}\,\|\mathcal{R}_h(z)\|_{p^*}^p\,.
\end{aligned}$$

Since

$$\limsup_{k\to\infty} \int_{\{|u_k|\geq h\}} \gamma\,dx \leq \int_{\{|u|\geq h\}} \gamma\,dx\,,$$

from Lemma 3.1 we infer that

$$\liminf_{h\to\infty} \left(\liminf_{k\to\infty} \|\nabla \mathcal{R}_h(u_k)\|_p\right) = 0\,, \tag{3.2}$$

provided that $r = r(n,p,\eta/\nu) > 0$ satisfies

$$\frac{2^{p^*+p-2}\,5^{p^*-p}}{S(n,p)}\,\eta\, r^{p^*-p} < \nu\,.$$

Up to a subsequence, we may also assume that there exists

$$\lim_{k\to\infty} \|u_k - u\|_{p^*}$$

and along such a subsequence we still have

$$\limsup_{k\to\infty} \langle H_{\mu_k}(u_k), u_k - u\rangle \leq 0\,.$$

Since

$$\|u_k - u\|_{p^*} \leq \|\mathcal{T}_h(u_k) - \mathcal{T}_h(u)\|_{p^*} + \|\mathcal{R}_h(u_k)\|_{p^*} + \|\mathcal{R}_h(u)\|_{p^*}\,,$$

it follows

$$\lim_{k\to\infty} \|u_k - u\|_{p^*} \leq \left(\liminf_{k\to\infty} \|\mathcal{R}_h(u_k)\|_{p^*}\right) + \|\mathcal{R}_h(u)\|_{p^*}\,.$$

Passing to the lower limit as $h \to \infty$ and taking into account (3.2) and Sobolev's theorem, we conclude that

$$\lim_{k\to\infty} \|u_k - u\|_{p^*} = 0\,.$$

It follows

$$\lim_{k\to\infty} \int_\Omega b_{\mu_k}(x, u_k, \nabla u_k)(u_k - u)\, dx = 0\,,$$

hence

$$\limsup_{k\to\infty} \int_\Omega a_{\mu_k}(x, u_k, \nabla u_k) \cdot \nabla(u_k - u)\, dx \le 0\,.$$

We also have

$$\lim_{k\to\infty} \|a_{\mu_k}(x, u_k, \nabla u) - a_\mu(x, u, \nabla u)\|_{p'} = 0\,,$$

hence

$$\limsup_{k\to\infty} \int_\Omega [a_{\mu_k}(x, u_k, \nabla u_k) - a_{\mu_k}(x, u_k, \nabla u)] \cdot \nabla(u_k - u)\, dx \le 0\,.$$

Since

$$[a_{\mu_k}(x, u_k, \nabla u_k) - a_{\mu_k}(x, u_k, \nabla u)] \cdot \nabla(u_k - u) \ge 0 \qquad \text{a.e. in } \Omega\,,$$

we conclude that

$$\lim_{k\to\infty} \|[a_{\mu_k}(x, u_k, \nabla u_k) - a_{\mu_k}(x, u_k, \nabla u)] \cdot \nabla(u_k - u)\|_1 = 0$$

and the proof is complete. □

A stronger result holds in the "subcritical case".

Lemma 3.3. *Assume $(UC)_1$, $(UC)_2$ and $(UC)_4$. Let (u_k) be a sequence weakly convergent to u in $W_0^{1,p}(\Omega)$ and (μ_k) a sequence convergent to μ in M such that*

$$\limsup_{k\to\infty} \langle H_{\mu_k}(u_k), u_k - u\rangle \le 0\,.$$

Then

$$\lim_{h\to\infty} \left(\liminf_{k\to\infty} \|\nabla \mathcal{R}_h(u_k)\|_p \right) = 0\,, \qquad \lim_{k\to\infty} \|u_k - u\|_{p^*} = 0\,,$$

$$\lim_{k\to\infty} \|[a_{\mu_k}(x, u_k, \nabla u_k) - a_{\mu_k}(x, u_k, \nabla u)] \cdot \nabla(u_k - u)\|_1 = 0\,.$$

Proof. Up to minor variants, we can argue as in the proof of Lemma 3.2. □

Finally, assume now a strict monotonicity condition, namely that:

$(UC)_5$ *we have*

$$[a_\mu(x, s, \xi) - a_\mu(x, s, \eta)] \cdot (\xi - \eta) > 0$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^n$, $\mu \in M$, with $\xi \ne \eta$.

The next result is concerned with the "critical case".

Theorem 3.4. *Assume $(UC)_1$, $(UC)_3$ and $(UC)_5$. Then there exists $r = r(n, p, \eta/\nu) > 0$ such that, for every $z \in L^{p^*}(\Omega)$, the continuous map*

$$H : \left\{ v \in W_0^{1,p}(\Omega) :\ \|v - z\|_{p^*} \le r \right\} \times M \to W^{-1,p'}(\Omega)$$

is of class $(S)_+$.

Proof. Let $r = r(n, p, \eta/\nu) > 0$ be as in Lemma 3.2. Let (u_k) be a sequence in

$$\left\{v \in W_0^{1,p}(\Omega) : \ \|v - z\|_{p^*} \le r\right\}$$

weakly convergent to u in $W_0^{1,p}(\Omega)$ and (μ_k) a sequence convergent to μ in M such that

$$\limsup_{k\to\infty} \langle H_{\mu_k}(u_k), u_k - u\rangle \le 0\,.$$

Up to a subsequence, (u_k) is convergent to u a.e. in Ω and, by Lemma 3.2, we also have

$$\lim_{k\to\infty} \|u_k - u\|_{p^*} = 0\,,$$

$$\lim_{k\to\infty} [a_{\mu_k}(x, u_k, \nabla u_k) - a_{\mu_k}(x, u_k, \nabla u)] \cdot \nabla(u_k - u) = 0 \qquad \text{a.e. in } \Omega\,.$$

Taking into account $(UC)_5$, from [**9**, Lemma 6] we deduce that

$$\lim_{k\to\infty} \nabla u_k = \nabla u \qquad \text{a.e. in } \Omega\,.$$

It follows

$$\lim_{k\to\infty} a_{\mu_k}(x, u_k, \nabla u_k) = a_\mu(x, u, \nabla u) \qquad \text{weakly in } L^{p'}(\Omega)\,,$$

$$\lim_{k\to\infty} b_{\mu_k}(x, u_k, \nabla u_k) = b_\mu(x, u, \nabla u) \qquad \text{weakly in } L^{(p^*)'}(\Omega)\,,$$

hence

$$\lim_{k\to\infty} \int_\Omega a_{\mu_k}(x, u_k, \nabla u_k) \cdot \nabla u\, dx = \int_\Omega a_\mu(x, u, \nabla u) \cdot \nabla u\, dx\,,$$

$$\lim_{k\to\infty} \int_\Omega b_{\mu_k}(x, u_k, \nabla u_k)\, u\, dx = \int_\Omega b_\mu(x, u, \nabla u)\, u\, dx\,,$$

which yields

$$\limsup_{k\to\infty} \int_\Omega \Big(a_{\mu_k}(x, u_k, \nabla u_k) \cdot \nabla u_k + b_{\mu_k}(x, u_k, \nabla u_k)\, u_k\Big)\, dx \le \int_\Omega \Big(a_\mu(x, u, \nabla u) \cdot \nabla u + b_\mu(x, u, \nabla u)\, u\Big)\, dx\,.$$

Since $\|u_k - u\|_{p^*} \to 0$, we can apply the (generalized) Fatou lemma to the sequence

$$a_{\mu_k}(x, u_k, \nabla u_k) \cdot \nabla u_k + b_{\mu_k}(x, u_k, \nabla u_k)\, u_k - \nu|\nabla u_k|^p \ge -\gamma - \eta\, |u_k|^{p^*}\,,$$

obtaining

$$\limsup_{k\to\infty} \|\nabla u_k\|_p^p \le \|\nabla u\|_p^p\,.$$

We infer that

$$\lim_{k\to\infty} \|\nabla u_k - \nabla u\|_p = 0$$

and the assertion follows. □

Finally, in the "subcritical case" we have a stronger result.

Theorem 3.5. *Assume $(UC)_1$, $(UC)_4$ and $(UC)_5$. Then the continuous map $H : W_0^{1,p}(\Omega) \times M \to W^{-1,p'}(\Omega)$ is of class $(S)_+$.*

Proof. Taking into account Lemma 3.3, the argument is the same. □

4. Quasilinear elliptic equations with natural growth conditions

Again, let Ω be a bounded open subset of $\mathbb{R}^n$, let M be a metrizable topological space and let

$$a : \Omega \times (\mathbb{R} \times \mathbb{R}^n \times M) \to \mathbb{R}^n ,$$

$$b : \Omega \times (\mathbb{R} \times \mathbb{R}^n \times M) \to \mathbb{R}$$

be two Carathéodory functions.

Now assume that:

$(UN)_1$ *there exist* $p \in]1, n[$, $\alpha_0 \in L^1(\Omega)$, $\alpha_1 \in L^{p'}(\Omega)$ *and* $\beta \in \mathbb{R}$ *such that*

$$|a_\mu(x, s, \xi)| \leq \alpha_1(x) + \beta\, |s|^{\frac{p^*}{p'}} + \beta\, |\xi|^{p-1} ,$$

$$|b_\mu(x, s, \xi)| \leq \alpha_0(x) + \beta\, |s|^{p^*} + \beta\, |\xi|^p ,$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ *and* $\mu \in M$;

$(UN)_2$ *we have*

$$[a_\mu(x, s, \xi) - a_\mu(x, s, \eta)] \cdot (\xi - \eta) > 0$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^n$ *and* $\mu \in M$ *with* $\xi \neq \eta$;

$(UN)_3$ *there exist* $R, \nu > 0$ *and, for every* $\varepsilon > 0$, $\gamma_\varepsilon \in L^1(\Omega)$ *such that*

$$a_\mu(x, s, \xi) \cdot \xi \geq \nu |\xi|^p - \gamma_\varepsilon(x) - \varepsilon\, |s|^{p^*} ,$$

$$|s| \geq R \implies b_\mu(x, s, \xi)\, s \geq -\gamma_\varepsilon(x) - \varepsilon\, |s|^{p^*} - \varepsilon\, |\xi|^p ,$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ *and* $\mu \in M$.

Then the map

$$H_\mu(u) = -\mathrm{div}\, [a_\mu(x, u, \nabla u)] + b_\mu(x, u, \nabla u)$$

is well defined from $W_0^{1,p}(\Omega) \times M$ into $W^{-1,p'}(\Omega) + L^1(\Omega) \subseteq \mathcal{D}'(\Omega)$. In particular, for any $\mu \in M$ and $u \in W_0^{1,p}(\Omega)$, we will write $H_\mu(u) = 0$, namely

$$-\mathrm{div}\, [a_\mu(x, u, \nabla u)] + b_\mu(x, u, \nabla u) = 0 ,$$

meaning that

$$\int_\Omega \big(a_\mu(x, u, \nabla u) \cdot \nabla v + b_\mu(x, u, \nabla u)\, v\big)\, dx = 0 \qquad \text{for every } v \in C_c^\infty(\Omega) .$$

In the line of the Brezis-Browder theorem [**5**], we can automatically enlarge the class of test functions.

Proposition 4.1. *Let* $\mu \in M$ *and* $u \in W_0^{1,p}(\Omega)$ *be such that*

$$-\mathrm{div}\, [a_\mu(x, u, \nabla u)] + b_\mu(x, u, \nabla u) = 0 .$$

Then, for every $v \in W_0^{1,p}(\Omega)$ *with* $\big(b_\mu(x, u, \nabla u)\, v\big)^- \in L^1(\Omega)$, *we have*

$$b_\mu(x, u, \nabla u)\, v \in L^1(\Omega) ,$$

$$\int_\Omega \big(a_\mu(x, u, \nabla u) \cdot \nabla v + b_\mu(x, u, \nabla u)\, v\big)\, dx = 0 .$$

In particular, we have $b_\mu(x, u, \nabla u)\, u \in L^1(\Omega)$ *and*

$$\int_\Omega \big(a_\mu(x, u, \nabla u) \cdot \nabla u + b_\mu(x, u, \nabla u)\, u\big)\, dx = 0 .$$

PROOF. First of all, an easy density argument shows that we have

$$\int_\Omega \big(a_\mu(x,u,\nabla u)\cdot\nabla w + b_\mu(x,u,\nabla u)\, w\big)\,dx = 0 \qquad \text{for every } w \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)\,.$$

Since $\mathcal{T}_h(v) \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, it follows

$$\int_\Omega b_\mu(x,u,\nabla u)\,\mathcal{T}_h(v)\,dx = -\int_\Omega a_\mu(x,u,\nabla u)\cdot\nabla \mathcal{T}_h(v)\,dx$$

with

$$b_\mu(x,u,\nabla u)\,\mathcal{T}_h(v) \geq -\big(b_\mu(x,u,\nabla u)\,v\big)^-\,.$$

From Fatou's lemma we infer that

$$\int_\Omega b_\mu(x,u,\nabla u)\,v\,dx \leq -\int_\Omega a_\mu(x,u,\nabla u)\cdot\nabla v\,dx\,,$$

whence $b_\mu(x,u,\nabla u)\,v \in L^1(\Omega)$. Since

$$\big|b_\mu(x,u,\nabla u)\,\mathcal{T}_h(v)\big| \leq \big|b_\mu(x,u,\nabla u)\,v\big|\,,$$

from Lebesgue's theorem now we infer that

$$\int_\Omega \big(a_\mu(x,u,\nabla u)\cdot\nabla v + b_\mu(x,u,\nabla u)\,v\big)\,dx = 0\,.$$

From $(UN)_1$ and $(UN)_3$ we deduce that $\big(b_\mu(x,u,\nabla u)\,u\big)^- \in L^1(\Omega)$, whence the second assertion. □

For any $h \in \mathbb{N}$ with $h \geq 1$ and $s \in \mathbb{R}$, we set as usual

$$T_h(s) = \min\{\max\{s,-h\},h\}\,.$$

Now we can prove the main result of the section.

THEOREM 4.2. *For every bounded and closed subset C of $W_0^{1,p}(\Omega)$, the set*

$$\{\mu \in M:\ -\mathrm{div}\,[a_\mu(x,u,\nabla u)] + b_\mu(x,u,\nabla u) = 0 \text{ for some } u \in C\}$$

is closed in M.

PROOF. Let (μ_k) be a sequence convergent to μ in M and (u_k) a sequence in C with

$$-\mathrm{div}\,[a_{\mu_k}(x,u_k,\nabla u_k)] + b_{\mu_k}(x,u_k,\nabla u_k) = 0\,.$$

Up to a subsequence, (u_k) is convergent to some u weakly in $W_0^{1,p}(\Omega)$ and a.e. in Ω.

By $(UN)_3$, for every $\varepsilon > 0$, there exists $\gamma_\varepsilon \in L^1(\Omega)$ such that

$$|s| \geq R \Longrightarrow a_\mu(x,s,\xi)\cdot\xi + b_\mu(x,s,\xi)\,s \geq \frac{\nu}{2}\,|\xi|^p - \gamma_\varepsilon(x) - \varepsilon\,|s|^{p^*}\,.$$

It follows, for every $h \geq R$,

$$
\begin{aligned}
a_\mu(x,s,\xi)\cdot(\mathcal{R}_h'(s)\xi) + b_\mu(x,s,\xi)\,\mathcal{R}_h(s) \\
&= \left[1-\vartheta\left(\frac{s}{h}\right)\right]\,[a_\mu(x,s,\xi)\cdot\xi + b_\mu(x,s,\xi)\,s] \\
&\qquad - a_\mu(x,s,\xi)\cdot\left\{\left[\vartheta'\left(\frac{s}{h}\right)\frac{s}{h}\right]\xi\right\} \\
&\geq \left[1-\vartheta\left(\frac{s}{h}\right)\right]\,\left(\frac{\nu}{2}\,|\xi|^p - \gamma_\varepsilon(x) - \varepsilon\,|s|^{p^*}\right) \\
&\qquad - 4\,\chi_{\{h<|t|<2h\}}(s)\,|a_\mu(x,s,\xi)|\,|\xi| \\
&= \frac{\nu}{2}\,|\mathcal{R}_h'(s)\xi|^p - \left[1-\vartheta\left(\frac{s}{h}\right)\right]\,\gamma_\varepsilon(x) - \varepsilon\,|\mathcal{R}_h(s)|^{p^*} \\
&\qquad + \frac{\nu}{2}\,\left\{\left[1-\vartheta\left(\frac{s}{h}\right)\right]\,|\xi|^p - |\mathcal{R}_h'(s)\xi|^p\right\} \\
&\qquad\qquad - \varepsilon\left\{\left[1-\vartheta\left(\frac{s}{h}\right)\right]\,|s|^{p^*} - |\mathcal{R}_h(s)|^{p^*}\right\} \\
&\qquad\qquad\qquad - 4\,\chi_{\{h<|t|<2h\}}(s)\,|a_\mu(x,s,\xi)|\,|\xi| \\
&\geq \frac{\nu}{2}\,|\mathcal{R}_h'(s)\xi|^p - \chi_{\{|t|>h\}}(s)\,\gamma_\varepsilon(x) - \varepsilon\,|\mathcal{R}_h(s)|^{p^*} \\
&\qquad - \chi_{\{h<|t|<2h\}}(s)\,\left[\frac{5^p}{2}\,\nu\,|\xi|^p + \varepsilon\,|s|^{p^*} + 4\,|a_\mu(x,s,\xi)|\,|\xi|\right].
\end{aligned}
$$

First of all, we infer that $\left(b_{\mu_k}(x,u_k,\nabla u_k)\,R_h(u_k)\right)^- \in L^1(\Omega)$, whence by Proposition 4.1

$$
\begin{aligned}
0 &= \int_\Omega \left(a_{\mu_k}(x,u_k,\nabla u_k)\cdot\nabla\mathcal{R}_h(u_k) + b_{\mu_k}(x,u_k,\nabla u_k)\,\mathcal{R}_h(u_k)\right)dx \\
&\geq \frac{\nu}{2}\,\|\nabla\mathcal{R}_h(u_k)\|_p^p - \int_{\{|u_k|>h\}}\gamma_\varepsilon\,dx - \varepsilon\,\|\mathcal{R}_h(u_k)\|_{p^*}^{p^*} \\
&\qquad - \int_{\{h<|u_k|<2h\}}\left[\frac{5^p}{2}\,\nu\,|\nabla u_k|^p + \varepsilon\,|u_k|^{p^*} + 4\,|a_{\mu_k}(x,u_k,\nabla u_k)|\,|\nabla u_k|\right]dx\,.
\end{aligned}
$$

As before, we have

$$
\liminf_{h\to\infty}\left(\liminf_{k\to\infty}\int_{\{h<|u_k|<2h\}}\left(|\nabla u_k|^p + |u_k|^{p^*}\right)dx\right) = 0\,,
$$

$$
\limsup_{k\to\infty}\int_{\{|u_k|\geq h\}}\gamma_\varepsilon\,dx \leq \int_{\{|u|\geq h\}}\gamma_\varepsilon\,dx\,.
$$

Taking into account $(UN)_1$, we deduce that

$$
\frac{\nu}{2}\,\liminf_{h\to\infty}\left(\liminf_{k\to\infty}\,\|\nabla\mathcal{R}_h(u_k)\|_p^p\right) \leq \varepsilon\,\sup_k\,\|u_k\|_{p^*}^{p^*}\,.
$$

By the arbitrariness of ε, we infer that

$$
\liminf_{h\to\infty}\left(\liminf_{k\to\infty}\,\|\nabla\mathcal{R}_h(u_k)\|_p\right) = 0\,, \tag{4.1}
$$

hence, as in the proof of Lemma 3.2, that

$$
\lim_{k\to\infty}\,\|u_k - u\|_{p^*} = 0\,.
$$

Let now $\varphi : \mathbb{R} \to \mathbb{R}$ be the solution of

$$\begin{cases} \varphi'(s) = 1 + \dfrac{\beta}{\nu}\,|\varphi(s)|\,, \\ \varphi(0) = 0\,, \end{cases}$$

where β is given in assumption $(UN)_1$. If we set $v_{h,k} = T_h(u_k) - T_h(u)$, then $\varphi(v_{h,k}) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$\int_\Omega \big(\varphi'(v_{h,k})\, a_{\mu_k}(x, u_k, \nabla u_k) \cdot \nabla v_{h,k} + b_{\mu_k}(x, u_k, \nabla u_k)\, \varphi(v_{h,k})\big)\, dx = 0\,. \tag{4.2}$$

It holds

(4.3)

$$\begin{aligned}
&\int_\Omega \varphi'(v_{h,k})\, a_{\mu_k}(x, u_k, \nabla u_k) \cdot \nabla v_{h,k}\, dx \\
&\quad = \int_\Omega \varphi'(v_{h,k})\, a_{\mu_k}(x, T_h(u_k), \nabla T_h(u_k)) \cdot \nabla (T_h(u_k) - T_h(u))\, dx \\
&\qquad + \int_\Omega \varphi'(v_{h,k}) \big(a_{\mu_k}(x, u_k, \nabla u_k) - a_{\mu_k}(x, T_h(u_k), \nabla T_h(u_k))\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \nabla (T_h(u_k) - T_h(u))\, dx \\
&\quad = \int_\Omega a_{\mu_k}(x, T_h(u_k), \nabla T_h(u_k)) \cdot \nabla (T_h(u_k) - T_h(u))\, dx \\
&\qquad - \int_\Omega \big(a_{\mu_k}(x, u_k, \nabla u_k) - a_{\mu_k}(x, T_h(u_k), \nabla T_h(u_k))\big) \cdot \big(\varphi'(v_{h,k})\, \nabla T_h(u)\big)\, dx \\
&\qquad - \frac{\beta}{\nu} \int_\Omega a_{\mu_k}(x, T_h(u_k), \nabla T_h(u_k)) \cdot \big(|\varphi(v_{h,k})|\, \nabla T_h(u)\big)\, dx \\
&\qquad + \frac{\beta}{\nu} \int_{\{|u_k| < h\}} |\varphi(v_{h,k})| a_{\mu_k}(x, u_k, \nabla u_k) \cdot \nabla u_k\, dx\,.
\end{aligned}$$

Now we have

$$\lim_{k\to\infty} \chi_{\{|u|<h\}} \big(a_{\mu_k}(x, u_k, \nabla u_k) - a_{\mu_k}(x, T_h(u_k), \nabla T_h(u_k))\big) = 0 \qquad \text{weakly in } L^{p'}(\Omega)\,,$$

as the sequence is bounded in $L^{p'}(\Omega)$ and goes to 0 a.e. in Ω. Since

$$\lim_{k\to\infty} \|\varphi'(v_{h,k})\, \nabla T_h(u) - \varphi'(0)\, \nabla T_h(u)\|_p = 0\,,$$

it follows

(4.4)

$$\lim_{k\to\infty} \int_\Omega \big(a_{\mu_k}(x, u_k, \nabla u_k) - a_{\mu_k}(x, T_h(u_k), \nabla T_h(u_k))\big) \cdot \big(\varphi'(v_{h,k})\, \nabla T_h(u)\big)\, dx = 0\,.$$

Moreover $(a_{\mu_k}(x, T_h(u_k), \nabla T_h(u_k)))$ is bounded in $L^{p'}(\Omega)$ and

$$\lim_{k\to\infty} \|\varphi(v_{h,k})\, \nabla T_h(u)\|_p = 0\,,$$

so that

$$\lim_{k\to\infty} \int_\Omega a_{\mu_k}(x, T_h(u_k), \nabla T_h(u_k)) \cdot \big(|\varphi(v_{h,k})|\, \nabla T_h(u)\big)\, dx = 0\,. \tag{4.5}$$

Where $|u_k| \geq h \geq R$ we also have $\dfrac{\varphi(v_{h,k})}{u_k} \geq 0$, hence

$$\begin{aligned} b_{\mu_k}(x, u_k, \nabla u_k)\, \varphi(v_{h,k}) &\geq -\frac{\varphi(v_{h,k})}{u_k} \left(\gamma_\varepsilon(x) - \varepsilon\, |u_k|^{p^*} - \varepsilon\, |\nabla u_k|^p\right) \\ &\geq -\frac{|\varphi(v_{h,k})|}{h}\, \gamma_\varepsilon(x) - \varepsilon\, \frac{|\varphi(v_{h,k})|}{h}\, |u_k|^{p^*} - \varepsilon\, \frac{\varphi(2h)}{h}\, |\nabla u_k|^p\,. \end{aligned}$$

Since $\|u_k - u\|_{p^*} \to 0$ and, by Lebesgue's theorem,

$$\lim_{k\to\infty} \int_\Omega \frac{|\varphi(v_{h,k})|}{h}\, \gamma_\varepsilon(x)\, dx = 0\,,$$

it follows

$$\liminf_{k\to\infty} \int_{\{|u_k|\geq h\}} b_{\mu_k}(x, u_k, \nabla u_k)\, \varphi(v_{h,k})\, dx \geq -\varepsilon\, \frac{\varphi(2h)}{h} \left(\sup_k\, \|\nabla u_k\|_p^p\right)\,.$$

From the arbitrariness of ε we infer that

$$\liminf_{k\to\infty} \int_{\{|u_k|\geq h\}} b_{\mu_k}(x, u_k, \nabla u_k)\, \varphi(v_{h,k})\, dx \geq 0 \qquad \text{for any } h \geq R\,. \tag{4.6}$$

Combining (4.2) with (4.3), (4.4), (4.5) and (4.6), we obtain

$$\begin{aligned} \limsup_{k\to\infty} \Bigg(\int_\Omega a_{\mu_k}(x, T_h(u_k), \nabla T_h(u_k)) \cdot \nabla(T_h(u_k) - T_h(u))\, dx \\ + \frac{\beta}{\nu} \int_{\{|u_k|<h\}} |\varphi(v_{h,k})| a_{\mu_k}(x, u_k, \nabla u_k) \cdot \nabla u_k\, dx \\ - \int_{\{|u_k|<h\}} |b_{\mu_k}(x, u_k, \nabla u_k)|\, |\varphi(v_{h,k})|\, dx \Bigg) \leq 0\,. \end{aligned}$$

By $(UN)_1$ and $(UN)_3$ we also have

$$\begin{aligned} &\frac{\beta}{\nu}\, a_{\mu_k}(x, u_k, \nabla u_k) \cdot \nabla u_k - |b_{\mu_k}(x, u_k, \nabla u_k)| \\ &\qquad \geq \beta\, |\nabla u_k|^p - \frac{\beta}{\nu}\, \gamma_\varepsilon - \frac{\varepsilon\beta}{\nu}\, |u_k|^{p^*} - \alpha_0 - \beta\, |u_k|^{p^*} - \beta\, |\nabla u_k|^p \\ &\qquad = -\frac{\beta}{\nu}\, \gamma_\varepsilon - \alpha_0 - \left(\frac{\varepsilon\beta}{\nu} + \beta\right) |u_k|^{p^*}\,, \end{aligned}$$

whence

$$\begin{aligned} \liminf_{k\to\infty} \Bigg(\frac{\beta}{\nu} \int_{\{|u_k|<h\}} |\varphi(v_{h,k})| a_{\mu_k}(x, u_k, \nabla u_k) \cdot \nabla u_k\, dx \\ - \int_{\{|u_k|<h\}} |b_{\mu_k}(x, u_k, \nabla u_k)|\, |\varphi(v_{h,k})|\, dx \Bigg) \geq 0\,. \end{aligned}$$

We infer that

$$\limsup_{k\to\infty} \int_\Omega a_{\mu_k}(x, T_h(u_k), \nabla T_h(u_k)) \cdot \nabla(T_h(u_k) - T_h(u))\, dx \leq 0\,,$$

hence that

$$\lim_{k\to\infty} \|\nabla T_h(u_k) - \nabla T_h(u)\|_p = 0$$

by Theorem 3.5. Since $|\nabla \mathcal{T}_h(u_k)| \leq 5|\nabla T_{2h}(u_k)|$ a.e. in Ω, we also have

$$\lim_{k\to\infty} \|\nabla \mathcal{T}_h(u_k) - \nabla \mathcal{T}_h(u)\|_p = 0\,.$$

Combining this fact with (4.1), we deduce that

$$\lim_{k\to\infty} \|\nabla u_k - \nabla u\|_p = 0\,.$$

In particular, we have $u \in C$ and

$$\int_\Omega \big(a_\mu(x,u,\nabla u)\cdot\nabla v + b_\mu(x,u,\nabla u)\,v\big)\,dx = 0 \qquad \text{for every } v \in W^{1,p}_0(\Omega)\cap L^\infty(\Omega)\,,$$

whence the assertion. □

5. Topological degree for quasilinear elliptic equations with natural growth conditions

Now let Ω be a bounded open subset of $\mathbb{R}^n$ and let

$$a : \Omega \times (\mathbb{R}\times\mathbb{R}^n) \to \mathbb{R}^n\,,$$

$$b : \Omega \times (\mathbb{R}\times\mathbb{R}^n) \to \mathbb{R}$$

be two Carathéodory functions such that

$(N)_1$ *there exist $p \in]1,n[$, $\alpha_0 \in L^1(\Omega)$, $\alpha_1 \in L^{p'}(\Omega)$ and $\beta \in \mathbb{R}$ such that*

$$|a(x,s,\xi)| \le \alpha_1(x) + \beta\,|s|^{\frac{p^*}{p'}} + \beta\,|\xi|^{p-1}\,,$$

$$|b(x,s,\xi)| \le \alpha_0(x) + \beta\,|s|^{p^*} + \beta\,|\xi|^{p}\,,$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$;

$(N)_2$ *we have*

$$[a(x,s,\xi) - a(x,s,\eta)]\cdot(\xi-\eta) > 0$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$ and $\xi,\eta \in \mathbb{R}^n$ with $\xi \ne \eta$;

$(N)_3$ *there exist $R,\nu > 0$ and, for every $\varepsilon > 0$, $\gamma_\varepsilon \in L^1(\Omega)$ such that*

$$a(x,s,\xi)\cdot\xi \ge \nu|\xi|^p - \gamma_\varepsilon(x) - \varepsilon\,|s|^{p^*}\,,$$

$$|s| \ge R \implies b(x,s,\xi)\,s \ge -\gamma_\varepsilon(x) - \varepsilon\,|s|^{p^*} - \varepsilon\,|\xi|^p\,,$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$.

Then the map

$$F(u) = -\mathrm{div}\,[a(x,u,\nabla u)] + b(x,u,\nabla u)$$

is well defined from $W^{1,p}_0(\Omega)$ into $W^{-1,p'}(\Omega) + L^1(\Omega) \subseteq \mathcal{D}'(\Omega)$.

We fix $\vartheta \in C^1(\mathbb{R})$ as in Section 3 and set, for any $\mu \in [0,1]$ and $s \in \mathbb{R}$,

$$\Theta_\mu(s) = \vartheta(\mu|s|)\,s\,.$$

We consider $M = [0,1]$ and set

$$a_\mu(x,s,\xi) = a(x,s,\xi)\,,$$
$$b_\mu(x,s,\xi) = \Theta_\mu\left(b(x,s,\xi)\right)\,.$$

It is easily seen that a_μ, b_μ satisfy $(UN)_1$–$(UN)_3$. Moreover, for every $\underline{\mu} \in]0,1[$, they satisfy $(UC)_1$, $(UC)_4$ and $(UC)_5$, if μ is restricted to $[\underline{\mu},1]$. In particular, we can define a continuous map $H : W^{1,p}_0(\Omega)\times]0,1] \to W^{-1,p'}(\Omega)$ by

$$H_\mu(u) = -\mathrm{div}\,[a_\mu(x,u,\nabla u)] + b_\mu(x,u,\nabla u)$$

and, by Theorem 3.5, this map is of class $(S)_+$.

Proposition 5.1. *Let U be an open and bounded subset of $W_0^{1,p}(\Omega)$ such that the equation*

$$-\mathrm{div}[a(x,u,\nabla u)] + b(x,u,\nabla u) = 0$$

has no solution $u \in \partial U$.

Then there exists $\overline{\mu} \in]0,1]$ such that

(a) *the equation $H_\mu(u) = 0$ has no solution $u \in \partial U$ for any $\mu \in]0,\overline{\mu}]$;*
(b) *the topological degree $\deg(H_\mu, U, 0)$ is constant for $\mu \in]0,\overline{\mu}]$.*

Proof. Since ∂U is closed and bounded in $W_0^{1,p}(\Omega)$, assertion (a) follows from Theorem 4.2. For any $\underline{\mu} \in]0,\overline{\mu}[$, we have that a_μ and b_μ satisfy $(UC)_1$, $(UC)_4$ and $(UC)_5$, if μ is restricted to $[\underline{\mu},\overline{\mu}]$. By Theorem 2.7, it follows that $\deg(H_\mu, U, 0)$ is constant for $\mu \in [\underline{\mu},\overline{\mu}]$, whence assertion (b). □

Definition 5.2. Let U be an open and bounded subset of $W_0^{1,p}(\Omega)$ such that the equation

$$-\mathrm{div}[a(x,u,\nabla u)] + b(x,u,\nabla u) = 0$$

has no solution $u \in \partial U$. We set

$$\deg(-\mathrm{div}[a(x,u,\nabla u)] + b(x,u,\nabla u), U, 0) = \deg(F,U,0) = \lim_{\mu\to 0} \deg(H_\mu, U, 0)\,.$$

Proposition 5.3. *Assume also that there exist $\tilde{\alpha}_0 \in L^{(p^*)'}(\Omega)$ and $\tilde{\beta} \geq 0$ such that*

$$|b(x,s,\xi)| \leq \tilde{\alpha}_0(x) + \tilde{\beta}\,|s|^{p^*-1} + \tilde{\beta}\,|\xi|^{\frac{p}{(p^*)'}}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$.

Then the map F is continuous and of class $(S)_+$ from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$. Moreover, if U is an open and bounded subset of $W_0^{1,p}(\Omega)$ such that the equation

$$-\mathrm{div}[a(x,u,\nabla u)] + b(x,u,\nabla u) = 0$$

has no solution $u \in \partial U$, then the degree of F, as a map of class $(S)_+$, agrees with that of Definition 5.2.

Proof. It is easily seen that this time a_μ and b_μ satisfy $(UC)_1$ and $(UC)_5$, for μ belonging to all $[0,1]$. It is also clear that a_μ and b_μ satisfy $(UC)_4$, for μ belonging to all $[0,1]$, provided that $|s| \geq R$. Otherwise, for every $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that

$$\begin{aligned} b(x,s,\xi)s &\geq -\tilde{\alpha}_0(x)\,|s| - \tilde{\beta}\,|s|^{p^*} - \tilde{\beta}|s|\,|\xi|^{\frac{p}{(p^*)'}} \\ &\geq -\tilde{\alpha}_0(x)\,|s| - \tilde{\beta}\,|s|^{p^*} - K_\varepsilon |s|^{p^*} - \varepsilon|\xi|^p \\ &\geq -R\,\tilde{\alpha}_0(x) - \tilde{\beta}\,R^{p^*} - K_\varepsilon R^{p^*} - \varepsilon|\xi|^p\,. \end{aligned}$$

Therefore $(UC)_4$ holds also for $|s| < R$.

Now the assertions follow from Theorems 3.5 and 2.7. □

Corollary 5.4. *Let $a(x,s,\xi) = \nu\,|\xi|^{p-2}\xi$ and $b(x,s,\xi) = 0$, so that*

$$-\mathrm{div}[a(x,u,\nabla u)] + b(x,u,\nabla u) = -\nu\,\Delta_p u\,.$$

Then, for every bounded and open subset U of $W_0^{1,p}(\Omega)$ with $0 \in U$, we have

$$\deg(-\nu\,\Delta_p u, U, 0) = 1\,.$$

Proof. It follows from Proposition 5.3 and Theorem 2.4. □

THEOREM 5.5. *Let U be an open and bounded subset of $W_0^{1,p}(\Omega)$ such that the equation*

$$-\mathrm{div}[a(x,u,\nabla u)] + b(x,u,\nabla u) = 0$$

has no solution $u \in \overline{U}$.

Then $\deg(F,U,0) = 0$.

PROOF. Since $\overline{U}$ is closed and bounded in $W_0^{1,p}(\Omega)$, by Theorem 4.2 there exists $\overline{\mu} \in]0,1]$ such that the equation $H_\mu(u) = 0$ has no solution $u \in \overline{U}$ for any $\mu \in]0,\overline{\mu}]$. By Theorem 2.3 it follows

$$\deg(F,U,0) = \deg(H_{\overline{\mu}},U,0) = 0\,.$$

□

Along the same line, also additivity and excision property can be proved taking advantage of Theorems 2.5, 2.6 and 4.2.

THEOREM 5.6. *Let U be an open and bounded subset of $W_0^{1,p}(\Omega)$ such that the equation*

$$-\mathrm{div}[a(x,u,\nabla u)] + b(x,u,\nabla u) = 0$$

has no solution $u \in \partial U$. Assume that $U = U_1 \cup U_2$, where U_1, U_2 are two disjoint open subsets of $W_0^{1,p}(\Omega)$.

Then $\deg(F,U,0) = \deg(F,U_1,0) + \deg(F,U_2,0)$.

THEOREM 5.7. *Let $V \subseteq U$ be two open and bounded subsets of $W_0^{1,p}(\Omega)$ such that the equation*

$$-\mathrm{div}[a(x,u,\nabla u)] + b(x,u,\nabla u) = 0$$

has no solution $u \in \overline{U} \setminus V$

Then $\deg(F,U,0) = \deg(F,V,0)$.

Let us see more in detail the homotopy invariance.

THEOREM 5.8. *Let*

$$a : \Omega \times (\mathbb{R} \times \mathbb{R}^n \times [0,1]) \to \mathbb{R}^n\,,$$

$$b : \Omega \times (\mathbb{R} \times \mathbb{R}^n \times [0,1]) \to \mathbb{R}$$

be two Carathéodory functions satisfying $(UN)_1$–$(UN)_3$ with respect to $M = [0,1]$ and let $H_t : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) + L^1(\Omega)$ be defined by

$$H_t(u) = -\mathrm{div}[a_t(x,u,\nabla u)] + b_t(x,u,\nabla u)\,.$$

Let U be an open and bounded subset of $W_0^{1,p}(\Omega)$ such that the equation

$$-\mathrm{div}[a_t(x,u,\nabla u)] + b_t(x,u,\nabla u) = 0$$

has no solution $u \in \partial U$, for any $t \in [0,1]$.

Then $\deg(H_t,U,0)$ is independent of $t \in [0,1]$.

PROOF. Consider

$$a_{t,\mu}(x,s,\xi) = a_t(x,s,\xi)\,,$$
$$b_{t,\mu}(x,s,\xi) = \Theta_\mu(b_t(x,s,\xi))\,,$$

for $(t,\mu) \in \widehat{M} = [0,1] \times [0,1]$, and define

$$H_{t,\mu}(u) = -\mathrm{div}[a_{t,\mu}(x,u,\nabla u)] + b_{t,\mu}(x,u,\nabla u)\,.$$

It is easily seen that $a_{t,\mu}$ and $b_{t,\mu}$ satisfy $(UN)_1$–$(UN)_3$ with respect to $\widehat{M}$. Since $[0,1]\times\{0\}$ is compact, by Theorem 4.2 there exists $\overline{\mu}\in]0,1]$ such that the equation

$$-\mathrm{div}[a_{t,\mu}(x,u,\nabla u)]+b_{t,\mu}(x,u,\nabla u)=0$$

has no solution $u\in\partial U$, for any $(t,\mu)\in[0,1]\times[0,\overline{\mu}]$.

For any $t,\tau\in[0,1]$, it follows

$$\begin{aligned}\deg(H_t,U,0)&=\deg(H_{t,0},U,0)=\deg(H_{t,\overline{\mu}},U,0)\\&=\deg(H_{\tau,\overline{\mu}},U,0)=\deg(H_{\tau,0},U,0)=\deg(H_\tau,U,0)\,.\end{aligned}$$

□

Let us point out that, by Theorem 5.8 and Proposition 5.3, the degree of F can now be calculated also by other approximation techniques, with respect to the one used in Definition 5.2.

THEOREM 5.9. *Let U be an open and bounded subset of $W^{1,p}_0(\Omega)$, with $0\in U$, such that*

$$\int_\Omega\big(a(x,u,\nabla u)\cdot\nabla u+b(x,u,\nabla u)\,u\big)\,dx>0$$
$$\text{for every } u\in\partial U \text{ with } b(x,u,\nabla u)\,u\in L^1(\Omega)\,.$$

Then the equation

$$-\mathrm{div}[a(x,u,\nabla u)]+b(x,u,\nabla u)=0$$

has no solution $u\in\partial U$ and $\deg(F,U,0)=1$.

In particular, there exists $u\in U$ such that

$$-\mathrm{div}[a(x,u,\nabla u)]+b(x,u,\nabla u)=0\,.$$

PROOF. If, by contradiction, there exists $u\in\partial U$ with

$$-\mathrm{div}[a(x,u,\nabla u)]+b(x,u,\nabla u)=0\,,$$

from Proposition 4.1 we deduce that $b(x,u,\nabla u)\,u\in L^1(\Omega)$ and

$$\int_\Omega\big(a(x,u,\nabla u)\cdot\nabla u+b(x,u,\nabla u)\,u\big)\,dx=0\,,$$

whence a contradiction.

If we set

$$a_t=(1-t)\,a(x,s,\xi)+t\,\nu\,|\xi|^{p-2}\xi\,,$$
$$b_t=(1-t)\,b(x,s,\xi)\,,$$

then a_t and b_t satisfy $(UN)_1$–$(UN)_3$ with respect to $M=[0,1]$ and we have

$$\begin{aligned}&\int_\Omega\big(a_t(x,u,\nabla u)\cdot\nabla u+b_t(x,u,\nabla u)\,u\big)\,dx\\&\qquad=(1-t)\int_\Omega\big(a(x,u,\nabla u)\cdot\nabla u+b(x,u,\nabla u)\,u\big)\,dx+t\nu\int_\Omega|\nabla u|^p\,dx>0\end{aligned}$$

for every $t\in[0,1]$ and $u\in\partial U$ with $b_t(x,u,\nabla u)\,u\in L^1(\Omega)$. Again from Proposition 4.1 we deduce that the equation

$$-\mathrm{div}[a_t(x,u,\nabla u)]+b_t(x,u,\nabla u)=0$$

has no solution $u \in \partial U$, for any $t \in [0,1]$. From Theorem 5.8 and Corollary 5.4 we infer that $\deg(F,U,0) = 1$.

The final assertion follows from Theorem 5.5. □

REMARK 5.10. If $\varphi \in W^{-1,p'}(\Omega)$, let $v \in W_0^{1,p}(\Omega)$ with $-\Delta_p v = \varphi$ and let $w = |\nabla v|^{p-2}\nabla v$. Then $w \in L^{p'}(\Omega)$, $-\mathrm{div}\, w = \varphi$ and

$$a_w(x,s,\xi) = a(x,s,\xi) - w(x)$$

still satisfies $(N)_1$–$(N)_3$. Therefore the equation

$$-\mathrm{div}[a(x,u,\nabla u)] + b(x,u,\nabla u) = \varphi$$

with $\varphi \in W^{-1,p'}(\Omega)$ can be easily reduced to the equation

$$-\mathrm{div}[a_w(x,u,\nabla u)] + b(x,u,\nabla u) = 0\,.$$

Finally, let us give an example of existence result, in the line of [**3**].

EXAMPLE 5.11. Assume also that there exist $\hat{\nu} > 0$ and, for every $\varepsilon > 0$, $\hat{\gamma}_\varepsilon \in L^1(\Omega)$ such that

$$a(x,s,\xi)\cdot\xi + b(x,s,\xi)\,s \geq \hat{\nu}\,|\xi|^p - \hat{\gamma}_\varepsilon(x) - \varepsilon\,|s|^p \tag{5.1}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$.

Then, for every $\varphi \in W^{-1,p'}(\Omega)$, there exists $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega \big(a(x,u,\nabla u)\cdot\nabla v + b(x,u,\nabla u)\,v\big)\,dx = \langle\varphi, v\rangle \qquad \text{for every } v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)\,.$$

PROOF. If we substitute $a(x,s,\xi)$ with $a_w(x,s,\xi) = a(x,s,\xi) - w(x)$ for some $w \in L^{p'}(\Omega)$, it is easily seen that (5.1) is still satisfied. By Remark 5.10 we may assume, without loss of generality, that $\varphi = 0$.

Because of (5.1), from Poincaré inequality we deduce that there exists $r > 0$ such that

$$\int_\Omega \big(a(x,u,\nabla u)\cdot\nabla u + b(x,u,\nabla u)\,u\big)\,dx > 0$$
$$\text{for every } u \in \partial B_r(0) \text{ with } b(x,u,\nabla u)\,u \in L^1(\Omega)\,.$$

From Theorem 5.9 and Proposition 4.1 the assertion follows. □

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