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Some results on the mathematical analysis of crack problems with forces applied on the fracture lips

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Declaration

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Introduction

This thesis is devoted to the study of some models of fracture growth in elastic materials, characterized by the presence of forces acting on the crack lips. The motivation is the following: there are lots of applications in which different kinds of surface forces, such as pressure or cohesive forces, affect the crack growth process. However, the majority of the mathematical results on fracture evolution achieved up to now deals with traction free cracks. Hence, the goal of this Ph.D. thesis is to discuss the role played by surface forces in the fracture evolution through the study of some model examples. The results presented in this work are contained in the papers [1, 2, 3, 4].

In the following discussions we focus our attention on *quasi-static rate-independent* processes. The term *quasi-static* means that we neglect all the inertial effects, so that, at every instant, the system is assumed to be at the equilibrium with the applied external loadings. This is a reasonable approximation when the data which drive the evolution, such as forces or prescribed boundary conditions, vary slowly in time. By *rate-independent* system we intend that, if the time-dependent data are rescaled by a strictly monotone increasing function, then the system reacts by rescaling the possible evolutions in the same way.

The starting point of our analysis is the famous Griffith's theory [43] of *brittle fracture* in elastic materials: the system presents a perfectly elastic behavior outside the cracked region and no force is transmitted across the cracks. The key assumption of Griffith's model is that the fracture growth is the result of the competition between the elastic energy released in the process of crack production and the energy dissipated in order to create a new portion of fracture. Therefore, the total energy of the system associated to a displacement u and a crack set Γ can be written, in its simplest form, as

$$\mathcal{E}(u,\Gamma) := \mathcal{E}^{el}(u,\Gamma) + \mathcal{K}(\Gamma)\,, \tag{1}$$

where $\mathcal{E}^{el}(u,\Gamma)$ and $\mathcal{K}(\Gamma)$ are the *stored elastic energy* (reversible) and the energy *dissipated* by the fracture growth, respectively. Usually, the term $\mathcal{K}(\Gamma)$ is supposed to be proportional to the crack length (or area in higher dimension), with a positive proportionality constant κ to which we refer as the *toughness* of the material. The last assumption on $\mathcal{K}$ is justified by the fact that, in the first model proposed by Griffith, no cohesive forces act between the fracture lips, i.e., once the crack is created the two faces of the fracture do not interact between each other. Therefore, the energy required to produce a new crack coincides with the energy spent to break the inter-atomic bonds, which are uniformly distributed in space.

Griffith's criterion of crack evolution is stated in terms of the *energy release rate*, i.e., the negative of the derivative of the elastic energy $\mathcal{E}^{el}$ with respect to the crack variation. Let us suppose that the elastic body is 2-dimensional and that the fracture set Γ evolves only along a prescribed smooth curve Λ. Then, the crack can be expressed as a function $\Gamma(s)$ of the arc-length s. As usual, the elastic energy at the equilibrium $\mathcal{E}^{el}_m(s)$ is defined by

$$\mathcal{E}^{el}_m(s) := \min\{\mathcal{E}^{el}(u, \Gamma(s)) : \ u \text{ admissible displacements}\},$$

and the energy release rate $\mathfrak{G}(s)$ by

$$\mathfrak{G}(s) := -\frac{\partial \mathcal{E}^{el}_m(s)}{\partial s}.$$

Assuming in (1) that $\mathcal{K}(\Gamma(s)) = \kappa s$, Griffith's criterion reads as follows: if $\mathfrak{G}(s)$ is less than the material toughness κ, then the crack is stable, otherwise it will grow. In formulas, a quasi-static evolution $t \mapsto \Gamma(s(t))$ has to satisfy:

(G1) *irreversibility*: $\dot{s}(t) \geq 0$;

(G2) *stability*: $\mathfrak{G}(s(t)) \leq \kappa$;

(G3) *activation*: $(\mathfrak{G}(s(t)) - \kappa)\dot{s}(t) = 0$,

where the dot denotes the derivative with respect to time. The Griffith's principle has been studied in several papers assuming at least $C^{1,1}$-regularity of the crack path (see, e.g., [46, 47, 48, 63, 66, 72] for the case of prescribed crack path, and [51, 52] for a more general setting in linearized antiplane elasticity).

Griffith's formulation (G1)-(G3) of the crack evolution problem can be interpreted as a *first order* condition: one seeks a solution among all the *critical points* of the energy (1). Since it is demanding to deal with stationarity in this context, Griffith's criterion typically works only in some particular situations, such as planar elasticity with prescribed and sufficiently smooth crack path. Indeed, in dimension 2 the fracture set can be parametrized by its length, while, in higher dimensions, such a strategy can not be easily generalized since the crack variation could be very "non-local", even in the case of prescribed path.

To deal with more general situations, in the late 90's Francfort & Marigo [38] proposed a new variational approach to the quasi-static crack evolution in brittle materials which shows, as a byproduct, that the problem of fracture growth fits in the general framework of rate-independent processes *à la* Mielke [56, 59]. Conditions (G1)-(G3) are indeed rephrased in a *derivative free* setting in the form of *global stability* and *energy-dissipation balance*:

(GS) at every instant of time t, a solution $\Gamma(t)$ has to minimize the total energy of the system (1) among all other admissible cracks $\Gamma \supseteq \Gamma(t)$;

(E) the increment of the elastic energy plus the energy dissipated by the crack production equals the work done by the external forces acting on the system.

In [38], the authors proposed also a *time-discretization* procedure to prove the existence of such an evolution: a continuous-time solution is approximated by discrete-time solutions obtained by solving incremental minimum problems. This technique is frequently used in the study of rate-independent processes [56, 59].

Francfort & Marigo's approach permits to overcome some restrictions of the Griffith's principle, such as being 2-dimensional with a prescribed path. Indeed, the energetic formulation (GS)-(E) is valid in any dimensions and allows the fracture set $\Gamma(t)$ to choose its way during the evolution process according to the variational principle of energy minimization: the solutions we look for are now *global minimizers* of the energy of the system, so that (GS) can be interpreted as a *zero order* condition. Therefore, the regularity of the crack set is not needed anymore, and the class of competitors for an evolution $t \mapsto \Gamma(t)$ may be very general (see, e.g., [22, 24, 37], where the admissible fractures are rectifiable sets with finite $\mathcal{H}^{n-1}$-measure).

On the other hand, it has to be noticed that the energetic formulation (GS)-(E) could produce unnatural discontinuities in the evolution. From a mathematical point of view, this is due to the fact that we want the minimality condition (GS) to be satisfied by an energy of the form (1) which is usually not convex with respect to the crack set variable. As a consequence, it could happen that a solution $t \mapsto \Gamma(t)$ of (GS)-(E) jumps instantaneously from a stable configuration to another one passing through an energetic barrier. On the contrary, Griffith's principle is a sort of differential condition on the fracture evolution, and thus is expected to produce a more physical solution, i.e., a solution which jumps later than a globally stable one. For this reason, the ongoing mathematical research on fracture mechanics still deals with weaker notions of Griffith's criterion obtained as limit of rate-dependent models (see, for instance, [46, 47, 48, 51, 52, 63, 66, 67, 72]).

Since Francfort & Marigo's variational approach was introduced, the mathematical community has given more and more attention to the investigation of various aspects of the mathematical model of brittle fracture. The key issue is, of course, the existence of quasi-static evolutions satisfying (GS)-(E). The first result in this direction was obtained by Dal Maso and Toader in [27] in antiplane linearized elasticity: the reference configuration is an open subset Ω of $\mathbb{R}^2$, the admissible cracks Γ are one dimensional closed sets with a finite number of connected components, and the displacements $u\colon \Omega \to \mathbb{R}$ are Sobolev functions in $\Omega \setminus \Gamma$. The energy they considered is of the form

$$\mathcal{E}(u, \Gamma) := \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla u|^2 \, \mathrm{d}x + \kappa \mathcal{H}^1(\Gamma) \, . \tag{2}$$

This existence result was then generalized by Chambolle [18] to the case of planar linearized elasticity, dealing with the energy

$$\mathcal{E}(u,\Gamma) := \frac{1}{2}\int_{\Omega\setminus\Gamma} \mathbb{C}\mathrm{E}u\cdot\mathrm{E}u\,\mathrm{d}x + \kappa\mathcal{H}^1(\Gamma)\,, \tag{3}$$

where $\mathbb{C}$ is the usual elasticity tensor and $\mathrm{E}u$ stands for the symmetric part of the gradient of the displacement $u\colon\Omega\to\mathbb{R}^2$.

Later on, Francfort and Larsen [37] presented a more "unified" formulation in the framework of SBV functions: working again in antiplane linearized elasticity with driving energy (2), the fracture Γ becomes a rectifiable set containing the discontinuity set S_u of the displacement $u\in SBV(\Omega)$. This new formulation allowed them to overcome the unphysical restrictions on the dimension (now $\Omega\subseteq\mathbb{R}^n$) and on the a priori bound on the number of connected components of the cracks used in [27].

In [22], Dal Maso, Francfort, and Toader generalized the above results, working with a total energy of the form

$$\mathcal{E}(u,\Gamma) := \int_\Omega W(x,\nabla u)\,\mathrm{d}x + \int_\Gamma \psi(x,\nu_\Gamma)\,\mathrm{d}\mathcal{H}^{n-1} + \text{external forces}, \tag{4}$$

where the density ψ of the energy dissipated by the crack production depends on the orientation of the crack set Γ through its unit normal vector ν_Γ (the usual Griffith's energy proportional to the measure of the fracture corresponds to $\psi(x,\nu)=\kappa|\nu|$), and the stored elastic energy density $W(x,\xi)$ is a quasi-convex function with a polynomial growth in ξ, uniformly with respect to x. Therefore, the bulk energy in (4) has a nonlinear dependence on the gradient of the displacement u, according to the rules of hyperelasticity. The functional setting they considered is the space of generalized special functions of bounded variation $GSBV$ which permits, with suitable modifications of the arguments developed in [37], to deal with vector valued displacements.

From the prototypical energies reported in formulas (2)-(4), we can deduce that most of the mathematical results obtained up to now holds only for traction free fractures: for instance, the equilibrium system resulting from (2) is

$$\begin{cases} \Delta u = 0 & \text{in } \Omega\setminus\Gamma,\\ \dfrac{\partial u}{\partial\nu_\Gamma} = 0 & \text{on } \Gamma,\\ + \text{ boundary conditions on } \partial\Omega. \end{cases}$$

The same holds for the energies (3) and (4) with suitable modifications. Our aim is to understand, through some applied models which are presented below, how the presence of surface forces applied on the crack lips may affect the evolution process. In Chapter 2 we are interested in quasi-static evolutions satisfying (GS)-(E) in the framework of hydraulic fracture. Chapter 3 is devoted to the application of Griffith's criterion (G1)-(G3) to a cohesive fracture model. Eventually, in Chapter 4 we deal with a static problem: given an elastic body $\Omega\subseteq\mathbb{R}^n$, we consider an energy of the form (4) and we study, from a "variational" point of view, the interaction between the energy dissipated by the crack production and the power spent by the surface forces applied on the boundary $\partial\Omega$ of the elastic system.

A more detailed presentation of these results is contained in the next paragraphs.

Hydraulic fracture

Hydraulic fracture studies the process of crack growth in rocks driven by the injection of high pressure fluids. This subject finds its main application in the extraction of natural gas or oil. In these cases, a fluid at high pressure is pumped into a preexisting fracture through a wellbore, causing the enlargement of the crack. A similar phenomenon has also been identified in epithelial tissues [54]. Here, an elastic body with initial cracks (a cell monolayer) is bonded and hydraulically connected to a poroelastic material, typically a hydrogel. The fracture growth is due to the motion of the solvent inside the poroelastic body: when the system is under tension or compression, the fluid experiences a change of pressure and is driven towards the existing cracks at cell-cell junctions.

Hydraulic fracture has been largely studied from an engeneering and numerical point of view, coupling the fluid equation, typically Reynolds' equation, and the elasticity system for the surrounding material (see for instance [42, 44, 53]). Particular attention has been given to the tip behavior of a fluid driven crack (see [29, 39]). Some numerical approaches (see, e.g., [19, 60, 61]) are inspired by Francfort & Marigo's variational model of brittle fracture [38] and characterized by the phase field approximation of the crack introduced by Ambrosio and Tortorelli [11].

In Chapter 2 we present a new energetic formulation (in dimension 2 and 3) of the problem of quasi-static evolution in hydraulic fracture, adapting Francfort & Marigo's mathematical model [38] to our purposes. Contrary to the results obtained in [19, 60, 61], the model presented in this work is built on the sharp-interface version originally developed in [38].

2-dimensional model. In Section 2.2 we study a 2-dimensional model of hydraulic fracture, starting from the key ideas of [54], in which the authors investigate the crack growth in epithelial tissues driven by the exchange of fluid between a cell monolayer (elastic) and the surrounding poroelastic material.

In order to describe such a phenomenon, we consider an unbounded linearly elastic body filling the whole $\mathbb{R}^2_+ := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ adhered and hydraulically connected to an infinite hydrogel substrate. The elastic part of the system is supposed to be homogeneous, isotropic, impermeable, and presents an initial crack Γ_0 starting from the origin.

According to [38], we do not assume to know a priori the crack path. Therefore, we are able to consider a sufficiently large class of admissible fractures, keeping some regularity properties: every crack has to be the graph of a $C^{1,1}$-function starting from the origin and with first and second derivatives uniformly bounded (see Definition 2.2.1 for further details and comments).

Let us briefly describe the physical behavior of the system. Given $T > 0$, for every $t \in [0,T]$ the system is supposed to be subject to a remote time-dependent strain field $\epsilon(t)\mathrm{I}$, where $\epsilon(t) \in \mathbb{R}$ and I denotes the identity matrix of order 2. Since the far strain $\epsilon(t)$ is stretching or compressing the whole system, a pressure gradient $\nabla p(t)$ is generated in the hydrogel, which drives the exchange of fluid volume $V(t)$ between the fracture and the poroelastic material according to Darcy's law $\dot{V}(t) = -\nabla p(t)$. Motivated by the small scale of the problem, we approximate the pressure gradient with the finite difference $(p_\infty(t) - p(t))/\ell$, where $p_\infty(t)$ is the fluid pressure generated by $\epsilon(t)$ far from the crack inlet, $p(t)$ is the pressure of the fluid inside the crack, and $\ell > 0$ is a length scale which, for simplicity, we will assume to be equal to 1. For technical reasons, we suppose $p_\infty, \epsilon \in C([0,T])$.

Let us present the energy which describes the elastic response of the body. The presence of the far strain field $\epsilon(t)\mathrm{I}$ is intended in the following way: the elastic body $\mathbb{R}^2_+$ has to accommodate for a displacement $u\colon \mathbb{R}^2_+ \to \mathbb{R}^2$ close to $\epsilon(t)\, id$ at infinity, where id stands for the identity map in $\mathbb{R}^2$. Equivalently, the strain field $\mathrm{E}u$ induced by u has to be close to $\epsilon(t)\mathrm{I}$ far from the origin. In our setting, we require $\mathrm{E}u - \epsilon(t)\mathrm{I}$ to be an L^2-function. This implies that the usual stored elastic energy

$$\frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}u \,\mathrm{d}x \tag{5}$$

can not be finite. In order to deal with a finite energy, (5) is replaced by the *renormalized* stored elastic energy

$$\mathcal{E}^{el}(u,\Gamma,\epsilon(t)) := \frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u - \epsilon(t)\mathrm{I})\cdot(\mathrm{E}u - \epsilon(t)\mathrm{I})\,\mathrm{d}x\,. \tag{6}$$

We refer to Proposition 2.2.3 for a rigorous derivation of (6). The crack evolution is governed by the following total energy, sum of the renormalized stored elastic energy and of the energy dissipated by the crack production:

$$\mathcal{E}(u,\Gamma,\epsilon(t)) := \mathcal{E}^{el}(u,\Gamma,\epsilon(t)) + \kappa\mathcal{H}^1(\Gamma)\,, \tag{7}$$

where κ is the toughness of the material.

We start by analyzing the static problem of a linearly elastic body filling $\mathbb{R}^2_+$, subject to a uniform strain field $\epsilon\mathrm{I}$, $\epsilon \in \mathbb{R}$, and with a fracture Γ filled by a volume $V \in [0,+\infty)$ of incompressible fluid. According to the variational principles of linear elasticity, the static problem is solved by minimizing the total energy (7) among a certain class of admissible displacements. We are able to show that a minimizer u solves the equilibrium system

$$\begin{cases} \operatorname{div}\mathbb{C}\mathrm{E}u = 0 & \text{in } \mathbb{R}^2_+\setminus\Gamma, \\ (\mathbb{C}\mathrm{E}u)\nu_\Gamma = -p\nu_\Gamma & \text{on } \Gamma, \end{cases}$$

where $p = p(\Gamma, V, \epsilon)$ is the pressure of the fluid inside the crack (see Section 2.2.1).

In this mathematical framework, a quasi-static evolution is described by two functions defined on the interval $[0,T]$: the fracture $t \mapsto \Gamma(t)$ and the volume $t \mapsto V(t)$ of the fluid inside the crack, to which corresponds a function $t \mapsto p(t)$ standing for the fluid pressure into the fracture. The notion of evolution (see Definition 2.2.17) is based on *global stability*, *energy-dissipation balance*, and the approximate *Darcy's law* $\dot{V}(t) = p_\infty(t) - p(t)$ for $t \in [0,T]$. The existence of such an evolution is shown in Theorem 2.2.18.

3-dimensional model. In Section 2.3 we discuss a 3-dimensional model for hydraulic fracture, focusing our attention on the main differences between 3D and 2D. We assume that the elastic body fills the whole space $\mathbb{R}^3$ and has an initial crack lying on a plane Λ passing through the origin. We allow the crack to grow only within Λ. For technical reasons, we will need some regularity of the relative boundary of the crack sets in Λ. This will be provided by the *interior ball property* (see Definition 2.3.1). In order to simplify the exposition, we assume that the far strain field $\epsilon(\cdot)$ is null and that the volume function $V(\cdot)$ is known, with $V \in AC([0,T];[0,+\infty))$, the space of absolutely continuous function from $[0,T]$ with values in $[0,+\infty)$. Also in this context, we prove the existence of a quasi-static evolution based on global stability and energy-dissipation balance (see Definition 2.3.3 and Theorem 2.3.4). We conclude Section 2.3 with an explicit example of quasi-static evolution in the particular case of circular fractures, the so-called *penny-shaped cracks*.

Quasi-static evolution via vanishing viscosity

In Chapter 3 we are interested in the application of the Griffith's criterion to a problem of quasi-static cohesive crack growth in the setting of planar linearized elasticity. We consider a linearly elastic body $\overline{\Omega}$, where $\Omega \subseteq \mathbb{R}^2$ is an open, bounded, connected set with Lipschitz boundary $\partial\Omega$. We assume that the crack can grow only along a prescribed simple $C^{2,1}$-curve $\Lambda \subseteq \overline{\Omega}$ with $\mathcal{H}^1(\Lambda) =: L$. We denote by $\lambda \in C^{2,1}([0,L];\Lambda)$ its arc-length parametrization and we consider admissible fractures of the form $\Gamma_s := \{\lambda(\sigma) : 0 \le \sigma \le s\}$ for $s \in [0,L]$, so that each crack Γ_s can be parametrized by its length s. We set also $\Omega_s := \Omega \setminus \Gamma_s$.

In this chapter we deal with *Barenblatt's cohesive model* of fracture (see, e.g., [13, 15]), whose main feature is, in contrast with Griffith's theory of brittle materials, the presence on the crack lips of the so-called cohesive forces, which describe a sort of residual interaction between the atoms of the material lying on the two faces of the evolving fracture. In the mathematical model, the density of the energy spent by the cohesive forces is represented by a function $\varphi\colon [0,+\infty) \to [0,+\infty)$ which depends, in its simplest form, only on the modulus of the jump of the displacement across Λ. Moreover, φ should be monotone increasing, concave, bounded by a constant $\mu > 0$, and such that

$$\varphi(0) = 0\,, \qquad \varphi'(0) < +\infty\,, \qquad \lim_{|\xi|\to+\infty} \varphi(|\xi|) = \mu\,.$$

We notice that, for our purposes, these further hypotheses on φ are not needed. Indeed, given $T>0$, we consider a C^1-function $\varphi\colon [0,T]\times\mathbb{R}^2\to\mathbb{R}$ such that $\varphi(t,0)=0$ and $\varphi(t,\xi)\le c\,(1+|\xi|^p)$ for some $c>0$ and some $p\in(1,+\infty)$. In particular, φ could be time dependent and negative. Thus, with the model we are going to present we are able to discuss also the case of an external time dependent force $h:[0,T]\to\mathbb{R}^2$ acting on both the fracture lips, namely $\varphi(t,\xi):=-h(t)\cdot\xi$.

Different from the Barenblatt's model, we assume, as in [17], that the energy spent by the cohesive forces is completely reversible. Moreover, as in (7) we add to the surface energy the dissipative term κs, that can be interpreted as an activation threshold, i.e., as the energy required to break the inter-atomic bonds along the fracture.

We stress that the coexistence of a cohesive term and of an activation threshold has been noticed in several papers related to fracture mechanics: in [32] in the approximation of fracture models via Γ-convergence of Ambrosio-Tortorelli type functionals, in [12, 28] in the study of the asymptotic behavior of composite materials through a homogenization procedure, and in [23, 45] in the framework of fracture models as Γ-limits of damage models.

Let us describe the main features of the evolution problem. Let $f\colon[0,T]\to L^2(\Omega;\mathbb{R}^2)$ and $w\colon[0,T]\to H^1(\Omega;\mathbb{R}^2)$ denote the volume forces and the Dirichlet boundary datum, respectively. For every $t\in[0,T]$, every $s\in[0,L]$, and every displacement $u\in H^1(\Omega_s;\mathbb{R}^2)$, the total energy of the system is given by

$$\mathcal{E}(t,s,u):=\frac{1}{2}\int_{\Omega_s}\mathbb{C}\mathrm{E}u\cdot\mathrm{E}u\,\mathrm{d}x-\int_{\Omega_s}f(t)\cdot u\,\mathrm{d}x+\int_{\Gamma_s}\varphi(t,[u])\,\mathrm{d}\mathcal{H}^1+s\,,$$

where $[u]$ denotes the jump of u across Λ.

For $t\in[0,T]$ and $s\in[0,L]$, we define the reduced energy:

$$\mathcal{E}_m(t,s):=\min\left\{\mathcal{E}(t,s,u):\,u\in H^1(\Omega_s,\mathbb{R}^2),\,u=w(t)\text{ on }\partial\Omega\right\}. \tag{8}$$

In order to give a definition of quasi-static evolution for our cohesive fracture model via Griffith's criterion (G1)-(G3), we first study the differentiability of $\mathcal{E}_m$ with respect to the crack length s. In Section 3.2 we show that, because of the non-convexity of $\varphi(t,\cdot)$, the solution to the minimum problem (8) is not unique and, as a consequence, the reduced energy $\mathcal{E}_m$ is not differentiable in s. However, we can still compute its right and left derivatives $\partial_s^+\mathcal{E}_m$ and $\partial_s^-\mathcal{E}_m$ (see Theorems 3.2.2 and 3.2.3). In particular, we are in a situation different from [47, 72], where the reduced energy is differentiable and has a continuous derivative, and similar in this aspect to [46, 48], where finite-strain elasticity in brittle fracture is considered.

In order to get a quasi-static evolution satisfying a weak version of the Griffith's principle, in Sections 3.3-3.6 we tackle the evolution problem by means of vanishing viscosity. This procedure has been studied for instance in [10, 30, 57, 58] in an abstract setting, and in [47, 48, 52, 72] for the problem of crack growth. It consists in the perturbation of minimum problems with a viscosity term driven by a small positive parameter ε, enforcing a local minimality of the solutions. Let us briefly discuss how we exploit this technique. Given a partition $\{t_i^k\}_{i=0}^k$ of the time interval $[0,T]$, we consider, for $i \geq 1$, the incremental minimum problem

$$\min \left\{ \mathcal{E}_m(t_i^k, s) + \frac{\varepsilon}{2} \frac{(s - s_\varepsilon^{k,i-1})^2}{t_i^k - t_{i-1}^k} \,:\, s \geq s_\varepsilon^{k,i-1} \right\}, \tag{9}$$

where $s_\varepsilon^{k,i-1}$ is a solution of (9) at time t_{i-1}^k and $s_\varepsilon^{k,0} := s_0$, the initial condition. In (9), we are penalizing the distance between the new and the previous cracks with the viscosity term driven by $\varepsilon > 0$. Having constructed the discrete time solutions for every $\varepsilon > 0$, the scheme is to pass to the limit as $k \to +\infty$, in order to find the so-called *viscous evolution* s_ε (Theorem 3.3.4), and, finally, let ε tend to zero. In this way, we obtain a quasi-static evolution for the cohesive fracture problem (Theorem 3.3.6) satisfying a weak notion of the Griffith's criterion stated in terms of left and right derivatives $\partial_s^\pm \mathcal{E}^m$ of the reduced energy (see Definition 3.3.5).

Finally, in Sections 3.7-3.8, we generalize the previous results to the case of many non-interacting cracks.

A free discontinuity functional with a boundary term

In the last chapter of this thesis we study a free discontinuity functional of the form

$$\mathcal{G}(u) := \int_\Omega W(x, \nabla u)\,\mathrm{d}x + \int_\Omega f(x,u)\,\mathrm{d}x + \int_{S_u} \psi(x, \nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial\Omega} g(x,u)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{10}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 1$, with Lipschitz boundary $\partial\Omega$, u belongs to $GSBV(\Omega;\mathbb{R}^m)$, S_u denotes the discontinuity set of u, ν_u is the approximate unit normal vector to S_u, and ∇u stands for the approximate gradient of u.

In the framework of fracture mechanics [22, 38] the functional (10) represents the energy of an elastic body Ω, with a crack S_u, subject to a displacement u and to external volume and surface forces whose potentials are given by f and g, respectively. In particular, W is the density of the stored elastic energy, while ψ stands for the energy per unit surface needed to extend the crack, as in (4).

As usual in elasticity, the equilibrium condition of such a body is expressed in terms of the minimum problem

$$\min\{\mathcal{G}(u) \,:\, u \in GSBV(\Omega;\mathbb{R}^m)\}. \tag{11}$$

To apply the direct method of the calculus of variations, we need to know the lower semicontinuity properties of $\mathcal{G}$.

The usual hypotheses on the first volume term of (10) (see, e.g., [22, Section 3]) are the ones given in Theorem 1.2.13, i.e., $W(x,\xi)$ is quasiconvex in ξ and satisfies a p-growth condition for some $p \in (1, +\infty)$. These assumptions on W imply that in (10) the approximate gradient ∇u is p-summable when $\mathcal{G}(u) < +\infty$, thus the domain of $\mathcal{G}$ is actually $GSBV^p(\Omega;\mathbb{R}^m)$. Moreover, they guarantee that

$$\mathcal{W}(u) := \int_\Omega W(x,\nabla u)\,\mathrm{d}x$$

is lower semicontinuous with respect to the weak convergence in $GSBV^p(\Omega;\mathbb{R}^m)$ (we refer to Definition 1.2.11 for this notion of weak convergence).

With mild hypotheses on f, such as continuity with respect to the second variable and a q-growth condition for some $q \in (1,+\infty)$, we may assume that the second volume integral in (10) is lower semicontinuous with respect to the same notion of convergence.

Therefore, to prove the existence of a solution to (11), we are led to study the lower semicontinuity of the surface part of (10), namely

$$\int_{S_u} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial\Omega} g(x,u)\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{12}$$

In this thesis, we consider a slightly more general free discontinuity functional of the form

$$\mathcal{F}(u) := \int_{S_u\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Sigma} g(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{13}$$

where Σ is an orientable Lipschitz manifold of dimension $n-1$ contained in $\overline{\Omega}$ with $\mathcal{H}^{n-1}(\Sigma) < +\infty$, $\mathcal{H}^{n-1}(\overline{\Sigma}\setminus\Sigma) = 0$, and $\mathcal{H}^{n-1}((\overline{\Sigma\cap\Omega})\cap\partial\Omega) = 0$, while u^+ and u^- are the traces of u on the positive and negative side of Σ (according to its orientation). To give a precise definition of $\mathcal{F}$ when $\Sigma\cap\partial\Omega \neq \emptyset$, the function u is extended to 0 out of Ω, so that u^+ and u^- are well defined $\mathcal{H}^{n-1}$-a.e. on Σ. The functional in (12) corresponds to the case $\Sigma = \partial\Omega$.

In Section 4.2 we prove that $\mathcal{F}$ is lower semicontinuous with respect to the weak convergence in $GSBV^p(\Omega;\mathbb{R}^m)$ under the following assumptions: ψ is a continuous function on $\overline{\Omega}\times\mathbb{R}^n$ such that

$$\begin{gathered}\psi(x,\cdot) \text{ is a norm on } \mathbb{R}^n \text{ for every } x\in\overline{\Omega}\,,\\ c_1|\nu| \le \psi(x,\nu) \le c_2|\nu| \text{ for every } (x,\nu)\in\overline{\Omega}\times\mathbb{R}^n\end{gathered} \tag{14}$$

for some $0 < c_1 \le c_2$, and g is a Borel function on $\Sigma\times\mathbb{R}^m\times\mathbb{R}^m$ satisfying

$$(s,t)\mapsto g(x,s,t) \text{ is lower semicontinuous on } \mathbb{R}^m\times\mathbb{R}^m \text{ for every } x\in\Sigma, \tag{15}$$

and, for $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma$ and every $s,s',t,t'\in\mathbb{R}^m$,

$$g(x,s,t) \le g(x,s',t) + \psi(x,\nu_\Sigma(x)) \quad\text{and}\quad g(x,s,t) \le g(x,s,t') + \psi(x,\nu_\Sigma(x))\,, \tag{16}$$

where $\nu_\Sigma(x)$ denotes the unit normal to Σ at x.

We notice that the hypotheses (14) on ψ are quite standard and guarantee that

$$\Psi(u) := \int_{S_u\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} \qquad \text{for } u\in GSBV^p(\Omega;\mathbb{R}^m)$$

is lower semicontinuous (see [7, 8]). The novelty of our result is the presence of an integral over a fixed surface Σ which is not lower semicontinuous on its own because of the lack of regularity of the function u near Σ. Indeed, we only know that the traces u^+ and u^- of u on the two sides of Σ are measurable functions, but we do not have any continuity or compactness property of the trace operator at our disposal, due to the presence of the jump set. As a matter of fact, it could happen that, along a sequence u_k converging to u weakly in $GSBV^p(\Omega;\mathbb{R}^m)$, the jump set S_{u_k} approaches Σ as $k \to +\infty$. In this case, we have no information on the convergence of the traces of u_k. Condition (16) will allow us to control the behavior of $\mathcal{F}$ along such sequences.

The proof of the lower semicontinuity theorem is divided into three steps. By the blow-up technique introduced in [14, 35, 36] we first prove that

$$\mathcal{F}(u) \leq \liminf_k \mathcal{F}(u_k) \tag{17}$$

whenever u_k converges to u pointwise and $u_k, u \in BV(\Omega;\mathrm{N})$ for some finite subset N of $\mathbb{R}^m$ (see Theorem 4.2.4). In Theorem 4.2.7 we extend (17) by approximation to functions belonging to $SBV^p(\Omega;\mathbb{R}^m)$. The third step is a truncation argument, which allows us to conclude in the general case $u \in GSBV^p(\Omega;\mathbb{R}^m)$. In Theorem 4.2.8 we show that condition (16) is also necessary for the lower semicontinuity of the functional $\mathcal{F}$ in $GSBV^p(\Omega;\mathbb{R}^m)$, provided that g is a Carathéodory function satisfying the following properties:

$$\begin{aligned}&\text{there exists } a \in L^1(\Sigma)^+ \text{ such that } g(x,s,t) \geq -a(x) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Sigma\\ &\text{and every } s,t \in \mathbb{R}^m,\end{aligned} \tag{18}$$

$$g(\cdot,s,t) \in L^1(\Sigma) \text{ for every } s,t \in \mathbb{R}^m.$$

We conclude Section 4.2 by proving that the minimum problem (11) admits a solution (Theorem 4.2.9).

Finally, in Section 4.3 we prove a relaxation result for a functional $\mathcal{F}$ of the form (13), i.e., we give an integral representation formula for $sc^-\mathcal{F}$, defined as the greatest sequentially weakly lower semicontinuous functional on $GSBV^p(\Omega;\mathbb{R}^m)$ which is less than or equal to $\mathcal{F}$. In (13) we still assume that ψ satisfies (14). As for g, instead of (15) and (16), we suppose that g is a Carathéodory function such that $g(x,\cdot,\cdot)$ is uniformly continuous on $\mathbb{R}^m \times \mathbb{R}^m$, (18) holds, and, for every $M > 0$, $g(x,s,t) \leq a_M(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ and every $s,t \in \mathbb{R}^m$ with $|s|$, $|t| \leq M$, where $a_M \in L^1(\Sigma)$.

In Theorem 4.3.3 we show that

$$sc^-\mathcal{F}(u) = \int_{S_u \setminus \Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g_{12}(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1},$$

where, for $(x,s,t) \in \Sigma \times \mathbb{R}^m \times \mathbb{R}^m$, we have set

$$g_{12}(x,s,t) := \min\Big\{g_1(x,s,t),\, \inf_{\tau\in\mathbb{R}^m} g_1(x,s,\tau) + \psi(x,\nu_\Sigma(x))\Big\},$$

$$g_1(x,s,t) := \min\Big\{g(x,s,t),\, \inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,t) + \psi(x,\nu_\Sigma(x))\Big\}.$$

In this theorem the uniform continuity of $g(x,\cdot,\cdot)$ is replaced by the weaker assumption of continuity of $g_{12}(x,\cdot,\cdot)$.

Therefore, the relaxed functional $sc^{-}\mathcal{F}$ is again of the form (13) and the density g_{12} on Σ is a Carathéodory function which satisfies properties (15) and (16). The mechanical interpretation of this result is that, if the potential g of the surface force is too strong, it is energetically more convenient to create a new crack near the surface Σ.

We conclude Chapter 4 with a relaxation result for the functional $\mathcal{G}$ introduced in (10). More precisely, we characterize the functional $sc^{-}\mathcal{G}$, defined this time as the greatest lower semicontinuous functional in $L^q(\Omega;\mathbb{R}^m)$ which is less than or equal to $\mathcal{G}$. Assuming that $W(x,\xi)$ is quasiconvex and has a p-growth with respect to ξ, and that $f(x,s)$ has a q-growth with respect to s, in Theorem 4.3.5 we prove that

$$sc^{-}\mathcal{G}(u)=\int_\Omega W(x,\nabla u)\mathrm{d}x+\int_\Omega f(x,u)\mathrm{d}x+\int_{S_u\setminus\Sigma}\psi(x,\nu_u)\mathrm{d}\mathcal{H}^{n-1}+\int_\Sigma g_{12}(x,u^+,u^-)\mathrm{d}\mathcal{H}^{n-1}$$

if $u\in GSBV^p(\Omega;\mathbb{R}^m)\cap L^q(\Omega;\mathbb{R}^m)$, and $sc^{-}\mathcal{G}(u)=+\infty$ otherwise in $L^q(\Omega;\mathbb{R}^m)$.

Chapter 1

Preliminaries and notation

1.1 Sets

For every set E, the symbol $\mathbf{1}_E$ stands for the characteristic function of E, i.e., the function defined by $\mathbf{1}_E(x) = 1$ for $x \in E$ and $\mathbf{1}_E(x) = 0$ for $x \notin E$. For every $\delta > 0$, we set

$$\mathcal{I}_\delta(E) := \{x \in \mathbb{R}^n : d(x, E) < \delta\}, \tag{1.1.1}$$

where $d(\cdot, E)$ is the usual distance function from the set E.

For every $r > 0$ and every $x \in \mathbb{R}^n$, we denote by $\mathrm{B}_r(x)$ the open ball of radius r and center x, and we set $\mathrm{B}_r^+(x) := \mathrm{B}_r(x) \cap \mathbb{R}^n_+$, where $\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$. When x is the origin, we use the shorter notation B_r and B_r^+.

For every $x \in \mathbb{R}^n$, every $\xi \in \mathbb{S}^{n-1}$, and every $\rho > 0$, on the hyperplane orthogonal to ξ and passing through the origin we denote by $\mathrm{Q}^{n-1}_{\rho,\xi}(x)$ an $(n-1)$-dimensional cube of side length ρ and centered in the projection $x - (x \cdot \xi)\xi$ of x onto that hyperplane. Given $C > 0$, we also define the n-dimensional rectangle centered in x by

$$\mathrm{R}^C_{\rho,\xi}(x) := \{y + t\xi : y \in \mathrm{Q}^{n-1}_{\rho,\xi}(x), \, |t - x \cdot \xi| < C\rho\}. \tag{1.1.2}$$

We denote by $\mathcal{K}$ the set of all compact subsets of $\mathbb{R}^n$. Given $K_1, K_2 \in \mathcal{K}$, the Hausdorff distance $d_H(K_1, K_2)$ between K_1 and K_2 is defined by

$$d_H(K_1, K_2) := \max\left\{\max_{x \in K_1} d(x, K_2), \max_{x \in K_2} d(x, K_1)\right\}.$$

We say that $K_h \to K$ in the Hausdorff metric if $d_H(K_h, K) \to 0$. We refer to [68] for the main properties of the Hausdorff metric. The following compactness theorem is well known (see, e.g., [68, Blaschke's Selection Theorem]).

Theorem 1.1.1. *Let K_h be a sequence in $\mathcal{K}$. Assume that there exists $H \in \mathcal{K}$ such that $K_h \subseteq H$ for every $h \in \mathbb{N}$. Then there exists $K \in \mathcal{K}$ such that, up to a subsequence, $K_h \to K$ in the Hausdorff metric.*

We say that a function $K\colon [0,T] \to \mathcal{K}$ is increasing if $K(s) \subseteq K(t)$ for every $0 \le s \le t \le T$. We recall two results concerning increasing set functions which can be found for instance in [27, Section 6].

Theorem 1.1.2. *Let $H \in \mathcal{K}$ and let $K\colon [0,T] \to \mathcal{K}$ be an increasing set function such that $K(t) \subseteq H$ for every $t \in [0,T]$. Let $K^-\colon (0,T] \to \mathcal{K}$ and $K^+\colon [0,T) \to \mathcal{K}$ be the functions defined by*

$$\begin{aligned} K^-(t) &:= \overline{\textstyle\bigcup_{s<t} K(s)} \quad \textit{for } 0 < t \le T, \\ K^+(t) &:= \textstyle\bigcap_{s<t} K(s) \quad \textit{fot } 0 \le t < T. \end{aligned}$$

Then

$$K^-(t) \subseteq K(t) \subseteq K^+(t) \quad \textit{for } 0 < t < T.$$

Let Θ be the set of points $t \in (0,T)$ such that $K^+(t) = K^-(t)$. Then $[0,T] \setminus \Theta$ is at most countable and $K(t_h) \to K(t)$ in the Hausdorff metric for every $t \in \Theta$ and every sequence t_h in $[0,T]$ converging to t.

Theorem 1.1.3. *Let K_h be a sequence of increasing set functions from $[0,T]$ to $\mathcal{K}$. Assume that there exists $H \in \mathcal{K}$ such that $K_h(t) \subseteq H$ for every $t \in [0,T]$ and every $h \in \mathbb{N}$. Then there exist a subsequence, still denoted by K_h, and an increasing set function $K\colon [0,T] \to \mathcal{K}$ such that $K_h(t) \to K(t)$ in the Hausdorff metric for every $t \in [0,T]$.*

Throughout the thesis, $\mathcal{L}^n$ and $\mathcal{H}^k$ stand for the Lebesgue and the k-dimensional Hausdorff measure in $\mathbb{R}^n$, respectively. For every $E \subseteq \mathbb{R}^n$, we denote by $\mathcal{H}^k \lfloor E$ the measure $\mathcal{H}^k$ restricted to E, which is defined by $\mathcal{H}^k \lfloor E\,(F) := \mathcal{H}^k(F \cap E)$ for every measurable set F.

A set $\Gamma \subseteq \mathbb{R}^n$ is said to be countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable if there exists a sequence Γ_j of $(n-1)$-dimensional C^1-manifolds such that $\Gamma = \bigcup \Gamma_j$ up to an $\mathcal{H}^{n-1}$-negligible set. It is well known that every countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set Γ admits an approximate unit normal vector $\nu_\Gamma(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ (see, for instance, [31, Sections 3.2.14-16]).

In Chapter 4 we will need the following definition.

Definition 1.1.4. A subset $\Sigma \subseteq \mathbb{R}^n$ is said to be a Lipschitz manifold of dimension $n-1$ with Lipschitz constant L if for every $x \in \Sigma$ there exist a vector $\xi(x) \in \mathbb{S}^{n-1}$, an $(n-1)$-dimensional rectangle Δ_x contained in the hyperplane orthogonal to $\xi(x)$ and passing through the origin, an interval I_x, and a Lipschitz function $\varphi_x\colon \Delta_x \to \mathrm{I}_x$ with Lipschitz constant L such that

$$\{y + t\xi(x) : y \in \Delta_x,\, t \in \mathrm{I}_x\} \cap \Sigma = \{y + \varphi_x(y)\xi(x) : y \in \Delta_x\}.$$

If Σ is a Lipschitz manifold with Lipschitz constant L, for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ there exists a unit normal vector $\nu_\Sigma(x)$. The tangent space to Σ at x is then

$$T_x(\Sigma) := \{y \in \mathbb{R}^n : y \cdot \nu_\Sigma(x) = 0\}. \tag{1.1.3}$$

Definition 1.1.5. An orientable Lipschitz manifold is a pair (Σ, ν_Σ), where Σ is a Lipschitz manifold of dimension $n-1$ and Lipschitz constant L and $\nu_\Sigma \colon \Sigma \to \mathbb{S}^{n-1}$ is a Borel vector field with the following properties:

- $\nu_\Sigma(x)$ is normal to Σ for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$;
- for every $x_0 \in \Sigma$ there exist $\xi(x_0)$, Δ_{x_0}, and I_{x_0} as in Definition 1.1.4 such that $\nu_\Sigma(x) \cdot \xi(x_0) > 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in \{y + t\xi(x_0) : y \in \Delta_{x_0},\, t \in \mathrm{I}_{x_0}\} \cap \Sigma$.

Every Lipschitz manifold Σ is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable (see, e.g., [9, Proposition 2.76]) and its approximate unit normal coincides $\mathcal{H}^{n-1}$-a.e. with the vector ν_Σ considered above.

If Ω is an open set in $\mathbb{R}^n$ with Lipschitz boundary, $\nu_\Omega(x)$ denotes the inner unit normal to Ω at x, which exists for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$. It is easy to see that $(\partial\Omega, \nu_\Omega)$ is an orientable Lipschitz manifold.

1.2 Function spaces

For every $m, n \in \mathbb{N}$, we denote by $\mathbb{M}^{m\times n}$ the space of $m \times n$ matrices with real coefficients. For every $\mathrm{F} \in \mathbb{M}^{m\times n}$, F_{ij} stands for the (i,j)-element of F. In the case $m = n$, we use the shorter notation $\mathbb{M}^n$ for $\mathbb{M}^{n\times n}$. The symbols $\mathbb{M}^n_{sym}$ and $\mathbb{M}^n_{skw}$ stand for the subspaces of $\mathbb{M}^n$ of symmetric and skew-symmetric matrices, respectively. For every $\mathrm{F} \in \mathbb{M}^n$, we denote by $\mathit{cof}\,\mathrm{F}$ the cofactor matrix of F. Finally, the scalar product between matrices is defined by

$$\mathrm{F} \cdot \mathrm{G} := tr(\mathrm{F}\mathrm{G}^T) \qquad \text{for every } \mathrm{F}, \mathrm{G} \in \mathbb{M}^n,$$

where the symbol tr stands for the trace of a matrix and G^T is the transpose matrix of G. Furthermore, we denote by I the identity matrix in $\mathbb{M}^n$.

For every $E \subseteq \mathbb{R}^n$ measurable and every $1 \le p < +\infty$, the space $L^p(E;\mathbb{R}^m)$ is defined as the set of functions $u \colon E \to \mathbb{R}^m$ measurable and p-integrable. For every function $u \in L^p(E;\mathbb{R}^m)$, u_i indicates the i-th component of u. As before, $L^p(E;\mathbb{M}^{m\times n})$ is the set of functions $u \colon E \to \mathbb{M}^{m\times n}$ measurable and p-integrable. In both cases, we denote by $\|\cdot\|_p$ or $\|\cdot\|_{p,E}$ the L^p-norm on E with respect to $\mathcal{L}^n$ or $\mathcal{H}^k$, according to the context.

For every open set $\Omega \subseteq \mathbb{R}^n$ and every $1 \le p < +\infty$, $W^{1,p}(\Omega;\mathbb{R}^m)$ is the set of functions $u \in L^p(\Omega;\mathbb{R}^m)$ whose gradient ∇u belongs to $L^p(\Omega;\mathbb{M}^{m\times n})$. The space $W^{1,p}(\Omega;\mathbb{R}^m)$ is a Banach space equipped with the norm $\|u\|_{W^{1,p}(\Omega)} := \|u\|_{p,\Omega} + \|\nabla u\|_{p,\Omega}$. In the case $p = 2$, the space $W^{1,2}(\Omega;\mathbb{R}^m)$ will be denoted by $H^1(\Omega;\mathbb{R}^m)$. In particular, $H^1(\Omega;\mathbb{R}^m)$ is a Hilbert space, and we denote its norm by $\|\cdot\|_{H^1(\Omega)}$. As usual, when the functions take values in $\mathbb{R}$ we will use the shorter notation $L^p(\Omega)$, $W^{1,p}(\Omega)$, and $H^1(\Omega)$.

We say that $u \in L^p_{loc}(\Omega;\mathbb{R}^m)$ (resp. $u \in W^{1,p}_{loc}(\Omega;\mathbb{R}^m)$) if $u \in L^p(\Omega';\mathbb{R}^m)$ (resp. $u \in W^{1,p}(\Omega';\mathbb{R}^m)$) for every $\Omega' \subset\subset \Omega$.

In the case $n = m = 2$, for every Ω open subset of $\mathbb{R}^2$ we define, as in [71] and [18],

$$LD^2(\Omega;\mathbb{R}^2) := \{u \in L^2_{loc}(\Omega;\mathbb{R}^2) : \mathrm{E}u \in L^2(\Omega;\mathbb{M}^2_{sym})\}\,, \tag{1.2.1}$$

where $\mathrm{E}u$ stands for the symmetric gradient of u, namely, $\mathrm{E}u = \frac{1}{2}(\nabla u + \nabla u^T)$. For every $i, j = 1, 2$, $\mathrm{E}_{ij}u$ stands for the (i, j)-component of $\mathrm{E}u$.

We recall the relationship between the spaces $LD^2(\Omega;\mathbb{R}^2)$ and $H^1(\Omega;\mathbb{R}^2)$.

Proposition 1.2.1. *Let Ω be a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary. Then $LD^2(\Omega;\mathbb{R}^2) = H^1(\Omega;\mathbb{R}^2)$. In particular, there exists a constant $C = C(\Omega)$ such that for every $u \in LD^2(\Omega;\mathbb{R}^2)$*

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x \le C\Big(\int_\Omega |u|^2 \,\mathrm{d}x + \int_\Omega |\mathrm{E}u|^2 \,\mathrm{d}x\Big)\,. \tag{1.2.2}$$

Moreover, if $E \subset\subset \Omega$ is open, $E \neq \emptyset$, then there exists $C' := C'(\Omega, E)$ such that

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x \le C' \int_\Omega |\mathrm{E}u|^2 \,\mathrm{d}x \tag{1.2.3}$$

for every $u \in LD^2(\Omega;\mathbb{R}^2)$ with

$$\int_E (\nabla u - \nabla u^T)\,\mathrm{d}x = 0\,.$$

Proof. See [33, Section 4] and [25, Appendix]. □

Since in the space $LD^2(\Omega;\mathbb{R}^2)$ we can control only the symmetric part of the gradient, we have that $\|\mathrm{E}u\|_{2,\Omega}$ is not a norm. Indeed, if we define

$$\mathcal{R} := \{v \colon \Omega \to \mathbb{R}^2 : v(x) = Ax + b \text{ with } b \in \mathbb{R}^2,\ A \in \mathbb{M}^2_{skw}\}\,,$$

the set of rigid motion in Ω, we have that $\mathcal{R} \subset LD^2(\Omega;\mathbb{R}^2)$ and $\|\mathrm{E}u\|_{2,\Omega} = 0$ for every $u \in \mathcal{R}$.

In Sections 2.2 and 2.2.3 we shall use the following subspace of $LD^2(\Omega;\mathbb{R}^2)$ on which $\|\mathrm{E}u\|_{2,\Omega}$ is a norm. Let Ω be an open subset of $\mathbb{R}^2_+$ such that $\mathcal{H}^1(\partial\Omega \cap \partial\mathbb{R}^2_+) > 0$. For every open set $E \subset\subset \Omega$ we define

$$LD^2_E(\Omega;\mathbb{R}^2) := \Big\{u \in LD^2(\Omega;\mathbb{R}^2) : \int_E u_1 \,\mathrm{d}x = 0 \text{ and } u_2 = 0 \text{ on } \partial\mathbb{R}^2_+\Big\}\,. \tag{1.2.4}$$

It is easy to see that $LD^2_E(\Omega;\mathbb{R}^2) \cap \mathcal{R} = \{0\}$.

In the following proposition, we prove that $\|\mathrm{E}u\|_{2,\Omega}$ is a norm on $LD^2_E(\Omega;\mathbb{R}^2)$.

Proposition 1.2.2. *Let Ω be an open bounded subset of $\mathbb{R}^2_+$ with Lipschitz boundary and let $E \subset\subset \Omega$ be open, $E \neq \emptyset$. Assume that $\mathcal{H}^1(\partial\Omega \cap \partial\mathbb{R}^2_+) > 0$. Then there exists $C = C(\Omega, E)$ such that*

$$\|u\|_{H^1(\Omega)} \le C\|\mathrm{E}u\|_{2,\Omega} \qquad \text{for every } u \in LD^2_E(\Omega;\mathbb{R}^2)\,. \tag{1.2.5}$$

Proof. By Proposition 1.2.1 we have that $LD^2_E(\Omega;\mathbb{R}^2) \subseteq H^1(\Omega;\mathbb{R}^2)$. To prove (1.2.5), in view of (1.2.2) it is enough to show that

$$\|u\|_{2,\Omega} \leq C\|\mathrm{E}u\|_{2,\Omega}$$

for some positive constant C.

Let us assume by contradiction that there exists a sequence u_k in $LD^2_E(\Omega;\mathbb{R}^2)$ such that $\|u_k\|_{2,\Omega} > k\|\mathrm{E}u_k\|_{2,\Omega}$. It is not restrictive to assume that $\|u_k\|_{2,\Omega} = 1$ for every k. From (1.2.2) we deduce that u_k is bounded in $H^1(\Omega;\mathbb{R}^2)$. Therefore there exists $u \in LD^2_E(\Omega;\mathbb{R}^2)$ such that, up to a subsequence, u_k converges to u weakly in $H^1(\Omega;\mathbb{R}^2)$ and strongly in $L^2(\Omega;\mathbb{R}^2)$. In particular $\|u\|_{2,\Omega} = 1$.

From the strong convergence of $\mathrm{E}u_n$ to 0 in $L^2(\Omega;\mathbb{M}^2_{sym})$, we deduce that $u \in \mathcal{R}$, and hence $u = 0$, which is a contradiction. □

Remark 1.2.3. Let Ω and E be as in Proposition 1.2.2. For every $\lambda > 0$ let us set $\Omega_\lambda := \lambda\Omega$ and $E_\lambda := \lambda E$. Then, for every $u \in LD^2_{E_\lambda}(\Omega_\lambda;\mathbb{R}^2)$ we have

$$\|u\|_{2,\Omega_\lambda} \leq C\lambda\|\mathrm{E}u\|_{2,\Omega_\lambda},$$

where $C = C(\Omega;E)$ is the constant found in (1.2.5).

As a straightforward consequence of Proposition 1.2.2 we have the following corollary.

Corollary 1.2.4. *Let Ω be an open subset of $\mathbb{R}^2_+$ with $\mathcal{H}^1(\partial\Omega\cap\partial\mathbb{R}^2_+) > 0$. Let $E \subset\subset \Omega$ be open, $E \neq \emptyset$. Then the space $LD^2_E(\Omega;\mathbb{R}^2)$ is a Hilbert space equipped with the norm $\|\mathrm{E}u\|_{2,\Omega}$.*

We now state a stability property of the Korn's inequality shown in Proposition 1.2.2.

Proposition 1.2.5. *Let Ω_k, Ω_∞ be bounded open subsets of $\mathbb{R}^2$ with Lipschitz boundaries. Assume that $\mathcal{H}^1(\partial\Omega_k \cap \partial\mathbb{R}^2_+) > 0$, $\mathcal{H}^1(\partial\Omega_\infty \cap \partial\mathbb{R}^2_+) > 0$, $\overline{\Omega}_k \to \overline{\Omega}_\infty$ in the Hausdorff metric and that $\partial\Omega_k$, $\partial\Omega_\infty$ have Lipschitz constant $L > 0$. Let, in addition, E be an open subset of $\bigcap \Omega_k$, $E \neq \emptyset$. Then, there exists $C = C(E)$ such that, for n sufficiently large,* (1.2.5) *holds for every $u \in LD^2_E(\Omega_k;\mathbb{R}^2)$.*

Proof. The proof can be carried out following the steps of [33, Theorem 4.2] using the results of Proposition 1.2.1. □

In the case $n = m = 3$, we will need (see Section 2.3) the following function space: for every open set $\Omega \subseteq \mathbb{R}^3$ we define, as in [55],

$$\mathrm{W}^1_{2,6}(\Omega;\mathbb{R}^3) := \{u \in L^6(\Omega;\mathbb{R}^3) : \nabla u \in L^2(\Omega;\mathbb{M}^3)\}$$

equipped with the norm

$$\|u\|_{\mathrm{W}^1_{2,6}(\Omega)} := \|u\|_{6,\Omega} + \|\nabla u\|_{2,\Omega}. \tag{1.2.6}$$

The choice of the exponent 6 is due to the fact that in dimension 3 the exponent 2^* in the Sobolev embedding theorem is equal to 6.

Proposition 1.2.6. *Let Σ be a plane in $\mathbb{R}^3$ and let $\Omega = \mathbb{R}^3$ or $\Omega = \mathbb{R}^3 \setminus \Sigma$. Then $\mathrm{W}^1_{2,6}(\Omega;\mathbb{R}^3)$ is a Banach space and the norms $\|\nabla u\|_{2,\Omega}$ and $\|\mathrm{E}u\|_{2,\Omega}$ are equivalent to the norm* (1.2.6)*, thus $\mathrm{W}^1_{2,6}(\Omega;\mathbb{R}^3)$ is a Hilbert space.*

Proof. When $\Omega = \mathbb{R}^3$ these results are proved in [55, Chapter 1.4], except for the equivalence of the norm (1.2.6) with $\|\mathrm{E}u\|_{2,\mathbb{R}^3}$, which is a consequence of Korn's inequality.

To prove the results for $\Omega = \mathbb{R}^3 \setminus \Sigma$, assume for simplicity that Σ is the plane $x_3 = 0$. Fix $u \in \mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma;\mathbb{R}^3)$. We have $u|_{\mathbb{R}^3_+} \in \mathrm{W}^1_{2,6}(\mathbb{R}^3_+;\mathbb{R}^3)$. Extending u by reflection with respect to Σ we obtain a function $\hat{u} \in \mathrm{W}^1_{2,6}(\mathbb{R}^3;\mathbb{R}^3)$. Hence, by the previous step,

$$\|u\|_{\mathrm{W}^1_{2,6}(\mathbb{R}^3_+)} \le \|\hat{u}\|_{\mathrm{W}^1_{2,6}(\mathbb{R}^3)} \le C\|\nabla\hat{u}\|_{2,\mathbb{R}^3} = 2C\|\nabla u\|_{2,\mathbb{R}^3_+}$$

By the same argument we obtain this estimate also for $u|_{\mathbb{R}^3_-}$.

The statement on $\|\mathrm{E}u\|_{2,\mathbb{R}^3\setminus\Sigma}$ can be obtained by Korn's inequality in a half-space. □

Let us now briefly present the function spaces used in Chapter 4. Given a bounded open subset Ω of $\mathbb{R}^n$, $\mathcal{B}(\Omega)$ denotes the set of Borel subsets of Ω and $\mathcal{M}_b(\Omega)$ stands for the set of bounded Radon measures on Ω. For every $\mu, \lambda \in \mathcal{M}_b(\Omega)$, we denote by $\mathrm{d}\mu/\mathrm{d}\lambda$ the Radon-Nikodym derivative of μ with respect to λ.

Let $n, m \in \mathbb{N}$. For every measurable function $u\colon \Omega \to \mathbb{R}^m$, we define the discontinuity set S_u of u as the set of $x \in \Omega$ such that u does not have an approximate limit at x (see [9, Section 4.5]).

The space $BV(\Omega;\mathbb{R}^m)$ of functions of bounded variation is the set of $u \in L^1(\Omega;\mathbb{R}^m)$ whose distributional gradient Du is a bounded Radon measure on Ω with values in the space $\mathbb{M}^{m\times n}$. Given $u \in BV(\Omega;\mathbb{R}^m)$, we can write $Du = D^a u + D^s u$, where $D^a u$ is absolutely continuous and $D^s u$ is singular with respect to $\mathcal{L}^n$. The function u is approximatively differentiable $\mathcal{L}^n$-a.e. in Ω and its approximate gradient ∇u belongs to $L^1(\Omega;\mathbb{M}^{m\times n})$ and coincides $\mathcal{L}^n$-a.e. in Ω with the density of $D^a u$ with respect to $\mathcal{L}^n$. Note that the discontinuity set S_u agrees with the complement of the set of Lebesgue points of u, up to an $\mathcal{H}^{n-1}$-negligible set. For all these notions we refer to [9, Sections 3.6 and 3.9].

The space $SBV(\Omega;\mathbb{R}^m)$ of special functions of bounded variation is defined as the set of all $u \in BV(\Omega;\mathbb{R}^m)$ such that $D^s u$ is concentrated on the discontinuity set S_u, i.e., $|D^s u|(\Omega \setminus S_u) = 0$.

As usual, $SBV_{loc}(\Omega;\mathbb{R}^m)$ denotes the space of functions which belong to $SBV(\Omega';\mathbb{R}^m)$ for every $\Omega' \subset\subset \Omega$.

For $p \in (1, +\infty)$, the space $SBV^p(\Omega;\mathbb{R}^m)$ is the set of functions $u \in SBV(\Omega;\mathbb{R}^m)$ with approximate gradient $\nabla u \in L^p(\Omega;\mathbb{M}^{m\times n})$ and $\mathcal{H}^{n-1}(S_u) < +\infty$. We now give the definition of weak convergence in $SBV^p(\Omega;\mathbb{R}^m)$.

Definition 1.2.7. Let $u_k, u \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$. The sequence u_k converges to u weakly in $SBV^p(\Omega;\mathbb{R}^m)$ if $u_k \to u$ pointwise $\mathcal{L}^n$-a.e. in Ω, $\nabla u_k \rightharpoonup \nabla u$

weakly in $L^p(\Omega;\mathbb{M}^{m\times n})$, and $\|u_k\|_\infty$ and $\mathcal{H}^{n-1}(S_{u_k})$ are uniformly bounded with respect to k.

The following compactness theorem is proved in [5].

Theorem 1.2.8. *Let $p\in(1,+\infty)$ and let u_k be a sequence in $SBV^p(\Omega;\mathbb{R}^m)$ such that $\|u_k\|_\infty$, $\|\nabla u_k\|_p$, and $\mathcal{H}^{n-1}(S_{u_k})$ are bounded uniformly with respect to k. Then there exists a subsequence which converges weakly in $SBV^p(\Omega;\mathbb{R}^m)$.*

This result is in general not enough for some applications since it requires an a priori bound on the L^∞-norm. To overcome this difficulty, we consider the larger space $GSBV(\Omega;\mathbb{R}^m)$ of generalized special functions of bounded variation, defined as the set of measurable functions $u\colon\Omega\to\mathbb{R}^m$ such that $\varphi(u)\in SBV_{loc}(\Omega;\mathbb{R}^m)$ for every $\varphi\in C^1(\mathbb{R}^m;\mathbb{R}^m)$ whose gradient has compact support. If $u\in GSBV(\Omega;\mathbb{R}^m)$, then the approximate gradient ∇u exists $\mathcal{L}^n$-a.e. in Ω and the jump set S_u is countably $(\mathcal{H}^{n-1},n-1)$-rectifiable. Its approximate unit normal vector is denoted by ν_u.

In the case $m=1$, we have that $u\in GSBV(\Omega)$ if and only if $T_h(u)\in SBV_{loc}(\Omega;\mathbb{R})$ for every $h\in\mathbb{N}$, where T_h is the truncation function defined by

$$T_h(s):=\min\{\max\{s,-h\},h\}\qquad\text{for } s\in\mathbb{R},$$

(see for instance [9, Section 4.5]).

For $p\in(1,+\infty)$, we define $GSBV^p(\Omega;\mathbb{R}^m)$ as the set of functions $u\in GSBV(\Omega;\mathbb{R}^m)$ such that $\nabla u\in L^p(\Omega;\mathbb{M}^{m\times n})$ and $\mathcal{H}^{n-1}(S_u)<+\infty$. In particular, if u belongs to $GSBV^p(\Omega;\mathbb{R}^m)$, then $\varphi(u)\in SBV^p(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$ for every $\varphi\in C^1(\mathbb{R}^m;\mathbb{R}^m)$ with $\operatorname{supp}(\nabla\varphi)\subset\subset\mathbb{R}^m$. We notice that $GSBV^p(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)=SBV^p(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$.

We now recall some basic properties of $GSBV^p(\Omega;\mathbb{R}^m)$, which can be found in [9, Section 4.5] and [22, Section 2].

Proposition 1.2.9. *$GSBV^p(\Omega;\mathbb{R}^m)$ is a vector space. A function $u\colon\Omega\to\mathbb{R}^m$ belongs to $GSBV^p(\Omega;\mathbb{R}^m)$ if and only if each component u_i belongs to $GSBV^p(\Omega;\mathbb{R})$.*

If Ω has a Lipschitz boundary, for every $u\in GSBV^p(\Omega;\mathbb{R}^m)$ there exists a function $\tilde u\colon\partial\Omega\to\mathbb{R}^m$ such that, for $\mathcal{H}^{n-1}$-a.e. $x\in\partial\Omega$, $\tilde u(x)$ is the approximate limit of u at x, and we write

$$\tilde u(x):=\operatorname*{ap\,lim}_{\substack{y\to x\\ y\in\Omega}}u(y)\tag{1.2.7}$$

(see, e.g., [31, Section 2.9.12]). The function $\tilde u$ is called the trace of u on $\partial\Omega$.

Remark 1.2.10. If (Σ,ν_Σ) is an orientable Lipschitz manifold of dimension $n-1$, with $\Sigma\subseteq\Omega$, for every $x\in\Sigma$ there exists an open neighborhood V of x contained in Ω such that $V\setminus\Sigma$ has two connected components V^+ and V^-, with Lipschitz boundaries and with $\nu_\Sigma(x)$ pointing towards V^+. For every function $u\in GSBV^p(\Omega;\mathbb{R}^m)$ the traces on $\Sigma\cap V$ of the restriction of u to $V^\pm$ are denoted by $u^\pm$. This allows us to define the traces $u^\pm$ of u $\mathcal{H}^{n-1}$-a.e. on Σ.

We now recall the notion of weak convergence in $GSBV^p(\Omega;\mathbb{R}^m)$.

Definition 1.2.11. Let $u_k, u \in GSBV^p(\Omega;\mathbb{R}^m)$. The sequence u_k converges to u weakly in $GSBV^p(\Omega;\mathbb{R}^m)$ if $u_k \to u$ pointwise $\mathcal{L}^n$-a.e. in Ω, $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(\Omega;\mathbb{M}^{m\times n})$, and $\mathcal{H}^{n-1}(S_{u_k})$ is uniformly bounded with respect to k.

The following compactness theorem has been proved in [6] (see also [9, Section 4.5]).

Theorem 1.2.12. *Let $p \in (1,+\infty)$ and let u_k be a sequence in $GSBV^p(\Omega;\mathbb{R}^m)$ such that $\|u_k\|_1$, $\|\nabla u_k\|_p$, and $\mathcal{H}^{n-1}(S_{u_k})$ are bounded uniformly with respect to k. Then there exists a subsequence which converges weakly in $GSBV^p(\Omega;\mathbb{R}^m)$.*

We recall a lower semicontinuity result in $GSBV^p(\Omega;\mathbb{R}^m)$, proved in [50, Theorem 1.2].

Theorem 1.2.13. *Let $W\colon \Omega\times\mathbb{M}^{m\times n}\to\mathbb{R}$ be a Carathéodory function such that*

$$W(x,\cdot) \text{ is quasiconvex for every } x\in\Omega, \tag{1.2.8}$$

$$a_1|\xi|^p - b_1(x) \le W(x,\xi) \le a_2|\xi|^p + b_2(x) \quad \text{for every } (x,\xi)\in\Omega\times\mathbb{M}^{m\times n} \tag{1.2.9}$$

for some $1<p<+\infty$, $0<a_1\le a_2$, and $b_1, b_2\in L^1(\Omega)$.

Then the functional $\mathcal{W}\colon GSBV^p(\Omega;\mathbb{R}^m)\to\mathbb{R}$ defined by

$$\mathcal{W}(u) := \int_\Omega W(x,\nabla u)\,\mathrm{d}x \tag{1.2.10}$$

is lower semicontinuous with respect to the weak convergence in $GSBV^p(\Omega;\mathbb{R}^m)$.

We conclude this preliminary section with a simple lemma on sets of finite perimeter which will be useful in Chapter 4.

We say that $E\subseteq\mathbb{R}^n$ is a set of finite perimeter if the distributional gradient of its characteristic function $\mathbf{1}_E$ is a bounded Radon measure on $\mathbb{R}^n$. The essential boundary ∂^*E of E is defined by

$$\partial^*E := \Big\{x\in\mathbb{R}^n:\ \limsup_{\rho\searrow 0}\frac{\mathcal{L}^n(\mathrm{B}_\rho(x)\cap E)}{\rho^n}>0 \quad\text{and}\quad \limsup_{\rho\searrow 0}\frac{\mathcal{L}^n(\mathrm{B}_\rho(x)\setminus E)}{\rho^n}>0\Big\}.$$

For $\mathcal{H}^{n-1}$-a.e. $x\in\partial^*E$, there exists the measure theoretical inner unit normal vector $\nu_E(x)$ to E at x. We refer to [9, Sections 3.3 and 3.5] for further properties of sets of finite perimeter.

Lemma 1.2.14. *Let Ω be an open set with Lipschitz boundary and let $E\subseteq\Omega$ be a set of finite perimeter. Let us set*

$$\mathrm{t}(E) := \{x\in\partial\Omega:\ \widetilde{\mathbf{1}_E}(x)=1\},$$

where $\widetilde{\mathbf{1}_E}$ is the trace on $\partial\Omega$ of the restriction of $\mathbf{1}_E$ to Ω. Then $\mathrm{t}(E)=\partial\Omega\cap\partial^*E$ up to an $\mathcal{H}^{n-1}$-negligible set.

Proof. We first notice that the trace of $\mathbf{1}_E$ on $\partial\Omega$ is either 1 or 0 for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$. Therefore, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega \setminus \mathrm{t}(E)$ we have that $\widetilde{\mathbf{1}_E}(x) = 0$, hence, by definition of trace,

$$\lim_{\rho \searrow 0} \frac{\mathcal{L}^n(\mathrm{B}_\rho(x) \cap E)}{\rho^n} = \lim_{\rho \searrow 0} \frac{1}{\rho^n} \int_{\mathrm{B}_\rho(x) \cap \Omega} \mathbf{1}_E(y) \, \mathrm{d}y = 0 \,. \tag{1.2.11}$$

This implies that $\partial\Omega \cap \partial^* E \subseteq \mathrm{t}(E)$ up to an $\mathcal{H}^{n-1}$-negligible set.

Viceversa, let $x \in \mathrm{t}(E)$ be such that the inner unit normal $\nu_\Omega(x)$ to Ω at x exists. As in (1.2.11), by the properties of the trace we have that

$$\lim_{\rho \searrow 0} \frac{\mathcal{L}^n(\mathrm{B}_\rho(x) \cap (\Omega \setminus E))}{\rho^n} = \lim_{\rho \searrow 0} \frac{1}{\rho^n} \int_{\mathrm{B}_\rho(x) \cap \Omega} |\mathbf{1}_E(y) - 1| \, \mathrm{d}y = 0 \,.$$

From the previous equality and the properties of $\nu_\Omega(x)$ we deduce that

$$\lim_{\rho \searrow 0} \frac{\mathcal{L}^n(\{y \in \mathrm{B}_\rho(x) \setminus E : \, (y - x) \cdot \nu_\Omega(x) > 0\})}{\rho^n} = 0 \,. \tag{1.2.12}$$

In view of (1.2.12) we obtain that

$$\limsup_{\rho \searrow 0} \frac{\mathcal{L}^n(\mathrm{B}_\rho(x) \cap E)}{\mathcal{L}^n(\mathrm{B}_\rho(x))} \geq \frac{1}{2} \,. \tag{1.2.13}$$

Moreover, since $E \subseteq \Omega$, by the properties of $\nu_\Omega(x)$ we get

$$\limsup_{\rho \searrow 0} \frac{\mathcal{L}^n(\mathrm{B}_\rho(x) \setminus E)}{\mathcal{L}^n(\mathrm{B}_\rho(x))} \geq \lim_{\rho \searrow 0} \frac{\mathcal{L}^n(\mathrm{B}_\rho(x) \setminus \Omega)}{\mathcal{L}^n(\mathrm{B}_\rho(x))} = \frac{1}{2} \,. \tag{1.2.14}$$

Inequalities (1.2.13) and (1.2.14) imply that $x \in \partial\Omega \cap \partial^* E$, and the proof is thus complete. $\square$

Chapter 2

Quasi-static evolution in hydraulic fracture

2.1 Overview of the chapter

In this chapter we present a variational formulation of the problem of quasi-static crack growth in hydraulic fracture based on the mathematical model of brittle fracture introduced in [38].

In Section 2.2, we study a 2-dimensional model of hydraulic fracture, starting from the key ideas of [54], where the authors investigate such a phenomenon in epithelial tissues. We refer to [54] for more details on the physical interpretation of the model (see also the Introduction). Here, we consider an unbounded linearly elastic body filling the whole $\mathbb{R}^2_+$ adhered and hydraulically connected to an infinite hydrogel substrate (poroelastic material). The elastic part of the system is supposed to be homogeneous, isotropic, impermeable, and presents an initial crack Γ_0 starting from the origin, while the fluid inside the hydrogel is assumed to be incompressible.

In dimension two, we are able to develop a model in which we do not assume to know a priori the crack path. However, for technical reasons we need to require some regularity of the fracture sets: in Definition 2.2.1 we define the set of admissible cracks $\mathcal{C}_\eta$ as the class of graphs of $C^{1,1}$-functions starting from the origin and with first and second derivatives uniformly bounded by a constant η. Hence, the family $\mathcal{C}_\eta$ depends on a positive parameter η which is fixed once and for all.

The evolution problem is driven by a remote strain field $\epsilon(t)\mathrm{I}$, $\epsilon(t) \in \mathbb{R}$, and by the pressure $p_\infty(t)$ of the fluid inside the hydrogel, far from the crack inlet. We assume $\epsilon(\cdot)$ and $p_\infty(\cdot)$ to be continuous functions from $[0,T]$, $T>0$, with values in $\mathbb{R}$.

The presence of the far strain field $\epsilon(t)\mathrm{I}$ is intended in the following way: the strain field $\mathrm{E}u$ associated to a displacement $u\colon \mathbb{R}^2_+ \to \mathbb{R}^2$ has to be close to $\epsilon(t)\mathrm{I}$ far from the origin. In our setting, we require $\mathrm{E}u - \epsilon(t)\mathrm{I}$ to be an L^2-function (see (2.2.4) and (2.2.18)). Therefore, the usual stored elastic energy

$$\frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}\mathrm{E}u\cdot\mathrm{E}u\,\mathrm{d}x \tag{2.1.1}$$

can not be finite. In Proposition 2.2.3 we rigorously prove that (2.1.1) has to be replaced by the renormalized stored elastic energy

$$\mathcal{E}^{el}(u,\Gamma,\epsilon(t)) := \frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u-\epsilon(t)\mathrm{I})\cdot(\mathrm{E}u-\epsilon(t)\mathrm{I})\,\mathrm{d}x\,. \tag{2.1.2}$$

According to the pioneering work by Griffith [43] and to the mathematical model developed in [38], given a displacement $u\colon\mathbb{R}^2_+\to\mathbb{R}^2$, a remote strain field $\epsilon(t)\mathrm{I}$, and a crack $\Gamma\in\mathcal{C}_\eta$, the total energy of the system is now of the form

$$\mathcal{E}(u,\Gamma,\epsilon(t)) := \mathcal{E}^{el}(u,\Gamma,\epsilon(t)) + \kappa\mathcal{H}^1(\Gamma)\,, \tag{2.1.3}$$

where κ is the toughness of the material.

In Section 2.2.1 we start by analyzing the static problem of a linearly elastic body filling $\mathbb{R}^2_+$, subject to a uniform strain field $\epsilon\mathrm{I}$, $\epsilon\in\mathbb{R}$, and with a fracture $\Gamma\in\mathcal{C}_\eta$ filled by a volume $V\in[0,+\infty)$ of incompressible fluid. According to the variational principles of linear elasticity, the static problem is solved by minimizing the total energy among a certain class of admissible displacements (see (2.2.18)). In Remarks 2.2.7 and 2.2.8 we determine the equilibrium system satisfied by a solution u of the static problem and make more precise the relation between the strain fields $\mathrm{E}u$ and $\epsilon\mathrm{I}$, showing that they are L^∞-close at infinity. Moreover, in Proposition 2.2.6 and Remarks 2.2.11 and 2.2.13 we determine the value of the pressure $p=p(\Gamma,V,\epsilon)$ of the fluid inside the crack.

In Definition 2.2.17 we define a quasi-static evolution for the hydraulic crack growth as a function $t\mapsto(\Gamma(t),V(t))$ from $[0,T]$ with values in $\mathcal{C}_\eta\times[0,+\infty)$ satisfying a global stability condition, an energy-dissipation balance, and the approximate Darcy's law

$$\dot V(t) = p_\infty(t)-p(t) \qquad \text{for } t\in[0,T]\,,$$

where $p(t):=p(\Gamma(t),V(t),\epsilon(t))$. The existence of such an evolution is proved in Theorem 2.2.5.

In Section 2.3 we briefly discuss a 3-dimensional model for hydraulic fracture, focusing our attention on the main differences between 3D and 2D. We assume that the elastic body fills the whole space $\mathbb{R}^3$ and has an initial crack lying on a plane Λ passing through the origin. We allow the crack to grow only within Λ. For technical reasons, we need some regularity of the relative boundary of the crack sets in Λ. This is provided by the *interior ball property* (see Definition 2.3.1). In order to simplify the exposition, we suppose that the far strain field $\epsilon(\cdot)$ is null and that the volume function $V(\cdot)$ is known, with $V\in AC([0,T];[0,+\infty))$, the space of absolutely continuous function from $[0,T]$ with values in $[0,+\infty)$. Also in this context, we prove the existence of a quasi-static evolution based on global stability and energy-dissipation balance (see Definition 2.3.3 and Theorem 2.3.4). We conclude Section 2.3 with an explicit example of quasi-static evolution in the particular case of circular fractures, the so-called *penny-shaped cracks*.

The results contained in this chapter have been presented in [2, 3].

2.2 2-dimensional model

We describe the mathematical framework we consider in our 2-dimensional model inspired by [54], to which we refer for more details on the physical interpretation.

To fix the simplest possible geometry, we consider a system made of an elastic body filling the whole $\mathbb{R}^2_+$ which is adhered to a poroelastic body occupying $\mathbb{R}^2 \setminus \mathbb{R}^2_+$. Throughout this section, we denote by Σ the set $\partial\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$ and by ν_Σ the unit vector $(0,1)$ normal to Σ.

As we have said in Section 2.1, we assume that the incompressible fluid inside the poroelastic material is subject to a pressure $p_\infty \in C([0,T])$ far from the crack inlet.

Let us concentrate on the main features of the elastic part of the system. We assume that it presents a regular enough initial crack Γ_0. More precisely, we suppose that there exists a $C^{1,1}$-function $\gamma_0 \colon [0, a_{\Gamma_0}] \to \mathbb{R}$, $a_{\Gamma_0} > 0$, defined on the x_2-axis and such that $\gamma_0(0) = 0$, $|\gamma_0'(0)| < +\infty$, and

$$\Gamma_0 = \mathrm{graph}(\gamma_0) = \{(\gamma_0(x_2), x_2) : x_2 \in [0, a_{\Gamma_0}]\}.$$

In particular, $\Gamma_0 \subseteq \overline{\mathbb{R}^2_+}$, $0 < \mathcal{H}^1(\Gamma_0) < +\infty$, $\Gamma_0 \cap \Sigma = \{(0,0)\}$, and $|\nu_{\Gamma_0} \cdot \nu_\Sigma| \neq 1$ at the origin, where ν_{Γ_0} denotes the unit normal to Γ_0 and the dot stands for the usual scalar product in $\mathbb{R}^2$. We refer to Remark 2.2.19 for further comments on Γ_0.

In our model, especially in the evolution problem studied in Section 2.2.2, we do not suppose to know a priori the crack path, which will be a result of an energy minimization procedure (see Definition 2.2.17), but we keep a technical regularity assumption on the fracture set, which is specified in the following definition of the class of admissible cracks.

Definition 2.2.1. Let $\eta > 0$. We define $\mathcal{C}_\eta$ to be the set of all closed curves Γ of class $C^{1,1}$ in $\overline{\mathbb{R}^2_+}$ such that the following properties hold:

(a) $\Gamma \supseteq \Gamma_0$ and $\Gamma \setminus \Gamma_0 \subset\subset \mathbb{R}^2_+$;

(b) there exist $a_\Gamma > 0$ and $\gamma \in C^{1,1}([0, a_\Gamma])$ such that $\|\gamma'\|_{\infty,[0,a_\Gamma]}, \|\gamma''\|_{\infty,[0,a_\Gamma]} \leq \eta$ and $\Gamma = \mathrm{graph}(\gamma) = \{(\gamma(x_2), x_2) : x_2 \in [0, a_\Gamma]\}$.

By definition of Γ_0, we can always find a sufficiently large η so that $\|\gamma_0'\|_{\infty,[0,a_{\Gamma_0}]} \leq \eta$ and $\|\gamma_0''\|_{\infty,[0,a_{\Gamma_0}]} \leq \eta$. Clearly, the requirements of Definition 2.2.1 ensure that for every $\Gamma \in \mathcal{C}_\eta$ there are no self-intersections. Moreover, for every $\Gamma \in \mathcal{C}_\eta$ it is convenient to fix an orientation and a unit normal vector ν_Γ to Γ.

We show a compactness property of the class $\mathcal{C}_\eta$ with respect to the Hausdorff convergence of sets.

Proposition 2.2.2. *Let Γ_k be a sequence in $\mathcal{C}_\eta$ such that $\mathcal{H}^1(\Gamma_k)$ is uniformly bounded with respect to k. Then there exists $\Gamma_\infty \in \mathcal{C}_\eta$ such that, up to a subsequence, $\Gamma_k \to \Gamma_\infty$ in the Hausdorff metric. Moreover, $\mathcal{H}^1(\Gamma_k) \to \mathcal{H}^1(\Gamma_\infty)$.*

Proof. Let $\Gamma_k \in \mathcal{C}_\eta$ be as in the statement of the proposition and let $a_{\Gamma_k} > 0$ and $\gamma_k \in C^{1,1}([0, a_{\Gamma_k}])$ be as in Definition 2.2.1. Since $\mathcal{H}^1(\Gamma_k)$ is bounded, we have that the sequence a_{Γ_k} is bounded in $\mathbb{R}$ and

$$\sup_k \|\gamma_k\|_{W^{2,\infty}([0,a_{\Gamma_k}])} < +\infty \,. \tag{2.2.1}$$

Therefore, we may assume that, up to a subsequence, $a_{\Gamma_k} \to a$. Moreover, we may rescale γ_k on the interval $[0, a]$ by

$$\tilde{\gamma}_k(x_2) := \gamma_k\Big(\frac{x_2 a_{\Gamma_k}}{a}\Big) \quad \text{for } x_2 \in [0, a]\,,$$

so that

$$\Gamma_k = \Big\{\Big(\tilde{\gamma}_k(x_2), \frac{x_2 a_{\Gamma_k}}{a}\Big) : \, x_2 \in [0, a]\Big\}\,.$$

By (2.2.1) we have that, up to a subsequence, $\tilde{\gamma}_k$ weakly*-converges in $W^{2,\infty}([0, a])$ to some γ. Let us set $\Gamma := \mathrm{graph}(\gamma)$. It is clear from the convergence of a_{Γ_k} to a and of $\tilde{\gamma}_k$ to γ that $\Gamma \in \mathcal{C}_\eta$ and that Γ_k converges to Γ in the Hausdorff metric. Moreover, since $\tilde{\gamma}'_k$ converges to γ' uniformly in the interval $[0, a]$, we get that

$$\lim_k \mathcal{H}^1(\Gamma_k) = \lim_k \int_0^a \sqrt{\Big(\frac{a_{\Gamma_k}}{a}\Big)^2 + \tilde{\gamma}'^2_k(y)}\,\mathrm{d}y = \int_0^a \sqrt{1 + \gamma'^2(y)}\,\mathrm{d}y = \mathcal{H}^1(\Gamma)\,,$$

and this concludes the proof of the proposition. □

We assume that outside the crack the elastic body is isotropic, homogeneous, and impermeable. Therefore, the behavior of the elastic body is fully characterized by the constant elasticity tensor $\mathbb{C} \colon \mathbb{M}^2_{sym} \to \mathbb{M}^2_{sym}$ defined by

$$\mathbb{C}\mathrm{F} := \lambda tr(\mathrm{F})\mathrm{I} + 2\mu \mathrm{F} \qquad \text{for every } \mathrm{F} \in \mathbb{M}^2_{sym}\,, \tag{2.2.2}$$

λ and μ being the Lamé coefficients of the body. As usual, we assume that $\mathbb{C}\mathrm{F} = 0$ for every $\mathrm{F} \in \mathbb{M}^2_{skw}$ and that $\mathbb{C}$ is positive definite, that is, there exist two constants $0 < \alpha \le \beta < +\infty$ such that

$$\alpha|\mathrm{F}|^2 \le \mathbb{C}\mathrm{F} \cdot \mathrm{F} \le \beta|\mathrm{F}|^2 \qquad \text{for every } \mathrm{F} \in \mathbb{M}^2_{sym}\,. \tag{2.2.3}$$

Our aim is now to define the set of admissible displacements and the energy of the elastic body $\mathbb{R}^2_+$ subject to a *remote strain field* $\epsilon\mathrm{I}$, $\epsilon \in \mathbb{R}$, and with a crack $\Gamma \in \mathcal{C}_\eta$ filled by a volume $V \in [0, +\infty)$ of incompressible fluid.

Let us start with a simpler case in which we do not consider the volume of fluid inside the crack. As we have already mentioned in Section 2.1, the action of the strain $\epsilon\mathrm{I}$ is intended in the following way: the displacement $u \colon \mathbb{R}^2_+ \to \mathbb{R}^2$ of the elastic body has to induce a strain field $\mathrm{E}u$ which is close to $\epsilon\mathrm{I}$ at infinity. The previous requirement is translated into the condition $u-\epsilon\, id \in LD^2(\mathbb{R}^2_+ \setminus \Gamma;\mathbb{R}^2)$, where $LD^2(\mathbb{R}^2_+ \setminus \Gamma;\mathbb{R}^2)$ is defined in (1.2.1) and id stands for the identity map in $\mathbb{R}^2$. For what follows, we notice that, for every Ω open bounded subset of $\mathbb{R}^2_+$ with Lipschitz boundary and every $\Gamma \in \mathcal{C}_\eta$ with $\Gamma \setminus \Gamma_0 \subset\subset \Omega$, Propositions 1.2.1-1.2.5 are still valid in $LD^2(\Omega \setminus \Gamma;\mathbb{R}^2)$.

In view of the previous comments, for every $\epsilon \in \mathbb{R}$ and every $\Gamma \in \mathcal{C}_\eta$ we introduce the set of admissible displacements (without volume constraint)

$$\begin{aligned}\mathcal{AD}(\Gamma,\epsilon) := \{u\colon \mathbb{R}^2_+ \to \mathbb{R}^2 : u-\epsilon\, id \in LD^2(\mathbb{R}^2_+ \setminus \Gamma;\mathbb{R}^2),\ u_2 = 0 \text{ on } \Sigma,\\ [u]\cdot\nu_\Gamma \geq 0 \text{ on } \Gamma\}\,,\end{aligned} \tag{2.2.4}$$

where $[u]$ stands for the jump of u through Γ, that is, $[u] := u^+ - u^-$, with u^+ and u^- denoting the traces of u on the two sides Γ^+ and Γ^- of Γ, defined according to the orientation of ν_Γ.

Let us give some comments on $\mathcal{AD}(\Gamma,\epsilon)$. The choice of the space $LD^2(\mathbb{R}^2_+ \setminus \Gamma;\mathbb{R}^2)$ has some important consequences. First of all, it says that every admissible displacement is Sobolev regular (see Proposition 1.2.1) outside of the curve Γ, hence the crack is actually contained in Γ. Furthermore, the fact that $\mathrm{E}u - \epsilon\mathrm{I} \in L^2(\mathbb{R}^2_+ \setminus \Gamma;\mathbb{M}^2_{sym})$ means, in a suitable weak sense, that $\mathrm{E}u$ has to coincide with the uniform strain $\epsilon\mathrm{I}$ at infinity. We refer to Remark 2.2.8 for further comments on the relation between $\mathrm{E}u$ and $\epsilon\mathrm{I}$. In what follows, we will assume, when needed, that $\mathrm{E}u - \epsilon\mathrm{I}$ is a function in $L^2(\mathbb{R}^2_+;\mathbb{M}^2_{sym})$. For instance, this is true if we extend it by zero on Γ.

The boundary condition $u_2 = 0$ on Σ reflects the fact that, according to the model studied in [54], the elastic body is adhered to the poroelastic substrate. Finally, the inequality in formula (2.2.4), which is assumed to hold $\mathcal{H}^1$-a.e. in Γ, takes into account the non-interpenetration condition: the fracture lips can not cross each other.

Let us now define the elastic energy of the body for a displacement $u \in \mathcal{AD}(\Gamma,\epsilon)$. Due to the summability hypothesis made on $\mathrm{E}u-\epsilon\mathrm{I}$, we get that $\mathrm{E}u \notin L^2(\mathbb{R}^2_+;\mathbb{M}^2_{sym})$ whenever $\epsilon \neq 0$. Hence, from (2.2.3) we deduce that the usual *stored elastic energy*

$$\frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}\mathrm{E}u\cdot\mathrm{E}u\,\mathrm{d}x$$

is not finite. Therefore, in order formulate our problem in the setting of rate independent processes [56], for every displacement $u \in \mathcal{AD}(\Gamma,\epsilon)$ we have to define the *renormalized* energy

$$\mathcal{F}^{el}(u,\Gamma,\epsilon) := \frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u-\epsilon\mathrm{I})\cdot(\mathrm{E}u-\epsilon\mathrm{I})\,\mathrm{d}x - \int_\Gamma \boldsymbol{\sigma}(\epsilon)\nu_\Gamma\cdot[u]\,\mathrm{d}\mathcal{H}^1 \tag{2.2.5}$$

where $\boldsymbol{\sigma}(\epsilon) := \epsilon\mathbb{C}\mathrm{I}$ is the far stress field associated to ϵ. For simplicity, we set also

$$\sigma(\epsilon) := 2\epsilon(\lambda+\mu)\,, \tag{2.2.6}$$

so that, by (2.2.2), $\boldsymbol{\sigma}(\epsilon)=\sigma(\epsilon)\mathrm{I}$ and (2.2.5) becomes

$$\mathcal{F}^{el}(u,\Gamma,\epsilon)=\frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma}\mathbb{C}(\mathrm{E}u-\epsilon\mathrm{I})\cdot(\mathrm{E}u-\epsilon\mathrm{I})\,\mathrm{d}x-\sigma(\epsilon)\int_\Gamma[u]\cdot\nu_\Gamma\,\mathrm{d}\mathcal{H}^1\,. \tag{2.2.7}$$

Besides $\mathcal{F}^{el}$, it is useful to introduce also the *renormalized stored elastic energy*

$$\mathcal{E}^{el}(u,\Gamma,\epsilon):=\frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma}\mathbb{C}(\mathrm{E}u-\epsilon\mathrm{I})\cdot(\mathrm{E}u-\epsilon\mathrm{I})\,\mathrm{d}x\,. \tag{2.2.8}$$

The definition of the renormalized energy given in (2.2.5) is also motivated by the fact that $\mathcal{F}^{el}(u,\Gamma,\epsilon)$ can be obtained as limit of the stored elastic energy on bounded domains which tend to $\mathbb{R}^2_+$, as we show below. Let us consider $R>0$ such that $\Gamma\subseteq\overline{\mathrm{B}}^+_R$ and let us set

$$\mathcal{E}^{el}_R(u,\Gamma):=\frac{1}{2}\int_{\mathrm{B}^+_R\setminus\Gamma}\mathbb{C}\mathrm{E}u\cdot\mathrm{E}u\,\mathrm{d}x$$

for every displacement $u\in\mathcal{AD}_R(\Gamma,\epsilon)$, where

$$\begin{aligned}\mathcal{AD}_R(\Gamma,\epsilon):=\{u\in H^1(\mathrm{B}^+_R\setminus\Gamma;\mathbb{R}^2):u=\epsilon\,id\text{ on }\partial\mathrm{B}^+_R\setminus\Sigma,\ u_2=0\text{ on }\partial\mathrm{B}^+_R\cap\Sigma,\\ [u]\cdot\nu_\Gamma\geq 0\text{ on }\Gamma\}\,.\end{aligned}$$

We notice that the Dirichlet condition $u=\epsilon\,id$ on $\partial\mathrm{B}^+_R\setminus\Sigma$ corresponds, in the bounded case, to the condition $u-\epsilon\,id\in LD^2(\mathbb{R}^2_+\setminus\Gamma;\mathbb{R}^2)$ in (2.2.4). Indeed, if we extend $u\in\mathcal{AD}_R(\Gamma,\epsilon)$ by $\epsilon\,id$ in $\mathbb{R}^2_+\setminus\mathrm{B}^+_R$, it is straightforward to see that we obtain an element of $\mathcal{AD}(\Gamma,\epsilon)$. In what follows, we will denote by $\bar{u}$ this extension.

An integration by parts shows that for every $u\in\mathcal{AD}_R(\Gamma,\epsilon)$ the following equality holds:

$$\mathcal{E}^{el}_R(u,\Gamma)-\mathcal{E}^{el}_R(\epsilon\,id,\Gamma)=\mathcal{E}^{el}_R(u-\epsilon\,id,\Gamma)-\int_\Gamma\boldsymbol{\sigma}(\epsilon)\nu_\Gamma\cdot[u]\,\mathrm{d}\mathcal{H}^1=:\mathcal{F}^{el}_R(u,\Gamma)\,. \tag{2.2.9}$$

The aim of the following proposition is to pass to the limit in (2.2.9) as $R\to+\infty$, recovering the renormalized energy defined in (2.2.5) and (2.2.7).

Proposition 2.2.3. *Let $\Gamma\in\mathcal{C}_\eta$ and $\epsilon\in\mathbb{R}$. Then the following facts hold:*

(a) *for every sequence u_R in $\mathcal{AD}_R(\Gamma,\epsilon)$ such that*

$$\sup_{R>0}\mathcal{F}^{el}_R(u_R,\Gamma)<+\infty \tag{2.2.10}$$

there exists $u\in\mathcal{AD}(\Gamma,\epsilon)$ such that, up to a subsequence, $\mathrm{E}\bar{u}_R-\epsilon\mathrm{I}\rightharpoonup\mathrm{E}u-\epsilon\mathrm{I}$ weakly in $L^2(\mathbb{R}^2_+;\mathbb{M}^2_{sym})$ and

$$\mathcal{F}^{el}(u,\Gamma,\epsilon)\leq\liminf_{R\to+\infty}\mathcal{F}^{el}_R(u_R-\epsilon\,id,\Gamma)\,;$$

(*b*) *for every* $u \in \mathcal{AD}(\Gamma, \epsilon)$ *there exists a sequence* v_R *in* $\mathcal{AD}_R(\Gamma, \epsilon)$ *such that* $\mathrm{E}\bar{v}_R - \epsilon\mathrm{I} \to \mathrm{E}u - \epsilon\mathrm{I}$ *in* $L^2(\mathbb{R}^2_+; \mathbb{M}^2_{sym})$ *and*

$$\mathcal{F}^{el}(u, \Gamma, \epsilon) = \lim_{R \to +\infty} \mathcal{F}^{el}_R(v_R, \Gamma)\,.$$

Proof. Let us prove (a). Let Γ, ϵ, and u_R be as in the statement of the proposition. It is easy to see from (2.2.7) and (2.2.9) that $\mathcal{F}^{el}_R(u_R, \Gamma) = \mathcal{F}^{el}(\bar{u}_R, \Gamma, \epsilon)$ for every $R > 0$ such that $\Gamma \setminus \Gamma_0 \subset\subset \mathrm{B}^+_R$.

Let $E \subset\subset \mathbb{R}^2_+ \setminus \Gamma$ be an open bounded set with Lipschitz boundary. For every $R > 0$ such that $E \subset\subset \mathrm{B}^+_R$, there exists a horizontal translation t_R such that $\bar{u}_R - \epsilon\, id - t_R \in LD^2_E(\mathbb{R}^2_+ \setminus \Gamma; \mathbb{R}^2)$. In view of Proposition 1.2.2, for $r > 0$ sufficiently large there exists a positive constant C_r satisfying

$$\|\bar{u}_R - \epsilon\, id - t_R\|_{H^1(\mathrm{B}^+_r \setminus \Gamma)} \leq C_r \|\mathrm{E}\bar{u}_R - \epsilon\mathrm{I}\|_{2, \mathbb{R}^2_+} \tag{2.2.11}$$

for every $R > 0$ with $E \subset\subset \mathrm{B}^+_R$. By (2.2.3), (2.2.10), and (2.2.11), we have that $\mathrm{E}\bar{u}_R - \epsilon\mathrm{I}$ is bounded in $L^2(\mathbb{R}^2_+)$. Hence, by Proposition 1.2.2 and by inequality (2.2.11), there exist $v \in H^1_{loc}(\mathbb{R}^2_+ \setminus \Gamma; \mathbb{R}^2)$ and $\psi \in L^2(\mathbb{R}^2_+; \mathbb{M}^2_{sym})$ such that, up to a subsequence, $\mathrm{E}\bar{u}_R - \epsilon\mathrm{I} \rightharpoonup \psi$ weakly in $L^2(\mathbb{R}^2_+)$ and $\bar{u}_R - \epsilon\, id - t_R \rightharpoonup v$ weakly in $H^1(\mathrm{B}^+_r \setminus \Gamma; \mathbb{R}^2)$ for every $r > 0$. Therefore, $\mathrm{E}v = \psi$ and $v \in LD^2(\mathbb{R}^2_+ \setminus \Gamma; \mathbb{R}^2)$. By continuity of the traces with respect to the weak convergence in H^1, we have that $v_2 = 0$ on Σ and

$$[u_R] \cdot \nu_\Gamma = [\bar{u}_R - \epsilon\, id - t_R] \cdot \nu_\Gamma \to [v] \cdot \nu_\Gamma \quad \text{in } L^2(\Gamma) \text{ as } R \to +\infty\,. \tag{2.2.12}$$

Let us set $u := v + \epsilon\, id$. From the previous convergences we deduce that $u \in \mathcal{AD}(\Gamma, \epsilon)$ and that, up to a subsequence, $\mathrm{E}\bar{u}_R - \epsilon\mathrm{I} \rightharpoonup \mathrm{E}u - \epsilon\mathrm{I}$ weakly in $L^2(\mathbb{R}^2_+; \mathbb{M}^2_{sym})$. Moreover, by (2.2.12) we get

$$\mathcal{F}^{el}(u, \Gamma, \epsilon) \leq \liminf_{R \to +\infty} \mathcal{F}^{el}(\bar{u}_R, \Gamma, \epsilon) = \liminf_{R \to +\infty} \mathcal{F}^{el}_R(u_R, \Gamma)\,,$$

which concludes the proof of (a).

Let us now prove (b). Let $u \in \mathcal{AD}(\Gamma, \epsilon)$ and let $E \subset\subset \mathrm{B}^+_{1/2} \setminus \overline{\mathrm{B}}^+_{1/4}$ be an open set with Lipschitz boundary. Let $\varphi \in C^\infty_c(\mathrm{B}_{1/2})$ be a cut-off function such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ on $\overline{\mathrm{B}}_{1/4}$. Let us set $E_R := RE$ and $\varphi_R(x) := \varphi(x/R)$ for every $x \in \mathbb{R}^2$ and every $R > 0$. It is clear that

$$\|\nabla \varphi_R\|_{\infty, \mathbb{R}^2} = \frac{\|\nabla \varphi\|_{\infty, \mathbb{R}^2}}{R}\,. \tag{2.2.13}$$

Let us restrict our attention to $R > 0$ such that $\Gamma \setminus \Gamma_0 \subset\subset \mathrm{B}^+_{R/4}$. Arguing as in point (a), for such R we find a horizontal translation t_R such that $u - \epsilon\, id - t_R \in LD^2_{E_R}(\mathbb{R}^2_+ \setminus \Gamma; \mathbb{R}^2)$. In particular, by Proposition 1.2.2 and Remark 1.2.3, there exists a positive constant $C = C(E)$ such that

$$\|u - \epsilon\, id - t_R\|_{2, \mathrm{B}^+_R \setminus \overline{\mathrm{B}}^+_{R/4}} \leq CR \|\mathrm{E}u - \epsilon\mathrm{I}\|_{2, \mathrm{B}^+_R \setminus \overline{\mathrm{B}}^+_{R/4}}\,. \tag{2.2.14}$$

We define $v_R := \varphi_R(u - t_R) + (1-\varphi_R)\epsilon\, id$. By construction, we have $v_R = u - t_R$ in $\mathrm{B}^+_{R/4}$ and $v_R = \epsilon id$ in $\mathbb{R}^2_+ \setminus \mathrm{B}^+_{R/2}$. Therefore, for every $R > 0$ such that $\Gamma \subseteq \overline{\mathrm{B}}^+_{R/4}$ we have $v_R \in \mathcal{AD}_R(\Gamma, \epsilon)$ and v_R coincides with $\bar{v}_R$. Moreover,

$$\|\mathrm{E}\bar{v}_R - \mathrm{E}u\|^2_{2,\mathbb{R}^2_+} \leq \|\mathrm{E}u - \epsilon \mathrm{I}\|^2_{2,\mathbb{R}^2_+ \setminus \mathrm{B}^+_{R/4}} + \int_{\mathrm{B}^+_R \setminus \mathrm{B}^+_{R/4}} |\nabla\varphi_R \odot (u - \epsilon\, id - t_R)|^2 \,\mathrm{d}x\,, \quad (2.2.15)$$

where the symbol $\odot$ denotes the symmetric tensor product. Combining (2.2.13)-(2.2.15) we obtain

$$\|\mathrm{E}\bar{v}_R - \mathrm{E}u\|^2_{2,\mathbb{R}^2_+} \leq \|\mathrm{E}u - \epsilon \mathrm{I}\|^2_{2,\mathbb{R}^2_+ \setminus \mathrm{B}^+_{R/4}} + C\|\mathrm{E}u - \epsilon \mathrm{I}\|^2_{2,\mathbb{R}^2_+ \setminus \mathrm{B}^+_{R/4}}\,, \quad (2.2.16)$$

for some constant $C > 0$ independent of R. Passing to the limit as $R \to +\infty$ in (2.2.16) we deduce that $\mathrm{E}\bar{v}_R - \epsilon\mathrm{I} \to \mathrm{E}u - \epsilon\mathrm{I}$ in $L^2(\mathbb{R}^2_+; \mathbb{M}^2_{sym})$. Finally, it is clear that

$$\mathcal{F}^{el}(u, \Gamma, \epsilon) = \lim_{R\to+\infty} \mathcal{F}^{el}_R(v_R, \Gamma) = \lim_{R\to+\infty} \mathcal{F}^{el}(\bar{v}_R, \Gamma, \epsilon)\,,$$

and the proof is thus concluded. □

We are now in a position to define the total energy of the system: for every $\Gamma \in \mathcal{C}_\eta$, every $\epsilon \in \mathbb{R}$, and every displacement $u \in \mathcal{AD}(\Gamma, \epsilon)$, we set

$$\mathcal{F}(u, \Gamma, \epsilon) := \mathcal{F}^{el}(u, \Gamma, \epsilon) + \kappa\mathcal{H}^1(\Gamma)\,, \quad (2.2.17)$$

where κ is a positive constant related to the fracture toughness.

We conclude this section considering the additional volume constraint in the definitions of admissible displacements (2.2.4) and of the total energy $\mathcal{F}$ in (2.2.17). Let us assume that the elastic body $\mathbb{R}^2_+$, subject to a far strain field $\epsilon\mathrm{I}$, $\epsilon \in \mathbb{R}$, has a crack $\Gamma \in \mathcal{C}_\eta$ filled by a volume $V \in [0, +\infty)$ of incompressible fluid. Since we are dealing with linearized elasticity, for the volume of the cavity determined by the crack lips we use the approximate formula

$$\int_\Gamma [u] \cdot \nu_\Gamma \,\mathrm{d}\mathcal{H}^1\,,$$

so that the class of admissible displacements becomes

$$\mathcal{A}(\Gamma, V, \epsilon) := \left\{ u \in \mathcal{AD}(\Gamma, \epsilon) : \int_\Gamma [u] \cdot \nu_\Gamma \,\mathrm{d}\mathcal{H}^1 = V \right\}. \quad (2.2.18)$$

It is clear that a result similar to Proposition 2.2.3 can be stated adding the volume constraint of (2.2.18). Therefore, also in this case the use of the energy (2.2.5) is fully justified. Moreover, thanks to the volume condition we have that

$$\mathcal{F}^{el}(u, \Gamma, \epsilon) = \mathcal{E}^{el}(u, \Gamma, \epsilon) - \sigma(\epsilon)V \qquad \text{for every } u \in \mathcal{A}(\Gamma, V, \epsilon)\,.$$

Since $\sigma(\epsilon)$ and V are given constants, as total energy of the system we consider

$$\mathcal{E}(u, \Gamma, \epsilon) := \mathcal{E}^{el}(u, \Gamma, \epsilon) + \kappa\mathcal{H}^1(\Gamma)\,, \quad (2.2.19)$$

for every displacement $u \in \mathcal{A}(\Gamma, V, \epsilon)$. In particular, the energy (2.2.19) is the sum of the renormalized stored elastic energy (2.2.8) and of the energy dissipated by the crack production.

2.2.1 Static problem

Here, we analyze the equilibrium condition for the elastic body $\mathbb{R}^2_+$ subject to a far strain field $\epsilon \mathrm{I}$, $\epsilon \in \mathbb{R}$, when a crack $\Gamma \in \mathcal{C}_\eta$ is filled by a prescribed volume $V \in [0, +\infty)$ of incompressible fluid.

According to the variational principles of linear elasticity, the equilibrium of the elastic body with a prescribed crack $\Gamma \in \mathcal{C}_\eta$ is achieved if the displacement u is a solution of the minimum problem

$$\min_{u \in \mathcal{A}(\Gamma, V, \epsilon)} \mathcal{E}(u, \Gamma, \epsilon)\,, \tag{2.2.20}$$

where the set $\mathcal{A}(\Gamma, V, \epsilon)$ of admissible displacements is defined in (2.2.18) and the energy $\mathcal{E}$ is given by (2.2.19). The existence of solutions of (2.2.20) follows from the direct method of the calculus of variations and Proposition 2.2.4 below, and is discussed in Corollary 2.2.5. Proposition 2.2.4 is stated in a more general form than the one needed here since we shall use it also in the study of the evolution problem in Section 2.2.2.

Proposition 2.2.4. *Let $\Gamma, \Gamma_k, \Gamma_\infty \in \mathcal{C}_\eta$ be such that $\Gamma \subseteq \Gamma_k$ and $\Gamma_k \to \Gamma_\infty$ in the Hausdorff metric. Let $V_k, V_\infty \in [0, +\infty)$ with $V_k \to V_\infty$, and let $\epsilon_k, \epsilon_\infty \in \mathbb{R}$ with $\epsilon_k \to \epsilon_\infty$. Assume that $u_k \in \mathcal{A}(\Gamma_k, V_k, \epsilon_k)$ is such that*

$$\sup_k \|\mathrm{E}u_k - \epsilon_k \mathrm{I}\|_{2, \mathbb{R}^2_+} < +\infty\,. \tag{2.2.21}$$

Then, there exists $u_\infty \in \mathcal{A}(\Gamma_\infty, V_\infty, \epsilon_\infty)$ such that, up to a subsequence, $\mathrm{E}u_k - \epsilon_k \mathrm{I}$ converges to $\mathrm{E}u_\infty - \epsilon_\infty \mathrm{I}$ weakly in $L^2(\mathbb{R}^2_+; \mathbb{M}^2_{sym})$.

Proof. By the Hausdorff convergence of Γ_k to Γ_∞, it is easy to see that $\Gamma \subseteq \Gamma_\infty$.

Since the sequence $\mathrm{E}u_k - \epsilon_k \mathrm{I}$ is bounded in $L^2(\mathbb{R}^2_+; \mathbb{M}^2_{sym})$, we may assume that there exists $\varphi \in L^2(\mathbb{R}^2_+; \mathbb{M}^2_{sym})$ such that, up to a subsequence, $\mathrm{E}u_k - \epsilon_k \mathrm{I} \rightharpoonup \varphi$ weakly in $L^2(\mathbb{R}^2_+; \mathbb{M}^2_{sym})$.

Let $r > 0$ be such that $\Gamma_\infty \setminus \Gamma \subset\subset \mathrm{B}^+_r$. Thanks to the regularity of the sets Γ_k, Γ_∞, and to the convergence of Γ_k to Γ_∞ in the Hausdorff metric, arguing as in the proof of Proposition 2.2.3 and applying Proposition 1.2.5 we have that there exist a positive constant C_r and a sequence t_k of horizontal translations such that, for k large enough, the following inequality holds:

$$\|u_k - \epsilon_k\, id - t_k\|_{H^1(\mathrm{B}^+_r \setminus \Gamma_k)} \leq C_r \|\mathrm{E}u_k - \epsilon_k \mathrm{I}\|_{2, \mathbb{R}^2_+}\,. \tag{2.2.22}$$

In view of (2.2.21) and (2.2.22), we may further assume that there exists a function $v \in H^1_{loc}(\mathbb{R}^2_+ \setminus \Gamma_\infty; \mathbb{R}^2)$ such that for $r, \delta > 0$

$$u_k - \epsilon_k\, id - t_k \rightharpoonup v \quad \text{weakly in } H^1(\mathrm{B}^+_r \setminus \overline{\mathcal{I}_\delta}(\Gamma_\infty \setminus \Gamma); \mathbb{R}^2)\,, \tag{2.2.23}$$

where $\mathcal{I}_\delta(\Gamma_\infty \setminus \Gamma)$ is defined in (1.1.1). Clearly, $\mathrm{E}v = \varphi$ and $v \in LD^2(\mathbb{R}^2_+ \setminus \Gamma_\infty; \mathbb{R}^2)$.

Let us show that v satisfies the non-interpenetration and the volume constraints appearing in (2.2.18). Let us fix Ω_k, Ω_∞ bounded open subsets of $\mathbb{R}^2_+$ with Lipschitz boundaries such that $\Gamma_k \setminus \Gamma \subset\subset \Omega_k$, $\Gamma_\infty \setminus \Gamma \subset\subset \Omega_\infty$, and $\overline{\Omega}_k \to \overline{\Omega}_\infty$ in the Hausdorff metric. By the convergence of Γ_k to Γ_∞, we may split Ω_k (resp. Ω_∞) in two open subsets $\Omega_k^\pm$ (resp. $\Omega_\infty^\pm$) with Lipschitz boundaries such that the following properties hold:

$$\Gamma_k \subseteq \partial\Omega_k^\pm \setminus \partial\Omega_k \qquad \text{and} \qquad \Gamma_\infty \subseteq \partial\Omega_\infty^\pm \setminus \partial\Omega_\infty\,, \tag{2.2.24}$$

$$\overline{\Omega}_k^\pm \to \overline{\Omega}_\infty^\pm \text{ in the Hausdorff metric}\,, \tag{2.2.25}$$

$$\nu_{\Gamma_k} \text{ points towards } \Omega_k^+ \text{ and } \nu_{\Gamma_\infty} \text{ points towards } \Omega_\infty^+\,. \tag{2.2.26}$$

By (2.2.22), (2.2.25), and by a simple reflection argument, we get that

$$(u_k - \epsilon_k\, id - t_k)\,\mathbf{1}_{\Omega_k^\pm} \to v\,\mathbf{1}_{\Omega_\infty^\pm} \quad \text{strongly in } L^2(\mathbb{R}^2_+)\,. \tag{2.2.27}$$

By Proposition 1.2.1, $u_k - \epsilon_k\, id - t_k \in H^1(\Omega_k \setminus \Gamma_k;\mathbb{R}^2)$ and $v \in H^1(\Omega_\infty \setminus \Gamma_\infty;\mathbb{R}^2)$. Thus, by the properties of the traces of Sobolev functions (see, e.g., [71]) and by (2.2.24) and (2.2.26), for every $k \in \mathbb{N}$ and every $\psi \in C^1(\mathbb{R}^2)$ with $supp(\psi) \cap \partial\Omega_k \setminus \Sigma = \emptyset$ we have

$$\begin{aligned}\sum_{i=1}^2 \int_{\Omega_k} (u_k - \epsilon_k\, id - t_k)_i (\nabla\psi)_i \,\mathrm{d}x + \sum_{i=1}^2 \int_{\Omega_k} \psi(\mathrm{E}_{ii}u_k - \epsilon_k)\,\mathrm{d}x \\ = -\int_{\Gamma_k} \psi[u_k]\cdot\nu_{\Gamma_k}\,\mathrm{d}\mathcal{H}^1 + \int_{\Sigma\cap\partial\Omega_k} \psi t_k\cdot\nu_\Sigma\,\mathrm{d}\mathcal{H}^1 = -\int_{\Gamma_k}\psi[u_k]\cdot\nu_{\Gamma_k}\,\mathrm{d}\mathcal{H}^1\,,\end{aligned} \tag{2.2.28}$$

where, in the last equality, we have used the fact that t_k is a horizontal translation and $\nu_\Sigma = (0,1)$ is the normal vector to Σ.

Let us consider $\psi \in C^1(\mathbb{R}^2)$ such that $supp(\psi) \cap \partial\Omega_\infty \setminus \Sigma = \emptyset$. Since $\overline{\Omega}_k \to \overline{\Omega}_\infty$ in the Hausdorff metric, for k large enough we have $supp(\psi) \cap \partial\Omega_k \setminus \Sigma = \emptyset$, so that (2.2.28) holds. Taking into account (2.2.27) and the weak convergence of $\mathrm{E}u_k - \epsilon_k\mathrm{I}$ to $\mathrm{E}v$ in $L^2(\mathbb{R}^2_+;\mathbb{M}^2_{sym})$, passing to the limit in (2.2.28) as $k \to +\infty$ we obtain

$$\begin{aligned}-\lim_k\Big(\int_{\Gamma_k}\psi[u_k]\cdot\nu_{\Gamma_k}\,\mathrm{d}\mathcal{H}^1 - \int_{\Sigma\cap\partial\Omega_k}\psi t_k\cdot\nu_\Sigma\,\mathrm{d}\mathcal{H}^1\Big) &= -\lim_k \int_{\Gamma_k}\psi[u_k]\cdot\nu_{\Gamma_k}\,\mathrm{d}\mathcal{H}^1 \\ &= \lim_k\Big(\sum_{i=1}^2\int_{\Omega_k}(u_k - \epsilon_k id - t_k)_i(\nabla\psi)_i\mathrm{d}x + \sum_{i=1}^2\int_{\Omega_k}\psi(\mathrm{E}_{ii}u_k - \epsilon_k)\mathrm{d}x\Big) \\ &= \sum_{i=1}^2\int_{\Omega_\infty} v_i(\nabla\psi_i)\mathrm{d}x + \sum_{i=1}^2\int_{\Omega_\infty}\psi\mathrm{E}_{ii}v\mathrm{d}x \\ &= -\int_{\Gamma_\infty}\psi[v]\cdot\nu_{\Gamma_\infty}\,\mathrm{d}\mathcal{H}^1 - \int_{\Sigma\cap\partial\Omega_\infty}\psi v\cdot\nu_\Sigma\,\mathrm{d}\mathcal{H}^1\,,\end{aligned} \tag{2.2.29}$$

where, in the last equality, we have used again the properties of the traces of Sobolev functions.

By (2.2.23) we have that

$$0 = \lim_k \int_{\Sigma\cap\partial\Omega_k} \psi t_k \cdot \nu_\Sigma \,\mathrm{d}\mathcal{H}^1 = \int_{\Sigma\cap\partial\Omega_\infty} \psi v \cdot \nu_\Sigma \,\mathrm{d}\mathcal{H}^1\,,$$

which implies, in view of (2.2.29), that

$$\lim_k \int_{\Gamma_k} \psi[u_k]\cdot \nu_{\Gamma_k}\,\mathrm{d}\mathcal{H}^1 = \int_{\Gamma_\infty} \psi[v]\cdot\nu_{\Gamma_\infty}\,\mathrm{d}\mathcal{H}^1\,. \tag{2.2.30}$$

for every $\psi \in C^1(\mathbb{R}^2)$ such that $supp(\psi)\cap\partial\Omega_\infty\setminus\Sigma=\emptyset$. By the hypotheses and the arbitrariness of ψ, from (2.2.30) we easily get that

$$[v]\cdot\nu_{\Gamma_\infty}\geq 0 \quad \text{on } \Gamma_\infty \qquad \text{and} \qquad \int_{\Gamma_\infty}[v]\cdot\nu_{\Gamma_\infty}\,\mathrm{d}\mathcal{H}^1 = V_\infty\,.$$

In view of (2.2.23), we also have that $v_2 = 0$ on Σ, hence $v \in \mathcal{A}(\Gamma_\infty, V_\infty, 0)$. Thus, it is clear that $u_\infty := v + \epsilon_\infty\, id \in \mathcal{A}(\Gamma_\infty, V_\infty, \epsilon_\infty)$. Since $\mathrm{E}u_\infty = \mathrm{E}v + \epsilon_\infty \mathrm{I}$, we finally get that $\mathrm{E}u_k - \epsilon_k\mathrm{I} \rightharpoonup \mathrm{E}u_\infty - \epsilon_\infty\mathrm{I}$ weakly in $L^2(\mathbb{R}^2_+;\mathbb{M}^2_{sym})$, and the proof is thus concluded.

□

We are now ready to discuss existence and uniqueness of solution of (2.2.20).

Corollary 2.2.5. *The minimum problem* (2.2.20) *admits a unique solution, up to a translation parallel to the* x_1*-axis.*

Proof. We apply the direct method of the calculus of variations. Let u_k be a minimizing sequence. It is clear that the sequence $\mathrm{E}u_k - \epsilon\mathrm{I}$ is bounded in $L^2(\mathbb{R}^2_+;\mathbb{M}^2_{sym})$. Hence, by Proposition 2.2.4, there exists $u \in \mathcal{A}(\Gamma, V, \epsilon)$ such that, up to a subsequence, $\mathrm{E}u_k - \epsilon\mathrm{I} \rightharpoonup \mathrm{E}u - \epsilon\mathrm{I}$ weakly in $L^2(\mathbb{R}^2_+;\mathbb{M}^2_{sym})$. Therefore,

$$\mathcal{E}(u,\Gamma,\epsilon) \leq \liminf_k \mathcal{E}(u_k,\Gamma,\epsilon)\,,$$

and this concludes the proof of existence.

The uniqueness of solution up to a horizontal translation follows by the strict convexity of the energy, by the convexity of the constraints on the crack Γ, and by the boundary condition $u_2 = 0$ on Σ.

□

In the following propositions and remarks we study some properties of a solution u of the minimum problem (2.2.20).

Proposition 2.2.6. *Let* $u \in \mathcal{A}(\Gamma, V, \epsilon)$ *be a solution of* (2.2.20) *with* $\Gamma \in \mathcal{C}_\eta$, $V \in [0,+\infty)$, *and* $\epsilon\in\mathbb{R}$. *Then, for every* $v \in LD^2(\mathbb{R}^2_+\setminus\Gamma;\mathbb{R}^2)$ *such that* $[v]\cdot\nu_\Gamma = 0$ *on* Γ *and* $v_2 = 0$ *on* Σ *it holds*

$$\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u-\epsilon\mathrm{I})\cdot\mathrm{E}v\,\mathrm{d}x = 0\,. \tag{2.2.31}$$

Moreover, there exists $q(\Gamma, V, \epsilon) \geq 0$ *such that for every* $\varphi \in C_c^1(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}(\varphi(u - \epsilon\, id))\, \mathrm{d}x = q(\Gamma, V, \epsilon) \int_\Gamma \varphi[u] \cdot \nu_\Gamma \, \mathrm{d}\mathcal{H}^1 \,. \tag{2.2.32}$$

Before proving Proposition 2.2.6, we briefly discuss some consequences of formula (2.2.31).

Remark 2.2.7 (Equilibrium system). Let u be a solution of (2.2.20) and let us set

$$\boldsymbol{\sigma}(u) := \mathbb{C}\mathrm{E}u \,, \tag{2.2.33}$$

the stress field associated to u. Formula (2.2.31) means that u is a weak solution of

$$\mathrm{div}(\boldsymbol{\sigma}(u) - \boldsymbol{\sigma}(\epsilon)) = 0 \qquad \text{in } \mathbb{R}^2_+ \setminus \Gamma, \tag{2.2.34}$$

which reduces to

$$\mathrm{div}\, \boldsymbol{\sigma}(u) = 0 \quad \text{in } \mathbb{R}^2_+ \setminus \Gamma \,, \tag{2.2.35}$$

since $\boldsymbol{\sigma}(\epsilon)$ is a constant matrix. Equation (2.2.35) says that u satisfies the usual balance of forces.

Moreover, integrating by parts in (2.2.31), we deduce that u fulfills also the condition $\boldsymbol{\sigma}(u)_{12} = 0$ on Σ, that is, the shear stress applied on the boundary of the elastic body is zero.

Remark 2.2.8 (Strain field). Since a solution u to (2.2.20) is also a weak solution of the system (2.2.34), applying Proposition 1.2.1 and the standard regularity theory for systems with constant coefficients (see, for instance, [40, Chapter 2]), we have that for every $R > 0$ there exists a constant $C = C(R)$ satisfying the following condition: for every $x_0 \in \mathbb{R}^2_+ \setminus \Gamma$ such that the open ball $\mathrm{B}_R(x_0)$ is compactly contained in $\mathbb{R}^2_+ \setminus \Gamma$

$$\|\mathrm{E}u - \epsilon \mathrm{I}\|_{\infty, \mathrm{B}_{R/2}(x_0)} \leq C \|\mathrm{E}u - \epsilon \mathrm{I}\|_{2, \mathrm{B}_R(x_0)} \,.$$

This implies that

$$\lim_{|x| \to +\infty} \|\mathrm{E}u - \epsilon \mathrm{I}\|_{\infty, \mathrm{B}_{R/2}(x)} = 0 \,,$$

which means that at infinity $\mathrm{E}u$ tends to coincide with the strain $\epsilon\mathrm{I}$. Therefore, the choice of the function space $LD^2(\mathbb{R}^2_+ \setminus \Gamma; \mathbb{R}^2)$ is fully justified.

Remark 2.2.9. From (2.2.31) we deduce that for every $v, w \in LD^2(\mathbb{R}^2_+ \setminus \Gamma; \mathbb{R}^2)$ such that $[v] \cdot \nu_\Gamma = [w] \cdot \nu_\Gamma$ on Γ and $v_2 = w_2$ on Σ the following equality holds:

$$\int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}v \, \mathrm{d}x = \int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}w \, \mathrm{d}x \,. \tag{2.2.36}$$

This property will be extensively used in the sequel.

Proof of Proposition 2.2.6. When $V = 0$ we have, up to a horizontal translation, $u = \epsilon\, id$, thus we can take $q(\Gamma, 0, \epsilon) = 0$.

Assume now $V > 0$. Let $v \in LD^2(\mathbb{R}^2_+ \setminus \Gamma; \mathbb{R}^2)$ be such that $[v] \cdot \nu_\Gamma = 0$ on Γ and $v_2 = 0$ on Σ. Then, for every $\delta \in \mathbb{R}$ the function $u + \delta v$ belongs to $\mathcal{A}(\Gamma, V, \epsilon)$. Therefore,

$$\mathcal{E}(u, \Gamma, \epsilon) \leq \mathcal{E}(u + \delta v, \Gamma, \epsilon)\,,$$

which implies

$$\mathcal{E}^{el}(u, \Gamma, \epsilon) \leq \mathcal{E}^{el}(u+\delta v, \Gamma, \epsilon) = \mathcal{E}^{el}(u, \Gamma, \epsilon) + \delta \int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}v \,\mathrm{d}x + \frac{\delta^2}{2} \int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}\mathrm{E}v \cdot \mathrm{E}v \,\mathrm{d}x\,,$$

where $\mathcal{E}^{el}$ is defined in (2.2.8). By the arbitrariness of δ, from the previous inequality we get (2.2.31).

Let us now prove (2.2.32). We define two linear operators L and M on $C^1_c(\mathbb{R}^2)$:

$$\begin{aligned} L(\varphi) &:= \int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}(\varphi u) \,\mathrm{d}x\,, \\ M(\varphi) &:= \int_\Gamma \varphi [u] \cdot \nu_\Gamma \,\mathrm{d}\mathcal{H}^1\,. \end{aligned}$$

For every $\varphi \in C^1_c(\mathbb{R}^2)$ with $M(\varphi) = 0$, we consider the function $(1 + \delta\varphi)u$. For $|\delta|$ small enough, we have $(1 + \delta\varphi)u \in \mathcal{A}(\Gamma, V, \epsilon)$. Arguing as in the previous step, we get that

$$\int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}(\varphi u) \,\mathrm{d}x = 0\,. \tag{2.2.37}$$

Let us denote by $\ker(L)$ and $\ker(M)$ the kernels of the linear operators L and M, respectively. Equality (2.2.37), which is satisfied for every $\varphi \in \mathcal{N}(M)$, implies that $\ker(M) \subseteq \ker(L)$. Therefore, there exists $q = q(\Gamma, V, \epsilon) \in \mathbb{R}$ such that $L = qM$.

It is clear that for every $\varphi \in C^1_c(\mathbb{R}^2)$ we have

$$[\varphi(u - \epsilon\, id)] \cdot \nu_\Gamma = [\varphi u] \cdot \nu_\Gamma \quad \text{on } \Gamma\,. \tag{2.2.38}$$

Recalling (2.2.31) and Remark 2.2.9, equality (2.2.38) implies that

$$\begin{aligned} \int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot (\mathrm{E}\varphi(u - \epsilon\, id)) \,\mathrm{d}x &= \int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}(\varphi u) \,\mathrm{d}x \\ &= q \int_\Gamma \varphi [u] \cdot \nu_\Gamma \,\mathrm{d}\mathcal{H}^1\,, \end{aligned} \tag{2.2.39}$$

which is (2.2.32). Taking in (2.2.39) a function $\varphi \in C^1_c(\mathbb{R}^2)$ such that $\varphi = 1$ on Γ and using again Remark 2.2.9, we get that

$$\int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot (\mathrm{E}u - \epsilon \mathrm{I}) \,\mathrm{d}x = qV\,, \tag{2.2.40}$$

which implies that $q > 0$. This concludes the proof of the proposition. □

Remark 2.2.10. In the case $V > 0$, from (2.2.40) we get immediately an explicit formula for $q(\Gamma, V, \epsilon)$ in terms of the elastic energy and of the volume V:

$$q(\Gamma, V, \epsilon) = \frac{1}{V} \int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot (\mathrm{E}u - \epsilon \mathrm{I}) \, \mathrm{d}x \,. \tag{2.2.41}$$

Remark 2.2.11 (Fluid pressure). Let us consider the constant

$$p(\Gamma, V, \epsilon) := q(\Gamma, V, \epsilon) - \sigma(\epsilon) \,, \tag{2.2.42}$$

where $q(\Gamma, V, \epsilon)$ and $\sigma(\epsilon)$ are defined in Proposition 2.2.6 and in formula (2.2.6), respectively. We want now to explain why $p(\Gamma, V, \epsilon)$ can be interpreted as a fluid pressure. It is clear that, if u is a solution of (2.2.20) without the non-interpenetration condition, then $q(\Gamma, V, \epsilon)$ is a Lagrange multiplier due to the volume constraint, and hence we have

$$\int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}v \, \mathrm{d}x = q(\Gamma, V, \epsilon) \int_\Gamma [v] \cdot \nu_\Gamma \, \mathrm{d}\mathcal{H}^1 \tag{2.2.43}$$

for every $v \in LD^2(\mathbb{R}^2_+ \setminus \Gamma; \mathbb{R}^2)$ such that $v_2 = 0$ on Σ. Thus, with the notation introduced in (2.2.33), u satisfies the condition

$$\begin{aligned} \boldsymbol{\sigma}(u)\nu_\Gamma = \boldsymbol{\sigma}(\epsilon)\nu_\Gamma - q(\Gamma, V, \epsilon)\nu_\Gamma &= (\sigma(\epsilon) - q(\Gamma, V, \epsilon))\nu_\Gamma \\ = -p(\Gamma, V, \epsilon)\nu_\Gamma \qquad &\text{on } \Gamma \,. \end{aligned} \tag{2.2.44}$$

Formula (2.2.44) means that the total force that the elastic body exerts on the crack Γ has modulus $-p(\Gamma, V, \epsilon)$ and is directed along ν_Γ. On the contrary, the fluid inside the crack exerts a force $p(\Gamma, V, \epsilon)\nu_\Gamma$ on the fracture lips. Therefore, we are allowed to interpret $p(\Gamma, V, \epsilon)$ as the fluid pressure. According to (2.2.44), the pressure $p(\Gamma, V, \epsilon)$ is acting on Γ along its normal ν_Γ in the reference configuration rather than in the deformed one. This does not affect our interpretation, since we are dealing with a linearized model.

To justify the same interpretation of $p(\Gamma, V, \epsilon)$ when the non-interpenetration condition is considered, we have to show that (2.2.43) holds for a sufficiently large class of functions in $LD^2(\mathbb{R}^2_+ \setminus \Gamma; \mathbb{R}^2)$.

Proposition 2.2.12. *Let u be the solution of* (2.2.20) *with $\Gamma \in \mathcal{C}_\eta$, $V \in [0, +\infty)$, and $\epsilon \in \mathbb{R}$. Then* (2.2.43) *holds for every $v \in LD^2(\mathbb{R}^2_+ \setminus \Gamma; \mathbb{R}^2)$ such that $supp(v) \subset\subset \mathbb{R}^2_+$ and $|[v] \cdot \nu_\Gamma| \leq C[u] \cdot \nu_\Gamma$ for some $C \geq 0$.*

Proof. When $V = 0$ we have $u = \epsilon\, id$ and the statement is true with $q(\Gamma, 0, \epsilon) = 0$.

Let us assume that $V > 0$. Let v be as in the statement of the proposition, and let us fix $\overline{\varphi} \in C^1_c(\mathbb{R}^2_+)$ such that $\overline{\varphi} = 1$ on $\mathrm{supp}(v)$ and

$$\int_\Gamma \overline{\varphi}^2 [u] \cdot \nu_\Gamma \, \mathrm{d}\mathcal{H}^1 > 0 \,. \tag{2.2.45}$$

If we set $\overline{u} := \overline{\varphi} u$, thanks to Proposition 1.2.1 we have that $\overline{u} \in H^1(\mathbb{R}^2_+ \setminus \Gamma; \mathbb{R}^2)$. In view of (2.2.36), we now modify the functions $\overline{u}$ and v, keeping the same values of

$[\overline{u}] \cdot \nu_\Gamma$ and $[v] \cdot \nu_\Gamma$ on Γ. Let us fix Ω a bounded open subset of $\mathbb{R}^2_+$ with smooth boundary such that $\Gamma \setminus \Gamma_0 \subset\subset \Omega$, $\mathrm{supp}(\overline{u}) \subset\subset \Omega$, and $\mathrm{supp}(v) \subset\subset \Omega$. We may assume that there exists an extension $\hat{\Gamma}$ of Γ in $\mathcal{C}_\eta$ such that $\nu_{\hat{\Gamma}} = \nu_\Gamma$ on Γ, $\Omega \setminus \hat{\Gamma}$ is the disjoint union of two open subsets $\Omega^\pm$ with Lipschitz boundaries and with $\nu_{\hat{\Gamma}}$ pointing towards Ω^+. We consider a scalar function $\tilde{u} \in H^1(\mathbb{R}^2_+ \setminus \Gamma)$ such that $\mathrm{supp}(\tilde{u}) \subset\subset \Omega$, $\tilde{u} \geq 0$ on $\mathbb{R}^2_+$, $\tilde{u} = 0$ on Ω^-, and $(\tilde{u})^+ = [\overline{u}] \cdot \nu_\Gamma$ on Γ. Similarly, we can find a scalar function $\tilde{v} \in H^1(\mathbb{R}^2_+ \setminus \Gamma)$ such that $\mathrm{supp}(\tilde{v}) \subset\subset \Omega$, $\tilde{v} = 0$ on Ω^-, $(\tilde{v})^+ = [v] \cdot \nu_\Gamma$ on Γ, and

$$|\tilde{v}| \leq C|\tilde{u}| \quad \text{a.e. on } \mathbb{R}^2_+ \,. \tag{2.2.46}$$

Besides $\tilde{u}$ and $\tilde{v}$, we also fix a $C^{0,1}$-extension $\tilde{\nu}_{\hat{\Gamma}}$ of the unit normal $\nu_{\hat{\Gamma}}$ to $\hat{\Gamma}$. We further assume that $\tilde{\nu}_{\hat{\Gamma}}$ has compact support in $\mathbb{R}^2$. In what follows, we will consider the functions $\tilde{u}$, $\tilde{v}$, $\tilde{u}\tilde{\nu}_{\hat{\Gamma}}$, and $\tilde{v}\tilde{\nu}_{\hat{\Gamma}}$. By construction, they belong to $H^1(\mathbb{R}^2_+ \setminus \Gamma)$ and have compact support in $\mathbb{R}^2_+$.

We now need to approximate $\tilde{u}$ and $\tilde{v}$ by truncation. Let $T_k \colon \mathbb{R} \to \mathbb{R}$ be the truncation function introduced in Chapter 1 and let $S_k \colon \mathbb{R} \to \mathbb{R}$ be defined by $S_k(s) := s - T_k(s)$.

From (2.2.46) it follows that for every $k \in \mathbb{N}$

$$|S_{1/k}(T_k(\tilde{v}))| \leq C T_k(\tilde{u}) \quad \text{a.e. on } \mathbb{R}^2_+ \,. \tag{2.2.47}$$

In particular, $S_{1/k}(T_k(\tilde{v})) = 0$ where $\tilde{u} < 1/(kC)$.

By the properties of $\tilde{u}$ and of $\tilde{\nu}_{\hat{\Gamma}}$, for every k we have

$$[T_k(\tilde{u})\tilde{\nu}_{\hat{\Gamma}}] \cdot \nu_\Gamma = [T_k(\tilde{u})]\nu_\Gamma \cdot \nu_\Gamma = T_k([\overline{u}] \cdot \nu_\Gamma) \quad \text{on } \Gamma \,.$$

The previous equality implies that

$$0 \leq [T_k(\tilde{u})\tilde{\nu}_{\hat{\Gamma}}] \cdot \nu_\Gamma \leq [\overline{u}] \cdot \nu_\Gamma \leq [u] \cdot \nu_\Gamma \quad \text{on } \Gamma \,. \tag{2.2.48}$$

Taking into account (2.2.48), with the same technique used to prove Proposition 2.2.6 we deduce that there exists $q_k \in \mathbb{R}$ such that for every $\varphi \in C^1_c(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}(\varphi T_k(\tilde{u})\tilde{\nu}_{\hat{\Gamma}}) \, \mathrm{d}x = q_k \int_\Gamma \varphi [T_k(\tilde{u})\tilde{\nu}_{\hat{\Gamma}}] \cdot \nu_\Gamma \, \mathrm{d}\mathcal{H}^1 \,. \tag{2.2.49}$$

We now show that $q_k \to q(\Gamma, V, \epsilon)$. Since $T_k(\tilde{u})\tilde{\nu}_{\hat{\Gamma}} \to \tilde{u}\tilde{\nu}_{\hat{\Gamma}}$ in $H^1(\mathbb{R}^2_+ \setminus \Gamma; \mathbb{R}^2)$, passing to the limit in (2.2.49) as $k \to +\infty$ and recalling (2.2.36), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}(\varphi \overline{u}) \, \mathrm{d}x &= \int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}(\varphi \tilde{u}\tilde{\nu}_{\hat{\Gamma}}) \, \mathrm{d}x \\
&= \lim_k \int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}(\varphi T_k(\tilde{u})\tilde{\nu}_{\hat{\Gamma}}) \, \mathrm{d}x = \lim_k q_k \int_\Gamma \varphi [T_k(\tilde{u})\tilde{\nu}_{\hat{\Gamma}}] \cdot \nu_\Gamma \, \mathrm{d}\mathcal{H}^1 \\
&= \lim_k q_k \int_\Gamma \varphi [\tilde{u}\tilde{\nu}_{\hat{\Gamma}}] \cdot \nu_\Gamma \, \mathrm{d}\mathcal{H}^1 = \lim_k q_k \int_\Gamma \varphi [\overline{u}] \cdot \nu_\Gamma \, \mathrm{d}\mathcal{H}^1 \\
&= \lim_k q_k \int_\Gamma \varphi \overline{\varphi} [u] \cdot \nu_\Gamma \, \mathrm{d}\mathcal{H}^1 \,.
\end{aligned} \tag{2.2.50}$$

Taking $\varphi = \overline{\varphi}$ in (2.2.50), by (2.2.32) of Proposition 2.2.6 we get

$$\begin{aligned} q(\Gamma, V, \epsilon) \int_\Gamma \overline{\varphi}^2 [u] \cdot \nu_\Gamma \, \mathrm{d}\mathcal{H}^1 &= \int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}(\overline{\varphi}^2 u) \, \mathrm{d}x \\ &= \lim_k q_k \int_\Gamma \overline{\varphi}^2 [u] \cdot \nu_\Gamma \, \mathrm{d}\mathcal{H}^1 \,. \end{aligned} \tag{2.2.51}$$

Since (2.2.45) holds, from (2.2.51) we deduce that $q_k \to q(\Gamma, V, \epsilon)$.

We now define the scalar function

$$w_k(x) := \begin{cases} \dfrac{S_{1/k}(T_k(\tilde{v}(x)))}{T_k(\tilde{u}(x))} & \text{if } \tilde{u}(x) \neq 0 \,, \\ 0 & \text{if } \tilde{u}(x) = 0 \,. \end{cases}$$

Then, by (2.2.47), $w_k \in H^1(\mathbb{R}^2_+ \setminus \Gamma) \cap L^\infty(\mathbb{R}^2_+)$ and $\mathrm{supp}(w_k) \subseteq \mathrm{supp}(\tilde{v}) \subset\subset \Omega$. In particular, $w_k = 0$ in Ω^-. Hence, for every k there exists a sequence $(\varphi^j_k)_j$ in $C^1_c(\mathbb{R}^2_+)$ such that $\|\varphi^j_k\|_{\infty, \mathbb{R}^2_+} \leq \|w_k\|_{\infty, \mathbb{R}^2_+}$ and $\varphi^j_k \to w_k$ strongly in $H^1(\Omega^+)$ as $j \to +\infty$.

We consider the sequence $\varphi^j_k T_k(\tilde{u}) \tilde{\nu}_{\hat{\Gamma}}$ in $H^1(\mathbb{R}^2_+ \setminus \Gamma; \mathbb{R}^2)$. By the dominated convergence theorem, we have $\varphi^j_k T_k(\tilde{u}) \tilde{\nu}_{\hat{\Gamma}} \to S_{1/k}(T_k(\tilde{v})) \tilde{\nu}_{\hat{\Gamma}}$ strongly in $H^1(\mathbb{R}^2_+ \setminus \Gamma; \mathbb{R}^2)$ as $j \to +\infty$. Since $S_{1/k}(T_k(\tilde{v})) \tilde{\nu}_{\hat{\Gamma}} \to \tilde{v} \tilde{\nu}_{\hat{\Gamma}}$ strongly in $H^1(\mathbb{R}^2_+ \setminus \Gamma; \mathbb{R}^2)$ as $k \to +\infty$, by a diagonal argument we find a sequence φ_k in $C^\infty_c(\mathbb{R}^2_+)$ such that $\varphi_k T_k(\tilde{u}) \tilde{\nu}_{\hat{\Gamma}} \to \tilde{v} \tilde{\nu}_{\hat{\Gamma}}$ strongly in $H^1(\mathbb{R}^2_+ \setminus \Gamma; \mathbb{R}^2)$. Therefore, we get

$$\begin{aligned} \int_{\mathbb{R}^2_+ \setminus \Gamma} & \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}v \, \mathrm{d}x = \int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}(\tilde{v} \tilde{\nu}_{\hat{\Gamma}}) \, \mathrm{d}x \\ &= \lim_k \int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}(\varphi_k T_k(\tilde{u}) \tilde{\nu}_{\hat{\Gamma}}) \, \mathrm{d}x = \lim_k q_k \int_\Gamma \varphi_k [T_k(\tilde{u}) \tilde{\nu}_{\hat{\Gamma}}] \cdot \nu_\Gamma \, \mathrm{d}\mathcal{H}^1 \\ &= q(\Gamma, V, \epsilon) \int_\Gamma [\tilde{v} \tilde{\nu}_{\hat{\Gamma}}] \cdot \nu_\Gamma \, \mathrm{d}\mathcal{H}^1 = q(\Gamma, V, \epsilon) \int_\Gamma [v] \cdot \nu_\Gamma \, \mathrm{d}\mathcal{H}^1 \,, \end{aligned}$$

and this concludes the proof. □

Remark 2.2.13. Integrating by parts, thanks to Proposition 2.2.12 we get that a solution u of (2.2.20) satisfies the condition $\boldsymbol{\sigma}(u) \nu_\Gamma = (\sigma(\epsilon) - q(\Gamma, V, \epsilon)) \nu_\Gamma$ on $\{[u] \cdot \nu_\Gamma \neq 0\}$, which is the part of the crack Γ occupied by the fluid. Therefore, we can repeat the argument of Remark 2.2.11 on the set $\{[u] \cdot \nu_\Gamma \neq 0\}$ and we conclude that $p(\Gamma, V, \epsilon) = q(\Gamma, V, \epsilon) - \sigma(\epsilon)$ can be interpreted as the fluid pressure.

We conclude this section considering another static problem. In view of Proposition 2.2.6 and of Remarks 2.2.11 and 2.2.13, we know that to every triple $(\Gamma, V, \epsilon) \in \mathcal{C}_\eta \times [0, +\infty) \times \mathbb{R}$ corresponds a pressure $p(\Gamma, V, \epsilon) = q(\Gamma, V, \epsilon) - \sigma(\epsilon)$, with $q(\Gamma, V, \epsilon) \in [0, +\infty)$.

What we want to do now is to briefly discuss the relationship between Γ, V, ϵ, and p studying the equilibrium problem of an elastic body filling $\mathbb{R}^2_+$ subject to a uniform strain $\epsilon \mathrm{I}$, $\epsilon \in \mathbb{R}$, and with a force $p \nu_\Gamma$ acting on the crack $\Gamma \in \mathcal{C}_\eta$. According to the result presented in Proposition 2.2.3, in this case the total energy of the system is of the form

$$\begin{aligned}\mathscr{E}(u,\Gamma,p,\epsilon) &:= \mathcal{F}(u,\Gamma,\epsilon) - p\int_\Gamma [u]\cdot\nu_\Gamma\,\mathrm{d}\mathcal{H}^1 \\ &= \mathcal{E}(u,\Gamma,\epsilon) - (p+\sigma(\epsilon))\int_\Gamma [u]\cdot\nu_\Gamma\,\mathrm{d}\mathcal{H}^1\,,\end{aligned} \tag{2.2.52}$$

where $\mathcal{F}$ is defined in (2.2.17). The class of admissible displacements is the set $\mathcal{AD}(\Gamma,\epsilon)$ given by formula (2.2.4). As in (2.2.20), the equilibrium condition is expressed by the minimum problem

$$\min_{u\in\mathcal{AD}(\Gamma,\epsilon)} \mathscr{E}(u,\Gamma,p,\epsilon)\,. \tag{2.2.53}$$

The existence of a solution of (2.2.53) follows by the arguments used to prove Proposition 2.2.4 and Corollary 2.2.5. The solution is unique up to a translation along the x_1-axis.

Given u a solution of (2.2.53), we set

$$V(\Gamma,p,\epsilon) := \int_\Gamma [u]\cdot\nu_\Gamma\,\mathrm{d}\mathcal{H}^1\,, \tag{2.2.54}$$

the volume between the crack lips. Then, the following proposition holds.

Proposition 2.2.14. *For every $\Gamma\in\mathcal{C}_\eta$, every $V\in[0,+\infty)$, and every $\epsilon\in\mathbb{R}$, we have*

$$V(\Gamma,p(\Gamma,V,\epsilon),\epsilon) = V\,. \tag{2.2.55}$$

Proof. During this proof, we denote by u_V a solution of (2.2.20) associated to (Γ,V,ϵ), and by u_p a solution of (2.2.53) corresponding to $(\Gamma,p(\Gamma,V,\epsilon),\epsilon)$.

First of all, we notice that, by (2.2.42), the energy defined in (2.2.52) reduces to

$$\mathscr{E}(u,\Gamma,p(\Gamma,V,\epsilon),\epsilon) = \mathcal{E}(u,\Gamma,\epsilon) - q(\Gamma,V,\epsilon)\int_\Gamma [u]\cdot\nu_\Gamma\,\mathrm{d}\mathcal{H}^1 \tag{2.2.56}$$

for every $u\in\mathcal{AD}(\Gamma,\epsilon)$.

If $V=0$, we have, by Remarks 2.2.11 and 2.2.13, that $p(\Gamma,V,\epsilon)=-\sigma(\epsilon)$. Hence, it is clear by (2.2.19) and (2.2.52) that we can take $u_V=u_p=\epsilon\,id$, and (2.2.55) is satisfied.

Assume now $V>0$. Let us first show that $V(\Gamma,p(\Gamma,V,\epsilon),\epsilon)>0$. By contradiction, if $V(\Gamma,p(\Gamma,V,\epsilon),\epsilon)=0$, then, up to a horizontal translation, $u_p=\epsilon\,id$. Thus, by (2.2.19), (2.2.41), (2.2.52), (2.2.56), and by the minimality of u_p, we get that

$$\begin{aligned}&\mathscr{E}(u_p,\Gamma,p(\Gamma,V,\epsilon),\epsilon) = \kappa\mathcal{H}^1(\Gamma)\\ &\quad\le \mathscr{E}(u_V,\Gamma,p(\Gamma,V,\epsilon),\epsilon) = \frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma}\mathbb{C}(\mathrm{E}u_V-\epsilon\mathrm{I})\cdot(\mathrm{E}u_V-\epsilon\mathrm{I})\,\mathrm{d}x - q(\Gamma,V,\epsilon)V + \kappa\mathcal{H}^1(\Gamma)\\ &\quad= -\frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma}\mathbb{C}(\mathrm{E}u_V-\epsilon\mathrm{I})\cdot(\mathrm{E}u_V-\epsilon\mathrm{I})\,\mathrm{d}x + \kappa\mathcal{H}^1(\Gamma)\,,\end{aligned}$$

which, in view of (2.2.3), leads to a contradiction. Hence, $V(\Gamma, p(\Gamma, V, \epsilon), \epsilon) > 0$.

Arguing as in Proposition 2.2.6 and Remark 2.2.10, we can prove that

$$q(\Gamma, V, \epsilon) = \frac{1}{V(\Gamma, p(\Gamma, V, \epsilon), \epsilon)} \int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u_p - \epsilon \mathrm{I}) \cdot (\mathrm{E}u_p - \epsilon \mathrm{I}) \, \mathrm{d}x \,. \tag{2.2.57}$$

Therefore, by the minimality of u_p and by formula (2.2.56) we have

$$\begin{aligned} \mathscr{E}(u_p, \Gamma, p(\Gamma, V, \epsilon), \epsilon) &= \mathcal{E}(u_p, \Gamma, \epsilon) - q(\Gamma, V, \epsilon) V(\Gamma, p(\Gamma, V, \epsilon), \epsilon) \\ &\leq \mathscr{E}(u_V, \Gamma, p(\Gamma, V, \epsilon), \epsilon) = \mathcal{E}(u_V, \Gamma, \epsilon) - q(\Gamma, V, \epsilon) V \,. \end{aligned} \tag{2.2.58}$$

Combining (2.2.41), (2.2.57), and (2.2.58), we get

$$\int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u_V - \epsilon \mathrm{I}) \cdot (\mathrm{E}u_V - \epsilon \mathrm{I}) \, \mathrm{d}x \leq \int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u_p - \epsilon \mathrm{I}) \cdot (\mathrm{E}u_p - \epsilon \mathrm{I}) \, \mathrm{d}x \,,$$

which implies, together with (2.2.58), that

$$V(\Gamma, p(\Gamma, V, \epsilon), \epsilon) \geq V \,. \tag{2.2.59}$$

Finally, let us set

$$v := \frac{V}{V(\Gamma, p(\Gamma, V, \epsilon), \epsilon)} (u_p - \epsilon \, id) + \epsilon \, id \,.$$

Then $v \in \mathcal{A}(\Gamma, V, \epsilon)$ and, by (2.2.8), (2.2.19), (2.2.41), (2.2.57), (2.2.59) and by definition of u_V,

$$\begin{aligned} \mathcal{E}(u_V, \Gamma, \epsilon) \leq \mathcal{E}(v, \Gamma, \epsilon) &= \left(\frac{V}{V(\Gamma, p(\Gamma, V, \epsilon), \epsilon)} \right)^2 \mathcal{E}^{el}(u_p, \Gamma, \epsilon) + \kappa \mathcal{H}^1(\Gamma) \\ &= \frac{V^2 \, q(\Gamma, V, \epsilon)}{2V(\Gamma, p(\Gamma, V, \epsilon), \epsilon)} + \kappa \mathcal{H}^1(\Gamma) = \frac{V}{V(\Gamma, p(\Gamma, V, \epsilon), \epsilon)} \mathcal{E}^{el}(u_V, \Gamma, \epsilon) + \kappa \mathcal{H}^1(\Gamma) \\ &\leq \mathcal{E}(u_V, \Gamma, \epsilon) \,. \end{aligned}$$

Therefore, the only possibility is $V(\Gamma, p(\Gamma, V, \epsilon), \epsilon) = V$, and this concludes the proof. □

Remark 2.2.15. With the notation used in Proposition 2.2.14, we also get that u_V and u_p coincide up to a horizontal translation.

Remark 2.2.16. Let us comment on the meaning of the result obtained in Proposition 2.2.14. When considering the equilibrium problem for the elastic body $\mathbb{R}^2_+$ subject to a far strain field $\epsilon \mathrm{I}$, $\epsilon \in \mathbb{R}$, with a crack Γ containing an incompressible fluid, we can, in principle, decide to work in two different settings: assume to know either the volume V or the pressure p of the fluid inside Γ. In the first case, we are led to study the minimum problem (2.2.20), finding, according to Proposition 2.2.6 and Remark 2.2.11, the fluid pressure $p(\Gamma, V, \epsilon)$. If, viceversa, we know the pressure p acting on Γ, we can solve the minimum problem (2.2.53) and deduce from formula (2.2.54) the volume $V(\Gamma, p, \epsilon)$ of the fluid between the crack lips. The equality (2.2.55) proved in Proposition 2.2.14 means that the solutions obtained considering either (2.2.20) or (2.2.52) coincide (same volumes, pressures, and displacements). Hence, we are considering the same problem from two different viewpoints. As it will be clear in Section 2.2.2 (see Remark 2.2.20), working with fixed fluid volume (2.2.20) is better for our purposes.

2.2.2 Quasi-static evolution problem

We now describe the quasi-static evolution for our model of hydraulic fracture. Given $T > 0$, for every $t \in [0, T]$ the elastic body is subject to a uniform strain field $\epsilon(t)\mathrm{I}$, $\epsilon(t) \in \mathbb{R}$, while a pressure $p_\infty(t) \in \mathbb{R}$ acts on the fluid far from the crack inlet. For technical reasons, we assume $\epsilon, p_\infty \in C([0,T])$. We denote by $V(t)$ the volume of fluid injected into the crack at time t.

It is convenient to introduce the reduced energy $\mathcal{E}_m(t, \Gamma, V)$ defined for every $t \in [0, T]$, every $\Gamma \in \mathcal{C}_\eta$, and every $V \in [0, +\infty)$ by

$$\mathcal{E}_m(t, \Gamma, V) := \min_{u \in \mathcal{A}(\Gamma, V, \epsilon(t))} \mathcal{E}(u, \Gamma, \epsilon(t)) = \min_{u \in \mathcal{A}(\Gamma, V, \epsilon(t))} \mathcal{E}^{el}(u, \Gamma, \epsilon(t)) + \kappa \mathcal{H}^1(\Gamma) . \quad (2.2.60)$$

Following [56] and [38], we state the problem in the general framework of rate-independent processes. The evolution is described by a crack set function $t \mapsto \Gamma(t)$ and a volume function $t \mapsto V(t)$. The Griffith's stability condition is here expressed in a derivative free setting in the following way: for every $t \in [0, T]$

$$\mathcal{E}_m(t, \Gamma(t), V(t)) \le \mathcal{E}_m(t, \Gamma, V(t)) \qquad \text{for every } \Gamma \in \mathcal{C}_\eta \text{ with } \Gamma \supseteq \Gamma(t).$$

Since the process is irreversible, we require $t \mapsto \Gamma(t)$ to be an increasing set function. Moreover, we impose an energy-dissipation balance: the rate of change of the reduced energy (2.2.60) of the system along a solution equals the power of the pressure forces exerted by the fluid plus the power expended by the far stress field $\sigma(\epsilon(t))$ generated by the strain $\epsilon(t)$ (see (2.2.6)).

Finally, we have to give an evolution law for the volume function $t \mapsto V(t)$. As we have seen in Proposition 2.2.6 and Remark 2.2.11, the presence of a strain $\epsilon(t)\mathrm{I}$ and of a volume $V(t)$ of fluid inside the crack $\Gamma(t)$ produces a pressure $p(t) := p(\Gamma(t), V(t), \epsilon(t))$ acting on the fracture lips, which is also interpreted as the fluid pressure inside the crack (see Remarks 2.2.11 and 2.2.13). As a consequence, a pressure difference $p_\infty(t) - p(t)$ is created into the fluid, which drives the evolution of $V(\cdot)$ according to the approximate Darcy's law: $\dot{V}(t) = p_\infty(t) - p(t)$.

This leads to the following definition.

Definition 2.2.17. Let $T > 0$, and let $\epsilon, p_\infty \in C([0,T])$. We say that a pair $(\Gamma, V)\colon [0, T] \to \mathcal{C}_\eta \times [0, +\infty)$ is an *irreversible quasi-static evolution for the hydraulic crack problem* if it satisfies the following conditions:

(*a*) *irreversibility*: Γ is increasing, i.e., $\Gamma(s) \subseteq \Gamma(t)$ for every $0 \le s \le t \le T$;

(*b*) *global stability*: for every $t \in [0,T]$,

$$\mathcal{E}_m(t,\Gamma(t),V(t)) \le \mathcal{E}_m(t,\Gamma,V(t)) \qquad \text{for every } \Gamma \in \mathcal{C}_\eta \text{ with } \Gamma \supseteq \Gamma(t)\,;$$

(*c*) *Darcy's law*: the function V is absolutely continuous on the interval $[0,T]$ and

$$\dot V(t) = (p_\infty(t) - p(t))\mathbf{1}_{\{V>0\}}(t)$$

for almost every $t \in [0,T]$, where $p(t) := q(\Gamma(t),V(t),\epsilon(t)) - \sigma(\epsilon(t))$ is the pressure introduced in Remark 2.2.11;

(*d*) *energy-dissipation balance*: the function $t \mapsto \mathcal{E}_m(t,\Gamma(t),V(t))$ is absolutely continuous on the interval $[0,T]$ and

$$\frac{d}{dt}\mathcal{E}_m(t,\Gamma(t),V(t)) = \big(p(t) + \sigma(\epsilon(t))\big)\dot V(t) \tag{2.2.61}$$

for almost every $t \in [0,T]$.

We are now in a position to state the main theorem of this paper.

Theorem 2.2.18. *Let $\epsilon, p_\infty \in C([0,T])$ and let $\Gamma_0 \in \mathcal{C}_\eta$ and $V_0 \in [0,+\infty)$. Assume that (stability at time $t=0$)*

$$\mathcal{E}_m(0,\Gamma_0,V_0) \le \mathcal{E}_m(0,\Gamma,V_0) \tag{2.2.62}$$

for every $\Gamma \in \mathcal{C}_\eta$ with $\Gamma \supseteq \Gamma_0$. Then, there exists an irreversible quasi-static evolution (Γ,V) of the hydraulic crack problem, with $\Gamma(0)=\Gamma_0$ and $V(0)=V_0$.

Let us comment on the initial condition of Theorem 2.2.18.

Remark 2.2.19. If the pair $(\Gamma_0,V_0) \in \mathcal{C}_\eta \times [0,+\infty)$ does not satisfy the stability condition (2.2.62), we define a new initial condition (Γ_0^*,V_0), with Γ_0^* solution of (2.2.62). In particular, Γ_0^* minimizes $\mathcal{E}_m(0,\Gamma,V_0)$ among all $\Gamma \in \mathcal{C}_\eta$ with $\Gamma \supseteq \Gamma_0^*$. Therefore, we can solve the evolution problem in Theorem 2.2.18 starting from (Γ_0^*,V_0).

A solution of (2.2.62) can be found by the direct method of the calculus of variations. Indeed, a minimizing sequence $\Gamma_k \in \mathcal{C}_\eta$ has bounded $\mathcal{H}^1$-measure, and thus is bounded in $\mathcal{C}_\eta$. By Proposition 2.2.2, we may assume that $\Gamma_k \to \Gamma$ in the Hausdorff metric, for a suitable $\Gamma \in \mathcal{C}_\eta$. For every $k \in \mathbb{N}$, there exists a unique (up to a horizontal translation) $u_k \in \mathcal{A}(\Gamma_k,V_0,\epsilon(0))$ solution of (2.2.20). Since $\mathrm{E}u_k - \epsilon(0)\mathrm{I}$ is bounded in $L^2(\mathbb{R}^2_+;\mathbb{M}^2_{sym})$, by Proposition 2.2.4 we have $\mathrm{E}u_k - \epsilon(0)\mathrm{I} \rightharpoonup \mathrm{E}v - \epsilon(0)\mathrm{I}$ weakly in $L^2(\mathbb{R}^2_+;\mathbb{M}^2_{sym})$ for some $v \in \mathcal{A}(\Gamma,V_0,\epsilon(0))$, and

$$\mathcal{E}_m(0,\Gamma,V_0) \le \mathcal{E}(v,\Gamma,\epsilon(0)) \le \liminf_k \mathcal{E}_m(0,\Gamma_k,V_0)\,.$$

Thus Γ is a minimizer.

The following remark explains why it is convenient to state the evolution problem in terms of the energy functional $\mathcal{E}$ defined in (2.2.19) rather than working with $\mathscr{E}$ of formula (2.2.52).

Remark 2.2.20. Let us assume for a moment to know a priori the pressure p of the fluid inside the crack $\Gamma \in \mathcal{C}_\eta$. Given $t \in [0,T]$, we may define the reduced energy

$$\mathscr{E}_m(t,\Gamma,p) := \min_{u \in \mathcal{AD}(\Gamma,\epsilon(t))} \mathscr{E}(u,\Gamma,p,\epsilon(t)), \tag{2.2.63}$$

where $\mathscr{E}$ and $\mathcal{AD}(\Gamma,\epsilon(t))$ are defined in (2.2.52) and (2.2.4), respectively. The non-interpenetration condition in (2.2.4) and the presence of the linear term

$$(p + \sigma(\epsilon(t))) \int_\Gamma [u] \cdot \nu_\Gamma \, \mathrm{d}\mathcal{H}^1$$

in (2.2.52) imply that the reduced energy $\mathscr{E}_m$ is not bounded from below with respect to the crack set variable. Indeed, when we try to repeat the argument of Remark 2.2.19, it is possible (when $p + \sigma(\epsilon(t)) > 0$) to construct a sequence Γ_k in $\mathcal{C}_\eta$ such that $\mathscr{E}_m(t,\Gamma_k,p) \to -\infty$ and $\mathcal{H}^1(\Gamma_k) \to +\infty$. This means that it would be energetically convenient to have a catastrophic rupture of the elastic body, which is in contrast with the quasi-static nature of the phenomenon we are studying.

On the contrary, the energy $\mathcal{E}_m$ defined in (2.2.60) is always positive, and this simplifies our analysis.

To prove Theorem 2.2.18, and in particular to obtain the global stability condition of Definition 2.2.17, we need the following two technical lemmas. The first one corresponds, in our setting, to the Jump Transfer Theorem [37, Theorem 2.1].

Lemma 2.2.21. *Let $\Gamma, \Gamma_k, \Gamma_\infty, \hat\Gamma_\infty \in \mathcal{C}_\eta$ be such that $\Gamma \subseteq \Gamma_k$, $\Gamma_k \to \Gamma_\infty$ in the Hausdorff metric, and $\Gamma_\infty \subseteq \hat\Gamma_\infty$. Let $V_k, V_\infty > 0$ and $t_k, t_\infty \in [0,T]$ with $V_k \to V_\infty$ and $t_k \to t_\infty$, and let $u \in \mathcal{A}(\hat\Gamma_\infty, V, \epsilon(t_\infty))$. Then there exist a sequence $\hat\Gamma_k$ in $\mathcal{C}_\eta$ and a sequence $u_k \in \mathcal{A}(\hat\Gamma_k, V_k, \epsilon(t_k))$ such that $\hat\Gamma_k \to \hat\Gamma_\infty$ in the Hausdorff metric, $\Gamma_k \subseteq \hat\Gamma_k$, $\mathrm{E}u_k - \epsilon(t_k)\mathrm{I} \to \mathrm{E}u - \epsilon(t_\infty)\mathrm{I}$ strongly in $L^2(\mathbb{R}^2_+;\mathbb{M}^2_{sym})$, and $\mathcal{E}(u_k,\hat\Gamma_k,\epsilon(t_k)) \to \mathcal{E}(u,\hat\Gamma_\infty,\epsilon(t_\infty))$.*

Proof. The proof is carried out following the steps of [64, Lemma 3.7]. The letter C will denote a positive constant, which can possibly change from line to line.

First, we construct the sets $\hat\Gamma_k$. Let $a_k, a_\infty > 0$, $\hat a_\infty > a_\infty$, $\gamma_k \in C^{1,1}([0,a_k])$, $\gamma_\infty \in C^{1,1}([0,a_\infty])$, and $\hat\gamma_\infty \in C^{1,1}([0,\hat a_\infty])$ be as in Definition 2.2.1. In particular, $\Gamma_k = \mathrm{graph}(\gamma_k)$, $\Gamma_\infty = \mathrm{graph}(\gamma_\infty)$, and $\hat\Gamma_\infty = \mathrm{graph}(\hat\gamma_\infty)$. It is also convenient to define a $W^{2,\infty}$-extension of $\hat\gamma_\infty$ to the interval $[0, \hat a_\infty + 2\delta]$, for some $\delta > 0$. For instance, this can be done in the following way:

$$\hat\gamma_\infty(x_2) := \begin{cases} \hat\gamma_\infty(x_2) & \text{if } x_2 \in [0,\hat a_\infty], \\ \hat\gamma_\infty(\hat a_\infty) + (x_2 - \hat a_\infty)\hat\gamma'_\infty(\hat a_\infty) & \text{if } x_2 \in (\hat a_\infty, \hat a_\infty + 2\delta]. \end{cases}$$

In view of the Hausdorff convergence of Γ_k to Γ_∞, we have that

$$a_k \to a_\infty, \quad \gamma_k(a_k) \to \gamma_\infty(a_\infty) = \hat\gamma_\infty(a_\infty), \quad \gamma'_k(a_k) \to \gamma'_\infty(a_\infty) = \hat\gamma'_\infty(a_\infty). \tag{2.2.64}$$

Without loss of generality, we may assume that $\gamma_k'(a_k) \geq \hat{\gamma}_\infty'(a_\infty) \geq 0$ (the other cases can be dealt in similar ways). Let $r \geq (1+\eta^2)/\eta$ and

$$z_k := (\gamma_k(a_k), a_k) - \frac{r}{\sqrt{1+|\gamma_k'(a_k)|^2}}(1, -\gamma_k'(a_k)) \ \in \mathbb{R}^2_+ \,.$$

Let us consider the ball $\mathrm{B}_r(z_k)$, which is tangent to Γ_k in $(\gamma_k(a_k), a_k)$. In a neighborhood of $(\gamma_k(a_k), a_k)$, the circle $\partial \mathrm{B}_r(z_k)$ can be seen as the graph of the function

$$\zeta_k(x_2) := \gamma_k(a_k) - \frac{r}{\sqrt{1+|\gamma_k'(a_k)|^2}} + \sqrt{r^2 - \left(x_2 - a_k - \frac{r\gamma_k'(a_k)}{\sqrt{1+|\gamma_k'(a_k)|^2}}\right)^2}\,.$$

We deduce that there exists $b_k \geq a_k$ such that $\hat{\gamma}_\infty'(a_\infty) = \zeta_k'(b_k)$ and $\hat{\gamma}_\infty'(a_\infty) \leq \zeta_k'(x_2) \leq \gamma_k'(a_k)$ for every $x_2 \in (a_k, b_k)$. Moreover, by the choice of r we have $|\zeta_k''(x_2)| \leq \eta$ in (a_k, b_k), and, by (2.2.64), $b_k \to a_\infty$.

We define

$$\hat{\gamma}_k(x_2) := \begin{cases} \gamma_k(x_2) & \text{if } x_2 \in [0, a_k], \\ \zeta_k(x_2) & \text{if } x_2 \in (a_k, b_k], \\ \hat{\gamma}_\infty(x_2 + a_\infty - b_k) + \zeta_k(b_k) - \hat{\gamma}_\infty(a_\infty) & \text{if } x_2 \in (b_k, \hat{a}_\infty + 2\delta + b_k - a_\infty]. \end{cases}$$

For k large enough, we have that $\hat{\gamma}_k$ is well-defined on the interval $[0, \hat{a}_\infty + \delta]$, $\hat{\gamma}_k \in C^{1,1}([0, \hat{a}_\infty + \delta])$ and, by construction of ζ_k,

$$\|\hat{\gamma}_k'\|_{\infty,[0,\hat{a}_\infty+\delta]} \leq \eta \qquad \text{and} \qquad \|\hat{\gamma}_k''\|_{\infty,[0,\hat{a}_\infty+\delta]} \leq \eta\,. \tag{2.2.65}$$

It is easy to see that $\hat{\gamma}_k \rightharpoonup \hat{\gamma}_\infty$ weakly* in $W^{2,\infty}([0, \hat{a}_\infty])$. Therefore, if we set $\hat{\Gamma}_k := \mathrm{graph}(\hat{\gamma}_k|_{[0,\hat{a}_\infty]})$, we deduce that $\Gamma_k \subseteq \hat{\Gamma}_k$ and $\hat{\Gamma}_k \to \hat{\Gamma}_\infty$ in the Hausdorff metric. Moreover, by (2.2.65), $\hat{\Gamma}_k \in \mathcal{C}_\eta$,

Let us fix $\rho > 0$ and let $d_k := \|\hat{\gamma}_k - \hat{\gamma}_\infty\|_{W^{1,\infty}([0,\hat{a}_\infty+\delta])}$. By the weak* convergence in $W^{2,\infty}$ of $\hat{\gamma}_k$ to $\hat{\gamma}_\infty$, we have that $d_k \to 0$. For k large enough (so that $\hat{\Gamma}_k \subseteq \mathcal{I}_\rho(\hat{\Gamma}_\infty)$), we want to construct a $C^{1,1}$-function $\Lambda_{k,\rho}$ such that $\Lambda_{k,\rho}(\hat{\Gamma}_\infty) = \hat{\Gamma}_k$ and $\Lambda_{k,\rho}(x) = x$ for $x \in \mathbb{R}^2 \setminus \mathrm{I}_\rho(\hat{\Gamma}_\infty)$. Let us first fix a function $\vartheta_\rho \in C_c^\infty(\mathbb{R}^2)$ such that $0 \leq \vartheta_\rho \leq 1$, $\vartheta_\rho = 1$ on $\mathcal{I}_{\rho/2}(\hat{\Gamma}_\infty \setminus \Gamma)$, and $supp(\vartheta_\rho) \subset\subset \mathcal{I}_\rho(\hat{\Gamma}_\infty \setminus \Gamma)$. For every $x = (x_1, x_2) \in \mathcal{I}_\rho(\hat{\Gamma}_\infty \setminus \Gamma)$, we define

$$\Lambda_{k,\rho}(x) := x + \begin{pmatrix} \vartheta_\rho(x)\big(\hat{\gamma}_k(x_2) - \hat{\gamma}_\infty(x_2)\big) \\ 0 \end{pmatrix}. \tag{2.2.66}$$

By the properties of ϑ_ρ, we have that $\Lambda_{k,\rho}(x) = x$ for every $x \notin \mathcal{I}_\rho(\hat{\Gamma}_\infty \setminus \Gamma)$, so that it makes sense to extend $\Lambda_{k,\rho}$ with the identity out of $\mathcal{I}_\rho(\hat{\Gamma}_\infty \setminus \Gamma)$. Moreover, we notice that, $\Lambda_{k,\rho} \in C^{1,1}(\mathbb{R}^2; \mathbb{R}^2)$ and $\Lambda_{k,\rho}(\hat{\Gamma}_\infty) = \hat{\Gamma}_k$.

From (2.2.66) and the definition of d_k, we deduce that

$$\lim_k \|\Lambda_{k,\rho} - id\|_{W^{1,\infty}(\mathbb{R}^2)} = 0\,, \tag{2.2.67}$$

$$\limsup_k \|\Lambda_{k,\rho} - id\|_{W^{2,\infty}(\mathbb{R}^2)} \leq C\,, \tag{2.2.68}$$

where $C > 0$ in (2.2.68) is independent of ρ. In particular, in view of (2.2.67), we can apply Hadamard Theorem (see [49, Theorem 6.2.3]), to deduce that $\Lambda_{k,\rho}$ is globally invertible with $\Lambda_{k,\rho}^{-1} \in C^{1,1}(\mathbb{R}^2;\mathbb{R}^2)$ and $\|\Lambda_{k,\rho}^{-1} - id\|_{W^{1,\infty}(\mathbb{R}^2)} \to 0$ as $k \to +\infty$.

We are now in a position to define the approximating functions. Let u belong to $\mathcal{A}(\hat{\Gamma}_\infty, V_\infty, \epsilon(t_\infty))$. We set

$$v_{k,\rho} := \big((cof\,\nabla\Lambda_{k,\rho})^{-T}(u - \epsilon(t_\infty)\,id)\big) \circ \Lambda_{k,\rho}^{-1}\,, \tag{2.2.69}$$

$$u_{k,\rho} := \frac{V_k}{V_\infty} v_{k,\rho} + \epsilon(t_k)\,id\,. \tag{2.2.70}$$

Thanks to [20, Section 1.7], $u_{k,\rho}$ satisfies the non-interpenetration condition and the volume constraint on $\hat{\Gamma}_k$, hence $u_{k,\rho} \in \mathcal{A}(\hat{\Gamma}_k, v_k, \epsilon(t_k))$. Moreover, (2.2.67)-(2.2.70) and Proposition 1.2.1 imply that

$$\limsup_k \|\mathrm{E}v_{k,\rho}\|_{2,\mathcal{I}_\rho(\hat{\Gamma}_\infty)} \le C\|u - \epsilon(t_\infty)\,id\|_{H^1(\mathcal{I}_\rho(\hat{\Gamma}_\infty)\setminus\hat{\Gamma}_\infty)}\,, \tag{2.2.71}$$

$$\mathrm{E}u_{k,\rho} - \epsilon(t_k)\mathrm{I} = \frac{V_k}{V_\infty}(\mathrm{E}u - \epsilon(t_\infty)\mathrm{I}) \qquad \text{in } \mathbb{R}^2_+ \setminus \mathcal{I}_\rho(\hat{\Gamma}_\infty). \tag{2.2.72}$$

In view of (2.2.70) and (2.2.72), we have that

$$\begin{aligned}
&|\mathcal{E}(u_{k,\rho}, \hat{\Gamma}_k, \epsilon(t_k)) - \mathcal{E}(u, \hat{\Gamma}_\infty, \epsilon(t_\infty))| \\
&\quad \le \frac{V_k^2}{2V_\infty^2} \int_{\mathcal{I}_\rho(\hat{\Gamma}_\infty)} \mathbb{C}\mathrm{E}v_{k,\rho} \cdot \mathrm{E}v_{k,\rho}\,\mathrm{d}x + \frac{1}{2}\int_{\mathcal{I}_\rho(\hat{\Gamma}_\infty)} \mathbb{C}(\mathrm{E}u - \epsilon(t_\infty)\mathrm{I}) \cdot (\mathrm{E}u - \epsilon(t_\infty)\mathrm{I})\,\mathrm{d}x \\
&\qquad + \frac{1}{2}\left(\frac{V_k^2}{V_\infty^2} - 1\right) \int_{\mathbb{R}^2_+\setminus\mathcal{I}_\rho(\hat{\Gamma}_\infty)} \mathbb{C}(\mathrm{E}u - \epsilon(t_\infty)\mathrm{I}) \cdot (\mathrm{E}u - \epsilon(t_\infty)\mathrm{I})\,\mathrm{d}x \\
&\qquad + |\mathcal{H}^1(\hat{\Gamma}_k) - \mathcal{H}^1(\hat{\Gamma}_\infty)|\,.
\end{aligned} \tag{2.2.73}$$

Recalling that $\mathcal{H}^1(\hat{\Gamma}_k) \to \mathcal{H}^1(\hat{\Gamma}_\infty)$, $V_k \to V_\infty$, and that (2.2.3) and (2.2.71) hold, we pass to the lim sup in (2.2.73) as $k \to +\infty$ obtaining

$$\limsup_k |\mathcal{E}(u_{k,\rho}, \hat{\Gamma}_k, \epsilon(t_k)) - \mathcal{E}(u, \hat{\Gamma}_\infty, \epsilon(t_\infty))| \le C\|u - \epsilon(t_\infty)\,id\|^2_{H^1(\mathcal{I}_\rho(\hat{\Gamma}_\infty))}\,. \tag{2.2.74}$$

Passing to the limit as $\rho \to 0$ in (2.2.74), we deduce that

$$\lim_{\rho\to 0} \limsup_k |\mathcal{E}(u_{k,\rho}, \hat{\Gamma}_k, \epsilon(t_k)) - \mathcal{E}(u, \hat{\Gamma}_\infty, \epsilon(t_\infty))| = 0\,. \tag{2.2.75}$$

Therefore, in view of (2.2.71) and (2.2.75), we can construct a sequence of functions $u_k \in \mathcal{A}(\hat{\Gamma}_k, V_k, \epsilon(t_k))$ such that $\mathcal{E}(u_k, \hat{\Gamma}_k, \epsilon(t_k)) \to \mathcal{E}(u, \hat{\Gamma}_\infty, \epsilon(t_\infty))$ and $\mathrm{E}u_k - \epsilon(t_k)\mathrm{I} \to \mathrm{E}u - \epsilon(t_\infty)\mathrm{I}$ strongly in $L^2(\mathbb{R}^2_+;\mathbb{M}^2_{sym})$. This concludes the proof of the lemma. □

The following lemma will be useful in the proof of the global stability condition (b) of Definition 2.2.17.

Lemma 2.2.22. *Let* $\Gamma, \Gamma_k, \Gamma_\infty \in \mathcal{C}_\eta$ *be such that* $\Gamma \subseteq \Gamma_k$ *and* $\Gamma_k \to \Gamma_\infty$ *in the Hausdorff metric. Let* $V_k, V_\infty \geq 0$ *and* $t_k, t_\infty \in [0,T]$ *with* $V_k \to V_\infty$ *and* $t_k \to t_\infty$. *Assume that*

$$\mathcal{E}_m(t_k, \Gamma_k, V_k) \leq \mathcal{E}_m(t_k, \hat{\Gamma}, V_k) \quad \text{for every } \hat{\Gamma} \in \mathcal{C}_\eta \text{ with } \hat{\Gamma} \supseteq \Gamma_k. \tag{2.2.76}$$

Then

$$\mathcal{E}_m(t_\infty, \Gamma_\infty, V_\infty) \leq \mathcal{E}_m(t_\infty, \hat{\Gamma}, V_\infty) \quad \text{for every } \hat{\Gamma} \in \mathcal{C}_\eta \text{ with } \hat{\Gamma} \supseteq \Gamma_\infty. \tag{2.2.77}$$

Moreover, let u_k, u_∞ *be solutions of* (2.2.20) *corresponding to the triples* $(\Gamma_k, V_k, \epsilon(t_k))$ *and* $(\Gamma_\infty, V_\infty, \epsilon(t_\infty))$, *and let* $p(\Gamma_k, V_k, \epsilon_k)$, $p(\Gamma_\infty, V_\infty, \epsilon_\infty)$ *be the corresponding pressures according to Remark 2.2.11. Then* $\mathrm{E}u_k - \epsilon(t_k)\mathrm{I} \to \mathrm{E}u_\infty - \epsilon(t_\infty)\mathrm{I}$ *in* $L^2(\mathbb{R}^2_+; \mathbb{M}^2_{sym})$, $p(\Gamma_k, V_k, \epsilon_k) \to p(\Gamma_\infty, V_\infty, \epsilon_\infty)$, *and* $\mathcal{E}_m(t_k, \Gamma_k, V_k) \to \mathcal{E}_m(t_\infty, \Gamma_\infty, V_\infty)$.

Proof. Let us fix $w_0 \in \mathcal{A}(\Gamma, 1, 0)$. Then,

$$w_k := V_k w_0 + \epsilon(t_k)\, id \in \mathcal{A}(\Gamma_k, V_k, \epsilon(t_k))$$

and, by definition of u_k,

$$\mathcal{E}^{el}(u_k, \Gamma_k, \epsilon(t_k)) \leq \mathcal{E}^{el}(w_k, \Gamma_k, \epsilon(t_k)) = V_k^2 \mathcal{E}^{el}(w_0, \Gamma_k, 0). \tag{2.2.78}$$

In view of (2.2.3), inequality (2.2.78) implies that the sequence $\mathrm{E}u_k - \epsilon(t_k)\mathrm{I}$ is bounded in $L^2(\mathbb{R}^2_+; \mathbb{M}^2_{sym})$. Hence, applying Proposition 2.2.4, we deduce that there exists $u_\infty \in \mathcal{A}(\Gamma_\infty, V_\infty, \epsilon(t_\infty))$ such that, up to a subsequence,

$$\mathrm{E}u_k - \epsilon(t_k)\mathrm{I} \rightharpoonup \mathrm{E}u_\infty - \epsilon(t_\infty)\mathrm{I} \quad \text{weakly in } L^2(\mathbb{R}^2_+). \tag{2.2.79}$$

Let us prove (2.2.77). Let $\hat{\Gamma} \in \mathcal{C}_\eta$, $\hat{\Gamma} \supseteq \Gamma_\infty$ be fixed. Let us denote by $u_{\hat{\Gamma}} \in \mathcal{A}(\hat{\Gamma}, V_\infty, \epsilon(t_\infty))$ a solution to (2.2.20) associated to $(\hat{\Gamma}, V_\infty, \epsilon(t_\infty))$. Applying Lemma 2.2.21 to $\Gamma_k, \Gamma_\infty, \hat{\Gamma}$, we can find a sequence $\hat{\Gamma}_k \in \mathcal{C}_\eta$ such that $\hat{\Gamma}_k \supseteq \Gamma_k$ and $\hat{\Gamma}_k \to \hat{\Gamma}$ in the Hausdorff metric, as well as a sequence of functions $v_k \in \mathcal{A}(\hat{\Gamma}_k, V_k, \epsilon(t_k))$ such that $\mathcal{E}(v_k, \hat{\Gamma}_k, \epsilon(t_k)) \to \mathcal{E}(u_{\hat{\Gamma}}, \hat{\Gamma}, \epsilon(t_\infty))$.

By (2.2.60), (2.2.76) and (2.2.79), we have that

$$\begin{aligned}
&\mathcal{E}_m(t_\infty, \Gamma_\infty, V_\infty) \leq \mathcal{E}(u_\infty, \Gamma_\infty, \epsilon(t_\infty)) \leq \liminf_k \mathcal{E}(u_k, \Gamma_k, \epsilon(t_k)) \\
&\quad = \liminf_k \mathcal{E}_m(t_k, \Gamma_k, V_k) \leq \limsup_k \mathcal{E}_m(t_k, \Gamma_k, V_k) \leq \limsup_k \mathcal{E}_m(t_k, \hat{\Gamma}_k, V_k) \\
&\quad \leq \lim_k \mathcal{E}(v_k, \hat{\Gamma}_k, \epsilon(t_k)) = \mathcal{E}(u_{\hat{\Gamma}}, \hat{\Gamma}, \epsilon(t_\infty)) = \mathcal{E}_m(t_\infty, \hat{\Gamma}, V_\infty),
\end{aligned} \tag{2.2.80}$$

from which we deduce (2.2.77). Moreover, taking $\hat{\Gamma} = \Gamma_\infty$ in (2.2.80), we get that $u_\infty \in \mathcal{A}(\Gamma_\infty, V_\infty, \epsilon(t_\infty))$ is a solution of (2.2.20), $\mathrm{E}u_k - \epsilon(t_k)\mathrm{I} \to \mathrm{E}u_\infty - \epsilon(t_\infty)\mathrm{I}$ strongly in $L^2(\mathbb{R}^2_+; \mathbb{M}^2_{sym})$, and $\mathcal{E}_m(t_k, \Gamma_k, V_k) \to \mathcal{E}_m(t_\infty, \Gamma_\infty, V_\infty)$. In view of these convergences, of Remark 2.2.10, and of formula (2.2.42), we deduce that $p(\Gamma_k, V_k, \epsilon_k) \to p(\Gamma_\infty, V_\infty, \epsilon_\infty)$, at least in the case $V_\infty > 0$.

It remains to prove that $p(\Gamma_k, V_k, \epsilon_k) \to -\sigma(\epsilon(t_\infty)) = p(\Gamma_\infty, V_\infty, \epsilon(t_\infty))$ if $V_\infty = 0$. Without loss of generality, we may assume $V_k > 0$ for every $k \in \mathbb{N}$. In view of (2.2.78), we have that

$$\int_{\mathbb{R}^2_+ \setminus \Gamma_k} \mathbb{C}(\mathrm{E}u_k - \epsilon(t_k)\mathrm{I}) \cdot (\mathrm{E}u_k - \epsilon(t_k)\mathrm{I})\,\mathrm{d}x \le V_k^2 \int_{\mathbb{R}^2_+ \setminus \Gamma_k} \mathbb{C}\mathrm{E}w_0 \cdot \mathrm{E}w_0\,\mathrm{d}x\,,$$

which implies, together with Remark 2.2.10 and formula (2.2.42), that

$$0 \le p(\Gamma_k, V_k, \epsilon_k) + \sigma(\epsilon(t_k)) \le V_k \int_{\mathbb{R}^2_+ \setminus \Gamma_k} \mathbb{C}\mathrm{E}w_0 \cdot \mathrm{E}w_0\,\mathrm{d}x\,.$$

Since $V_k \to V_\infty = 0$ and $\epsilon(t_k) \to \epsilon(t_\infty)$, we get $p(\Gamma_k, V_k, \epsilon_k) \to -\sigma(\epsilon(t_\infty))$. □

We are now ready to prove Theorem 2.2.18

Proof of Theorem 2.2.18. Let ϵ, p_∞, Γ_0, and V_0 be as in the statement of the theorem and let ν_{Γ_0} be the unit normal vector to Γ_0.

The proof is based on a time discretization process, see [38, 56]. For every $k \in \mathbb{N}$, we introduce the time step $\tau_k := T/k$ and a subdivision of the interval $[0,T]$ of the form $t_i^k := i\tau_k$ for $i = 0, \dots, k$. Let us describe the discrete problems. For every k we define V_i^k and Γ_i^k recursively with respect to i. For $i = 0$, we set $V_0^k := V_0$, $\Gamma_0^k := \Gamma_0$, and $p_0^k := p(\Gamma_0, V_0, \epsilon(0))$ the pressure introduced in Remark 2.2.11. For $i > 0$, assume that we already know V_{i-1}^k, Γ_{i-1}^k, and $p_{i-1}^k := p(\Gamma_{i-1}^k, V_{i-1}^k, \epsilon(t_{i-1}^k))$. We define

$$V_i^k := \max\{V_{i-1}^k + \big(p_\infty(t_{i-1}^k) - p_{i-1}^k\big)\tau_k, 0\}\,. \tag{2.2.81}$$

We notice that (2.2.81) is the discrete approximation of the Darcy's law of Definition 2.2.17. Then, we set Γ_i^k to be a solution of

$$\min\{\mathcal{E}_m(t_i^k, \Gamma, V_i^k) : \Gamma \in \mathcal{C}_\eta,\ \Gamma \supseteq \Gamma_{i-1}^k\}\,, \tag{2.2.82}$$

which can be found arguing as in Remark 2.2.19. In particular, (2.2.82) is the discrete form of the global stability condition in Definition 2.2.17.

Finally, we denote by u_i^k a solution of (2.2.20) with $\Gamma = \Gamma_i^k$, $V = V_i^k$, and $\epsilon = \epsilon(t_i^k)$, and we set $p_i^k := p(\Gamma_i^k, V_i^k, \epsilon(t_i^k))$ to be the corresponding pressure, according to Proposition 2.2.6 and Remark 2.2.11. Arguing as in the proof of Lemma 2.2.22, it is possible to prove that

$$\begin{gathered} \|\mathrm{E}u_i^k - \epsilon(t_i^k)\mathrm{I}\|_{2,\mathbb{R}^2_+} \le CV_i^k\,, \\ -\sigma(\epsilon(t_i^k)) \le p_i^k \le CV_i^k - \sigma(\epsilon(t_i^k))\,, \end{gathered} \tag{2.2.83}$$

for some constant $C > 0$ independent of k and i.

We introduce the following piecewise constant interpolation functions: for $t \in [t_i^k, t_{i+1}^k)$

$$\begin{aligned} &u_k(t) := u_i^k\,, \quad \Gamma_k(t) := \Gamma_i^k\,, \quad V_k(t) := V_i^k\,, \quad \epsilon_k(t) := \epsilon(t_i^k)\,, \\ &p_k(t) := p_i^k\,, \quad p_\infty^k(t) := p_\infty(t_i^k)\,, \quad \sigma_k(t) := \sigma(\epsilon(t_i^k))\,, \end{aligned} \tag{2.2.84}$$

and, for $t \in (t_i^k, t_{i+1}^k]$, $\overline{V}_k(t) := V_{i+1}^k$. Furthermore, we will also use the piecewise affine function

$$V^k(t) := V_{i-1}^k + \frac{V_i^k - V_{i-1}^k}{\tau_k}(t - t_{i-1}^k) \qquad \text{for } t \in (t_{i-1}^k, t_i^k]. \tag{2.2.85}$$

Since $p_i^k \geq -\sigma(\epsilon(t_i^k))$ for every k and every i, from (2.2.81) we easily deduce that

$$V_i^k \leq V_{i-1}^k + |p_\infty(t_{i-1}^k) + \sigma(\epsilon(t_{i-1}^k))|\tau_k\,. \tag{2.2.86}$$

Iterating inequality (2.2.86), we get

$$V_i^k \leq V_0 + \tau_k \sum_{j=1}^{i} |p_\infty(t_{j-1}^k) + \sigma(\epsilon(t_{j-1}^k))|\,. \tag{2.2.87}$$

Taking into account the regularity of $t \mapsto p_\infty(t)$ and of $t \mapsto \sigma(\epsilon(t))$, inequality (2.2.87) implies that

$$\sup_k \|V_k\|_{\infty,[0,T]} < +\infty\,. \tag{2.2.88}$$

Therefore, from (2.2.83) and (2.2.88) we obtain that

$$\sup_{i,k} \|\mathrm{E}u_i^k - \epsilon(t_i^k)\mathrm{I}\|_{2,\mathbb{R}^2_+} < \infty \qquad \text{and} \qquad \sup_k \|p_k\|_{\infty,[0,T]} < +\infty\,. \tag{2.2.89}$$

Moreover, thanks to (2.2.81) we have that

$$V_{i-1}^k - V_i^k \leq |p_\infty(t_{i-1}^k)|\tau_k + |p_{i-1}^k|\tau_k\,. \tag{2.2.90}$$

Combining (2.2.86), (2.2.89), and (2.2.90), we get that

$$\sup_k \|V^k\|_{W^{1,\infty}(0,T)} < +\infty\,. \tag{2.2.91}$$

We now prove a discrete energy inequality. By (2.2.82) we have that

$$\mathcal{E}_m(t_i^k, \Gamma_i^k, V_i^k) \leq \mathcal{E}_m(t_i^k, \Gamma_{i-1}^k, V_i^k)\,. \tag{2.2.92}$$

In order to estimate the right-hand side of (2.2.92), we fix $w_0 \in \mathcal{A}(\Gamma_0, 1, 0)$ and we define the functions

$$v_i^k := \begin{cases} \dfrac{u_i^k - \epsilon(t_i^k)\, id}{V_i^k} & \text{if } V_i^k \neq 0, \\ w_0 & \text{if } V_i^k = 0. \end{cases}$$

Notice that $v_i^k \in \mathcal{A}(\Gamma_i^k, 1, 0)$ and, by (2.2.83),

$$\|\mathrm{E}v_i^k\|_{2,\mathbb{R}^2_+} \leq M\,, \tag{2.2.93}$$

where $M \geq \|\mathrm{E}w_0\|_{2,\mathbb{R}^2_+}$.

Since $u^k_{i-1} + (\epsilon(t^k_i) - \epsilon(t^k_{i-1}))id + (V^k_i - V^k_{i-1})v^k_{i-1} \in \mathcal{A}(\Gamma^k_{i-1}, V^k_i, \epsilon(t^k_i))$, by (2.2.92) we get

$$\begin{aligned}\mathcal{E}_m(t^k_i, \Gamma^k_i, V^k_i) &\le \mathcal{E}(u^k_{i-1} + (\epsilon(t^k_i) - \epsilon(t^k_{i-1}))id + (V^k_i - V^k_{i-1})v^k_{i-1}, \Gamma^k_{i-1}, \epsilon(t^k_i)) \\ &= \mathcal{E}(u^k_{i-1}, \Gamma^k_{i-1}, \epsilon(t^k_{i-1})) + (V^k_i - V^k_{i-1}) \int_{\mathbb{R}^2_+ \setminus \Gamma^k_{i-1}} \mathbb{C}(\mathrm{E}u^k_{i-1} - \epsilon(t^k_{i-1})\mathrm{I}) \cdot \mathrm{E}v^k_{i-1}\,\mathrm{d}x \\ &\quad + \frac{(V^k_i - V^k_{i-1})^2}{2} \int_{\mathbb{R}^2_+ \setminus \Gamma^k_{i-1}} \mathbb{C}\mathrm{E}v^k_{i-1} \cdot \mathrm{E}v^k_{i-1}\,\mathrm{d}x\end{aligned} \tag{2.2.94}$$

Recalling (2.2.3), (2.2.93), and formula (2.2.41) which relates p^k_i to $\sigma(\epsilon(t^k_i))$ and to the quantity $q(\Gamma^k_i, V^k_i, \epsilon(t^k_i))$ introduced in Proposition 2.2.6, we can continue in (2.2.94) obtaining

$$\begin{aligned}\mathcal{E}_m(t^k_i, \Gamma^k_i, V^k_i) \le{}& \mathcal{E}_m(t^k_{i-1}, \Gamma^k_{i-1}, V^k_{i-1}) + (p^k_{i-1} + \sigma(\epsilon(t^k_{i.-1}))) \int_{t^k_{i-1}}^{t^k_i} \dot{V}^k(s)\,\mathrm{d}s \\ &+ \beta \widetilde{V}_k M^2 \int_{t^k_{i-1}}^{t^k_i} |\dot{V}_k(s)|\,\mathrm{d}s\,,\end{aligned} \tag{2.2.95}$$

where we have set

$$\widetilde{V}_k := \frac{1}{2} \sup_{j=1,\dots,k} |V^k_j - V^k_{j-1}|\,.$$

Iterating inequality (2.2.95) we obtain, for $t \in [t^k_i, t^k_{i+1})$,

$$\begin{aligned}\mathcal{E}_m(t^k_i, \Gamma_k(t), V_k(t)) \le{}& \mathcal{E}_m(0, \Gamma_0, V_0) + \int_0^{t^k_i} (p_k(s) + \sigma_k(s))\dot{V}^k(s)\,\mathrm{d}s \\ &+ \beta \widetilde{V}_k M^2 \int_0^T |\dot{V}^k(s)|\,\mathrm{d}s\,.\end{aligned} \tag{2.2.96}$$

In particular, (2.2.96) implies that $\mathcal{H}^1(\Gamma_k(t))$ is bounded uniformly with respect to $t \in [0,T]$ and $k \in \mathbb{N}$.

By Theorem 1.1.3 and Proposition 2.2.2, we have that, up to a subsequence, $\Gamma_k(t) \to \Gamma(t)$ in the Hausdorff metric for every $t \in [0,T]$, $\mathcal{H}^1(\Gamma_k(t)) \to \mathcal{H}^1(\Gamma(t))$, and the set function $\Gamma\colon [0,T] \to \mathcal{C}_\eta$ is bounded and increasing. Moreover, in view of (2.2.88) and (2.2.91), there exists a nonnegative function $V \in W^{1,\infty}([0,T])$ such that, up to a further subsequence, $V^k \rightharpoonup V$ weakly* in $W^{1,\infty}([0,T])$ and $V^k, V_k, \overline{V}_k \to V$ strongly in $L^\infty([0,T])$. Let us also denote by $u(t)$ a solution (unique up to a horizontal translation) to (2.2.20) associated to the triple $(\Gamma(t), V(t), \epsilon(t))$, and let $p(t) := p(\Gamma(t), V(t), \epsilon(t))$ be the corresponding pressure, according to Proposition 2.2.6 and Remark 2.2.11.

Thanks to the previous convergences, from Lemma 2.2.22 we deduce that for every $t \in [0,T]$ the pair $(\Gamma(t), V(t))$ satisfies the global stability condition (b) of

Definition 2.2.17, that $\mathrm{E}u_k(t) - \epsilon_k(t)\mathrm{I} \to \mathrm{E}u(t) - \epsilon(t)\mathrm{I}$ in $L^2(\mathbb{R}^2_+;\mathbb{M}^2_{sym})$, and that $p_k(t) \to p(t)$.

In order to prove the energy-dissipation balance, we first pass to the limit in (2.2.96) as $k \to +\infty$. The third term in the right-hand side of (2.2.96) tends to zero because of (2.2.91). In view of (2.2.89), of the pointwise convergence of p_k to p, of the continuity of $\sigma(\epsilon(\cdot))$, and of the weak* convergence in $L^\infty([0,T])$ of $\dot{V}^k$ to $\dot{V}$, we get that

$$\mathcal{E}_m(t,\Gamma(t),V(t)) \le \mathcal{E}_m(0,\Gamma_0,V_0) + \int_0^t \big(p(s) + \sigma(\epsilon(s))\big)\dot{V}(s)\,\mathrm{d}s\,. \tag{2.2.97}$$

For the opposite inequality, for every $t \in [0,T]$ we consider a subdivision of the interval $[0,t]$ of the form $s^k_h := \frac{ht}{k}$ for $k,h \in \mathbb{N}$, $k \neq 0$, and $h \le k$. For every $h = 0,\dots,k$ we set

$$v^k_h := \begin{cases} \dfrac{u(s^k_h) - \epsilon(s^k_h)\,id}{V(s^k_h)} & \text{if } V(s^k_h) \neq 0,\\ w_0 & \text{if } V(s^k_h) = 0. \end{cases}$$

Therefore, $\|\mathrm{E}v^k_h\|_{2,\mathbb{R}^2_+} \le M$ and $u(s^k_{h+1}) + (\epsilon(s^k_h) - \epsilon(s^k_{h+1}))id + (V(s^k_h) - V(s^k_{h+1}))v^k_{h+1}$ belongs to $\mathcal{A}(\Gamma(s^k_{h+1}), V(s^k_h), \epsilon(s^k_h))$. Since $\Gamma(\cdot)$ is increasing and satisfies the global stability condition, we have

$$\begin{aligned}
&\mathcal{E}_m(s^k_h,\Gamma(s^k_h),V(s^k_h)) \le \mathcal{E}_m(s^k_h,\Gamma(s^k_{h+1}),V(s^k_h))\\
&\le \mathcal{E}(u(s^k_{h+1}) + (\epsilon(s^k_h) - \epsilon(s^k_{h+1}))id + (V(s^k_h) - V(s^k_{h+1}))v^k_{h+1}, \Gamma(s^k_{h+1}), \epsilon(s^k_h))\\
&= \mathcal{E}_m(s^k_{h+1},\Gamma(s^k_{h+1}),V(s^k_{h+1})) + (V(s^k_h) - V(s^k_{h+1}))\int_{\mathbb{R}^2_+\setminus\Gamma(s^k_{h+1})} \mathbb{C}(\mathrm{E}u(s^k_{h+1}) - \epsilon(s^k_{h+1})\mathrm{I})\cdot \mathrm{E}v^k_{h+1}\mathrm{d}x\\
&\quad + \frac{(V(s^k_h) - V(s^k_{h+1}))^2}{2}\int_{\mathbb{R}^2_+\setminus\Gamma(s^k_{h+1})} \mathbb{C}\mathrm{E}v^k_{h+1}\cdot \mathrm{E}v^k_{h+1}\,\mathrm{d}x\\
&\le \mathcal{E}_m(s^k_{h+1},\Gamma(s^k_{h+1}),V(s^k_{h+1})) - \int_{s^k_h}^{s^k_{h+1}} \big(p(s^k_{h+1}) + \sigma(\epsilon(s^k_{h+1}))\big)\dot{V}(s)\mathrm{d}s + \beta\widehat{V}_k M^2 \int_{s^k_h}^{s^k_{h+1}} |\dot{V}(s)|\mathrm{d}s,
\end{aligned}$$

where β is the constant defined in (2.2.3) and

$$\widehat{V}_k := \frac{1}{2}\sup_{h=1,\dots,k} |V(s^k_h) - V(s^k_{h-1})|\,.$$

Iterating the previous inequality for $h = 0,\dots,k$ and setting $p^k(s) := p(s^k_{h+1})$, $\sigma^k(s) := \sigma(\epsilon(s^k_{h+1}))$ for $s \in (s^k_h, s^k_{h+1}]$, we get

$$\begin{aligned}
\mathcal{E}_m(0,\Gamma_0,V_0) \le \mathcal{E}_m(t,\Gamma(t),V(t)) &- \int_0^t (p^k(s) + \sigma^k(s))\dot{V}(s)\,\mathrm{d}s\\
&+ \beta\widehat{V}_k M^2 \int_0^t |\dot{V}(s)|\,\mathrm{d}s\,.
\end{aligned} \tag{2.2.98}$$

Since $\Gamma\colon [0,T] \to \mathcal{C}_\eta$ is an increasing set function, according to Theorem 1.1.2 there exists a set $\Theta \subseteq [0,T]$ such that $[0,T]\setminus\Theta$ is at most countable and $\Gamma(\cdot)$ is continuous at every point in Θ. By Lemma 2.2.22, we have that $s \mapsto \mathrm{E}u(s) - \epsilon(s)\mathrm{I}$ is strongly continuous in $L^2(\mathbb{R}^2_+)$ at every point of Θ and $s \mapsto p(s)$ is continuous at the same points. Thus $p^k(s) \to p(s)$ for every $s \in \Theta$. By the dominated convergence theorem $(p^k+\sigma^k)\dot V \to (p+\sigma(\epsilon))\dot V$ in $L^1([0,t])$ and, passing to the limit in (2.2.98) as $k \to +\infty$, we obtain

$$\mathcal{E}_m(0,\Gamma_0,V_0) \le \mathcal{E}_m(t,\Gamma(t),V(t)) - \int_0^t \big(p(s)+\sigma(\epsilon(s))\big)\dot V(s)\,\mathrm{d}s\,.$$

Recalling (2.2.97), this concludes the proof of the energy-dissipation balance (d) of Definition 2.2.17.

It remains to prove the Darcy's law (c) of Definition 2.2.17. Let us fix $j \in \mathbb{N}$, $j \neq 0$, and let us set $E_j := \{t \in [0,T] : V(t) \ge 1/j\}$. By the uniform convergences, for k large enough we may assume that $V_k(t), V^k(t), \overline{V}_k(t) > 0$ for every $t \in E_j$. Therefore, in view of (2.2.81) and of (2.2.85), for such t we get, using the notation introduced in (2.2.84),

$$\dot V^k(t) = p^k_\infty(t) - p_k(t)\,. \tag{2.2.99}$$

In view of (2.2.99), for every $t \in [0,T]$ we have

$$V^k(t) = V_0 + \int_0^t \dot V^k(s)\,\mathrm{d}s = V_0 + \int_{[0,t]\setminus E_j} \dot V^k(s)\,\mathrm{d}s + \int_{E_j} (p^k_\infty(s) - p_k(s))\,\mathrm{d}s\,. \tag{2.2.100}$$

Passing to the limit as $k \to +\infty$ in (2.2.100), by the continuity of p_∞ and by L^1-convergence of p_k to p we obtain that

$$V(t) = V_0 + \int_{[0,t]\setminus E_j} \dot V(s)\,\mathrm{d}s + \int_{E_j} (p_\infty(s) - p(s))\,\mathrm{d}s\,,$$

from which we deduce, passing to the limit as $j \to +\infty$ and recalling that $\dot V = 0$ a.e. in $\{V = 0\}$, that

$$V(t) = V_0 + \int_0^t (p_\infty(s) - p(s))\mathbf{1}_{\{V>0\}}(s)\,\mathrm{d}s\,.$$

This concludes the proof of condition (c) of Definition 2.2.17. □

2.2.3 Derivatives of the energy and Griffith's principle

In this section we discuss some properties of a quasi-static evolution $(\Gamma,V)\colon [0,T] \to \mathcal{C}_\eta \times [0,+\infty)$ given by Definition 2.2.17. In Theorem 2.2.25 we show that, under suitable regularity assumptions on the crack set, the reduced energy (2.2.60) is differentiable with respect to time, to the crack length, and to the fluid volume. The main result of this section is Theorem 2.2.31, where we prove that the evolution (Γ,V) satisfies the Griffith's criterion (see [43]).

Let us start with the computation of the derivatives of the reduced energy (2.2.60). We do it in a quite general setting, assuming that the crack path is known a priori: the crack set can only evolve along a curve $\Lambda \in \mathcal{C}_\eta$. For technical reasons, we need Λ to be of class $C^{2,1}$.

Remark 2.2.23. Since we are interested in the (a posteriori) properties of a quasi-static evolution (Γ, V), we notice that it is not so strange to assume that the crack can only move along a prescribed path. Indeed, once the crack set function $\Gamma\colon [0,T] \to \mathcal{C}_\eta$ is given, it is clear that the fracture grows following $\Gamma(T)$. Hence, the true assumption is that $\Gamma(T)$ (or Λ) is a $C^{2,1}$-curve.

Let $L := \mathcal{H}^1(\Lambda) > 0$, and let $\lambda\colon [0,L] \to \mathbb{R}^2$ be an arc-length parametrization of Λ of class $C^{2,1}$ such that $\lambda(0) = (0,0)$. In what follows, we denote by λ_1 and λ_2 the components of λ. Moreover, for every $s \in [0,L]$, we define

$$\Lambda_s := \{\lambda(\sigma) : 0 \le \sigma \le s\}. \tag{2.2.101}$$

In order to do our computations, we will need to slightly move the crack tip along the prescribed curve Λ. Thus, for $s \in (0,L)$ and δ such that $s+\delta \in [0,L]$, we construct a $C^{2,1}$-diffeomorphism $F_{s,\delta}$ such that $F_{s,\delta}(\mathbb{R}^2_+) = \mathbb{R}^2_+$, $F_{s,\delta}|_\Sigma = id|_\Sigma$, and $F_{s,\delta}(\Lambda_s) = \Lambda_{s+\delta}$. Indeed, by definition of the class $\mathcal{C}_\eta$ and by our regularity assumption, there exists $\lambda_g\colon [0,\lambda_2(L)] \to \mathbb{R}$ of class $C^{2,1}$ such that $\Lambda = \mathrm{graph}(\lambda_g)$.

Let us fix $\zeta > 0$ small and let $\vartheta \in C_c^\infty(\mathrm{B}_{\zeta/2}(0))$ be a cut-off function with $\vartheta = 1$ on $\overline{\mathrm{B}}_{\zeta/3}(0)$. We define $F_{s,\delta}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F_{s,\delta}(x) := x + \begin{pmatrix} \lambda_g(x_2 + (\lambda_2(s+\delta) - \lambda_2(s))\vartheta(\lambda(s)-x)) - \lambda_g(x_2) \\ (\lambda_2(s+\delta) - \lambda_2(s))\vartheta(\lambda(s)-x) \end{pmatrix} \tag{2.2.102}$$

if $x \in \mathrm{B}_{\zeta/2}(\lambda(s))$, while $F_{s,\delta}(x) := x$ if $x \in \mathbb{R}^2 \setminus \overline{\mathrm{B}}_{\zeta/2}(\lambda(s))$.

In the following lemma, we give some properties of $F_{s,\delta}$ (see, e.g., [47]).

Lemma 2.2.24. *For every $s \in (0,L)$, there exists $\delta_0 > 0$ such that:*

(*a*) $F_{s,\cdot} \in C^{2,1}((-\delta_0,\delta_0) \times \mathbb{R}^2; \mathbb{R}^2)$ *and, for every $|\delta| < \delta_0$, the map $F_{s,\delta}$ is a $C^{2,1}$-diffeomorphism. Moreover, $F_{s,\delta}(\mathbb{R}^2_+) = \mathbb{R}^2_+$, $F_{s,\delta}(\lambda(s)) = \lambda(s+\delta)$, $F_{s,\delta}(\Lambda_s) = \Lambda_{s+\delta}$, and $F_{s,\delta}(x) = x$ for every $x \in \mathbb{R}^2 \setminus \mathrm{B}_{\zeta/2}(\lambda(s))$;*

(*b*) *the norms $\|F_{s,\delta}\|_{C^{2,1}}$ and $\|F_{s,\delta}^{-1}\|_{C^{2,1}}$ are uniformly bounded with respect to δ and there exist c_1, $c_2 > 0$ such that, for every $|\delta| < \delta_0$ and every $x \in \mathbb{R}^2$, we have $c_1 \le \det \nabla F_{s,\delta}(x) \le c_2$;*

(*c*) $\|id - F_{s,\delta}\|_{C^2} \to 0$ *as $\delta \to 0$;*

(*d*) *some derivatives:*

$$\begin{gathered}\rho_s(x) := \partial_\delta(F_{s,\delta}(x))|_{\delta=0} = \lambda_2'(s)\vartheta(\lambda(s)-x)\begin{pmatrix}\lambda_g'(x_2)\\ 1\end{pmatrix},\\ \partial_\delta(\det\nabla F_{s,\delta})|_{\delta=0} = \mathrm{div}\rho_s\,,\\ \partial_\delta(\nabla F_{s,\delta})|_{\delta=0} = -\partial_\delta(\nabla F_{s,\delta})^{-1}|_{\delta=0} = \nabla\rho_s\,,\\ \partial_\delta(\mathit{cof}\,\nabla F_{s,\delta})^T|_{\delta=0} = -\partial_\delta(\mathit{cof}\,\nabla F_{s,\delta})^{-T}|_{\delta=0} = \mathrm{div}\rho_s\,\mathrm{I}-\nabla\rho_s\,.\end{gathered} \tag{2.2.103}$$

Proof. See [41] for the proof of (a), (b), and (d) in the case of C^∞ maps. The same arguments are applicable with the $C^{2,1}$ regularity of $F_{s,\delta}$. Property (c) follows immediately from the definition (2.2.102) of $F_{s,\delta}$. □

As we have seen in Corollary 2.2.5, a solution to the minimum problem (2.2.20) which defines the reduced energy $\mathcal{E}_m$ exists and is unique up to a horizontal translation. In order to compute the derivatives of $\mathcal{E}_m$ with respect to the crack length s and to the volume V, it is convenient to slightly modify the set of admissible displacements $\mathcal{A}$ defined in (2.2.18) in such a way that the minimizer of (2.2.20) is unique. To do this, it is enough, for instance, to fix the mean value of the first component of the displacement in an open set $E \subset\subset \mathbb{R}^2_+\setminus\Lambda$ with Lipschitz boundary, $E \neq \emptyset$. Thus, for every $s\in[0,L]$, every $V\in[0,+\infty)$, and every $\epsilon\in\mathbb{R}$ we define

$$\widetilde{\mathcal{A}}(\Lambda_s,V,\epsilon) := \Big\{u\in\mathcal{A}(\Lambda_s,V,\epsilon) : \int_E u_1\,\mathrm{d}x = 0\Big\}\,. \tag{2.2.104}$$

For simplicity of notation, when $\epsilon=0$ we set $\widetilde{\mathcal{A}}(\Lambda_s,V) := \widetilde{\mathcal{A}}(\Lambda_s,V,0)$. We notice that

$$\widetilde{\mathcal{A}}(\Lambda_s,V) = \Big\{u\in LD^2_E(\mathbb{R}^2_+\setminus\Lambda_s;\mathbb{R}^2) : [u]\cdot\nu_{\Lambda_s}\geq 0 \text{ on } \Lambda_s,\ \int_{\Lambda_s}[u]\cdot\nu_{\Lambda_s}\,\mathrm{d}\mathcal{H}^1 = V\Big\}\,.$$

In view of Corollary 2.2.5, for every $s\in[0,L]$ and every $V\in[0,+\infty)$ there exists a unique $u_V^s\in\widetilde{\mathcal{A}}(\Lambda_s,V)$ solution of (2.2.20) for the triple $(\Lambda_s,V,0)$. In particular, for every $\epsilon\in\mathbb{R}$ we have that

$$\mathcal{E}(u_V^s,\Lambda_s,0) = \mathcal{E}(u_V^s+\epsilon\,id,\Lambda_s,\epsilon)\,.$$

This implies that, for every $t\in[0,T]$,

$$\begin{aligned}\mathcal{E}_m(t,\Lambda_s,V) &= \min_{u\in\mathcal{A}(\Lambda_s,V,\epsilon(t))}\mathcal{E}(u,\Lambda_s,\epsilon(t))\\ &= \min_{u\in\widetilde{\mathcal{A}}(\Lambda_s,V)}\mathcal{E}(u,\Lambda_s,0) =: \widetilde{\mathcal{E}}_m(\Lambda_s,V)\,.\end{aligned} \tag{2.2.105}$$

For every $s\in(0,L)$ and every $V\in[0,+\infty)$, we set

$$\begin{aligned}\mathfrak{G}(s,V) := &\int_{\mathbb{R}^2_+\setminus\Lambda_s}\mathbb{C}\mathrm{E}u_V^s\cdot\nabla\big((\mathrm{div}\rho_s\,\mathrm{I}-\nabla\rho_s)\,u_V^s\big)\,\mathrm{d}x\\ &+\int_{\mathbb{R}^2_+\setminus\Lambda_s}\mathbb{C}\mathrm{E}u_V^s\cdot\big(\nabla u_V^s\nabla\rho_s\big)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Lambda_s}\mathbb{C}\mathrm{E}u_V^s\cdot\mathrm{E}u_V^s\mathrm{div}\rho_s\,\mathrm{d}x\,.\end{aligned}$$

In Theorem 2.2.25 we show that $\mathfrak{G}(s,V)$ corresponds, in our context, to the so-called *energy release rate*, that is, the derivative of the renormalized stored elastic energy with respect to the crack length parameter s (see (2.2.107)).

In the following theorem we give explicit formulas for the derivatives of the reduced energy (2.2.60) with respect to t, s, and V.

Theorem 2.2.25. *Let $t \in [0,T]$, $s \in (0,L)$, and $V \in [0,+\infty)$. Then*

$$\frac{\partial \mathcal{E}_m}{\partial t}(t,\Lambda_s,V) = \frac{\partial \widetilde{\mathcal{E}}_m}{\partial t}(\Lambda_s,V) = 0\,, \tag{2.2.106}$$

$$\frac{\partial \mathcal{E}_m}{\partial s}(t,\Lambda_s,V) = \frac{\partial \widetilde{\mathcal{E}}_m}{\partial s}(\Lambda_s,V) = \kappa - \mathfrak{G}(s,V)\,, \tag{2.2.107}$$

where κ is defined in (2.2.19)*.*

If, in addition, $V > 0$, then

$$\frac{\partial \mathcal{E}_m}{\partial V}(t,\Lambda_s,V) = \frac{\partial \widetilde{\mathcal{E}}_m}{\partial V}(\Lambda_s,V) = p(\Lambda_s,V,\epsilon(t)) + \sigma(\epsilon(t))\,. \tag{2.2.108}$$

To prove Theorem 2.2.25 we need to introduce, for every $s \in (0,L)$ and $\delta \in (-\delta_0,\delta_0)$ (see Lemma 2.2.24), the Piola transformation $P_{s,\delta}$ associated to $F_{s,\delta}$:

$$P_{s,\delta}\,u := (cof\,\nabla F_{s,\delta})^T u \circ F_{s,\delta} \qquad \text{for every } u \in \widetilde{\mathcal{A}}(\Lambda_{s+\delta},V)\,. \tag{2.2.109}$$

We refer to [20, Section 1.7] for the main properties of $P_{s,\delta}$. We notice that, at least for $|\delta|$ small, $P_{s,\delta}$ is an isomorphism between $\widetilde{\mathcal{A}}(\Lambda_{s+\delta},V)$ and $\widetilde{\mathcal{A}}(\Lambda_s,V)$ whose inverse is given by

$$P^{-1}_{s,\delta}\,u := ((cof\,\nabla F_{s,\delta})^{-T} u) \circ F^{-1}_{s,\delta} \qquad \text{for every } u \in \widetilde{\mathcal{A}}(\Lambda_s,V)\,. \tag{2.2.110}$$

Lemma 2.2.26. *Let $s \in (0,L)$ and let $u_\delta \in LD^2_E(\mathbb{R}^2_+ \setminus \Lambda;\mathbb{R}^2)$. Assume that there exists $u_0 \in LD^2_E(\mathbb{R}^2_+ \setminus \Lambda;\mathbb{R}^2)$ such that $u_\delta \to u_0$ in $LD^2_E(\mathbb{R}^2_+ \setminus \Lambda;\mathbb{R}^2)$ as $\delta \to 0$. Then the sequences $u_\delta \circ F_{s,\delta}$, $u_\delta \circ F^{-1}_{s,\delta}$, $P_{s,\delta}\,u_\delta$, and $P^{-1}_{s,\delta}\,u_\delta$ converge to u_0 strongly in $LD^2_E(\mathbb{R}^2_+ \setminus \Lambda;\mathbb{R}^2)$ as $\delta \to 0$.*

Proof. Thanks to Proposition 1.2.2 and to the properties stated in Lemma 2.2.24, the lemma can be easily proved by using the changes of coordinates $x = F^{-1}_{s,\delta}(y)$ and $x = F_{s,\delta}(y)$. □

Before proving Theorem 2.2.25, we show the continuity of u^s_V with respect to the parameters s and V.

Lemma 2.2.27. *Let $s_k, s \in (0,L)$ and $V_k, V \in [0,+\infty)$ be such that $s_k \to s$ and $V_k \to V$. Let $u^{s_k}_{V_k} \in \widetilde{\mathcal{A}}(\Lambda(s_k),V_k)$ be the sequence of solutions of* (2.2.20) *corresponding to s_k and V_k. Then $u^{s_k}_{V_k} \to u^s_V$ in $LD^2_E(\mathbb{R}^2_+ \setminus \Lambda;\mathbb{R}^2)$.*

Proof. Arguing as in the proof of Lemma 2.2.22, we can show that

$$\|\mathrm{E}u^{s_k}_{V_k}\|_{2,\mathbb{R}^2_+} \leq M V_k \tag{2.2.111}$$

for some $M \in \mathbb{R}$. Hence, by Propositions 1.2.2 and 2.2.4, there exists $u \in \widetilde{\mathcal{A}}(\Lambda_s, V)$ such that, up to a subsequence, $\mathrm{E}u^{s_k}_{V_k} \rightharpoonup \mathrm{E}u$ weakly in $L^2(\mathbb{R}^2_+;\mathbb{M}^2_{sym})$. If $V = 0$ we have that $u^s_V = 0$ and, by (2.2.111), $u^{s_k}_{V_k} \to 0$ in $LD^2_E(\mathbb{R}^2_+ \setminus \Lambda; \mathbb{R}^2)$.

Assume now that $V > 0$. Let us prove that $u = u^s_V$. By Lemma 2.2.24 and by the properties of the Piola transformation (2.2.109), for k large enough we have

$$v_k := \frac{V_k}{V} P^{-1}_{s,s_k - s}\, u^s_V \in \widetilde{\mathcal{A}}(\Lambda(s_k), V_k)\,.$$

Thanks to Lemma 2.2.26, $v_k \to u^s_V$ in $LD^2_E(\mathbb{R}^2_+ \setminus \Lambda; \mathbb{R}^2)$ as $k \to +\infty$. Thus, by the minimality of $u^{s_k}_{V_k}$ we obtain

$$\begin{aligned}
\widetilde{\mathcal{E}}_m(\Lambda_s, V) &\leq \mathcal{E}(u, \Lambda_s, 0) \leq \liminf_{k\to+\infty} \mathcal{E}(u^{s_k}_{V_k}, \Lambda(s_k), 0) \\
&\leq \limsup_{k\to+\infty} \mathcal{E}(u^{s_k}_{V_k}, \Lambda(s_k), 0) \leq \lim_{k\to+\infty} \mathcal{E}(v_k, \Lambda(s_k), 0) \\
&= \mathcal{E}(u^s_V, \Lambda_s, 0) = \widetilde{\mathcal{E}}_m(\Lambda_s, V)\,.
\end{aligned} \tag{2.2.112}$$

From (2.2.112) we deduce that $u = u^s_V$ and that $u^{s_k}_{V_k} \to u^s_V$ in $LD^2_E(\mathbb{R}^2_+ \setminus \Lambda; \mathbb{R}^2)$. □

We are now ready to prove Theorem 2.2.25.

Proof of Theorem 2.2.25. In view of (2.2.105), it is clear that $\mathcal{E}_m$ and $\widetilde{\mathcal{E}}_m$ do not depend on t, hence (2.2.106) holds.

Let us prove (2.2.107). Let $s \in (0, L)$ and $V \in [0, +\infty)$. Recalling the notation introduced in (2.2.109) and (2.2.110), we set

$$u^{s,\delta}_V := (cof\, \nabla F_{s,\delta})^{-T} u^s_V = (P^{-1}_{s,\delta}\, u^s_V) \circ F_{s,\delta}\,. \tag{2.2.113}$$

By (2.2.110), we have that $P^{-1}_{s,\delta} u^s_V \in \widetilde{\mathcal{A}}(\Lambda_{s+\delta}, V)$. Hence, by definition of $\widetilde{\mathcal{E}}_m$ and by the change of variables $x = F^{-1}_{s,\delta}(y)$, for $\delta > 0$ small enough we have

$$\begin{aligned}
\frac{\widetilde{\mathcal{E}}_m(\Lambda_{s+\delta}, V) - \widetilde{\mathcal{E}}_m(\Lambda_s, V)}{\delta} &\leq \frac{\mathcal{E}(P^{-1}_{s,\delta} u^s_V, \Lambda_{s+\delta}, 0) - \mathcal{E}(u^s_V, \Lambda_s, 0)}{\delta} \\
= \frac{1}{2\delta}\bigg(\int_{\mathbb{R}^2_+\setminus\Lambda_s} &\mathbb{C}\big(\nabla u^{s,\delta}_V\, (\nabla F_{s,\delta})^{-1}\big) \cdot \nabla u^{s,\delta}_V\, (\nabla F_{s,\delta})^{-1} \det \nabla F_{s,\delta}\, \mathrm{d}x \\
- \int_{\mathbb{R}^2_+\setminus\Lambda_s} &\mathbb{C}\mathrm{E}u^s_V \cdot \mathrm{E}u^s_V\, \mathrm{d}x \bigg) + \kappa\,.
\end{aligned}$$

Thanks to the properties of $F_{s,\delta}$ stated in Lemma 2.2.24, applying the dominated convergence theorem we easily get that

$$\limsup_{\delta\searrow 0} \frac{\widetilde{\mathcal{E}}_m(\Lambda_{s+\delta}, V) - \widetilde{\mathcal{E}}_m(\Lambda_s, V)}{\delta} \leq \kappa - \mathfrak{G}(s, V)\,. \tag{2.2.114}$$

On the other hand, if we set $U_V^{s,\delta} := u_V^{s+\delta} \circ F_{s,\delta}$, for $\delta > 0$ small we have, in view of (2.2.109),

$$\begin{aligned}\frac{\widetilde{\mathcal{E}}_m(\Lambda_{s+\delta}, V) - \widetilde{\mathcal{E}}_m(\Lambda_s, V)}{\delta} &\geq \frac{\mathcal{E}(u_V^{s+\delta}, \Lambda_{s+\delta}, 0) - \mathcal{E}(P_{s,\delta} u_V^{s+\delta}, \Lambda_s, 0)}{\delta} \\ = \frac{1}{2\delta}\bigg(\int_{\mathbb{R}^2_+ \setminus \Lambda_s} &\mathbb{C}\big(\nabla U_V^{s,\delta} (\nabla F_{s,\delta})^{-1}\big) \cdot \nabla U_V^{s,\delta} (\nabla F_{s,\delta})^{-1} \det \nabla F_{s,\delta} \, \mathrm{d}x \qquad (2.2.115) \\ - \int_{\mathbb{R}^2_+ \setminus \Lambda_s} &\mathbb{C}\nabla(P_{s,\delta}\, u_V^{s+\delta}) \cdot \nabla (P_{s,\delta}\, u_V^{s+\delta}) \, \mathrm{d}x \bigg) + \kappa \,.\end{aligned}$$

By Lemmas 2.2.26 and 2.2.27 we have that $U_V^{s,\delta}$ and $P_{s,\delta} u_V^{s+\delta}$ converge to u_V^s in $LD^2_E(\mathbb{R}^2_+ \setminus \Lambda; \mathbb{R}^2)$. Thus, by the dominated convergence theorem, passing to the limit in (2.2.115) as $\delta \searrow 0$ and recalling (2.2.114), we obtain

$$\lim_{\delta \searrow 0} \frac{\widetilde{\mathcal{E}}_m(\Lambda_{s+\delta}, V) - \widetilde{\mathcal{E}}_m(\Lambda_s, V)}{\delta} = \kappa - \mathfrak{G}(s, V) \,. \qquad (2.2.116)$$

With the same argument we can prove that

$$\lim_{\delta \nearrow 0} \frac{\widetilde{\mathcal{E}}_m(\Lambda_{s+\delta}, V) - \widetilde{\mathcal{E}}_m(\Lambda_s, V)}{\delta} = \kappa - \mathfrak{G}(s, V) \,, \qquad (2.2.117)$$

which, together with (2.2.116), implies (2.2.107).

Equality (2.2.108) can be proved with the same technique. For every $V > 0$, let us show that

$$\lim_{\delta \searrow 0} \frac{\widetilde{\mathcal{E}}_m(\Lambda_s, V + \delta) - \widetilde{\mathcal{E}}_m(\Lambda_s, V)}{\delta} \leq p(\Lambda_s, V, \epsilon(t)) + \sigma(\epsilon(t)) \,. \qquad (2.2.118)$$

Since $\frac{V+\delta}{V} u_V^s \in \widetilde{\mathcal{A}}(\Lambda_s, V + \delta)$, from (2.2.105) we deduce that

$$\frac{\widetilde{\mathcal{E}}_m(\Lambda_s, V + \delta) - \widetilde{\mathcal{E}}_m(\Lambda_s, V)}{\delta} \leq \frac{1}{2\delta}\left[\left(\frac{V+\delta}{V}\right)^2 - 1\right] \int_{\mathbb{R}^2_+ \setminus \Lambda_s} \mathbb{C}\mathrm{E}u_V^s \cdot \mathrm{E}u_V^s \, \mathrm{d}x \,. \qquad (2.2.119)$$

Passing to the $\limsup$ in (2.2.119) as $\delta \searrow 0$ and taking into account Remarks 2.2.10 and 2.2.11, we get (2.2.118). The rest of the proof can be carried out in a similar way. □

Before stating a Griffith's criterion for our model, we make a comment on formula (2.2.107) of Theorem 2.2.25.

Remark 2.2.28. As we have seen in Proposition 2.2.6 and Remark 2.2.11, to every $t \in [0, T]$, $s \in [0, L]$, and $V \in [0, +\infty)$, is associated a pressure $p(\Lambda_s, V, \epsilon(t)) \in [0, +\infty)$ which acts on the fracture lips along the normal ν_{Λ_s}. In order to determine the energy release rate, what is usually done in fracture mechanics (see, e.g., [69]) when a force p is applied to the crack is to compute the derivative of the reduced energy $\mathscr{E}_m$ of (2.2.63) with respect to the crack length s, keeping p fixed. On the contrary, in (2.2.107) we have computed the derivative of the reduced energy $\mathcal{E}_m$ of (2.2.60) with respect to s, keeping the fluid (or crack) volume V fixed.

Let us show that, at least formally, the two derivatives coincide. Indeed, by definition (2.2.52) of $\mathscr{E}_m$, we notice that, for every $t \in [0,T]$, every $s \in [0,L]$, and every $p \in \mathbb{R}$,

$$\mathscr{E}_m(t,\Lambda_s,p) = \mathcal{E}_m(t,\Lambda_s,V(\Lambda_s,p,\epsilon(t))) - (p+\sigma(\epsilon))V(\Lambda_s,p,\epsilon(t)) \,. \tag{2.2.120}$$

Since $p(\Lambda_s, V(\Lambda_s,p,\epsilon(t)),\epsilon(t)) = p$, computing the derivative of formula (2.2.120) with respect to s and using (2.2.107) and (2.2.108) we obtain

$$\frac{\partial \mathscr{E}_m}{\partial s}(t,\Lambda_s,p) = \kappa - \mathfrak{G}(s,V(\Lambda_s,p,\epsilon(t))) = \frac{\partial \mathcal{E}_m}{\partial s}(t,\Lambda_s,V(\Lambda_s,p,\epsilon(t))) \,.$$

We are now ready to state a Griffith's criterion for a quasi-static evolution (Γ,V) of the hydraulic crack growth problem given by Definition 2.2.17. In view of the regularity assumption of Theorem 2.2.25, we have to suppose that the curve $\Gamma(T)$ is of class $C^{2,1}$. Let $L_\Gamma := \mathcal{H}^1(\Gamma(T))$ and let $\gamma\colon [0,L_\Gamma] \to \mathbb{R}^2_+$ be an arc-length parametrization of $\Gamma(T)$ of class $C^{2,1}$. As in (2.2.101), we set $(\Gamma(T))_s := \gamma([0,s])$ for every $s \in [0,L_\Gamma]$. We introduce the concept of *failure time*, which will be used also in Chapter 3.

Definition 2.2.29. Let $a, b > 0$ and let $s\colon [0,a] \to [0,b]$ be a monotone non-decreasing function. We define the *failure time* $\mathcal{T}(s)$ of s by

$$\mathcal{T}(s) := \sup\{t \in [0,a] : \, s(t) < b\} \,.$$

Remark 2.2.30. We notice that $\mathcal{T}$ is lower semicontinuous with respect to the pointwise convergence, that is, if $s_k \to s$ pointwise, then

$$\mathcal{T}(s) \le \liminf_k \, \mathcal{T}(s_k) \,.$$

With the notation introduced above, we have the following theorem.

Theorem 2.2.31. *Let $(\Gamma,V)\colon [0,T] \to \mathcal{C}_\eta \times [0,+\infty)$ be a quasi-static evolution of the hydraulic crack growth problem with the properties stated above. Let $s\colon [0,T] \to [0,L_\Gamma]$ be the function defined by $s(t) := \mathcal{H}^1(\Gamma(t))$ for every $t \in [0,T]$, and let $\mathcal{T}_f := \sup\{t \in [0,T] : \, s(t) < L_\Gamma\}$. Then the following conditions hold:*

(1) $\dot s(t) \ge 0$ *for a.e.* $t \in [0,T]$*;*

(2) $\mathfrak{G}(s(t),V(t)) - \kappa \le 0$ *for every* $t \in [0,\mathcal{T}_f)$*;*

(3) $(\mathfrak{G}(s(t),V(t)) - \kappa)\dot s(t) = 0$ *for a.e.* $t \in [0,\mathcal{T}_f)$*.*

The first condition reflects the irreversibility condition of Definition 2.2.17. The second condition says that the energy release rate has to be less than or equal to κ during the evolution. Finally, the last condition means that the energy release rate has to be equal to κ when the crack tip moves with a positive velocity. This is the so-called Griffith's criterion in our model.

Proof. Since $t \mapsto s(t)$ is a monotone nondecreasing function, property (1) is clearly satisfied.

Property (2) follows by the global stability condition of Definition 2.2.17: indeed, for every $t \in [0, \mathcal{T}_f)$ we have that, for $s(t) < \sigma \leq L_\Gamma$,

$$\mathcal{E}_m(t, \Gamma(t), V(t)) \leq \mathcal{E}_m(t, (\Gamma(T))_\sigma, V(t)) . \tag{2.2.121}$$

Since (2.2.107) holds, dividing (2.2.121) by $\sigma - s(t)$ and passing to the limit as $\sigma \searrow s(t)$ we deduce (2).

In order to prove (3), we make more explicit the energy-dissipation balance (2.2.61): for a.e. $t \in [0, \mathcal{T}_f)$ we have

$$\begin{aligned}\big(p(t) + \sigma(\epsilon(t))\big)\dot{V}(t) &= \frac{d}{dt}\mathcal{E}_m(t, \Gamma(t), V(t)) = \frac{d}{dt}\mathcal{E}_m(t, (\Gamma(T))_{s(t)}, V(t)) \\ &= \big(\kappa - \mathfrak{G}(s(t), V(t))\big)\dot{s}(t) + \big(p(t) + \sigma(\epsilon(t))\big)\dot{V}(t) ,\end{aligned}$$

where, in the last equality, we have used the results of Theorem 2.2.25. □

2.3 3-dimensional model

In this section we present a 3-dimensional model of hydraulic fracture. Since the aim is to stress the main differences between 3D and 2D, we now assume that the far strain field $\epsilon(\cdot)$ is null and that the volume function $V\colon [0, T] \to [0, +\infty)$ is known a priori (see also Section 2.3.1). We notice that the evolution result proved in Section 2.2 (Darcy's law) can be obtained also in this context.

As in dimension 2, the body is linearly elastic, impermeable, unbounded, for simplicity filling all of $\mathbb{R}^3$. Here, we suppose that the crack path is prescribed a priori: the admissible cracks lie on the horizontal plane Λ passing through the origin. First of all, we need to define a new class of admissible cracks. For technical reasons, we need some regularity of the relative boundary of the crack sets in Λ. This is provided by the *interior ball property* (see condition (c) below).

Definition 2.3.1. Fix $\eta > 0$. We say that $\Gamma \in Adm_\eta(\Lambda)$ if it satisfies:

(a) Γ is a compact and connected subset of Λ;

(b) $0 \in \Gamma$;

(c) for every $x \in \partial\Gamma$ there exists $y \in \mathring{\Gamma}$ such that $x \in \partial \mathrm{B}_\eta(y)$ and $\mathrm{B}_\eta(y) \subseteq \Gamma$.

In this section, all topological notions (boundary, interior part, balls, etc.) are considered with respect to the relative topology of Λ.

In [65] it is shown that condition (c) implies the existence of a radius $0 < \eta' < \eta$ such that every $\Gamma \in Adm_\eta(\Lambda)$ can be written as the closure of a union of balls of radius η'. In particular, η' can be taken equal to $\eta/2$. By the Lindelöff's theorem, this union can be assumed to be countable. This fact will be useful in the proof of the continuity of the Hausdorff measure $\mathcal{H}^2$ with respect to the Hausdorff metric in $Adm_\eta(\Lambda)$ (see Proposition 2.3.6).

As in Section 2.2, the evolution is governed by linearized elasticity. Because of the lack of homogeneity, the elasticity tensor is a function of the space variable, which will be assumed to be measurable. As usual, for almost every $x \in \mathbb{R}^3$ $\mathbb{C}(x)$ is symmetric and positive definite. Hence, we assume that there exist two constants $0 < \alpha \leq \beta < +\infty$ such that for almost every $x \in \mathbb{R}^3$

$$\alpha|\mathrm{F}|^2 \leq \mathbb{C}(x)\mathrm{F}\cdot\mathrm{F} \leq \beta|\mathrm{F}|^2 \qquad \text{for every } \mathrm{F} \in \mathbb{M}^3_{sym}\,. \tag{2.3.1}$$

Therefore, recalling that we assume that the far strain field $\epsilon(\cdot)$ is null, the total energy of the system is of the form

$$\mathcal{E}(u,\Gamma) := \frac{1}{2}\int_{\mathbb{R}^3\setminus\Lambda} \mathbb{C}\mathrm{E}u\cdot\mathrm{E}u\,dx + \kappa\mathcal{H}^2(\Gamma)\,, \tag{2.3.2}$$

where $u \in \mathrm{W}^1_{2,6}(\mathbb{R}^3\setminus\Lambda;\mathbb{R}^3)$ is the displacement and κ is a positive constant related to the fracture toughness.

We now describe the equilibrium condition for the elastic body with a crack $\Gamma \in Adm_\eta(\Lambda)$ assuming that the region between the crack lips in the deformed configuration is partially filled by a prescribed volume V of an incompressible fluid. As in the two dimensional case, for the volume of the cavity determined by the crack we use the approximate formula

$$\int_\Gamma [u]\cdot\nu_\Lambda\,\mathrm{d}\mathcal{H}^2\,,$$

where ν_Λ is the unit normal vector to Λ and $[u]$ denotes the jump of u through Λ. Again, we consider the non-interpenetration condition $[u]\cdot\nu_\Lambda \geq 0$ on Λ.

The equilibrium of the elastic body with a crack $\Gamma \in Adm_\eta(\Lambda)$ is achieved if the displacement u is the solution of the minimum problem

$$\min_{u\in\mathcal{A}(\Gamma,V)} \mathcal{E}(u,\Gamma)\,, \tag{2.3.3}$$

where

$$\mathcal{A}(\Gamma,V) := \Big\{u \in \mathrm{W}^1_{2,6}(\mathbb{R}^3\setminus\Lambda;\mathbb{R}^3) : \{[u]\neq 0\} \subseteq \Gamma,\ [u]\cdot\nu_\Lambda \geq 0,\ \int_\Lambda [u]\cdot\nu_\Lambda\,\mathrm{d}\mathcal{H}^2 = V\Big\}\,.$$

is the set of admissible displacements. The choice of the function space $\mathrm{W}^1_{2,6}(\mathbb{R}^3\setminus\Lambda;\mathbb{R}^3)$ implies, in a suitable weak sense, that the displacement is zero at infinity. In particular, we notice that in dimension 3 we can work with Sobolev spaces since we have Sobolev inequalities at our disposal.

The inclusion in the previous formula reflects the fact that the crack is contained in Γ. Finally, the last equality takes into account the volume constraint.

The existence of a solution of (2.3.3) can be obtained by the direct method of the calculus of variations, taking into account Proposition 1.2.6. The uniqueness follows from the strict convexity of the functional and the convexity of the constraints.

Remark 2.3.2. With the same techniques used to prove Propositions 2.2.6 and 2.2.12, we have that if u be the solution of (2.3.3) with $V \in [0,+\infty)$ and $\Gamma \in Adm_\eta(\Lambda)$, then there exists a $p(\Gamma,V) \geq 0$ such that (2.2.43) holds for every $v \in \mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Lambda; \mathbb{R}^3)$ such that $\{[v] \neq 0\} \subseteq \Gamma$ and $|[v]\cdot\nu_\Lambda| \leq C[u]\cdot\nu_\Lambda$ for some constant $C \geq 0$ (recall that $\epsilon(\cdot) = 0$). The constant $p(\Gamma,V)$ can be interpreted as the fluid pressure, and, in the case $V > 0$, we have the explicit formula

$$p(\Gamma,V) = \frac{1}{V}\int_{\mathbb{R}^3\setminus\Lambda} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}u \,\mathrm{d}x\,.$$

Finally, arguing as in Remarks 2.2.10, 2.2.11, and 2.2.13, we have that u is also a weak solution of $\operatorname{div}\boldsymbol{\sigma}(u) = 0$ in $\mathbb{R}^3 \setminus \Lambda$, where $\boldsymbol{\sigma}(u)$ is defined in (2.2.33).

2.3.1 Quasi-static evolution

Let us now describe the quasi-static evolution of hydraulic cracks in this setting. Fixed $T > 0$, for every $t \in [0,T]$ we denote by $V(t)$ the volume of the fluid present in the crack at time t. In order to present the simplest possible model, we suppose that the volume function is a datum. For technical reasons, we assume $V \in AC([0,T];[0,+\infty))$, the space of absolutely continuous functions from $[0,T]$ with values in $[0,+\infty)$. By the way, we notice that all the results presented in Section 2.2 can be stated also in the three dimensional case following the lines we are going to discuss here.

It is convenient to introduce the reduced energy $\mathcal{E}_m(\Gamma,V)$ which is defined for every $\Gamma \in Adm_\eta(\Lambda)$ and $V \in [0,+\infty)$ by

$$\mathcal{E}_m(\Gamma,V) := \min_{u\in\mathcal{A}(\Gamma,V)} \mathcal{E}(u,\Gamma)\,.$$

We notice that, with respect to (2.2.60), we have dropped the explicit dependence on time t since we assume $\epsilon(t) = 0$ on the interval $[0,T]$.

Similarly to Definition 2.2.17, we define a quasi-static evolution as follows.

Definition 2.3.3. Let $T > 0$ and $V \in AC([0,T],[0,+\infty))$. We say that a set function $\Gamma\colon [0,T] \to Adm_\eta(\Lambda)$ is an *irreversible quasi-static evolution* of the *3D-hydraulic crack problem* if it satisfies the following conditions:

(a) *irreversibility*: Γ is increasing, i.e., $\Gamma(s) \subseteq \Gamma(t)$ for every $0 \leq s \leq t \leq T$;

(b) *global stability*: for every $t \in [0,T]$,

$$\mathcal{E}_m(\Gamma(t),V(t)) \leq \mathcal{E}_m(\Gamma,V(t)) \qquad \text{for every } \Gamma \in Adm_\eta(\Lambda) \text{ with } \Gamma \supseteq \Gamma(t)\,;$$

(c) *energy-dissipation balance*: the function $t \mapsto \mathcal{E}_m(\Gamma(t),V(t))$ is absolutely continuous on the interval $[0,T]$ and

$$\frac{d}{dt}\mathcal{E}_m(\Gamma(t),V(t)) = p(t)\dot{V}(t)$$

for almost every $t \in [0,T]$, where $p(t) := p(\Gamma(t),V(t))$ is the pressure introduced in Remark 2.3.2.

We are now in a position to state the main theorem of this section.

Theorem 2.3.4. *Let* $V \in AC([0,T],[0,+\infty))$ *and* $\Gamma_0 \in Adm_\eta(\Lambda)$. *Assume that (stability at time* $t=0$*)*

$$\mathcal{E}_m(\Gamma_0, V(0)) \leq \mathcal{E}_m(\Gamma, V(0)) \tag{2.3.4}$$

for every $\Gamma \in Adm_\eta(\Lambda)$ *such that* $\Gamma \supseteq \Gamma_0$. *Then there exists an irreversible quasi-static evolution* Γ *of the 3D-hydraulic crack problem, with* $\Gamma(0) = \Gamma_0$.

In order to prove Theorem 2.3.4, we have to establish some properties of the admissible cracks. In particular, we are interested in the continuity of the $\mathcal{H}^2$ measure with respect to the Hausdorff convergence of sets in the class $Adm_\eta(\Lambda)$ (Proposition 2.3.6).

Proposition 2.3.5. *The following facts hold:*

(*a*) $\Gamma = \overline{\mathring{\Gamma}}$ *for every* $\Gamma \in Adm_\eta(\Lambda)$;

(*b*) $\Gamma_1, \Gamma_2 \in Adm_\eta(\Lambda) \Longrightarrow \Gamma_1 \cup \Gamma_2 \in Adm_\eta(\Lambda)$.

Proof. Property (a) follows immediately from the definition.

Let us prove property (b). Given $\Gamma_1, \Gamma_2 \in Adm_\eta(\Lambda)$, the set $\Gamma_1 \cup \Gamma_2$ contains 0 and is closed and connected. Since for every $x \in \partial(\Gamma_1 \cup \Gamma_2)$, there exists $i = 1, 2$ such that $x \in \partial\Gamma_i$, by Definition 2.3.1, there exists $y_x \in \mathring{\Gamma}_i$ such that $\mathrm{B}_\eta(y_x) \subseteq \Gamma_i \subseteq \Gamma_1 \cup \Gamma_2$ and $x \in \partial\mathrm{B}_\eta(y_x)$. Hence $\Gamma_1 \cup \Gamma_2 \in Adm_\eta(\Lambda)$. □

Proposition 2.3.6. *Let* Γ_k *be a sequence in* $Adm_\eta(\Lambda)$ *and let* K, Γ *be compact subsets of* Λ *such that* $\Gamma, \Gamma_n \subseteq K$ *for every* $k \in \mathbb{N}$ *and* $\Gamma_k \to \Gamma$ *in the Hausdorff metric. Then* $\Gamma \in Adm_\eta(\Lambda)$ *and* $\mathcal{H}^2(\Gamma_k) \to \mathcal{H}^2(\Gamma)$.

Proof. Let us first prove that, if $\Gamma_k \to \Gamma$ in the Hausdorff metric, then

$$\lim_k \sup_{y \in \partial\Gamma} d(y, \partial\Gamma_k) = 0. \tag{2.3.5}$$

By contradiction, suppose that (2.3.5) is false, then there exist $\varepsilon > 0$ and a subsequence, still denoted by Γ_k, such that $\sup_{y\in\partial\Gamma} d(y, \partial\Gamma_k) > 2\varepsilon$ for every $k \in \mathbb{N}$. We can choose $y_k \in \partial\Gamma$ such that $d(y_k, \partial\Gamma_k) > 2\varepsilon$. Up to another subsequence, we can suppose $y_k \to \overline{y} \in \partial\Gamma$. By the triangle inequality, we can easily prove that $d(\overline{y}, \partial\Gamma_k) > \varepsilon$ for k large enough, hence

$$\mathrm{B}_\varepsilon(\overline{y}) \cap \partial\Gamma_k = \emptyset. \tag{2.3.6}$$

To show that this is a contradiction, let us fix $z \in \mathrm{B}_\varepsilon(\overline{y}) \setminus \Gamma$. Since $\Gamma_k \to \Gamma$ in the Hausdorff metric, we have $z \notin \Gamma_k$ for k large enough. On the other hand, since $\overline{y} \in \Gamma$, there exists a sequence $\overline{y}_k \to \overline{y}$ with $\overline{y}_k \in \Gamma_k$. For k large enough, $\overline{y}_k \in \mathrm{B}_\varepsilon(\overline{y})$. Since $z \notin \Gamma_k$, in the segment between $\overline{y}_k$ and z there exists a point of $\partial\Gamma_k$ for k large enough. This contradicts (2.3.6) and proves (2.3.5).

It is easy to see that Γ contains 0 and is closed and connected. By (2.3.5), for every $y \in \partial\Gamma$, there exists a sequence $y_k \in \partial\Gamma_k$ such that $y_k \to y$. For every $k \in \mathbb{N}$, by Definition 2.3.1 we can find $x_k \in \mathring{\Gamma}_k$ such that $\mathrm{B}_\eta(x_k) \subseteq \Gamma_k$ and $y_k \in \partial\mathrm{B}_\eta(x_k)$. Up to a subsequence, $x_k \to x \in \Gamma$ and $\overline{\mathrm{B}}_\eta(x_k) \to \overline{\mathrm{B}}_\eta(x)$ in the Hausdorff metric. Hence $y \in \partial \mathrm{B}_\eta(x)$ and $\mathrm{B}_\eta(x) \subseteq \Gamma$, which gives $\Gamma \in Adm_\eta(\Lambda)$.

It remains to prove that $\mathcal{H}^2(\Gamma_k) \to \mathcal{H}^2(\Gamma)$. The measure is upper semicontinuous with respect to the Hausdorff metric, so we have only to prove

$$\mathcal{H}^2(\Gamma) \leq \liminf_k \mathcal{H}^2(\Gamma_k)\,.$$

Thanks to [65], we have

$$\Gamma_k = \overline{\bigcup_{h\in\mathbb{N}} \overline{\mathrm{B}}_{\frac{\eta}{2}}(z_h^k)}\,, \tag{2.3.7}$$

for some $z_k^k \in \Gamma_k$.

Consider $\{x_h\} \subseteq \mathring{\Gamma}$ a countable dense set in Γ. By the Hausdorff convergence, for every $h \in \mathbb{N}$ there exists a sequence $x_h^k \in \Gamma_k$ such that $x_h^k \to x_h$. Using (2.3.7) we deduce that there exists a sequence y_h^k such that $y_h^k \in \Gamma_k$, $\mathrm{B}_{\frac{\eta}{2}}(y_h^k) \subseteq \Gamma_k$, and $x_h^k \in \overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h^k)$. Up to a subsequence, we can assume that $y_h^k \to y_h \in \Gamma$ for every $h \in \mathbb{N}$, so that $\overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h^k) \to \overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h)$ in the Hausdorff metric and $x_h \in \overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h) \subseteq \Gamma$. Therefore

$$\Gamma = \overline{\bigcup_{h\in\mathbb{N}} \overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h)} = \overline{\bigcup_{h\in\mathbb{N}} \mathrm{B}_{\frac{\eta}{2}}(y_h)}\,.$$

Let us consider the sets $\Gamma^N := \bigcup_{h=0}^N \overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h)$, $\Gamma_k^N := \bigcup_{h=0}^N \overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h^k)$ and the functions

$$\varphi^N := \sum_{h=0}^N \mathbf{1}_{\overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h)}\,, \qquad \varphi_k^N := \sum_{h=0}^N \mathbf{1}_{\overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h^k)}\,.$$

By the dominated convergence theorem

$$\begin{aligned}\mathcal{H}^2(\Gamma^N) &= \sum_{h=0}^N \int_{\overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h)} \frac{1}{\varphi^N(x)}\,\mathrm{d}\mathcal{H}^2(x) = \lim_k \sum_{h=0}^N \int_{\overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h^k)} \frac{1}{\varphi_k^N(x)}\,\mathrm{d}\mathcal{H}^2(x)\\ &= \lim_k \mathcal{H}^2(\Gamma_k^N) \leq \liminf_k \mathcal{H}^2(\Gamma_k)\,.\end{aligned}$$

If we pass to the limit as $N \to +\infty$, we get $\mathcal{H}^2(\Gamma^N) \to \mathcal{H}^2\big(\bigcup_{h\in\mathbb{N}} \overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h)\big)$, so we are led to prove that

$$\mathcal{H}^2\Big(\Gamma \setminus \bigcup_{h\in\mathbb{N}} \overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h)\Big) = 0\,. \tag{2.3.8}$$

Assume, by contradiction, that (2.3.8) is false. Then there exists $x \in \Gamma \setminus \bigcup_{h\in\mathbb{N}} \overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h)$ such that

$$\lim_{\rho\to 0^+} \frac{\mathcal{H}^2\big(\mathrm{B}_\rho(x) \cap \Gamma \setminus \bigcup_{h\in\mathbb{N}} \overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h)\big)}{\mathcal{H}^2(\mathrm{B}_\rho(x))} = 1\,. \tag{2.3.9}$$

We can find a ball $\mathrm{B}_{\frac{\eta}{2}}(y) \subseteq \bigcup_{h\in\mathbb{N}} \overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h)$ such that $x \in \overline{\mathrm{B}}_{\frac{\eta}{2}}(y)$, hence

$$\mathrm{B}_\rho(x) \cap \Gamma \setminus \bigcup_{h\in\mathbb{N}} \overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h) \subseteq \mathrm{B}_\rho(x) \setminus \mathrm{B}_{\frac{\eta}{2}}(y)\,,$$

so we get

$$\lim_{\rho\to 0^+} \frac{\mathcal{H}^2\big(\mathrm{B}_\rho(x)\cap\Gamma\setminus\bigcup_{h\in\mathbb{N}}\overline{\mathrm{B}}_{\frac{\eta}{2}}(y_h)\big)}{\mathcal{H}^2(\mathrm{B}_\rho(x))} \le \lim_{\rho\to 0^+} \frac{\mathcal{H}^2(\mathrm{B}_\rho(x)\setminus \mathrm{B}_{\frac{\eta}{2}}(y))}{\mathcal{H}^2(\mathrm{B}_\rho(x))} = \frac{1}{2}\,,$$

which contradicts (2.3.9). □

Remark 2.3.7. In this way we get also $\mathbf{1}_{\Gamma_k} \to \mathbf{1}_\Gamma$ in $L^1(\Lambda)$. Indeed, since

$$\mathcal{H}^2\Big(\Gamma\setminus\bigcup_{h\in\mathbb{N}}\mathrm{B}_{\frac{\eta}{2}}(y_h)\Big)=0\,,$$

we have that $\mathbf{1}_{\Gamma_k}(x) \to \mathbf{1}_\Gamma(x)$ for a.e. $x\in\Lambda$ and, by the dominated convergence theorem, we obtain the convergence in $L^1(\Lambda)$.

Proposition 2.3.8. *Let* $\Gamma \in Adm_\eta(\Lambda)$. *Then* $diam(\Gamma) \le \frac{8}{\pi\eta}\mathcal{H}^2(\Gamma)+\eta$.

Proof. First we prove that $\Gamma\in Adm_\eta(\Lambda)$ is path-connected. Indeed we can follow the standard proof for open sets and show by contradiction that for every two points $x,y\in\Gamma$, there exists a chain of balls joining them, i.e., there exist $\mathrm{B}_{\frac{\eta}{2}}(\xi_1),\dots,\mathrm{B}_{\frac{\eta}{2}}(\xi_k)\subseteq\Gamma$ such that $x\in\overline{\mathrm{B}}_{\frac{\eta}{2}}(\xi_0)$, $y\in\overline{\mathrm{B}}_{\frac{\eta}{2}}(\xi_k)$ and $\overline{\mathrm{B}}_{\frac{\eta}{2}}(\xi_i)\cap\overline{\mathrm{B}}_{\frac{\eta}{2}}(\xi_{i+1})\neq\emptyset$ for every $i=0,\dots,k-1$. Assume that this is not true, then there are two points $x,y\in\Gamma$ for which there is no chain. We define

$$\begin{aligned}\Gamma_1 &:= \{z\in\Gamma : \text{there exists a chain joining } z,y\},\\ \Gamma_2 &:= \{z\in\Gamma : \text{there is no chain joining } z,y\}.\end{aligned}$$

Of course $\Gamma=\Gamma_1\cup\Gamma_2$ and $\Gamma_1\cap\Gamma_2=\emptyset$. The set Γ_1 is nonempty, since $y\in\Gamma_1$, and closed. Indeed, given z_h in Γ_1 such that $z_h\to z$, then $z\in\Gamma$ and there exists a sequence ξ_h in Γ such that $z_h\in\overline{\mathrm{B}}_{\frac{\eta}{2}}(\xi_h)\subseteq\Gamma$. We can assume $\xi_h\to\xi$, $\xi\in\Gamma$, hence $\overline{\mathrm{B}}_{\frac{\eta}{2}}(\xi_h)\to\overline{\mathrm{B}}_{\frac{\eta}{2}}(\xi)$ in the Hausdorff metric. This implies $z\in\overline{\mathrm{B}}_{\frac{\eta}{2}}(\xi)$, so $z\in\Gamma_1$. Also the set Γ_2 is nonempty, since $x\in\Gamma_2$, and closed. Let z_h be a sequence in Γ_2 such that $z_h\to z$. We have $z\in\Gamma$. By contradiction, assume that $z\notin\Gamma_2$, then $z\in\Gamma_1$, which implies the existence of a chain joining z and y. For every $h\in\mathbb{N}$ we can find $\xi_h\in\Gamma$ such that $z_h\in\overline{\mathrm{B}}_{\frac{\eta}{2}}(\xi_h)\subseteq\Gamma$ and $\xi_h\to\xi$. Then $\overline{\mathrm{B}}_{\frac{\eta}{2}}(\xi_h)\to\overline{\mathrm{B}}_{\frac{\eta}{2}}(\xi)$ in the Hausdorff metric and $z\in\overline{\mathrm{B}}_{\frac{\eta}{2}}(\xi)\subseteq\Gamma$. We deduce that $z_h\in\Gamma_1\cap\Gamma_2$ for h large enough. Hence Γ_2 is closed and Γ is the union of two closed, disjoint and nonempty subset of Γ, which is in contradiction with the fact that Γ is connected. Therefore Γ is path-connected.

Given $x,y\in\Gamma$, we have to estimate the distance $l:=d(x,y)$ in term of $\mathcal{H}^2(\Gamma)$. Let $\gamma:[0,1]\to\Gamma$ be a continuous curve such that $\gamma(0)=x$ and $\gamma(1)=y$. We take the lines perpendicular to the segment $[x,y]$ at distance from x a multiple of η and intersecting $[x,y]$. They intersect the segment $[x,y]$ in $x=x_0,x_1,\dots,x_n$. Let us define the segments $I_k:=[x_{k-1},x_k]$ for $k=1,\dots,n$. For $h\in[0,(n+1)/2]\cap\mathbb{N}$, let ξ_{2h+1} be the middle point of the segment I_{2h+1} and let s_{2h+1} be the line perpendicular to $[x,y]$ passing through ξ_{2h+1}. These lines intersect the curve γ in ζ_{2h+1}.

For every h, there exists a ball $\mathrm{B}_{\frac{\eta}{2}}(y_{2h+1}) \subseteq \Gamma$ such that $\zeta_{2h+1} \in \overline{\mathrm{B}}_{\frac{\eta}{2}}(y_{2h+1})$. These balls are mutually disjoint, hence we have

$$\frac{l}{8}\pi\eta - \frac{\pi}{8}\eta^2 \leq \left[\frac{l}{2\eta} + \frac{1}{2}\right]\frac{\pi}{4}\eta^2 \leq \mathcal{H}^2(\Gamma)\,,$$

which implies

$$l \leq \frac{8}{\pi\eta}\mathcal{H}^2(\Gamma) + \eta\,,$$

and the proof is thus concluded. □

Let us now comment on the initial condition of Theorem 2.3.4.

Remark 2.3.9. If the set $\Gamma_0 \in Adm_\eta(\Lambda)$ does not satisfy the stability condition (2.3.4), we define Γ_0^* to be a solution of (2.3.4). In particular, Γ_0^* minimizes $\mathcal{E}_m(\Gamma, V(0))$ among all $\Gamma \in Adm_\eta(\Lambda)$ with $\Gamma \supseteq \Gamma_0^*$.

Therefore we can solve the problem considered in Theorem 2.3.4 with initial condition $\Gamma(0) = \Gamma_0^*$. A solution of (2.3.4) can be obtained by the direct method of the calculus of variations. Indeed, a minimizing sequence Γ_k is bounded by Proposition 2.3.6, so, by Theorem 1.1.1, we can assume $\Gamma_k \to \Gamma$ in the Hausdorff metric. For every $k \in \mathbb{N}$ there exists a unique $u_k \in \mathcal{A}(\Gamma_k, V(0))$ solution of (2.3.3). Since u_k is bounded in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Lambda; \mathbb{R}^3)$ by Proposition 1.2.6, we have $u_k \rightharpoonup v$ weakly in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Lambda; \mathbb{R}^3)$, hence $v \in \mathcal{A}(\Gamma, V(0))$ and

$$\mathcal{E}_m(\Gamma, V(0)) \leq \mathcal{E}(v, \Gamma) \leq \liminf_k \mathcal{E}_m(\Gamma_k, V(0))\,,$$

which shows that Γ is a minimizer.

To prove Theorem 2.3.4 we need the following lemma.

Lemma 2.3.10. *Let $\Gamma_0, \Gamma_k, \Gamma_\infty \in Adm_\eta(\Lambda)$ be such that $\Gamma_0 \subseteq \Gamma_k$ and $\Gamma_k \to \Gamma_\infty$ in the Hausdorff metric. Let $V_k, V_\infty \geq 0$ with $V_k \to V_\infty$. Assume that*

$$\mathcal{E}_m(\Gamma_k, V_k) \leq \mathcal{E}_m(\Gamma, V_k) \qquad \text{for every } \Gamma \in Adm_\eta(\Lambda) \text{ with } \Gamma \supseteq \Gamma_k\,.$$

Then

$$\mathcal{E}_m(\Gamma_\infty, V_\infty) \leq \mathcal{E}_m(\Gamma, V_\infty) \qquad \text{for every } \Gamma \in Adm_\eta(\Lambda) \text{ with } \Gamma \supseteq \Gamma_\infty\,. \tag{2.3.10}$$

Let u_k, u_∞ be the solutions of (2.3.3) corresponding to Γ_k, V_k and Γ_∞, V_∞ and let $p(\Gamma_k, V_k), p(\Gamma_\infty, V_\infty)$ be the corresponding pressures according to Remark 2.3.15. Then $u_k \to u_\infty$ in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Lambda; \mathbb{R}^3)$, $p(\Gamma_k, V_k) \to p(\Gamma_\infty, V_\infty)$, and $\mathcal{E}_m(\Gamma_k, V_k) \to \mathcal{E}_m(\Gamma_\infty, V_\infty)$.

Proof. Let us fix $w_0 \in \mathcal{A}(\Gamma_0, 1)$.

For every $\Gamma \in Adm_\eta(\Lambda)$ such that $\Gamma \supseteq \Gamma_\infty$, let $v_\Gamma \in \mathcal{A}(\Gamma, V_\infty)$ be the solution of (2.3.3). For every k we define $\hat{\Gamma}_k := \Gamma \cup \Gamma_k$ and

$$v_k := \begin{cases} v_\Gamma + (V_k - V_\infty)\dfrac{v_\Gamma}{V_\infty} & \text{if } V_\infty > 0\,, \\ V_k w_0 & \text{if } V_\infty = 0\,. \end{cases}$$

Then $\hat{\Gamma}_k \in Adm_\eta(\Lambda)$ by Proposition 2.3.5, $\Gamma_k \subseteq \hat{\Gamma}_k$, $\hat{\Gamma}_k \to \Gamma$ in the Hausdorff metric, and, by Proposition 2.3.6, $\mathcal{H}^2(\hat{\Gamma}_k) \to \mathcal{H}^2(\Gamma)$, $v_k \in \mathcal{A}(\hat{\Gamma}_k, V_k)$ and $v_k \to v_\Gamma$ in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Lambda; \mathbb{R}^3)$. By hypothesis we have

$$\mathcal{E}_m(\Gamma_k, V_k) \leq \mathcal{E}_m(\hat{\Gamma}_k, V_k) \leq \mathcal{E}(v_k, \hat{\Gamma}_k)\,. \tag{2.3.11}$$

This implies that u_k is bounded in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Lambda; \mathbb{R}^3)$, hence, up to a subsequence, $u_k \rightharpoonup u$ weakly in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Lambda; \mathbb{R}^3)$ and $u \in \mathcal{A}(\Gamma_\infty, V_\infty)$. Taking into account (2.3.11), we have

$$\begin{aligned} \mathcal{E}_m(\Gamma_\infty, V_\infty) &\leq \mathcal{E}(u, \Gamma_\infty) \leq \liminf_k \mathcal{E}_m(\Gamma_k, V_k) \\ &\leq \limsup_k \mathcal{E}_m(\Gamma_k, V_k) \leq \limsup_k \mathcal{E}(v_k, \hat{\Gamma}_k) = \mathcal{E}(v_\Gamma, \Gamma) = \mathcal{E}_m(\Gamma, V_\infty)\,, \end{aligned} \tag{2.3.12}$$

which proves (2.3.10). In particular, taking $\Gamma = \Gamma_\infty$, (2.3.12) shows that u satisfies

$$\mathcal{E}(u, \Gamma_\infty) = \mathcal{E}_m(\Gamma_\infty, V_\infty) = \lim_k \mathcal{E}_m(\Gamma_k, V_k) = \lim_k \mathcal{E}(u_k, \Gamma_k)\,.$$

By the uniqueness of the solution of (2.3.3), the whole sequence u_k converges to u_∞ strongly in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Lambda; \mathbb{R}^3)$. From this convergence and Remark 2.3.2, it follows that $p(\Gamma_k, V_k) \to p(\Gamma_\infty, V_\infty)$, when $V_\infty > 0$.

As in the proof of Lemma 2.2.22, we have that $p(\Gamma_k, V_k) \to 0 = p(\Gamma_\infty, V_\infty)$ if $V_\infty = 0$. This concludes the proof of the lemma. □

Remark 2.3.11. By the same argument we can show that the function $V \mapsto \mathcal{E}_m(\Gamma, V)$ is continuous for every $\Gamma \in Adm_\eta(\Lambda)$.

We are now ready to prove Theorem 2.3.4.

Proof of Theorem 2.3.4. We proceed following the lines of Theorem 2.2.18. We choose a subdivision of the interval $[0,T]$ of the form $t^k_i := \frac{iT}{k}$ for $i = 0, \dots, k$. For every k we define Γ^k_i recursively with respect to i: we set $\Gamma^k_0 := \Gamma_0$ and, for $i > 0$, Γ^k_i to be a solution of

$$\min\{\mathcal{E}_m(\Gamma, V(t^k_i)) : \Gamma \in Adm_\eta(\Lambda), \text{ with } \Gamma \supseteq \Gamma^k_{i-1}\}\,, \tag{2.3.13}$$

whose existence can be proved as in Remark 2.3.9. We denote by u^k_i the solution of (2.3.3) for $\Gamma = \Gamma^k_i$ and $V = V(t^k_i)$.

As in the proof of Lemma 2.2.22, we get that u^k_i are uniformly bounded in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Lambda; \mathbb{R}^3)$. Moreover, the pressure $p(\Gamma^k_i, V(t^k_i))$ associated to u^k_i according to Remark 2.3.15 is bounded.

We define the step functions

$$u_k(t) := u^k_i\,, \quad \Gamma_k(t) := \Gamma^k_i\,, \quad p_k(t) := p(\Gamma^k_i, V(t^k_i))\,, \quad \text{for } t^k_i \leq t < t^k_{i+1}\,.$$

We now prove a discrete energy inequality. As in the proof of Theorem 2.2.18 (recall that $\epsilon(t)=0$), we can prove that

$$\begin{aligned}&\mathcal{E}_m\big(\Gamma_i^k, V(t_i^k)\big)\\&\quad\le \mathcal{E}_m\big(\Gamma_{i-1}^k, V(t_{i-1}^k)\big)+C\beta V_k\int_{t_{i-1}^k}^{t_i^k}|\dot V(s)|\,\mathrm{d}s+p(\Gamma_{i-1}^k, V(t_{i-1}^k))\int_{t_{i-1}^k}^{t_i^k}\dot V(s)\,\mathrm{d}s\,,\end{aligned}$$

where $\beta>0$ is the constant defined in (2.3.1), $C>0$ does not depend on i,k, and

$$V_k := \frac{1}{2}\max_{j=1,\dots,k}|V(t_j^k)-V(t_{j-1}^k)|\,. \tag{2.3.14}$$

Iterating the previous inequality we get

$$\mathcal{E}_m\big(\Gamma_k(t), V(t_i^k)\big)\le \mathcal{E}_m\big(\Gamma_0, V(0)\big)+\beta M^2 V_k\int_0^T|\dot V(s)|\,ds+\int_0^{t_i^k}p_k(s)\dot V(s)\,\mathrm{d}s\,. \tag{2.3.15}$$

In particular, (2.3.15) implies that $\mathcal{H}^2(\Gamma_k(t))$ is uniformly bounded in time, hence, by Proposition 2.3.8, $\Gamma_k(t)$ is uniformly bounded.

By Theorem 1.1.3 and Proposition 2.3.6, up to a subsequence we have $\Gamma_k(t)\to\Gamma(t)$ in the Hausdorff metric for every $t\in[0,T]$ and the set function $\Gamma\colon[0,T]\to Adm_\eta(\Lambda)$ is bounded and increasing. Let $u(t)$ be the solution of (2.3.3) and $p(t)$ be the corresponding pressure. By Lemma 2.3.10, Γ satisfies the global stability condition (b) and, in addition, $u_k(t)\to u(t)$ strongly in $\mathrm{W}^1_{2,6}(\mathbb{R}^3\setminus\Lambda;\mathbb{R}^3)$ and $p_k(t)\to p(t):=p(\Gamma(t),V(t))$ for every $t\in[0,T]$.

To prove the energy-dissipation balance, we first pass to the limit in (2.3.15) as $k\to+\infty$. The second term in the right-hand side of (2.3.15) tends to zero as $k\to+\infty$ because V is absolutely continuous. Since p_k is bounded in $L^\infty([0,T])$ and converges pointwise to p, we have $p_k\dot V\to p\dot V$ in $L^1([0,T])$ and

$$\mathcal{E}_m(\Gamma(t),V(t))\le\mathcal{E}_m(\Gamma_0,V(0))+\int_0^t p(s)\dot V(s)\,ds\,.$$

For the opposite inequality, for every $t\in[0,T]$ we consider a subdivision of the interval $[0,t]$ of the form $\tau_h^k:=\frac{ht}{k}$ defined for every $k,h\in\mathbb{N}$, $k\neq 0$, such that $h\le k$. Since $t\mapsto\Gamma(t)$ is increasing, arguing as in the proof of Theorem 2.2.18 we obtain

$$\begin{aligned}&\mathcal{E}_m\big(\Gamma(\tau_h^k), V(\tau_h^k)\big)\\&\quad\le \mathcal{E}_m\big(\Gamma(\tau_{h+1}^k), V(\tau_{h+1}^k)\big)+C\beta V_k\int_{\tau_h^k}^{\tau_{h+1}^k}|\dot V(s)|\,\mathrm{d}s-\int_{\tau_h^k}^{\tau_{h+1}^k}p(\tau_{h+1}^k)\dot V(s)\,\mathrm{d}s\,,\end{aligned}$$

where V_k has been defined in (2.3.14) and C is some positive constant independent of h,k. Iterating the previous inequality and defining $p^k(s):=p(\tau_{h+1}^k)$ for $\tau_h^k<s\le\tau_{h+1}^k$, we get

$$\mathcal{E}_m(\Gamma_0,V(0))\le\mathcal{E}_m(\Gamma(t),V(t))+C\beta V_k\int_0^T|\dot V(s)|\,\mathrm{d}s-\int_0^t p^k(s)\dot V(s)\,\mathrm{d}s\,. \tag{2.3.16}$$

Since $\Gamma(\cdot)$ is an increasing function, combining Theorem 1.1.2 and Lemma 2.3.10 we deduce that $p^k(s) \to p(s)$ for a.e. $s \in [0,t]$. Hence, passing to the limit in (2.3.16) we obtain

$$\mathcal{E}_m(\Gamma_0, V(0)) \leq \mathcal{E}_m(\Gamma(t), V(t)) - \int_0^t p(s)\dot{V}(s)\,\mathrm{d}s\,.$$

This concludes the proof of the energy-dissipation balance (c). □

Let $\Gamma\colon [0,T] \to Adm_\eta(\Lambda)$ satisfy Theorem 2.3.4. For every $t \in (0,T]$ we consider $\Gamma^-(t)$ defined, as in Theorem 1.1.2, by

$$\Gamma^-(t) := \overline{\bigcup_{s<t} \Gamma(s)}\,. \tag{2.3.17}$$

We have $\Gamma(t) = \Gamma^-(t)$ and $\mathcal{E}_m(\Gamma^-(t), V(t)) = \mathcal{E}_m(\Gamma(t), V(t))$ for every $t \in (0,T]$ out of a countable set.

Proposition 2.3.12. *Let $\Gamma\colon [0,T] \to Adm_\eta(\Lambda)$ satisfy Theorem 2.3.4 and let $\Gamma^-(t)$ be given by* (2.3.17) *for every $t \in (0,T]$. Then*

$$\mathcal{E}_m(\Gamma^-(t), V(t)) = \mathcal{E}_m(\Gamma(t), V(t)) \qquad \text{for every } t \in (0,T]\,. \tag{2.3.18}$$

Moreover,

$$\mathcal{E}_m(\Gamma(t), V(t)) \leq \mathcal{E}_m(\Gamma, V(t)) \qquad \text{for every } \Gamma \in Adm_\eta(\Lambda) \text{ with } \Gamma \supseteq \Gamma^-(t)\,.$$

Proof. Since $\Gamma(s) \to \Gamma^-(t)$ in the Hausdorff metric as $s \nearrow t$, by Lemma 2.3.10 we get

$$\mathcal{E}_m(\Gamma^-(t), V(t)) = \lim_{s \nearrow t} \mathcal{E}_m(\Gamma(s), V(s))\,.$$

By the continuity of $s \mapsto \mathcal{E}_m(\Gamma(s), V(s))$ we obtain (2.3.18).

Fixed $\Gamma \in Adm_\eta(\Lambda)$ with $\Gamma^-(t) \subseteq \Gamma$, we have $\Gamma(s) \subseteq \Gamma$ for every $0 \leq s < t$, hence

$$\mathcal{E}_m(\Gamma(s), V(s)) \leq \mathcal{E}_m(\Gamma, V(s))\,. \tag{2.3.19}$$

and passing to the limit as $s \nearrow t$ we get the thesis. □

Remark 2.3.13. Thanks to Proposition 2.3.12 we have that if $\Gamma\colon [0,T] \to Adm_\eta(\Lambda)$ satisfies Theorem 2.3.4, the same is true for the function

$$t \mapsto \begin{cases} \Gamma(0) & \text{for } t = 0\,, \\ \Gamma^-(t) & \text{for } 0 < t \leq T\,, \end{cases}$$

where $\Gamma^-(t)$ is defined in (2.3.17). We notice that in the energy-dissipation balance we have to replace $p(t)$ with $p^-(t)$ which satisfies Proposition **??**, extending it at $t = 0$ by $p^-(0) := p(0)$. Then the quasi-static hydraulic crack problem has a left-continuous solution.

Remark 2.3.14. Repeating the same steps, for every $t \in [0,T)$ we define $\Gamma^+(t)$ as in Theorem 1.1.2. As in Proposition 2.3.12 we obtain $\mathcal{E}_m(\Gamma(t),V(t)) = \mathcal{E}_m(\Gamma^+(t),V(t))$ for every $t \in [0,T)$ and finally, as in Remark 2.3.13, we define the function

$$t \mapsto \begin{cases} \Gamma^+(t) & \text{for } 0 \le t < T\,, \\ \Gamma(T) & \text{for } t = T\,, \end{cases}$$

which satisfies properties (a), (b), and (c) of Definition 2.3.3. Therefore, we get a right-continuous solution of the problem. Note however that the right-continuous solution does not necessarily satisfy the initial condition.

2.3.2 The case of penny-shaped crack

Let us now briefly discuss a simplified 3D-model for which we can give an explicit formula for the evolution: the case of *penny-shaped* cracks. The body is supposed to be unbounded, filling $\mathbb{R}^3$. As before, we prescribe a priori the crack path: the admissible cracks lie on the horizontal plane Λ passing through the origin.

We assume that the initial crack is a circle centered at the origin and contained in Λ and that the body outside the crack is isotropic, homogeneous, and impermeable. Due to the symmetry conditions, we also assume that the crack is circular and centered at the origin at every time, so that every crack set is parametrized by its radius $R > 0$.

As in Section 2.3 the total energy of the body is defined by

$$\mathcal{E}(u,R) := \frac{1}{2}\int_{\mathbb{R}^3\setminus\Lambda} \mathbb{C}\mathrm{E}u\cdot\mathrm{E}u\,\mathrm{d}x + \kappa\pi R^2\,, \tag{2.3.20}$$

for a displacement $u \in \mathrm{W}^1_{2,6}(\mathbb{R}^3\setminus\Lambda;\mathbb{R}^3)$.

The equilibrium condition for the body with a prescribed crack of radius R is expressed by the minimum problem

$$\min_{u\in\mathcal{A}(R,V)} \mathcal{E}(u,R)\,, \tag{2.3.21}$$

where

$$\mathcal{A}(R,V) := \Big\{u \in \mathrm{W}^1_{2,6}(\mathbb{R}^3\setminus\Lambda) : \{[u]\neq 0\} \subseteq \overline{\mathrm{B}}_R,\ [u]\cdot\nu_\Lambda \ge 0,\ \int_{\mathrm{B}_R}[u]\cdot\nu_\Lambda\,\mathrm{d}\mathcal{H}^2 = V\Big\}\,.$$

The existence of a solution of (2.3.21) can be obtained by the direct method of the calculus of variations, taking into account Proposition 1.2.6. The uniqueness follows from the strict convexity of the functional and the convexity of the constraints.

Remark 2.3.15. All the results stated in Remark 2.3.2 holds also in this particular case. We denote by $p(R,V)$ the pressure of the fluid associated to the minimum problem (2.3.21). for every

Let us now consider the quasi-static evolution problem. Let $T > 0$ be fixed and for every $t \in [0,T]$ let $V(t)$ be the volume of the fluid present in the crack at time t. We assume that $V \in AC([0,T];[0,+\infty))$.

To describe the quasi-static evolution it is convenient to introduce the reduced energy $\mathcal{E}_m(R,V)$ defined for every $R \in [0,+\infty)$ and every $V \in [0,+\infty)$ by

$$\mathcal{E}_m(R,V) := \min_{u \in \mathcal{A}(R,V)} \mathcal{E}(u,R)\,. \tag{2.3.22}$$

In order to make explicit the dependence of $\mathcal{E}_m(R,V)$ on R and V, let us denote by u_R the solution to the minimum problem defining $\mathcal{E}_m(R,1)$. It is then easy to see that Vu_R is the solution of (2.3.22) and

$$\mathcal{E}_m(R,V) = \frac{V^2}{2}\int_{\mathbb{R}^3\setminus\Lambda} \mathbb{C}\mathrm{E}u_R \cdot \mathrm{E}u_R \,\mathrm{d}x + \kappa\pi R^2\,.$$

Moreover, by the uniqueness of the solution to (2.3.22) it follows that

$$u_R(x) = \frac{1}{R^2}u_1\Big(\frac{x}{R}\Big) \quad \text{and} \quad \int_{\mathbb{R}^3\setminus\Lambda} \mathbb{C}\mathrm{E}u_R \cdot \mathrm{E}u_R \,\mathrm{d}x = \frac{1}{R^3}\int_{\mathbb{R}^3\setminus\Lambda} \mathbb{C}\mathrm{E}u_1 \cdot \mathrm{E}u_1 \,\mathrm{d}x\,. \tag{2.3.23}$$

Therefore

$$\mathcal{E}_m(R,V) = K\frac{V^2}{R^3} + \kappa\pi R^2\,, \tag{2.3.24}$$

where $K := \frac{1}{2}\int_{\mathbb{R}^3\setminus\Lambda} \mathbb{C}\mathrm{E}u_1 \cdot \mathrm{E}u_1 \,\mathrm{d}x$. Since

$$\frac{d}{dR}\mathcal{E}_m(R,V) = -3K\frac{V^2}{R^4} + 2\kappa\pi R\,, \tag{2.3.25}$$

we note that the unique minimum point of $R \mapsto \mathcal{E}_m(R,V)$ is $R = \left(\frac{3KV^2}{2\kappa\pi}\right)^{1/5}$.

Hence, if we fix $\hat{R} > 0$, the unique solution to the minimum problem

$$\min_{R \geq \hat{R}} \mathcal{E}_m(R,V)$$

is given by

$$R_* = \max\left\{\hat{R}, \left(\tfrac{3KV^2}{2\kappa\pi}\right)^{1/5}\right\}.$$

In this simplified setting, since the function $R \mapsto \mathcal{E}_m(R,V)$ is convex, Griffith's stability condition expressed by the inequality

$$\frac{d}{dR}\mathcal{E}_m(R(t),V(t)) \geq 0 \qquad \text{for every } t \in [0,T]$$

is equivalent to the global minimality condition: for every $t \in [0,T]$

$$\mathcal{E}_m(R(t),V(t)) \leq \mathcal{E}_m(R,V(t)) \qquad \text{for every } R \geq R(t)\,,$$

which in this case reduces to

$$R(t) \geq \left(\frac{3KV^2(t)}{2\kappa\pi}\right)^{1/5} \qquad \text{for every } t \in [0,T]\,. \tag{2.3.26}$$

Since the fracture process is irreversible, we require that $R(\cdot)$ is increasing. Finally, we impose an energy-dissipation balance: the rate of change of the total energy (stored elastic energy plus energy dissipated by the crack) along a solution equals the power of the pressure forces exerted by the fluid.

This leads to the following definition.

Definition 2.3.16. Let $T > 0$ and $V \in AC([0,T];[0,+\infty))$. We say that a function $R\colon [0,T] \to (0,+\infty)$ is an *irreversible quasi-static evolution* of the *penny-shaped hydraulic crack problem* if it satisfies the following conditions:

(a) *irreversibility*: R is increasing, i.e., $R(s) \leq R(t)$ for every $0 \leq s \leq t \leq T$;

(b) *global stability*: for every $t \in [0,T]$,

$$\mathcal{E}_m(R(t),V(t)) \leq \mathcal{E}_m(R,V(t)) \qquad \text{for every } R \geq R(t)\,;$$

(c) *energy-dissipation balance*: the function $t \mapsto \mathcal{E}_m(R(t),V(t))$ is absolutely continuous on the interval $[0,T]$ and

$$\frac{d}{dt}\mathcal{E}_m(R(t),V(t)) = p(t)\dot{V}(t)$$

for almost every $t \in [0,T]$, where $p(t) := p(R(t),V(t))$ is the pressure introduced in Remark 2.3.15.

While in the technological applications to hydraulic fracture it is natural to suppose that V is increasing, the problem makes sense even without this assumption. For instance, if in a time interval V is decreasing, which means that some liquid is removed from the cavity, by the irreversibility assumption we expect that R remains constant in that interval and that the crack opening decreases to accommodate to the volume constraint. This is a direct consequence of the formula (2.3.28) proved in the next theorem.

We are now ready to state the main result of this section.

Theorem 2.3.17. *Let $V \in AC([0,T];[0,+\infty))$ and $R_0 > 0$. Assume that (stability at time $t = 0$)*

$$\mathcal{E}_m(R_0,V(0)) \leq \mathcal{E}_m(R,V(0)) \tag{2.3.27}$$

for every $R \geq R_0$. Then the unique irreversible quasi-static evolution $R_*\colon [0,T] \to (0,+\infty)$ of the penny-shaped hydraulic crack problem, with $R(0) = R_0$, is given by

$$R_*(t) = \max\left\{R_0, \left(\tfrac{3K}{2\kappa\pi}\right)^{1/5} V_*^{2/5}(t)\right\}, \tag{2.3.28}$$

where $V_*(t)$ is the smallest monotone increasing function which is greater than or equal to $V(t)$, i.e., $V_*(t) = \max\limits_{0 \leq s \leq t} V(s)$.

When V is increasing we recover the explicit solution considered, e.g., in [19], see also [69].

Remark 2.3.18. In view of (2.3.26) condition (2.3.27) amounts to

$$R_0 \geq \left(\frac{3KV^2(0)}{2\kappa\pi}\right)^{1/5} .$$

To prove Theorem 2.3.17 we need the following lemmas. In the first one we prove the absolute continuity of the function V_*.

Lemma 2.3.19. *Let* $V \in AC([0,T];[0,+\infty))$ *and for every* $t \in [0,T]$ *set* $V_*(t) = \max\limits_{0\leq s\leq t} V(s)$. *Then* $V_* \in AC([0,T];[0,+\infty))$ *and*

$$\dot{V}_*(t) = \dot{V}(t)1_{\{V=V_*\}}(t) \qquad \textit{for a.e. } t \in [0,T]\,. \tag{2.3.29}$$

Proof. As $V \in AC([0,T];[0,+\infty))$, there exist two increasing absolutely continuous functions $V_1, V_2\colon [0,T] \to [0,+\infty)$ such that $V = V_1 - V_2$. Note that

$$V_*(t_2) - V_*(t_1) \leq V_1(t_2) - V_1(t_1) \qquad \text{for every } 0 \leq t_1 \leq t_2 \leq T\,. \tag{2.3.30}$$

Indeed, for every $t_1 \leq s \leq t_2$

$$\begin{aligned} V(s) - V_*(t_1) &\leq V(s) - V(t_1) = V_1(s) - V_2(s) - V_1(t_1) + V_2(t_1) \\ &\leq V_1(t_2) - V_1(t_1) - (V_2(s) - V_2(t_1)) \leq V_1(t_2) - V_1(t_1)\,, \end{aligned}$$

and by the definition of V_* this implies (2.3.30). As V_1 is absolutely continuous, from (2.3.30) we deduce the absolute continuity of V_*.

Since the function V_* is locally constant on the open set $\{t \in [0,T] : V_*(t) > V(t)\}$, we have $\dot{V}_* = 0$ on this set, while $\dot{V}_*(\bar{t}) = \dot{V}(\bar{t})$ for a.e. $\bar{t} \in \{t \in [0,T] : V_*(t) = V(t)\}$. Therefore (2.3.29) holds. □

Lemma 2.3.20. *Let* $V \in AC([0,T];[0,+\infty))$ *and* $R_0 > 0$. *Assume that* R_0 *satisfies* (2.3.27). *Then* $R_*\colon [0,T] \to (0,+\infty)$ *given by* (2.3.28) *is the smallest increasing function which satisfies the global stability condition (b), with* $R(0) = R_0$.

Proof. Let $R(t)$ be an increasing function with $R(0) = R_0$ that satisfies the global stability condition (b). In view of (2.3.26) we have $R(t) \geq \left(\frac{3KV^2(t)}{2\kappa\pi}\right)^{1/5}$ for every $t \in [0,T]$. Since $R(t) \geq R(s)$ for every $s, t \in [0,T]$ with $s \leq t$, we get

$$R(t) \geq \max_{0\leq s\leq t}\left(\frac{3KV^2(s)}{2\kappa\pi}\right)^{1/5} = \left(\frac{3KV_*^2(t)}{2\kappa\pi}\right)^{1/5} ,$$

which implies $R(t) \geq R_*(t)$.

As $R_*(t)$ satisfies (2.3.26) for every $t \in [0,T]$, the function $t \mapsto R_*(t)$ satisfies the global stability condition (b). □

We now prove that $R_*\colon [0,T] \to (0,+\infty)$ defined by (2.3.28) is an irreversible quasi-static evolution of the penny-shaped hydraulic crack problem.

Proof of Theorem 2.3.17 (existence). It remains to prove the energy-dissipation balance (c). Let us set

$$\alpha_0 = \left(\frac{2\kappa\pi}{3K}\right)^{1/2} R_0^{5/2}.$$

By (2.3.28), if $V_*(t) \le \alpha_0$ then $R_*(t) = R_0$. Assume there exists $t \in [0,T]$ such that $V(t) \ge \alpha_0$ and let $\bar t := \inf\{t \in [0,T] : V(t) \ge \alpha_0\}$. Then $R_*(t) = R_0$ for $t \in [0,\bar t\,]$, while $R_*(t) = \left(\frac{3KV_*^2(t)}{2\kappa\pi}\right)^{1/5}$ and $V_*(t) \in [\alpha_0,+\infty)$ for $t \in [\,\bar t, T]$.

By Lemma 2.3.19 and the Lipschitz continuity of the function $a \mapsto a^{2/5}$ on $[\alpha_0,+\infty)$ we deduce that $R_*(\cdot)$ is absolutely continuous on $[0,T]$. Then, as

$$\mathcal{E}_m(R_*(t),V(t)) = K\frac{V^2(t)}{R_*^3(t)} + \kappa\pi R_*^2(t) \qquad \text{for every } t \in [0,T]\,,$$

it follows that $\mathcal{E}_m(R_*(\cdot),V(\cdot))$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\mathcal{E}_m(R_*(t),V(t)) = 2K\frac{V(t)\dot V(t)}{R_*^3(t)} + \dot R_*(t)\Big(2\kappa\pi R_*(t) - 3K\frac{V^2(t)}{R_*^4(t)}\Big) \qquad \text{for a.e. } t \in [0,T]\,.$$

Since, by Remark 2.3.2 and (2.3.24), $p(t) = 2K\frac{V(t)}{R_*^3(t)}$ for every $t \in [0,T]$, while by (2.3.28) and Lemma 2.3.19 the product $\dot R_*(t)\Big(2\kappa\pi R_*(t) - 3K\frac{V^2(t)}{R_*^4(t)}\Big)$ is equal to 0, we get

$$\frac{d}{dt}\mathcal{E}_m(R_*(t),V(t)) = p(t)\dot V(t)\,,$$

and this concludes the proof of the existence of an irreversible quasi-static evolution for the penny-shaped hydraulic crack problem. □

The next result establishes some regularity properties of a solution that will be used to prove the uniqueness.

Lemma 2.3.21. *Let $V \in AC([0,T];[0,+\infty))$ and let $R\colon [0,T] \to (0,+\infty)$ be an irreversible quasi-static evolution of the penny-shaped hydraulic crack problem with $R(0) = R_0$. Then $R(\cdot)$ is continuous on $[0,T]$ and is absolutely continuous on every compact set contained in*

$$I := \{t \in [0,T] : R(t) > R_*(t) \text{ and } V(t) > 0\}\,.$$

Proof. Let $R : [0,T] \to (0,+\infty)$ be an irreversible quasi-static evolution of the penny-shaped hydraulic crack problem with $R(0) = R_0$. By condition (c) of Definition 2.3.16 the function $t \mapsto \mathcal{E}_m(R(t),V(t))$ is absolutely continuous and by Lemma 2.3.20 $R(t) \ge R_*(t)$. Let us show by contradiction that R is continuous. Assume $\tilde t \in [0,T]$ is a discontinuity point. Since $t \mapsto R(t)$ is increasing and, by (2.3.24), the function $R \mapsto \mathcal{E}_m(R,V(t))$ is strictly increasing for $R \ge R_*(t)$, we have

$$\lim_{s\nearrow\tilde t}\mathcal{E}_m(R(s),V(s)) = \mathcal{E}_m(R(\tilde t^-),V(\tilde t)) < \mathcal{E}_m(R(\tilde t^+),V(\tilde t)) = \lim_{s\searrow\tilde t}\mathcal{E}_m(R(s),V(s))$$

which contradicts the continuity of $t \mapsto \mathcal{E}_m(R(t), V(t))$.

Let us define the function $e_R : [0,T] \to \mathbb{R}$ as

$$e_R(t) := K\frac{V^2(0)}{R_0^3} + \kappa\pi R_0^2 + \int_0^t p_R(s)\dot V(s)\,\mathrm{d}s\,, \tag{2.3.31}$$

where $p_R(t)$ is the pressure function introduced in Proposition 2.3.2 in the case $R = R(t)$ and $V = V(t)$. By Remark 2.3.2 and (2.3.23) we get

$$e_R(t) = K\frac{V^2(0)}{R_0^3} + \kappa\pi R_0^2 + 2K\int_0^t \frac{V(s)\dot V(s)}{R^3(s)}\,\mathrm{d}s\,.$$

Since $R(t) \geq R_0$ on $[0,T]$ and $V \in AC([0,T];[0,+\infty))$, it follows that $e_R \in AC([0,T])$. By the energy-dissipation balance condition (c) of Definition 2.3.16 and by (2.3.24)

$$\mathcal{E}_m(R(t), V(t)) = K\frac{V^2(t)}{R^3(t)} + \kappa\pi R^2(t) = e_R(t) \qquad \text{for every } t \in [0,T]\,. \tag{2.3.32}$$

Let $F(B) := \frac{K}{B^3} + \kappa\pi B^2$ for every $B \in (0,+\infty)$. It is easy to see that F belongs to $C^\infty((0,+\infty))$, it is strictly increasing and strictly convex on $J := \Big(\big(\frac{3K}{2\kappa\pi}\big)^{1/5}, +\infty\Big)$. Therefore, $F|_J$ is invertible and F^{-1}, the inverse of $F|_J$, is C^1.

For every $t \in I$ let $B(t) := \frac{R(t)}{V^{2/5}(t)}$. Thus, by (2.3.32)

$$F(B(t)) = \frac{K}{B^3(t)} + \kappa\pi B^2(t) = \frac{e_R(t)}{V^{4/5}(t)}\,.$$

Since $t \in I$ we have $B(t) > \frac{R_*(t)}{V^{2/5}(t)} \geq \big(\frac{3K}{2\kappa\pi}\big)^{1/5}$, hence

$$B(t) = F^{-1}\Big(\frac{e_R(t)}{V^{4/5}(t)}\Big) \qquad \text{for every } t \in I\,. \tag{2.3.33}$$

Since $\frac{e_R(\cdot)}{V^{4/5}(\cdot)}$ is bounded and absolutely continuous on every compact set contained in I, we deduce that $B(\cdot)$ is absolutely continuous on the same sets and so is $R(\cdot)$. $\square$

To prove the uniqueness of the quasi-static evolution, we need the following lemma on absolutely continuous functions.

Lemma 2.3.22. *Let $f, g\colon [a,b] \to \mathbb{R}$ be two functions satisfying the following properties: f is absolutely continuous on $[a,b]$, g is continuous on $[a,b]$, and there exists an open set $A \subset (a,b)$ such that $f = g$ on $(a,b)\setminus A$, and g is constant on each connected component of A. Then g is absolutely continuous on $[a,b]$.*

Proof. Let us fix $\varepsilon > 0$ and choose $\delta > 0$ such that for every finite family of pairwise disjoint intervals $\{(s_i,t_i)\}_{i\in I}$, with $s_i, t_i \in (a,b)$ and $\Sigma_{i\in I}(t_i - s_i) < \delta$, we have $\Sigma_{i\in I}|f(t_i) - f(s_i)| < \epsilon$.

If the interval (s_i, t_i) is contained in a connected component of A then, by our hypotheses on g, we have $g(t_i) = g(s_i)$. Let $I' := \{i \in I : (s_i, t_i) \not\subset A\}$ and let $i \in I'$. Then there exist $s_i', t_i' \in (a,b) \setminus A$ such that $s_i \le s_i' \le t_i' \le t_i$ and $(s_i, s_i'), (t_i', t_i) \subset A$. Indeed, if $s_i \notin A$ we take $s_i' = s_i$. In the opposite case s_i belongs to a connected component (α_i, β_i) of A and we take $s_i' = \beta_i$.

It follows that g is constant on (s_i, s_i') and, by continuity, $g(s_i) = g(s_i') = f(s_i')$, where the last equality holds since $s_i' \in (a,b) \setminus A$. Analogously we get $g(t_i) = g(t_i') = f(t_i')$. Hence $\Sigma_{i\in I}|g(t_i) - g(s_i)| = \Sigma_{i\in I'}|g(t_i) - g(s_i)| = \Sigma_{i\in I'}|f(t_i') - f(s_i')| < \epsilon$, which shows the absolute continuity of the function g on (a,b), and, by continuity, on $[a,b]$. □

We are now ready to prove that R_* defined by (2.3.28) is the unique irreversible quasi-static evolution of the penny-shaped hydraulic crack problem with $R_*(0) = R_0$.

Proof of Theorem 2.3.17 (uniqueness). Let $R\colon [0,T] \to (0,+\infty)$ be an irreversible quasi-static evolution of the penny-shaped hydraulic crack problem with $R(0) = R_0$. By Lemma 2.3.20 $R(t) \ge R_*(t)$ for every $t \in [0,T]$ and by Lemma 2.3.21, R is continuous on $[0,T]$.

Let us assume by contradiction that $R \ne R_*$. Then there exists an interval $(a,b) \subset [0,T]$ such that $R(a) = R_*(a)$ and $R(t) > R_*(t)$ for every $t \in (a,b)$. Let

$$A := \{t \in (a,b) : V(t) > 0\}\,. \tag{2.3.34}$$

By Lemma 2.3.21, the function R is absolutely continuous on every compact set contained in A, hence it is almost everywhere differentiable on A. Recalling (2.3.24) and the energy-dissipation balance condition (c) of Definition 2.3.16, we get

$$\frac{d}{dt}\mathcal{E}_m(R(t),V(t)) = p_R(t)\dot V(t) + \dot R(t)\Big(2\kappa\pi R(t) - 3K\frac{V^2(t)}{R^4(t)}\Big) = p_R(t)\dot V(t)$$

for a.e. $t \in A$, hence

$$\dot R(t)\Big(2\kappa\pi R(t) - 3K\frac{V^2(t)}{R^4(t)}\Big) = 0 \qquad \text{for a.e. } t \in A\,. \tag{2.3.35}$$

Since $R(t) > R_*(t)$ for every $t \in (a,b)$, by the definition of R_* we have

$$2\kappa\pi R(t) - 3K\frac{V^2(t)}{R^4(t)} = \frac{d}{dR}\mathcal{E}_m(R(t),V(t)) > 0 \quad \text{on } A\,.$$

The previous inequality and (2.3.35) imply that $\dot R(t) = 0$ for a.e. $t \in A$, thus R is constant on each connected component of A.

Moreover, by (2.3.32), we have $\kappa\pi R^2(t) = e_R(t)$ for every $t \in (a,b) \setminus A$. Hence applying Lemma 2.3.22 with

$$f = \frac{e_R}{\kappa\pi}, \quad g = R^2, \text{ and the set } A \text{ defined in (2.3.34)}\,,$$

we obtain that R^2 is absolutely continuous on $[a,b]$.

By (2.3.31), for every $t \in (a,b) \setminus A$ we have

$$R^2(t) = \frac{e_R(t)}{\kappa\pi} = \frac{1}{\kappa\pi}\Big(K\frac{V^2(0)}{R_0^3} + \kappa\pi R_0^2 + \int_0^t p_R(s)\dot{V}(s)\,\mathrm{d}s\Big)\,.$$

Since $V \in AC([0,T],[0,+\infty))$ and $V(t) = 0$ for every $t \in (a,b) \setminus A$, we obtain

$$\frac{d}{dt}R^2(t) = \frac{1}{\kappa\pi}\,p_R(t)\dot{V}(t) = 0 \qquad \text{for a.e. } t \in (a,b) \setminus A\,. \tag{2.3.36}$$

As $\dot{R}(t) = 0$ for a.e. $t \in A$, we deduce that $\dot{R}(t) = 0$ for a.e. $t \in (a,b)$, and therefore, being continuous, the function $R(\cdot)$ has to be constant on $[a,b]$. As a consequence, for every $t \in (a,b)$ we have

$$R(a) = R(t) > R_*(t) \geq R_*(a) = R(a)\,,$$

which is a contradiction. Therefore, $R = R_*$ and the proof of uniqueness is concluded. □

Chapter 3

Energy release rate and quasi-static evolution via vanishing viscosity in a fracture model depending on the crack opening

3.1 Introduction and setting of the problem

In this chapter we are interested in the application of the Griffith's criterion to a problem of quasi-static cohesive crack growth in the setting of planar linearized elasticity. We consider a linearly elastic body $\overline{\Omega}$, where $\Omega \subseteq \mathbb{R}^2$ is an open, bounded, connected set with Lipschitz boundary $\partial\Omega$. We assume that the crack can grow only along a prescribed simple $C^{2,1}$-curve $\Lambda \subseteq \overline{\Omega}$ with $\mathcal{H}^1(\Lambda) =: L$. Let $\lambda \in C^{2,1}([0, L]; \Lambda)$ be its arc-length parametrization and ν, τ be its unit normal and unit tangent vectors, respectively. We make the following assumptions on the geometry of the model:

- $\partial\Omega \cap \Lambda = \{\lambda(0), \lambda(L)\}$;
- $\Omega \setminus \Lambda = \Omega^+ \cup \Omega^-$, where Ω^+, Ω^- are two connected open subsets of $\mathbb{R}^2$ with Lipschitz boundary, defined according to the orientation of the normal vector ν, with $\Omega^+ \cap \Omega^- = \emptyset$.

The admissible fractures are of the form

$$\Gamma_s := \{\lambda(\sigma) : 0 \le \sigma \le s\} \tag{3.1.1}$$

for $s \in [0, L]$. We set $\Omega_s := \Omega \setminus \Gamma_s$ and we denote by $H^1(\Omega_s; \mathbb{R}^2)$ the set

$$\{u \in H^1(\Omega \setminus \Lambda; \mathbb{R}^2) : [u] = 0 \ \mathcal{H}^1\text{-a.e. on } \Lambda \setminus \Gamma_s\}.$$

The body outside the crack is supposed to be linearly elastic, with elasticity tensor $\mathbb{C}$. Here, $\mathbb{C}$ is assumed to be C^1 on $\Omega \setminus \Lambda$ and to be positive definite, in the sense of (2.3.1). For simplicity of notation, we will not specify the dependence on $x \in \Omega$ of the elasticity tensor.

The main feature of the Barenblatt's cohesive model (see, e.g., [13, 15]) is the presence of the so-called cohesive forces acting on the fracture lips. Given $T > 0$, in our model the density of the energy spent by the cohesive forces is represented by a time-dependent function $\varphi\colon [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ satisfying the following properties:

- $t \mapsto \varphi(t,\xi)$ is continuous for every $\xi \in \mathbb{R}^2$;
- $\xi \mapsto \varphi(t,\xi)$ is $C^1(\mathbb{R}^2)$ for every $t \in [0,T]$;
- $\varphi(t,0) = 0$ for every $t \in [0,T]$;
- there exist $p \in (1,+\infty)$ and $a_2 > 0$ such that for every $t \in [0,T]$ and every $\xi \in \mathbb{R}^2$
$$\begin{aligned} &\varphi(t,\xi) \le a_2(1 + |\xi|^p)\,, \\ &|D_\xi \varphi(t,\xi)| \le a_2(1 + |\xi|^{p-1})\,; \end{aligned} \tag{3.1.2}$$
- for every $\varepsilon > 0$, there exists $b_\varepsilon > 0$ such that for every $t \in [0,T]$ and every $\xi \in \mathbb{R}^2$
$$\varphi(t,\xi) \ge -b_\varepsilon - \varepsilon|\xi|^2\,. \tag{3.1.3}$$

Remark 3.1.1. At t fixed, the function $\varphi(t,\cdot)$ represents the density of the energy spent by the inter-atomic forces on the crack lips. It is concentrated on Λ and depends only on the jump of the displacement across Λ.

We stress that in our model the function φ is time-dependent and, possibly, negative (see (3.1.3)). This means that we are able to discuss also the case of a given force $h\colon [0,T] \to \mathbb{R}^2$ acting on both the fracture lips, namely $\varphi(t,\xi) := -h(t)\cdot\xi$.

Different from the Barenblatt's model, we assume that the energy spent by the cohesive forces is completely reversible. Moreover, we add to the surface energy the dissipative term κs, κ being the material toughness, that can be interpreted as an activation treshold.

Besides φ, we also consider a function $g\colon [0,T] \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$ with the following properties:

- $t \mapsto g(t,x,\xi)$ is continuous for every $\xi \in \mathbb{R}^2$ and a.e. $x \in \Omega$;
- $x \mapsto g(t,x,\xi)$ is measurable for every $t \in [0,T]$ and every $\xi \in \mathbb{R}^2$;
- $\xi \mapsto g(t,x,\xi)$ is $C^1(\mathbb{R}^2)$ for every $t \in [0,T]$ and a.e. $x \in \Omega$;
- $t \mapsto D_\xi g(t,x,\xi)$ is continuous for every $\xi \in \mathbb{R}^2$ and a.e. $x \in \Omega$;
- $x \mapsto D_\xi g(t,x,\xi)$ is measurable for every $t \in [0,T]$ and every $\xi \in \mathbb{R}^2$;
- for every $\varepsilon > 0$, there exists $a_\varepsilon > 0$ such that for a.e. $x \in \Omega$, every $t \in [0,T]$, and every $\xi \in \mathbb{R}^2$
$$|g(t,x,\xi)| \le a_\varepsilon + \varepsilon|\xi|^2\,; \tag{3.1.4}$$

- there exists $a_1 > 0$ such that for a.e. $x \in \Omega$, every $t \in [0,T]$, and every $\xi \in \mathbb{R}^2$

$$|D_\xi g(t,x,\xi)| \le a_1(1+|\xi|)\,. \tag{3.1.5}$$

Remark 3.1.2. We point out that the function g is a nonlinear generalization of the power spent by the volume forces. Indeed, in Section 3.3 we will set

$$g(t,x,\xi) := f(t,x)\cdot\xi\,, \tag{3.1.6}$$

with $f \in AC([0,T];L^2(\Omega;\mathbb{R}^2))$. The function f will represent the body forces applied on Ω. In particular, g as in (3.1.6) satisfies all the properties previously listed.

We are now ready to define the total energy of the system which is considered in this chapter: fixed $t \in [0,T]$, $s \in [0,L]$, and $u \in H^1(\Omega_s;\mathbb{R}^2)$, we set

$$\mathcal{E}(t,s,u) := \frac{1}{2}\int_{\Omega_s} \mathbb{C}\mathrm{E}u\cdot\mathrm{E}u\,\mathrm{d}x - \int_{\Omega_s} g(t,x,u)\,\mathrm{d}x + \int_{\Gamma_s}\varphi(t,[u])\,\mathrm{d}\mathcal{H}^1 + \kappa s\,. \tag{3.1.7}$$

Hence, the energy is the sum of the *stored elastic energy*, a term which generalizes the power spent by the volume forces, a surface term which can be interpreted as the energy spent by the cohesive forces on the fracture Γ_s, and an activation threshold κs proportional to the crack length.

Let us now briefly discuss the equilibrium condition of the system. Fix $t \in [0,T]$, $s \in [0,L]$, and the Dirichlet boundary datum $w \in H^1(\Omega;\mathbb{R}^2)$ on $\partial\Omega$. As usual, the body is in equilibrium with an assigned crack Γ_s if the displacement u is a solution of the minimum problem

$$\min_{u\in\mathcal{A}(s,w)} \mathcal{E}(t,s,u)\,, \tag{3.1.8}$$

where

$$\mathcal{A}(s,w) := \{u \in H^1(\Omega_s;\mathbb{R}^2) : [u]\cdot\nu \ge 0 \text{ on } \Lambda,\ u = w \text{ on } \partial\Omega\} \tag{3.1.9}$$

is the set of all admissible displacements associated to the crack Γ_s and the Dirichlet boundary datum w. In (3.1.9), the inequality $[u]\cdot\nu \ge 0$ takes into account the non-interpenetration condition, while the equality $u = w$ has to be intended in the trace sense on $\partial\Omega$.

We now state a general lemma which proves the lower semicontinuity of $\mathcal{E}$ and will be useful also in next sections.

Lemma 3.1.3. *Let $t_k, t \in [0,T]$, $s_k, s \in [0,L]$, $w_k, w \in H^1(\Omega;\mathbb{R}^2)$, $u_k \in \mathcal{A}(s_k,w_k)$ for every k, and $u \in \mathcal{A}(s,w)$. Assume that $t_k \to t$, $s_k \to s$, $w_k \to w$ in $H^1(\Omega;\mathbb{R}^2)$, and $u_k \rightharpoonup u$ weakly in $H^1(\Omega\setminus\Lambda;\mathbb{R}^2)$. Then*

$$\begin{aligned}
&\mathcal{E}(t,s,u) \le \liminf_k \mathcal{E}(t_k,s_k,u_k)\,,\\
&\int_{\Gamma_s}\varphi(t,[u])\,\mathrm{d}\mathcal{H}^1 = \lim_k \int_{\Gamma_{s_k}}\varphi(t_k,[u_k])\,\mathrm{d}\mathcal{H}^1\,,\\
&\int_{\Omega_s} g(t,x,u)\,\mathrm{d}x = \lim_k \int_{\Omega_{s_k}} g(t_k,x,u_k)\,\mathrm{d}x\,.
\end{aligned} \tag{3.1.10}$$

If, in addition, we assume that

$$\mathcal{E}(t, s, u) = \lim_k \mathcal{E}(t_k, s_k, u_k)\,, \tag{3.1.11}$$

then $u_k \to u$ *strongly in* $H^1(\Omega \setminus \Lambda; \mathbb{R}^2)$.

Proof. By compactness, we have that $u_k \to u$ strongly in $L^p(\Omega; \mathbb{R}^2)$ and in $L^p(\Lambda; \mathbb{R}^2)$ for every $p \in [1, +\infty)$. Up to a subsequence, we can assume that $u_k \to u$ pointwise in Ω and on Λ.

By the continuity properties of φ and g, we have the pointwise convergences

$$\varphi(t_k, [u_k]) \to \varphi(t, [u]) \qquad \text{and} \qquad g(t_k, x, u_k) \to g(t, x, u)\,.$$

Thanks to the hypotheses (3.1.2)-(3.1.4), applying the dominated convergence theorem we get the two equalities in (3.1.10). Since the stored elastic energy is lower semicontinuous, we obtain also the first inequality in (3.1.10).

If we assume (3.1.11), then, by (3.1.10), we deduce that

$$\int_{\Omega_s} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}u \,\mathrm{d}x = \lim_k \int_{\Omega_{s_k}} \mathbb{C}\mathrm{E}u_k \cdot \mathrm{E}u_k \,\mathrm{d}x\,.$$

Hence, we have that $u_k \to u$ strongly in $H^1(\Omega \setminus \Lambda; \mathbb{R}^2)$. □

Thanks to Lemma 3.1.3, to the hypotheses (2.3.1), (3.1.2)-(3.1.4), and to the application of Korn's inequality in $\Omega^\pm$, the minimum problem (3.1.8) admits a solution $u \in \mathcal{A}(s, w)$. As in Chapter 2, we introduce the reduced energy

$$\mathcal{E}_m(t, s, w) := \min_{u \in \mathcal{A}(s,w)} \mathcal{E}(t, s, u)\,. \tag{3.1.12}$$

Since we are interested in the notion of quasi-static evolution via Griffith's criterion for our cohesive fracture model, in Section 3.2 we first have to study the differentiability of $\mathcal{E}_m$ with respect to the crack length s. To this end, we notice that, because of the non-convexity of $\varphi(t, \cdot)$ and $g(t, x, \cdot)$, the solution to the minimum problem (3.1.12) could be not unique. This affects the computation of the derivative of the reduced energy $\mathcal{E}_m$ with respect to s. Indeed, in Section 3.2 we will see that in general $\mathcal{E}_m$ is not differentiable in s. However, we can still compute its right and left derivatives $\partial_s^+ \mathcal{E}_m$ and $\partial_s^- \mathcal{E}_m$ (see Theorems 3.2.2 and 3.2.3). In particular, we are in a situation different from [47, 72], where the reduced energy is differentiable and has a continuous derivative, and similar in this aspect to [46, 48], where finite-strain elasticity in brittle fracture is considered. In Proposition 3.2.10 we prove that the two derivatives $\partial_s^+ \mathcal{E}_m$ and $\partial_s^- \mathcal{E}_m$ satisfy a semicontinuity property which will play a key role in the proof of existence of a quasi-static evolution for the cohesive crack growth problem, (see Definition 3.3.5 and the proof of Theorem 3.3.6).

In Sections 3.3-3.6 we move to the evolution problem. In this context, the evolution is governed by a Dirichlet boundary datum $w \in AC([0, T]; H^1(\Omega; \mathbb{R}^2))$ and by the volume forces $f \in AC([0, T]; L^2(\Omega; \mathbb{R}^2))$. In particular, as mentioned in Remark 3.1.2, in the energy (3.1.7) we consider g of the form (3.1.6). In order to get a quasi-static evolution satisfying a weak version of the Griffith's principle, we tackle the problem by means of vanishing viscosity, as already discussed in the Introduction. We refer to Definitions 3.3.3 and 3.3.5 for the notions of viscous and quasi-static evolution, respectively. The existence of such evolutions is obtained in Theorems 3.3.4 and 3.3.6.

Finally, in Sections 3.7-3.8, we generalize the previous results to the case of many non-interacting cracks, in the spirit of [52]. In order to get the same properties of Definition 3.3.5, we use the notion of parametrized solution introduced in [58].

All the results contained in this chapter can be found in [1].

3.2 Energy release rate

The purpose of this section is to give precise formulas for the derivative of the energy $\mathcal{E}_m$ with respect to the crack length s. In order to do this, as in Section 2.2.3 we need to slightly move the crack tip along the prescribed curve Λ. Hence, fixed $t \in [0,T]$, $s \in (0,L)$, and δ such that $s+\delta \in [0,L]$, we construct a $C^{2,1}$-diffeomorphism $F_{s,\delta}$ such that $F_{s,\delta}(\Omega_s) = \Omega_{s+\delta}$, and $F_{s,\delta}|_{\partial\Omega} = id|_{\partial\Omega}$. Indeed, by our regularity assumption, in a neighborhood of the crack tip $\lambda(s)$ the curve Λ can be seen, up to a rotation, as the graph of a $C^{2,1}$-function, i.e., there exist $\eta > 0$ and $\psi_s \in C^{2,1}((\lambda_1(s)-\eta, \lambda_1(s)+\eta))$ such that

$$\Lambda = \{(x_1, \psi_s(x_1)) : x_1 \in (\lambda_1(s)-\eta, \lambda_1(s)+\eta)\},$$

where x_1 and λ_1 are the first components of $x=(x_1,x_2) \in \mathbb{R}^2$ and of the arc-length parametrization $\lambda = (\lambda_1, \lambda_2)$, respectively.

Choose a cut-off function $\vartheta \in C_c^\infty(\mathrm{B}_{\eta/2}(0))$ with $\vartheta = 1$ on $\overline{\mathrm{B}}_{\eta/3}(0)$. We define $F_{s,\delta} \colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F_{s,\delta}(x) := x + \begin{pmatrix} (\lambda_1(s+\delta)-\lambda_1(s))\vartheta(\lambda(s)-x) \\ \psi_s(x_1 + (\lambda_1(s+\delta)-\lambda_1(s))\vartheta(\lambda(s)-x)) - \psi_s(x_1) \end{pmatrix} \tag{3.2.1}$$

if $x \in \mathrm{B}_{\eta/2}(\lambda(s))$, while $F_{s,\delta}(x) := x$ if $x \in \mathbb{R}^2 \setminus \mathrm{B}_{\eta/2}(\lambda(s))$.

Remark 3.2.1. The properties stated in Lemma 2.2.24 still hold in this case for $F_{s,\delta}$ as in (3.2.1) with

$$\rho_s(x) := \partial_\delta(F_{s,\delta}(x))|_{\delta=0} = \lambda_1'(s)\vartheta(\lambda(s)-x)\begin{pmatrix} 1 \\ \psi_s'(x_1) \end{pmatrix}.$$

In particular, formulas (2.2.103) will appear in the expressions of the right and left derivatives of the reduced energy $\mathcal{E}_m$ with respect to s, see (3.2.2), (3.2.6)-(3.2.9).

Let $t \in [0,T]$, $s \in (0,L)$, $u \in H^1(\Omega_s;\mathbb{R}^2)$, and let ϑ be a cut-off function as in (3.2.1). We set

$$
\begin{aligned}
G(t,u,\vartheta) := & -\frac{1}{2}\int_{\Omega_s} (D\mathbb{C}\,\rho_s)\nabla u\cdot\nabla u\,\mathrm{d}x \\
& -\int_{\Omega_s} \mathbb{C}\nabla\big((\nabla\rho_s - \mathrm{div}\rho_s\,\mathrm{I})u\big)\cdot\nabla u\,\mathrm{d}x \\
& +\int_{\Omega_s} \mathbb{C}(\nabla u\nabla\rho_s)\cdot\nabla u\,\mathrm{d}x - \frac{1}{2}\int_{\Omega_s}\mathbb{C}\nabla u\cdot\nabla u\,\mathrm{div}\rho_s\,\mathrm{d}x \\
& +\int_{\Omega_s} D_\xi g(t,x,u)\cdot\big[(\nabla\rho_s - \mathrm{div}\rho_s\,\mathrm{I})u - \nabla u\,\rho_s\big]\,\mathrm{d}x \\
& -\int_{\Gamma_s} D_\xi\varphi(t,[u])\cdot\big((\nabla\rho_s - \mathrm{div}\rho_s\,\mathrm{I})u\big)\,\mathrm{d}\mathcal{H}^1 \\
& -\int_{\Gamma_s} \varphi(t,[u])\,\nu\otimes\tau\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\cdot\nabla\rho_s\,\mathrm{d}\mathcal{H}^1 \\
& -\int_{\Gamma_s} \varphi(t,[u])\,\mathrm{div}\rho_s\,\mathrm{d}\mathcal{H}^1\,,
\end{aligned}
\tag{3.2.2}
$$

where ν and τ are the unit normal and unit tangent vectors to Λ, respectively, I is the identity matrix in $\mathbb{M}^2$, and $D\mathbb{C}\,\rho_s$ is a fourth order tensor given by

$$
(D\mathbb{C}\,\rho_s)_{ijkl} := \sum_{m=1}^{2}\frac{\partial\mathbb{C}_{ijkl}}{\partial x_m}\,\rho_{s,m}\,,\qquad \rho_s = (\rho_{s,1},\rho_{s,2})\,. \tag{3.2.3}
$$

In particular, we notice that G depends on ϑ through the definition of ρ_s. We refer to Proposition 3.2.11 and Remark 3.2.12 for some comments on G.

We introduce the right and left derivatives of $\mathcal{E}_m$ with respect to the arc-length of the crack s: for every $t\in[0,T]$ and every $w\in H^1(\Omega;\mathbb{R}^2)$ we define

$$
\partial_s^+\mathcal{E}_m(t,s,w) := \lim_{\delta\searrow 0}\frac{\mathcal{E}_m(t,s+\delta,w)-\mathcal{E}_m(t,s,w)}{\delta}\qquad\text{for every } s\in[0,L)\,, \tag{3.2.4}
$$

and

$$
\partial_s^-\mathcal{E}_m(t,s,w) := \lim_{\delta\nearrow 0}\frac{\mathcal{E}_m(t,s+\delta,w)-\mathcal{E}_m(t,s,w)}{\delta}\qquad\text{for every } s\in(0,L]\,, \tag{3.2.5}
$$

if the two limits exist.

From now on, for every $t\in[0,T]$, every $s\in[0,L]$, and every $w\in H^1(\Omega;\mathbb{R}^2)$, we denote by u_s a solution to the minimum problem (3.1.8) in $A(s,w)$.

We are now ready to state the main results of this section.

Theorem 3.2.2. *For every $t\in[0,T]$, every $s\in(0,L)$, and every $w\in H^1(\Omega;\mathbb{R}^2)$, the limit in* (3.2.4) *exists and*

$$
\partial_s^+\mathcal{E}_m(t,s,w) = \kappa - \mathfrak{G}^+(t,s,w)\,, \tag{3.2.6}
$$

where we have set

$$\mathfrak{G}^+(t,s,w) := \max\{G(t,u_s,\vartheta) : u_s \in \mathcal{A}(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}, \quad (3.2.7)$$

for a given cut-off function ϑ as in (3.2.1)*.*

Moreover, $\mathfrak{G}^+(t,s,w)$ does not depend on the choice of ϑ.

Theorem 3.2.3. *For every $t \in [0,T]$, every $s \in (0,L)$, and every $w \in H^1(\Omega;\mathbb{R}^2)$, the limit in* (3.2.5) *exists and*

$$\partial_s^- \mathcal{E}_m(t,s,w) = \kappa - \mathfrak{G}^-(t,s,w), \quad (3.2.8)$$

where we have set

$$\mathfrak{G}^-(t,s,w) := \min\{G(t,u_s,\vartheta) : u_s \in \mathcal{A}(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}, \quad (3.2.9)$$

for a given cut-off function ϑ as in (3.2.1)*.*

Moreover, $\mathfrak{G}^-(t,s,w)$ does not depend on the choice of ϑ.

Remark 3.2.4. We notice that formulas (3.2.6)-(3.2.9) say that the function $s \mapsto \mathcal{E}_m(t,s,w)$ is not differentiable in the interval $(0,L)$. This is due to the lack of uniqueness of solution to (3.1.8) and, more in general, to the fact that a minimizer of $\mathcal{E}(t,s,\cdot)$ might not be approximated by minima of $\mathcal{E}(t,s+\delta,\cdot)$ as $\delta \to 0$. The consequences of this "non-approximability" will be clear in the proofs of Theorems 3.2.2 and 3.2.3, and will be stressed in Remark 3.2.8.

Let us anticipate, as stated in Proposition 3.2.10 below, that we can not expect to have the continuity of $\partial_s^+\mathcal{E}_m$ and $\partial_s^-\mathcal{E}_m$ as functions of t, s, and w, thus the arguments used in [47, 52] have to be modified as in [48] in order to find a quasi-static evolution as limit of viscous solutions (see Sections 3.3-3.6).

We finally notice that the terms $\mathfrak{G}^+$ and $\mathfrak{G}^-$ appearing in (3.2.6) and (3.2.8) are the generalization of the energy release rate (see, e.g., [46, 51]). To be consistent with the existent literature dealing with Griffith's criterion, the definitions of viscous and quasi-static evolutions will involve $\mathfrak{G}^+$ and $\mathfrak{G}^-$ (see Definitions 3.3.3 and 3.3.5).

As in Section 2.2.3, $P_{s,\delta}$ denotes the Piola transformation associated to $F_{s,\delta}$, see (2.2.109) and (2.2.110). In particular, $P_{s,\delta}$ is an isomorphism between $\mathcal{A}(s+\delta,w)$ and $\mathcal{A}(s,w)$ with inverse $P_{s,\delta}^{-1}$.

Before starting the proofs of Theorems 3.2.2 and 3.2.3, we show some properties concerning the behavior of $\mathcal{E}_m$ with respect to time t, the parameter s, and the Dirichlet boundary datum w. We notice that Lemma 2.2.26 still holds in this context, simply replacing $LD^2(\mathbb{R}^2_+;\mathbb{R}^2)$ with $H^1(\Omega\setminus\Lambda;\mathbb{R}^2)$.

In the next two lemmas, we prove the continuity of the energy $\mathcal{E}_m$ in $[0,T]\times(0,L)\times H^1(\Omega;\mathbb{R}^2)$.

Lemma 3.2.5. *The reduced energy $\mathcal{E}_m : [0,T]\times[0,L]\times H^1(\Omega;\mathbb{R}^2) \to \mathbb{R}$ is lower semicontinuous.*

Proof. Let $t_k, t \in [0,T]$, $s_k, s \in [0,L]$, $w_k, w \in H^1(\Omega;\mathbb{R}^2)$ be such that $t_k \to t$, $s_k \to s$, and $w_k \to w$ in $H^1(\Omega;\mathbb{R}^2)$ as $k \to +\infty$. For every k, let us fix $u_k \in \mathcal{A}(s_k, w_k)$ minimizer of $\mathcal{E}(t_k, s_k, \cdot)$. Then, by Korn's inequality and by the hypotheses (2.3.1) and (3.1.2)-(3.1.4), we have, for some $\varepsilon > 0$ small enough and some $c_1, c_2 > 0$,

$$c_1 \|u_k\|^2_{H^1(\Omega\setminus\Lambda)} - a_\varepsilon - b_\varepsilon \le \mathcal{E}(t_k, s_k, u_k) \le \mathcal{E}(t_k, s_k, w_k) \le c_2 \|w_k\|^2_{H^1(\Omega)} + a_\varepsilon + L\,.$$

The previous inequality and the convergence $w_k \to w$ in $H^1(\Omega;\mathbb{R}^2)$ imply that the sequence u_k is bounded in $H^1(\Omega\setminus\Lambda;\mathbb{R}^2)$. Therefore, there exists $u \in H^1(\Omega\setminus\Lambda;\mathbb{R}^2)$ such that, up to a subsequence, $u_k \rightharpoonup u$ weakly in $H^1(\Omega\setminus\Lambda;\mathbb{R}^2)$. By the compactness of the traces, we deduce that $u \in \mathcal{A}(s,w)$. Moreover, (3.1.10) holds. Hence

$$\mathcal{E}_m(t,s,w) \le \mathcal{E}(t,s,u) \le \liminf_k \mathcal{E}(t_k, s_k, u_k) = \liminf_k \mathcal{E}_m(t_k, s_k, w_k)\,,$$

and this concludes the proof. □

Lemma 3.2.6. *Let $t_k, t \in [0,T]$, $s_k, s \in (0,L)$, $w_k, w \in H^1(\Omega;\mathbb{R}^2)$ be such that $t_k \to t$, $s_k \to s$, and $w_k \to w$ in $H^1(\Omega;\mathbb{R}^2)$ as $k \to +\infty$. Let $u_k \in \mathcal{A}(s_k, w_k)$ be a sequence of minimizers of $\mathcal{E}(t_k, s_k, \cdot)$. Then, there exists $u \in \mathcal{A}(s,w)$ minimizer of $\mathcal{E}(t,s,\cdot)$ such that, up to a subsequence, $u_k \to u$ in $H^1(\Omega\setminus\Lambda;\mathbb{R}^2)$.*

In particular, the reduced energy $\mathcal{E}_m$ is continuous on $[0,T]\times(0,L)\times H^1(\Omega;\mathbb{R}^2)$.

Proof. As in the proof of Lemma 3.2.5, we can find $u \in \mathcal{A}(s,w)$ such that, up to a subsequence, $u_k \rightharpoonup u$ weakly in $H^1(\Omega\setminus\Lambda;\mathbb{R}^2)$.

In order to prove that u is a minimizer of $\mathcal{E}(t,s,\cdot)$ in $\mathcal{A}(s,w)$, we argue as in the proof of Lemma 2.2.22. Fix $u_s \in \mathcal{A}(s,w)$ minimizer of $\mathcal{E}(t,s,\cdot)$ and, for k large enough, take as a competitor $P^{-1}_{s,s_k-s}\, u_s + w_k - w \in \mathcal{A}(s_k, w_k)$. Repeating the argument of (2.2.80), we deduce that u is a minimizer of $\mathcal{E}(t,s,\cdot)$ in $\mathcal{A}(s,w)$ and that

$$\mathcal{E}(t,s,u) = \mathcal{E}_m(t,s,w) = \lim_{k\to+\infty} \mathcal{E}_m(t_k, s_k, w_k) = \lim_{k\to+\infty} \mathcal{E}(t_k, s_k, u_k)\,. \tag{3.2.10}$$

Therefore, by Lemma 3.1.3 we get that $u_k \to u$ strongly in $H^1(\Omega\setminus\Lambda;\mathbb{R}^2)$. Moreover, (3.2.10) implies that $\mathcal{E}_m$ is continuous on $[0,T]\times(0,L)\times H^1(\Omega;\mathbb{R}^2)$. □

In the proof of Theorem 3.2.2 we will need the following lemma.

Lemma 3.2.7. *Let $\Omega \subseteq \mathbb{R}^2$ be an open, bounded, and connected set with Lipschitz boundary. Let $\vartheta \in C_c^\infty(\Omega)$ and $\delta_0 > 0$ be fixed as in (3.2.1) and in Lemma 3.2.1. Then the following facts hold true:*

(a) *there exists $c = c(\vartheta) > 0$ such that for every $u \in H^1(\Omega;\mathbb{R}^2)$:*

$$\left\| \delta^{-1}\big(u \circ F^{-1}_{s,\delta} - u\big) \right\|_{2,\Omega} \le c(\vartheta) \|\nabla u\|_{2,\Omega}\,. \tag{3.2.11}$$

Moreover, $\delta^{-1}(u \circ F^{-1}_{s,\delta} - u) \to -\nabla u\, \rho_s$ in $L^2(\Omega;\mathbb{R}^2)$ as $\delta \to 0$;

(b) *assume that there exist* $\delta_k \to 0$, $|\delta_k| < \delta_0$, *and* $u_{\delta_k}, u \in H^1(\Omega;\mathbb{R}^2)$ *such that* $u_{\delta_k} \rightharpoonup u$ *weakly in* $H^1(\Omega;\mathbb{R}^2)$ *as* $k \to +\infty$. *Then* $\delta_k^{-1}(u_{\delta_k} - u_{\delta_k} \circ F_{s,\delta_k}) \rightharpoonup -\nabla u\, \rho_s$ *weakly in* $L^2(\Omega;\mathbb{R}^2)$ *as* $k \to +\infty$.

Proof. We adapt the proof of [46, Lemma 4.1] to the case of a curved prescribed crack path Λ.

Let us fix $u \in H^1(\Omega;\mathbb{R}^2)$. For $|\delta| < \delta_0$ we define $L_\delta(u) := \delta^{-1}(u \circ F_{s,\delta}^{-1} - u)$ and $L_0(u) := -\nabla u\, \rho_s$. The function $L_\delta \colon H^1(\Omega;\mathbb{R}^2) \to L^2(\Omega;\mathbb{R}^2)$ is a linear operator for every $|\delta| < \delta_0$. We want to prove that they are uniformly bounded.

To this end, for $|\delta| < \delta_0$ and $h \in \mathbb{R}$ small enough, we set $x_h := F_{s,\delta+h}^{-1}(y)$ and $x := F_{s,\delta}^{-1}(y)$ for $y \in \Omega$. We compute

$$\lim_{h\to 0} \frac{x_h - x}{h}\,.$$

By definition of $F_{s,\cdot}$, we have

$$\begin{aligned} 0 &= \frac{1}{h}(F_{s,\delta+h}(x_h) - F_{s,\delta}(x)) \\ &= \frac{1}{h}(F_{s,\delta+h}(x_h) - F_{s,\delta+h}(x)) + \frac{1}{h}(F_{s,\delta+h}(x) - F_{s,\delta}(x))\,. \end{aligned} \tag{3.2.12}$$

By the mean value theorem, there exists $t_h \in (0,1)$ such that

$$F_{s,\delta+h}(x_h) - F_{s,\delta+h}(x) = \nabla F_{s,\delta+h}(x_{t_h})(x_h - x)\,,$$

where $x_{t_h} := x + t_h(x_h - x)$. Since $F_{s,\delta+h}$ is a $C^{2,1}$-diffeomorphism, for every h there exists $(\nabla F_{s,\delta+h}(x_{t_h}))^{-1}$. Hence, (3.2.12) becomes

$$0 = \frac{x_h - x}{h} + (\nabla F_{s,\delta+h}(x_{t_h}))^{-1}\, \frac{F_{s,\delta+h}(x) - F_{s,\delta}(x)}{h}\,. \tag{3.2.13}$$

Passing to the limit in (3.2.13) as $h \to 0$, since $x_{t_h} \to x$ we get

$$\rho_{s,\delta}(x) := \lim_{h\to 0} \frac{x_h - x}{h} = -(\nabla F_{s,\delta}(x))^{-1} \partial_\delta F_{s,\delta}(x)\,. \tag{3.2.14}$$

Let now $u \in C^\infty(\overline{\Omega};\mathbb{R}^2)$ be fixed. For every $y \in \Omega$, by (3.2.14) we have

$$L_\delta(u)(y) = \frac{1}{\delta}\int_0^1 \frac{d}{dh} u(F_{s,h\delta}^{-1}(y))\,\mathrm{d}h = \int_0^1 \nabla u(F_{s,h\delta}^{-1}(y))\, \rho_{s,h\delta}(F_{s,h\delta}^{-1}(y))\,\mathrm{d}h\,. \tag{3.2.15}$$

Taking the L^2 norm of $L_\delta(u)$ in (3.2.15) and applying Hölder's inequality and the change of coordinates $y = F_{s,h\delta}(x)$, we obtain

$$\|L_\delta(u)\|_{2,\Omega}^2 \le \int_0^1 \int_\Omega |\nabla u\, \rho_{s,h\delta}|^2 \det \nabla F_{s,h\delta}\,\mathrm{d}x \le c(\vartheta)\|\nabla u\|_{2,\Omega}^2\,, \tag{3.2.16}$$

for some constant $c(\vartheta) > 0$ independent of δ. Since $C^\infty(\overline{\Omega};\mathbb{R}^2)$ is dense in $H^1(\Omega;\mathbb{R}^2)$, we deduce that (3.2.16) holds for every $u \in H^1(\Omega;\mathbb{R}^2)$, which is exactly (3.2.11).

Moreover, thanks to (3.2.15), for every $u \in C^\infty(\overline{\Omega};\mathbb{R}^2)$ we have

$$\|L_\delta(u) - L_0(u)\|^2_{2,\Omega} \leq \int_0^1 \int_\Omega |\nabla u(F^{-1}_{s,h\delta}(y))\rho_{s,h\delta}(F^{-1}_{s,h\delta}(y)) + \nabla u\, \rho_s(y)|^2 \, \mathrm{d}y \, \mathrm{d}h\,. \tag{3.2.17}$$

For $(h,y) \in [0,1] \times \Omega$ fixed, the integrand in (3.2.17) converges to 0 pointwise as $\delta \to 0$, thus, by the dominated convergence theorem, we get that $L_\delta(u) \to L_0(u)$ strongly in $L^2(\Omega;\mathbb{R}^2)$ for every $u \in C^\infty(\overline{\Omega};\mathbb{R}^2)$. By (3.2.16) and a density argument, the same is true for $u \in H^1(\Omega;\mathbb{R}^2)$. This concludes the proof of point (a).

Let us now prove (b). For every $v \in C^\infty_c(\Omega;\mathbb{R}^2)$, it holds

$$\begin{aligned}
&\int_\Omega \delta_k^{-1}(u_{\delta_k} - u_{\delta_k} \circ F_{s,\delta_k}) \cdot v \, \mathrm{d}x \\
&= - \int_\Omega u_{\delta_k} \cdot L_{\delta_k}(v) \, \mathrm{d}x + \delta_k^{-1} \int_\Omega u_{\delta_k} \cdot (v \circ F^{-1}_{s,\delta_k})(1 - \det \nabla F^{-1}_{s,\delta_k}) \, \mathrm{d}x \\
&= - \int_\Omega u_{\delta_k} \cdot L_{\delta_k}(v) \, \mathrm{d}x + \delta_k^{-1} \int_\Omega u_{\delta_k} \cdot (v \circ F^{-1}_{s,\delta_k}) \, \frac{\det \nabla F_{s,\delta_k}(F^{-1}_{s,\delta_k}(x)) - 1}{\det \nabla F_{s,\delta_k}(F^{-1}_{s,\delta_k}(x))} \, \mathrm{d}x\,.
\end{aligned} \tag{3.2.18}$$

In the last integral of (3.2.18) we perform the change of coordinates $x = F_{s,\delta}(y)$, thus we obtain

$$\begin{aligned}
&\int_\Omega \delta_k^{-1}(u_{\delta_k} - u_{\delta_k} \circ F_{s,\delta_k}) \cdot v \, \mathrm{d}x \\
&\qquad = - \int_\Omega u_{\delta_k} \cdot L_{\delta_k}(v) \, \mathrm{d}x + \int_\Omega (u_{\delta_k} \circ F_{s,\delta_k}) \cdot v \, \frac{\det \nabla F_{s,\delta_k} - 1}{\delta_k} \, \mathrm{d}x\,.
\end{aligned} \tag{3.2.19}$$

Passing to the limit in (3.2.18) as $k \to +\infty$, taking into account point (a), Lemma 2.2.24, and the weak convergence $u_{\delta_k} \circ F_{s,\delta_k} \rightharpoonup u$ in $H^1(\Omega;\mathbb{R}^2)$, we get

$$\begin{aligned}
\lim_k \int_\Omega \delta_k^{-1}(u_{\delta_k} - u_{\delta_k} \circ F_{s,\delta_k}) \cdot v \, \mathrm{d}x &= \int_\Omega u \cdot \nabla v \rho_s \, \mathrm{d}x + \int_\Omega u \cdot v \mathrm{div} \rho_s \, \mathrm{d}x \\
&= \int_\Omega u \cdot \mathrm{div}(v \otimes \rho_s) \, \mathrm{d}x = - \int_\Omega v \cdot \nabla u \rho_s \, \mathrm{d}x,
\end{aligned} \tag{3.2.20}$$

where, in the last equality, we have used the divergence theorem.

Since

$$\delta_k^{-1}(u_{\delta_k} - u_{\delta_k} \circ F_{s,\delta_k}) = L_{\delta_k}(u_{\delta_k} \circ F_{s,\delta_k})\,,$$

estimate (3.2.11) and the weak convergence of u_{δ_k} imply that there exists $C > 0$ such that for every k

$$\|\delta_k^{-1}(u_{\delta_k} - u_{\delta_k} \circ F_{s,\delta_k})\|_{2,\Omega} \leq C\,.$$

Therefore, taking into account the density of $C^\infty_c(\Omega;\mathbb{R}^2)$ in $L^2(\Omega;\mathbb{R}^2)$, we deduce that (3.2.20) holds for every $v \in L^2(\Omega;\mathbb{R}^2)$, hence $\delta_k^{-1}(u_{\delta_k} - u_{\delta_k} \circ F_{s,\delta_k}) \rightharpoonup -\nabla u\, \rho_s$ weakly in $L^2(\Omega;\mathbb{R}^2)$ as $k \to +\infty$, and the proof of the lemma is thus concluded. □

We are now ready to prove Theorem 3.2.2.

Proof of Theorem 3.2.2. Fix $t \in [0,T]$, $s \in (0,L)$, and $w \in H^1(\Omega;\mathbb{R}^2)$. Let $u_s \in \mathcal{A}(s,w)$ be a solution of (3.1.8) and let $0 < \delta < \delta_0$. For simplicity of notation, let us set

$$u^\delta := (cof\, \nabla F_{s,\delta})^{-T} u = (P_{s,\delta}^{-1}\, u) \circ F_{s,\delta} \qquad \text{for every } u \in \mathcal{A}(s,w).$$

By definition of $\mathcal{E}_m$ and by the change of variables $x = F_{s,\delta}^{-1}(y)$, we have

$$\begin{aligned}
&\frac{\mathcal{E}_m(t,s+\delta,w) - \mathcal{E}_m(t,s,w)}{\delta} \le \frac{\mathcal{E}(t,s+\delta,P_{s,\delta}^{-1}\, u_s) - \mathcal{E}(t,s,u_s)}{\delta} \\
&= \frac{1}{2\delta}\Big(\int_{\Omega_s} \mathbb{C}(F_{s,\delta}(x))\nabla u_s^\delta\,(\nabla F_{s,\delta})^{-1}\cdot \nabla u_s^\delta\,(\nabla F_{s,\delta})^{-1}\ \det \nabla F_{s,\delta}\,\mathrm{d}x \\
&\quad - \int_{\Omega_s} \mathbb{C}\mathrm{E}u_s \cdot \mathrm{E}u_s\,\mathrm{d}x\Big) - \frac{1}{\delta}\Big(\int_{\Omega_{s+\delta}} g(t,x,P_{s,\delta}^{-1}\, u_s)\,\mathrm{d}x - \int_{\Omega_s} g(t,x,u_s)\,\mathrm{d}x\Big) \qquad (3.2.21)\\
&\quad + \frac{1}{\delta}\Big(\int_{\Gamma_s} \varphi(t,[u_s^\delta])\,\frac{\sqrt{1+(\psi_s'\circ F_{s,\delta})^2}}{\sqrt{1+\psi_s'^2}}\ \det\nabla F_{s,\delta}\,\mathrm{d}\mathcal{H}^1 \\
&\quad - \int_{\Gamma_s} \varphi(t,[u_s])\,\mathrm{d}\mathcal{H}^1\Big) + \kappa = \frac{1}{\delta}I_1 - \frac{1}{\delta}I_2 + \frac{1}{\delta}I_3 + \kappa\,.
\end{aligned}$$

As in the proof of Theorem 2.2.25, by the dominated convergence theorem we get

$$\begin{aligned}
\lim_{\delta\searrow 0}\frac{1}{\delta}\,I_1 = \frac{1}{2}\int_{\Omega_s}(D\mathbb{C}\,\rho_s)\nabla u_s\cdot\nabla u_s \mathrm{d}x + \int_{\Omega_s}\mathbb{C}\nabla\big((\nabla\rho_s - \mathrm{div}\rho_s\,\mathrm{I})u_s\big)\cdot\nabla u_s\mathrm{d}x \\
- \int_{\Omega_s}\mathbb{C}(\nabla u_s\nabla\rho_s)\cdot\nabla u_s\,\mathrm{d}x + \frac{1}{2}\int_{\Omega_s}\mathbb{C}\nabla u_s\cdot\nabla u_s\,\mathrm{div}\rho_s\,\mathrm{d}x\,.
\end{aligned} \qquad (3.2.22)$$

We now deal with the term I_2 of (3.2.21). In view of the regularity properties of g, we can apply the mean value theorem: for a.e. $x \in \Omega$ there exists $\zeta_\delta(x) \in (0,1)$ such that

$$\begin{aligned}
&g(t,x,P_{s,\delta}^{-1}\, u_s(x)) - g(t,x,u_s(x)) \\
&\quad = D_\xi g(t,x,P_{s,\delta}^{-1}\, u_s(x) + \zeta_\delta(x)(P_{s,\delta}^{-1}\, u_s(x) - u_s(x)))\cdot(P_{s,\delta}^{-1}\, u_s(x) - u_s(x))\,.
\end{aligned} \qquad (3.2.23)$$

Let us set $\bar u_\delta := P_{s,\delta}^{-1}\, u_s + \zeta_\delta(P_{s,\delta}^{-1}\, u_s - u_s)$, where ζ_δ is as in (3.2.23). We can continue in (3.2.23), obtaining

$$\begin{aligned}
&g(t,x,P_{s,\delta}^{-1}\, u_s(x)) - g(t,x,u_s(x)) \\
&\quad = D_\xi g(t,x,\bar u_\delta(x))\cdot\big[(P_{s,\delta}^{-1}\, u_s - u_s\circ F_{s,\delta}^{-1}) + (u_s\circ F_{s,\delta}^{-1} - u_s)\big]\,.
\end{aligned} \qquad (3.2.24)$$

By Lemma 2.2.26, $u_s \circ F_{s,\delta}^{-1}$ and $P_{s,\delta}^{-1}\, u_s$ converge to u_s in $H^1(\Omega\setminus\Lambda;\mathbb{R}^2)$ as $\delta \searrow 0$. Hence, we also have, up to a subsequence, $\bar u_\delta \to u_s$ pointwise. Thanks to Lemmas 2.2.24 and 3.2.7, to condition (3.1.5) on g, and to the dominated convergence theorem, we get

$$\lim_{\delta\searrow 0}\frac{1}{\delta}\,I_2 = \int_{\Omega_s} D_\xi g(t,x,u_s)\cdot\big[(\nabla\rho_s - \mathrm{div}\rho_s\,\mathrm{I})\,u_s - \nabla u_s\,\rho_s\big]\,\mathrm{d}x\,. \qquad (3.2.25)$$

We now consider the term I_3 in (3.2.21). We can write it as

$$
\begin{aligned}
I_3 = & \int_{\Gamma_s} \varphi(t,[u_s^\delta]) \frac{\sqrt{1+(\psi_s'\circ F_{s,\delta})^2}}{\sqrt{1+\psi_s'^2}} (\det \nabla F_{s,\delta} - 1)\, \mathrm{d}\mathcal{H}^1 \\
& + \int_{\Gamma_s} \varphi(t,[u_s^\delta]) \bigg(\frac{\sqrt{1+(\psi_s'\circ F_{s,\delta})^2}}{\sqrt{1+\psi_s'^2}} - 1 \bigg)\, \mathrm{d}\mathcal{H}^1 \\
& + \int_{\Gamma_s} (\varphi(t,[u_s^\delta]) - \varphi(t,[u_s]))\, \mathrm{d}\mathcal{H}^1 = I_{1,3} + I_{2,3} + I_{3,3}\,.
\end{aligned}
\tag{3.2.26}
$$

For the first two terms in (3.2.26) it is easy to see that

$$
\begin{aligned}
& \lim_{\delta\searrow 0} \frac{1}{\delta}\, I_{1,3} + \frac{1}{\delta}\, I_{2,3} \\
& = \int_{\Gamma_s} \varphi(t,[u_s])\, \mathrm{div}\rho_s \, \mathrm{d}\mathcal{H}^1 + \int_{\Gamma_s} \varphi(t,[u_s])\, \frac{\psi_s'\psi_s''}{1+\psi_s'^2}\, \gamma_1'(s)\vartheta(\gamma(s)-x)\, \mathrm{d}\mathcal{H}^1 \\
& = \int_{\Gamma_s} \varphi(t,[u_s])\, \mathrm{div}\rho_s \, \mathrm{d}\mathcal{H}^1 + \int_{\Gamma_s} \varphi(t,[u_s])\, \nu\otimes\tau \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \nabla\rho_s \, \mathrm{d}\mathcal{H}^1 \,.
\end{aligned}
\tag{3.2.27}
$$

For the last term in (3.2.26), we exploit again the mean value theorem: for $\mathcal{H}^1$-a.e. $x\in\Gamma_s$ there exists $\zeta_\delta(x)\in(0,1)$ such that

$$
\varphi(t,[u_s^\delta](x)) - \varphi(t,[u_s](x)) = D_\xi\varphi(t,[u_s^\delta](x) + \zeta_\delta(x)([u_s^\delta](x)-[u_s](x)))\cdot([u_s^\delta](x)-[u_s](x))
$$

Arguing as in (3.2.25) and taking into account hypothesis (3.1.2) on φ, we get

$$
\lim_{\delta\searrow 0} \frac{1}{\delta}\, I_{3,3} = \int_{\Gamma_s} D_\xi\varphi(t,[u_s])\cdot\big((\nabla\rho_s - \mathrm{div}\rho_s\, \mathrm{I})u_s\big)\, \mathrm{d}\mathcal{H}^1\,. \tag{3.2.28}
$$

Collecting (3.2.21)-(3.2.22) and (3.2.25)-(3.2.28) we deduce

$$
\begin{aligned}
& \limsup_{\delta\searrow 0} \frac{\mathcal{E}_m(t,s+\delta,w)-\mathcal{E}_m(t,s,w)}{\delta} \\
& \qquad \le \lim_{\delta\searrow 0}\ \frac{\mathcal{E}(t,s+\delta,u_s^\delta)-\mathcal{E}(t,s,u_s)}{\delta} = \kappa - G(t,u_s,\vartheta)\,.
\end{aligned}
\tag{3.2.29}
$$

Since we can repeat the previous argument for every $u_s\in\mathcal{A}(s,w)$ minimizer of $\mathcal{E}(t,s,\cdot)$, taking the infimum in the right-hand side of (3.2.29) we get

$$
\begin{aligned}
& \limsup_{\delta\searrow 0} \frac{\mathcal{E}_m(t,s+\delta,w)-\mathcal{E}_m(t,s,w)}{\delta} \\
& \qquad \le \kappa - \sup\{G(t,u_s,\vartheta) :\ u_s\in\mathcal{A}(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}\,.
\end{aligned}
\tag{3.2.30}
$$

In particular, since the set of minimizers $\{u_s\}$ is bounded in $H^1(\Omega_s;\mathbb{R}^2)$ for every $s\in(0,L)$, the supremum in (3.2.30) is finite.

To prove the converse inequality for the $\liminf$, we argue in a similar way taking into account Lemmas 2.2.24, 2.2.26, 3.2.6, and point (b) of Lemma 3.2.7. Indeed, for every $\delta > 0$ we fix $u_{s+\delta} \in \mathcal{A}(s+\delta, w)$ minimizer of $\mathcal{E}(t, s+\delta, \cdot)$. By Lemma 3.2.6, we deduce that there exist a subsequence $\delta_k \searrow 0$ and $u_s \in \mathcal{A}(s,w)$ minimizer of $\mathcal{E}(t,s,\cdot)$ such that $u_{s+\delta_k} \to u_s$ in $H^1(\Omega \setminus \Lambda; \mathbb{R}^2)$. Lemma 2.2.26 implies that $u_{s+\delta_k} \circ F_{s,\delta_k} \to u_s$ in $H^1(\Omega_s; \mathbb{R}^2)$. For simplicity, we set $U_{s,\delta_k} := u_{s+\delta_k} \circ F_{s,\delta_k}$ and notice that $P_{s,\delta_k} u_{s+\delta_k} = (cof\, \nabla F_{s,\delta_k})^T U_{s,\delta_k}$.

We can write

$$
\begin{aligned}
&\frac{\mathcal{E}_m(t, s+\delta_k, w) - \mathcal{E}_m(t,s,w)}{\delta_k} \geq \frac{\mathcal{E}(t, s+\delta_k, u_{s+\delta_k}) - \mathcal{E}(t,s,P_{s,\delta_k} u_{s+\delta_k})}{\delta_k} \\
&= \frac{1}{2\delta_k}\bigg(\int_{\Omega_s} \mathbb{C}(F_{s,\delta_k}(x)) \nabla U_{s,\delta_k} (\nabla F_{s,\delta_k})^{-1} \cdot \nabla U_{s,\delta_k} (\nabla F_{s,\delta_k})^{-1} \det \nabla F_{s,\delta_k} \,\mathrm{d}x \\
&\quad - \int_{\Omega_s} \mathbb{C} \nabla (P_{s,\delta_k} u_{s+\delta_k}) \cdot \nabla (P_{s,\delta_k} u_{s+\delta_k}) \,\mathrm{d}x \bigg) \\
&\quad - \frac{1}{\delta_k}\bigg(\int_{\Omega_s} g(t,x,u_{s+\delta_k}) \,\mathrm{d}x - \int_{\Omega_s} g(t,x,P_{s,\delta_k} u_{s+\delta_k}) \,\mathrm{d}x \bigg) \\
&\quad + \frac{1}{\delta_k}\bigg(\int_{\Gamma_s} \varphi(t, [U_{s,\delta_k}]) \frac{\sqrt{1 + (\psi_s' \circ F_{s,\delta_k})^2}}{\sqrt{1+\psi_s'^2}} \det \nabla F_{s,\delta_k} \,\mathrm{d}\mathcal{H}^1 \\
&\quad - \int_{\Gamma_s} \varphi(t, [P_{s,\delta_k} u_{s+\delta_k}]) \,\mathrm{d}\mathcal{H}^1 \bigg) + \kappa
\end{aligned}
\tag{3.2.31}
$$

Following step by step the proof of (3.2.29), in view of Lemma 2.2.24, of point (b) of Lemma 3.2.7, and of the previous observations, we can pass to the limit as $k \to +\infty$ in (3.2.31) getting

$$
\begin{aligned}
\lim_k \frac{\mathcal{E}(t, s+\delta_k, u_{s+\delta_k}) - \mathcal{E}(t,s,P_{s,\delta_k} u_{s+\delta_k})}{\delta_k} &= \kappa - G(t, u_s, \vartheta) \\
\geq \kappa - \sup\{G(t,u_s,\vartheta) :\, u_s \in \mathcal{A}(s,w) &\text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}\,.
\end{aligned}
\tag{3.2.32}
$$

By a contradiction argument, from inequality (3.2.32) it follows that

$$
\begin{aligned}
&\liminf_{\delta \searrow 0} \frac{\mathcal{E}_m(t, s+\delta, w) - \mathcal{E}_m(t,s,w)}{\delta} \\
&\qquad \geq \kappa - \sup\{G(t,u_s,\vartheta) :\, u_s \in \mathcal{A}(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}\,.
\end{aligned}
\tag{3.2.33}
$$

Thus, collecting inequalities (3.2.30) and (3.2.33), we get that the limit in (3.2.4) exists and

$$
\begin{aligned}
&\partial_s^+ \mathcal{E}_m(t,s,w) \\
&\qquad = \kappa - \sup\{G(t,u_s,\vartheta) :\, u_s \in \mathcal{A}(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}\,.
\end{aligned}
\tag{3.2.34}
$$

It remains to prove that the supremum in (3.2.34) is attained. Let us consider a sequence of minimizers u_s^n of $\mathcal{E}(t,s,\cdot)$ in $\mathcal{A}(s,w)$ such that

$$
\lim_n G(t, u_s^n, \vartheta) = \sup\{G(t,u_s,\vartheta) : u_s \in \mathcal{A}(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}.
$$

Since Lemma 3.2.6 holds, there exist a subsequence, not relabeled, and a minimizer $u \in \mathcal{A}(s,w)$ of $\mathcal{E}(t,s,\cdot)$ such that $u_s^n \to u$ in $H^1(\Omega_s;\mathbb{R}^2)$. Since G is continuous with respect to the strong convergence in $H^1(\Omega \setminus \Lambda;\mathbb{R}^2)$, we have

$$\lim_n G(t,u_s^n,\vartheta) = G(t,u,\vartheta) = \sup\{G(t,u_s,\vartheta) : u_s \in \mathcal{A}(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}.$$

This concludes the proof of (3.2.6).

Finally, in view of the definition (3.2.4) of $\partial_s^+\mathcal{E}_m$, we notice that $\mathfrak{G}^+$ does not depend on the cut-off function ϑ. □

Exploiting the arguments of Theorem 3.2.2, we can also prove Theorem 3.2.3.

Proof of Theorem 3.2.3. We just have to follow step by step the proof of Theorem 3.2.2.

In this case, since we are dealing with $\delta < 0$, we have

$$\begin{aligned} &\frac{\mathcal{E}_m(t,s+\delta,w)-\mathcal{E}_m(t,s,w)}{\delta} \le \frac{\mathcal{E}(t,s+\delta,u_{s+\delta})-\mathcal{E}(t,s,P_{s,\delta}\,u_{s+\delta})}{\delta}, \\ &\frac{\mathcal{E}(t,s+\delta,P_{s,\delta}^{-1}\,u_s)-\mathcal{E}(t,s,u_s)}{\delta} \le \frac{\mathcal{E}_m(t,s+\delta,w)-\mathcal{E}_m(t,s,w)}{\delta}, \end{aligned} \tag{3.2.35}$$

for every $u_s \in \mathcal{A}(s,w)$ minimizer of $\mathcal{E}(t,s,\cdot)$ and every $u_{s+\delta} \in \mathcal{A}(s+\delta,w)$ minimizer of $\mathcal{E}(t,s+\delta,\cdot)$.

The second inequality in (3.2.35) can be treated as the corresponding one in the first part of the proof of Theorem 3.2.2. This time, it leads us to

$$\liminf_{\delta\nearrow 0} \frac{\mathcal{E}_m(t,s+\delta,w)-\mathcal{E}_m(t,s,w)}{\delta} \ge \kappa - G(t,u_s,\vartheta). \tag{3.2.36}$$

Since (3.2.36) holds for every $u_s \in \mathcal{A}(s,w)$ minimizer of $\mathcal{E}(t,s,\cdot)$, taking the supremum we obtain

$$\begin{aligned} &\liminf_{\delta\nearrow 0} \frac{\mathcal{E}_m(t,s+\delta,w)-\mathcal{E}_m(t,s,w)}{\delta} \\ &\qquad \ge \kappa - \inf\{G(t,u_s,\vartheta) : u_s \in \mathcal{A}(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}. \end{aligned} \tag{3.2.37}$$

For the first inequality in (3.2.35), we argue again as in the proof of (3.2.33). In this case, we get

$$\begin{aligned} &\limsup_{\delta\nearrow 0} \frac{\mathcal{E}_m(t,s+\delta,w)-\mathcal{E}_m(t,s,w)}{\delta} \\ &\qquad \le \kappa - \inf\{G(t,u_s,\vartheta) : u_s \in \mathcal{A}(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}. \end{aligned} \tag{3.2.38}$$

Collecting the inequalities (3.2.37) and (3.2.38), we have that the limit in (3.2.5) exists. Moreover, we have that

$$\begin{aligned} &\partial_s^-\mathcal{E}_m(t,s,w) \\ &\qquad = \kappa - \inf\{G(t,u_s,\vartheta) : u_s \in \mathcal{A}(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}. \end{aligned} \tag{3.2.39}$$

As in the proof of Theorem 3.2.2, the infimum in (3.2.39) is actually a minimum, thus (3.2.8) is proved. Finally, $\mathfrak{G}^-$ does not depend on the cut-off function ϑ. This concludes the proof of Theorem 3.2.3. □

Remark 3.2.8. As we have already noticed in Remark 3.2.4, the general non-existence of the derivative of $\mathcal{E}_m$ with respect to the crack-length s is due to the lack of approximability of the minimizers $u_s \in \mathcal{A}(s,w)$ of $\mathcal{E}(t,s,\cdot)$, that is, it is not true that for every u_s and every $\delta > 0$ there exist $u_{s+\delta} \in \mathcal{A}(s+\delta,w)$ minimizer of $\mathcal{E}(t,s+\delta,\cdot)$ and $u_{s-\delta} \in \mathcal{A}(s-\delta,w)$ minimizer of $\mathcal{E}(t,s-\delta,\cdot)$ such that $u_{s+\delta}, u_{s-\delta} \to u_s$ in $H^1(\Omega\setminus\Lambda;\mathbb{R}^2)$ as $\delta \searrow 0$. If this approximation property were true, then, in the inequalities (3.2.30), (3.2.33), (3.2.37), and (3.2.38), we could take both the infimum and the supremum. As a consequence, it would be $\partial_s^+\mathcal{E}_m = \partial_s^-\mathcal{E}_m$ and the reduced energy would be differentiable with respect to $s \in (0,L)$. For instance, this is true if the functions $\xi \mapsto \varphi(t,\xi)$ and $\xi \mapsto g(t,x,\xi)$ are convex. Indeed, in this case the minimum problem (3.1.8) has a unique solution $u_s \in \mathcal{A}(s,w)$ and the function $s \mapsto u_s$ is continuous.

Remark 3.2.9. We briefly notice that if we drop the non-interpenetration condition in the definition (3.1.9) of the admissible displacements $\mathcal{A}(s,w)$, Theorems 3.2.2 and 3.2.3 hold with a simpler formula for G, namely

$$\begin{aligned} G(t,u,\vartheta) := &-\frac{1}{2}\int_{\Omega_s}(D\mathbb{C}\,\rho_s)\nabla u\cdot\nabla u\,\mathrm{d}x + \int_{\Omega_s}\mathbb{C}(\nabla u\nabla\rho_s)\cdot\nabla u\,\mathrm{d}x\\ &-\frac{1}{2}\int_{\Omega_s}\mathbb{C}\nabla u\cdot\nabla u\,\mathrm{div}\rho_s\,\mathrm{d}x - \int_{\Omega_s}D_\xi g(t,x,u)\cdot\nabla u\,\rho_s\,\mathrm{d}x\\ &-\int_{\Gamma_s}\varphi(t,[u])\,\nu\otimes\tau\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\nabla\rho_s\,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma_s}\varphi(t,[u])\,\mathrm{div}\rho_s\,\mathrm{d}\mathcal{H}^1\,. \end{aligned}$$

The proofs present minor changes due to the fact that we do not need the Piola transformation $P_{s,\delta}$ anymore. Indeed, $u\circ F_{s,\delta} \in \mathcal{A}(s,w)$ for every $u \in \mathcal{A}(s+\delta,w)$ in this case.

Moreover, we stress that a C^2-regularity of the curve Λ is enough, and that we do not need the differentiability hypothesis on φ.

Thanks to Theorems 3.2.2 and 3.2.3, we are allowed to define the functions

$$\mathfrak{G}^+,\mathfrak{G}^-\colon [0,T]\times(0,L)\times H^1(\Omega;\mathbb{R}^2)\to\mathbb{R}\,,$$

whose expressions are given by (3.2.7) and (3.2.9), respectively.

We now state a property of semicontinuity of $\mathfrak{G}^+$ and $\mathfrak{G}^-$ which will be useful in the next sections.

Proposition 3.2.10. *The following facts hold:*

(*a*) *for every* $t\in[0,T]$, *every* $s\in(0,L)$, *and every* $w\in H^1(\Omega;\mathbb{R}^2)$

$$\mathfrak{G}^+(t,s,w) \geq \mathfrak{G}^-(t,s,w) \geq 0\,;$$

(*b*) *the function* $\mathfrak{G}^+$ *is upper semicontinuous with respect to the strong topology of* $\mathbb{R}\times\mathbb{R}\times H^1(\Omega;\mathbb{R}^2)$*;*

(c) *the function* $\mathfrak{G}^-$ *is lower semicontinuous with respect to the strong topology of* $\mathbb{R}\times\mathbb{R}\times H^1(\Omega;\mathbb{R}^2)$.

Proof. To prove property (a), we just notice that $\mathfrak{G}^+(t,s,w)$ and $\mathfrak{G}^-(t,s,w)$ are the negative of the right and left derivatives of the function

$$s\mapsto \mathcal{E}_m(t,s,w)-s\,.$$

Since this function is monotone non-increasing and Theorems 3.2.2, 3.2.3 hold, we get (a).

Let us prove (b). We consider a sequence $(t_k,s_k,w_k)\to(t,s,w)$ in $[0,T]\times(0,L)\times H^1(\Omega;\mathbb{R}^2)$ and ϑ a cut-off function defined as in (3.2.1). By Theorem 3.2.2, for every $k\in\mathbb{N}$ there exists $u_{s_k}\in\mathcal{A}(s_k,w_k)$ minimizer of $\mathcal{E}(t_k,s_k,\cdot)$ such that $\mathfrak{G}^+(t_k,s_k,w_k)=G(t_k,u_{s_k},\vartheta)$. By Lemma 3.2.6, there exists $u_s\in\mathcal{A}(s,w)$ minimizer of $\mathcal{E}(t,s,\cdot)$ such that, up to a subsequence, $u_{s_k}\to u_s$ in $H^1(\Omega\setminus\Lambda;\mathbb{R}^2)$. Formula (3.2.2), together with the hypotheses on g and on φ, implies that

$$G(t,u_s,\vartheta)=\lim_k G(t_k,u_{s_k},\vartheta)\,.$$

By (3.2.7), $G(t,u_s,\vartheta)\le\mathfrak{G}^+(t,s,w)$, thus we deduce the upper semicontinuity of $\mathfrak{G}^+$.

In the same way, taking into account (3.2.9), we obtain the lower semicontinuity of $\mathfrak{G}^-$, and this concludes the proof. ☐

We conclude this section with a proposition which helps us to give an interpretation to G defined in (3.2.2). Let $t\in[0,T]$, $s\in(0,L)$, $w\in H^1(\Omega;\mathbb{R}^2)$, $u\in H^1(\Omega\setminus\Lambda;\mathbb{R}^2)$, and $\eta>0$. We define

$$\mathcal{E}^\eta_{loc}(t,s,u):=\inf\{\mathcal{E}(t,s,v):\,v\in\mathcal{A}(s,w),\,\|v-u\|_{H^1}\le\eta\}\,. \tag{3.2.40}$$

By the direct method of the calculus of variations, we can prove that the infimum in (3.2.40) is attained.

Proposition 3.2.11. *Let* $t\in[0,T]$, $s\in(0,L)$, $w\in H^1(\Omega;\mathbb{R}^2)$, $u_s\in\mathcal{A}(s,w)$ *a minimizer of* $\mathcal{E}(t,s,\cdot)$, *and let* ϑ *be a cut-off function as in* (3.2.1). *Then*

$$\begin{aligned} G(t,u_s,\vartheta)-\kappa &= \lim_{\eta\searrow 0}\,\liminf_{\delta\searrow 0}\,\frac{\mathcal{E}(t,s,u_s)-\mathcal{E}^\eta_{loc}(t,s+\delta,u_s)}{\delta}\\ &= \lim_{\eta\searrow 0}\,\limsup_{\delta\searrow 0}\,\frac{\mathcal{E}(t,s,u_s)-\mathcal{E}^\eta_{loc}(t,s+\delta,u_s)}{\delta}\,. \end{aligned} \tag{3.2.41}$$

In particular, $G(t,u_s,\vartheta)=:G(t,u_s)$ *does not depend on* ϑ.

Proof. Let t, s, w, and u_s be as in the statement of the proposition. Let $\eta>0$ be fixed. With the notation introduced in Lemma 3.2.7, for $\delta>0$ small enough we have $P^{-1}_{s,\delta}\,u_s\in\mathcal{A}(s+\delta,w)$ and, by Lemma 2.2.26, $\|P^{-1}_{s,\delta}\,u_s-u_s\|_{H^1}\le\eta$. Thus, the following estimate from below holds:

$$\frac{\mathcal{E}(t,s,u_s)-\mathcal{E}(t,s+\delta,P^{-1}_{s,\delta}\,u_s)}{\delta}\le\frac{\mathcal{E}(t,s,u_s)-\mathcal{E}^\eta_{loc}(t,s+\delta,u_s)}{\delta}\,. \tag{3.2.42}$$

Therefore, as in the proof of Theorem 3.2.2, passing to the lim inf as $\delta \searrow 0$ in (3.2.42) we get

$$G(t, u_s, \vartheta) - \kappa \leq \liminf_{\delta \searrow 0} \frac{\mathcal{E}(t, s, u_s) - \mathcal{E}^{\eta}_{loc}(t, s + \delta, u_s)}{\delta}\,. \tag{3.2.43}$$

We now prove that

$$\begin{aligned} &\limsup_{\delta \searrow 0} \frac{\mathcal{E}(t, s, u_s) - \mathcal{E}^{\eta}_{loc}(t, s + \delta, u_s)}{\delta} \\ &\quad \leq \sup \{G(t, u_\eta, \vartheta) : \, u_\eta \in A(s, w) \text{ is a minimizer of } \mathcal{E}^{\eta}_{loc}(t, s, u_s)\} - \kappa\,. \end{aligned} \tag{3.2.44}$$

Let us fix a sequence $\delta_k \searrow 0$. Since, for every k, $\mathcal{E}^{\eta+1/k}_{loc}(t, s + \delta_k, u_s) \leq \mathcal{E}^{\eta}_{loc}(t, s + \delta_k, u_s)$, the following chain of inequalities holds:

$$\begin{aligned} \frac{\mathcal{E}(t, s, u_s) - \mathcal{E}^{\eta}_{loc}(t, s + \delta_k, u_s)}{\delta_k} &\leq \frac{\mathcal{E}(t, s, u_s) - \mathcal{E}^{\eta+1/k}_{loc}(t, s + \delta_k, u_s)}{\delta_k} \\ &= \frac{\mathcal{E}(t, s, u_s) - \mathcal{E}(t, s + \delta_k, u^k_\eta)}{\delta_k}\,, \end{aligned} \tag{3.2.45}$$

where we denote by $u^k_\eta \in A(s + \delta_k, w)$ a minimizer of $\mathcal{E}^{\eta+1/k}_{loc}(t, s + \delta_k, u_s)$. Since $\mathcal{E}(t, s, u_s) = \mathcal{E}_m(t, s, w)$ and $P_{s,\delta_k}\, u^k_\eta \in \mathcal{A}(s, w)$, we can continue in (3.2.45) getting

$$\frac{\mathcal{E}(t, s, u_s) - \mathcal{E}^{\eta}_{loc}(t, s + \delta_k, u_s)}{\delta_k} \leq \frac{\mathcal{E}(t, s, P_{s,\delta_k}\, u^k_\eta) - \mathcal{E}(t, s + \delta_k, u^k_\eta)}{\delta_k}\,. \tag{3.2.46}$$

Up to a subsequence, we can assume that

$$\limsup_{k} \frac{\mathcal{E}(t, s, P_{s,\delta_k}\, u^k_\eta) - \mathcal{E}(t, s + \delta_k, u^k_\eta)}{\delta_k} = \lim_{k} \frac{\mathcal{E}(t, s, P_{s,\delta_k}\, u^k_\eta) - \mathcal{E}(t, s + \delta_k, u^k_\eta)}{\delta_k}\,.$$

By construction, we have that u^k_η is bounded in $H^1(\Omega \setminus \Lambda; \mathbb{R}^2)$. Thus, we may assume that, up to a subsequence, $u^k_\eta \rightharpoonup u$ weakly in $H^1(\Omega \setminus \Lambda; \mathbb{R}^2)$ as $k \to +\infty$ for some $u \in H^1(\Omega \setminus \Lambda; \mathbb{R}^2)$. By the compactness of the trace operator and by the lower semicontinuity of the H^1-norm, we have $u \in A(s, w)$ and $\|u - u_s\|_{H^1} \leq \eta$.

Let us prove that u is a minimizer of $\mathcal{E}^{\eta}_{loc}(t, s, u_s)$: given $v_\eta \in \mathcal{A}(s, w)$ a minimum of $\mathcal{E}^{\eta}_{loc}(t, s, u_s)$, thanks to Lemma 2.2.26 we can find a sequence ε_k such that $0 < \varepsilon_k < \delta_k$, $\varepsilon_{k+1} < \varepsilon_k$, and $\|P^{-1}_{s,\varepsilon_k}\, v_\eta - v_\eta\|_{H^1} \leq 1/k$ for every $k \in \mathbb{N}$. Therefore, by the triangle inequality we get

$$\|P^{-1}_{s,\varepsilon_k}\, v_\eta - u_s\|_{H^1} \leq \eta + 1/k\,.$$

Moreover, by our choice of ε_k, $P^{-1}_{s,\varepsilon_k}\, v_\eta \in \mathcal{A}(s + \varepsilon_k, w) \subseteq \mathcal{A}(s + \delta_k, w)$. Hence, in view of (3.1.10) in Lemma 3.1.3 and of the definition of v_η, we obtain

$$\begin{aligned} \mathcal{E}(t, s, v_\eta) = \mathcal{E}^{\eta}_{loc}(t, s, u_s) &\leq \mathcal{E}(t, s, u) \leq \liminf_{k} \mathcal{E}(t, s + \delta_k, u^k_\eta) \\ &\leq \limsup_{k} \mathcal{E}(t, s + \delta_k, u^k_\eta) \leq \lim_{k} \mathcal{E}(t, s + \delta_k, P^{-1}_{s,\varepsilon_k}\, v_\eta) = \mathcal{E}(t, s, v_\eta)\,. \end{aligned} \tag{3.2.47}$$

where, in the last equality, we have used the strong convergence of $P^{-1}_{s,\varepsilon_k} v_\eta$ to v_η in $H^1(\Omega \setminus \Lambda; \mathbb{R}^2)$ as $k \to +\infty$. The chain of inequalities (3.2.47) implies that $u \in \mathcal{A}(s,w)$ is a minimizer of $\mathcal{E}^\eta_{loc}(t,s,u_s)$ and that

$$\mathcal{E}(t,s,u) = \lim_k \mathcal{E}(t, s+\delta_k, u^k_\eta)\,.$$

Thus, by Lemma 3.1.3 we get that $u^k_\eta \to u$ strongly in $H^1(\Omega \setminus \Lambda; \mathbb{R}^2)$ as $k \to +\infty$. By Lemma 2.2.26, we also have $P_{s,\delta_k} u^k_\eta \to u$ in $H^1(\Omega \setminus \Lambda; \mathbb{R}^2)$.

Passing to the lim sup in (3.2.46) as $k \to +\infty$ and taking into account the previous convergences, we get, as in the proofs of Theorems 3.2.2 and 3.2.3,

$$\begin{aligned}\limsup_k \, & \frac{\mathcal{E}(t,s,u_s) - \mathcal{E}^\eta_{loc}(t, s+\delta_k, u_s)}{\delta_k} \\ & \leq \lim_k \frac{\mathcal{E}(t,s,P_{s,\delta_k} u^k_\eta) - \mathcal{E}(t,s+\delta_k,u^k_\eta)}{\delta_k} = G(t,u,\vartheta) - \kappa\,.\end{aligned} \tag{3.2.48}$$

Taking the supremum in (3.2.48) among all the functions u minimizer of $\mathcal{E}^\eta_{loc}(t,s,u_s)$, we deduce that

$$\begin{aligned}\limsup_k \, & \frac{\mathcal{E}(t,s,u_s) - \mathcal{E}^\eta_{loc}(t, s+\delta_k, u_s)}{\delta_k} \\ & \leq \sup\{G(t,u_\eta,\vartheta) : \, u_\eta \in \mathcal{A}(s,w) \text{ is a minimizer of } \mathcal{E}^\eta_{loc}(t,s,u_s)\} - \kappa\,.\end{aligned} \tag{3.2.49}$$

By a contradiction argument, (3.2.49) implies (3.2.44). It is easy to see that, as in Theorem 3.2.2, the supremum in (3.2.44) is actually a maximum.

Finally, passing to the limit in inequalities (3.2.43) and (3.2.44) as $\eta \searrow 0$, we get (3.2.41), and the proof is thus concluded. $\square$

Remark 3.2.12. In view of Proposition 3.2.11, we can interpret $G(t,u_s)$ as a "local" energy release rate, in the sense that it takes into account only deformations which are close to u_s in the H^1-norm, while $\mathfrak{G}^\pm$ are "global" energy release rates.

Since we have explicit formulas for the right and left derivatives of the reduced energy $\mathcal{E}_m$ in terms of the generalized energy release rates $\mathfrak{G}^+$ and $\mathfrak{G}^-$, we are now in a position to study the problem of existence of a quasi-static evolution of our cohesive fracture model with an activation threshold. Following the ideas of [48], we look for an evolution satisfying a weak form of Griffith's criterion.

3.3 Quasi-static evolution

We provide a notion of quasi-static evolution based on the technique of vanishing viscosity. The solution is defined through a process of time discretization: we first solve some incremental problems and then pass to the limit as the time step vanishes. In order to enforce local minimality, the incremental problems are perturbed with a viscous parameter $\varepsilon > 0$ which tends to zero more slowly than the time step. This approach was employed in [10, 30, 57, 58] in an abstract setting and in [47, 48, 52, 72] for the problem of crack growth.

First of all, let us fix some notation which will be used from now on: the reference configuration is described by $\overline{\Omega}$, where $\Omega \subseteq \mathbb{R}^2$ is an open, bounded, connected set with Lipschitz boundary. The crack path is given by the $C^{2,1}$-curve $\Lambda \subseteq \overline{\Omega}$. See Section 3.1 for the properties of Ω and Λ and (3.1.1) for the definition of admissible cracks. Given $T > 0$, we consider

$$w \in AC([0,T]; H^1(\Omega;\mathbb{R}^2)) \qquad \text{and} \qquad f \in AC([0,T]; L^2(\Omega;\mathbb{R}^2)) \tag{3.3.1}$$

which represent the Dirichlet boundary datum and the volume forces applied to Ω, respectively. In particular, $f(t,x)\cdot\xi$ will substitute the function $g(t,x,\xi)$ defined in Section 3.1. For simplicity of notation, we will not indicate the dependence of f and w on the space variable x.

Finally, we assume that the function $\varphi\colon [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ satisfies a further property of differentiability: we suppose that $\varphi(\cdot,\xi) \in AC([0,T];\mathbb{R})$ for every $\xi \in \mathbb{R}^2$ and that there exist $p \in (1,+\infty)$ and $a_3 \in L^1([0,T])$ with $a_3 \geq 0$ such that

$$|D_t\varphi(t,\xi)| \leq a_3(t)(1+|\xi|^p) \qquad \text{for a.e. } t \in [0,T] \text{ and every } \xi \in \mathbb{R}^2\,. \tag{3.3.2}$$

Fixed $s \in [0,L]$ and $t \in [0,T]$, the energy of the system is, similar to (3.1.7),

$$\mathcal{E}(t,s,u) := \frac{1}{2}\int_{\Omega_s} \mathbb{C}\mathrm{E}u\cdot\mathrm{E}u\,\mathrm{d}x - \int_{\Omega_s} f(t)\cdot u\,\mathrm{d}x + \int_{\Gamma_s} \varphi(t,[u])\,\mathrm{d}\mathcal{H}^1 + \kappa s\,, \tag{3.3.3}$$

for every $u \in \mathcal{A}(s,w(t))$, the set of admissible displacements at time t, defined as in (3.1.9).

Since the boundary datum is a function of $t \in [0,T]$, we slightly change the notation for the reduced energy $\mathcal{E}_m$ and for the energy release rates with respect to Section 3.2: for every $s \in [0,L]$ and every $t \in [0,T]$, we define, similar to (3.1.12),

$$\mathcal{E}_m(t,s) := \min_{u\in\mathcal{A}(s,w(t))} \mathcal{E}(t,s,u)\,. \tag{3.3.4}$$

Remark 3.3.1. By (3.3.1), all the results about $\mathcal{E}_m$ proved in Section 3.2 hold: by Lemmas 3.2.5 and 3.2.6 the reduced energy $\mathcal{E}_m$ is lower semicontinuous on $[0,T]\times[0,L]$ and continuous on $[0,T]\times(0,L)$. By Theorems 3.2.2 and 3.2.3, it has right and left derivatives with respect to the crack length s which are now denoted by $\partial_s^+\mathcal{E}_m(t,s)$ and $\partial_s^-\mathcal{E}_m(t,s)$ for every $(t,s) \in [0,T]\times(0,L)$. Moreover,

$$\begin{aligned}\partial_s^+\mathcal{E}_m(t,s) &= \kappa - \mathfrak{G}^+(t,s,w(t))\,,\\ \partial_s^-\mathcal{E}_m(t,s) &= \kappa - \mathfrak{G}^-(t,s,w(t))\,,\end{aligned}$$

where $\mathfrak{G}^\pm$ are defined as in (3.2.7) and in (3.2.9).

With an abuse of notation, we now set

$$\mathfrak{G}^\pm(t,s) := \mathfrak{G}^\pm(t,s,w(t))\,,$$

where, in the formulas (3.2.2), (3.2.7), and (3.2.9) for $\mathfrak{G}^\pm(t,s,w(t))$, the function $g(t,x,u)$ is replaced by $f(t,x)\cdot u$ for an admissible displacement u.

Remark 3.3.2. Since w and f are continuous in time, a simple application of Proposition 3.2.10 shows that $\mathfrak{G}^+$ is upper semicontinuous and $\mathfrak{G}^-$ is lower semicontinuous on $[0,T] \times (0,L)$.

We now discuss briefly the time incremental minimum problems and then give our definitions of *viscous* and *quasi-static evolutions*.

For every $k \in \mathbb{N}$ we fix a subdivision $\{t_i^k\}_{i=0}^k$ of the time interval $[0,T]$ with $t_i^k := i\tau_k$ and $\tau_k := T/k$. Given $\varepsilon > 0$, we define recursively the solution $s_\varepsilon^{k,i}$ to incremental minimum problems: let $s_\varepsilon^{k,0} := s_0$, where $s_0 \in (0,L)$ is the initial condition, and, for $i \geq 1$, let $s_\varepsilon^{k,i}$ be a solution to

$$\min\left\{\mathcal{E}_m(t_i^k, s) + \frac{\varepsilon}{2}\,\frac{(s - s_\varepsilon^{k,i-1})^2}{\tau_k} \,:\, s \in [s_\varepsilon^{k,i-1}, L]\right\}. \tag{3.3.5}$$

We postpone the proof of existence of a solution to (3.3.5) to the next section, see Proposition 3.4.1, to comment briefly on the function which appears in (3.3.5). This function is the sum of two terms: the reduced energy $\mathcal{E}_m$ defined by (3.3.4), which represents the energy of the system at the equilibrium for a fixed $s \in [0,L]$, and a perturbation term driven by $\varepsilon > 0$ which enforces a local minimization of the energy with respect to s. This kind of approximation should guarantee that the evolution in the limit follows "local minimizers" of the energy (see [26, 30, 48, 52, 57, 58, 70] for further discussions and applications).

The passage to the limit will be performed in two steps: we let first $k \to +\infty$ and find a viscous evolution for every $\varepsilon > 0$, and, finally, we obtain a quasi-static evolution as the parameter ε tends to zero.

We now give a definition of *viscous evolution* and *quasi-static evolution* for the cohesive crack growth problem. We refer to Definition 2.2.29 for the definition of the failure time $\mathcal{T}(s)$ of a monotone non-decreasing function $s\colon [0,T] \to [0,L]$.

Definition 3.3.3. Let $\varepsilon > 0$ and $s_0 \in (0,L)$. We say that a monotone non-decreasing function $s_\varepsilon \in H^1([0,T])$ is a *viscous evolution for the cohesive crack growth problem* with $s_\varepsilon(0) = s_0$ if it satisfies the following rate-dependent Griffith's criterion:
for a.e. $t \in [0, \mathcal{T}(s_\varepsilon))$

(1) $\dot{s}_\varepsilon(t) \geq 0$;

(2) $\mathfrak{G}^-(t, s_\varepsilon(t)) - \kappa - \varepsilon\dot{s}_\varepsilon(t) \leq 0$;

(3) $(\mathfrak{G}^+(t, s_\varepsilon(t)) - \kappa - \varepsilon\dot{s}_\varepsilon(t))\,\dot{s}_\varepsilon(t) \geq 0$.

In Section 3.5 we prove the following existence theorem.

Theorem 3.3.4. *Let $\varepsilon > 0$, $f \in AC([0,T]; L^2(\Omega;\mathbb{R}^2))$, and $w \in AC([0,T]; H^1(\Omega;\mathbb{R}^2))$. Then, for every $s_0 \in (0,L)$ there exists a viscous evolution $s_\varepsilon \in H^1([0,T])$ for the cohesive crack growth problem with $s_\varepsilon(0) = s_0$.*

Given $s\colon [0,T] \to [0,L]$ monotone non-decreasing, we define the jump set of s by

$$J(s) := \{t \in [0,T] \,:\, s(t^-) < s(t^+)\}.$$

Definition 3.3.5. Let $s_0 \in (0,L)$. We say that a monotone non-decreasing function $s \in BV([0,T])$ is a *quasi-static evolution for the cohesive crack growth problem* with $s(0) = s_0$ if it satisfies:

(1) for every $t \in [0, \mathcal{T}(s)) \setminus J(s)$:

$$\mathfrak{G}^-(t, s(t)) \leq \kappa\,;$$

(2) for every $t \in [0, \mathcal{T}(s)) \cap J(s)$:

$$\mathfrak{G}^+(t,\sigma) \geq \kappa \qquad \text{for every } \sigma \in [s(t^-), s(t^+)]\,;$$

(3) if $t \in [0, \mathcal{T}(s))$ and $\mathfrak{G}^+(t, s(t)) < \kappa$, then s is differentiable at t and $\dot{s}(t) = 0$.

We can now state the main theorem of this section.

Theorem 3.3.6. *Let $f \in AC([0,T]; L^2(\Omega;\mathbb{R}^2))$ and $w \in AC([0,T]; H^1(\Omega;\mathbb{R}^2))$. Then, for every $s_0 \in (0,L)$ there exists a quasi-static evolution $s \in BV([0,T])$ for the cohesive crack growth problem with $s(0) = s_0$.*

Remark 3.3.7. We notice that in the proof of Theorem 3.3.6 we also show that if $\{s_\varepsilon\}_{\varepsilon>0}$ is a sequence of viscous evolutions for the cohesive crack growth problem with $s_\varepsilon(0) = s_0$, then, up to a subsequence, s_ε converges pointwise to a quasi-static evolution $s \in BV([0,T])$.

3.4 The discrete-time problems

We now discuss the properties of the discrete-time solutions $s_\varepsilon^{k,i}$ introduced in Section 3.3. First of all, we have to prove that they are well defined.

Proposition 3.4.1. *For every $\varepsilon > 0$, $k \in \mathbb{N}$, and $i = 1, \dots, k$, there exists a solution to* (3.3.5)*.*

Proof. We exploit the direct method of the calculus of variations. Let $\varepsilon > 0$, $k \in \mathbb{N}$, and $i = 1, \dots, k$ be fixed. Let $s_j \in [s_\varepsilon^{k,i-1}, L]$ be a minimizing sequence for the minimum problem (3.3.5). Up to a subsequence, we may assume that there exists $s \in [s_\varepsilon^{k,i-1}, L]$ such that $s_j \to s$. Taking into account Lemma 3.2.5, we have that

$$\mathcal{E}_m(t_i^k, s) \leq \liminf_j \, \mathcal{E}_m(t_i^k, s_j)\,,$$

hence s is a solution to (3.3.5). ☐

We now provide some a priori bounds on the incremental solutions. In what follows, $w_i^k := w(t_i^k)$ and $f_i^k := f(t_i^k)$.

Proposition 3.4.2. *There exists* $C > 0$ *such that, for every* $k \in \mathbb{N}$ *and every* $\varepsilon > 0$, *the following inequality holds*

$$\frac{\varepsilon}{2} \sum_{j=1}^{k} \frac{(s_\varepsilon^{k,j} - s_\varepsilon^{k,j-1})^2}{\tau_k} \leq C\,. \tag{3.4.1}$$

Proof. During the proof of this proposition, we will denote by u_i^k a minimizer of $\mathcal{E}(t_i^k, s_\varepsilon^{k,i}, \cdot)$ in $\mathcal{A}(s_\varepsilon^{k,i}, w_i^k)$ and by Ω_i^k, Γ_i^k the sets $\Omega_{s_\varepsilon^{k,i}}$, $\Gamma_{s_\varepsilon^{k,i}}$, respectively.

First, let us prove that the minimizers u_i^k are bounded in $H^1(\Omega \setminus \Lambda; \mathbb{R}^2)$ uniformly with respect to $k \in \mathbb{N}$, $i = 1, \ldots, k$, and $\varepsilon > 0$. Indeed, $w_i^k \in \mathcal{A}(s_\varepsilon^{k,i}, w_i^k)$ and, by (2.3.1), (3.1.2), the hypothesis $\varphi(t_i^k, 0) = 0$, and Hölder's inequality, we get

$$\mathcal{E}_m(t_i^k, s_\varepsilon^{k,i}) \leq \mathcal{E}(t_i^k, s_\varepsilon^{k,i}, w_i^k) \leq \frac{\beta}{2} \|w_i^k\|^2_{H^1(\Omega)} + \|f_i^k\|_{2,\Omega} \|w_i^k\|_{H^1(\Omega)} + L\,. \tag{3.4.2}$$

From (3.3.1) and (3.4.2), we deduce that, for some $c > 0$,

$$\mathcal{E}_m(t_i^k, s_\varepsilon^{k,i}) = \mathcal{E}(t_i^k, s_\varepsilon^{k,i}, u_i^k) \leq c\,. \tag{3.4.3}$$

Therefore, since (2.3.1) holds and φ satisfies (3.1.3) uniformly in t, applying Hölder's and Korn's inequalities to (3.4.3) we obtain

$$c_1 \|u_i^k\|^2_{H^1(\Omega)} - \|f_i^k\|_{2,\Omega} \|u_i^k\|_{H^1(\Omega)} - c_2 \leq \mathcal{E}_m(t_i^k, s_\varepsilon^{k,i}) \leq c \tag{3.4.4}$$

for some c_1, $c_2 > 0$. By the absolute continuity of f and by Young's inequality, from (3.4.4) it follows that there exists $M > 0$ such that for every k, every $i = 1, \ldots, k$, and every $\varepsilon > 0$:

$$\|u_i^k\|_{H^1(\Omega)} \leq M \qquad \text{and} \qquad \mathcal{E}_m(t_i^k, s_\varepsilon^{k,i}) \geq -M\,. \tag{3.4.5}$$

Let $k \in \mathbb{N}$, $i = 1, \ldots, k$, and $\varepsilon > 0$ be fixed. Since $u_{i-1}^k + w_i^k - w_{i-1}^k \in \mathcal{A}(s_\varepsilon^{k,i-1}, w_i^k)$, we have, by definition of $s_\varepsilon^{k,i}$ and of the reduced energy $\mathcal{E}_m$,

$$\begin{aligned}
&\mathcal{E}_m(t_i^k, s_\varepsilon^{k,i}) + \frac{\varepsilon}{2} \frac{(s_\varepsilon^{k,i} - s_\varepsilon^{k,i-1})^2}{\tau_k} \leq \mathcal{E}_m(t_i^k, s_\varepsilon^{k,i-1}) \\
&\quad \leq \mathcal{E}(t_i^k, s_\varepsilon^{k,i-1}, u_{i-1}^k + w_i^k - w_{i-1}^k) \\
&\quad = \mathcal{E}_m(t_{i-1}^k, s_\varepsilon^{k,i-1}) + \int_{\Omega_{i-1}^k} \mathbb{C}\mathrm{E}u_{i-1}^k \cdot \mathrm{E}(w_i^k - w_{i-1}^k)\, \mathrm{d}x \\
&\qquad + \frac{1}{2} \int_\Omega \mathbb{C}\mathrm{E}(w_i^k - w_{i-1}^k) \cdot \mathrm{E}(w_i^k - w_{i-1}^k)\, \mathrm{d}x - \int_{\Omega_{i-1}^k} (f_i^k - f_{i-1}^k) \cdot u_{i-1}^k\, \mathrm{d}x \\
&\qquad - \int_\Omega f_i^k \cdot (w_i^k - w_{i-1}^k)\, \mathrm{d}x + \int_{t_{i-1}^k}^{t_i^k} \int_{\Gamma_{i-1}^k} D_t \varphi(\tau, [u_{i-1}^k])\, \mathrm{d}\mathcal{H}^1\, \mathrm{d}\tau\,.
\end{aligned} \tag{3.4.6}$$

Thanks to (3.3.1), (3.3.2), (3.4.5), to Hölder's inequality, and to the continuity of the trace operator, (3.4.6) becomes

$$\begin{aligned}&\mathcal{E}_m(t_i^k, s_\varepsilon^{k,i}) + \frac{\varepsilon}{2}\frac{(s_\varepsilon^{k,i}-s_\varepsilon^{k,i-1})^2}{\tau_k}\\ &\quad\le \mathcal{E}_m(t_{i-1}^k, s_\varepsilon^{k,i-1}) + \beta M\int_{t_{i-1}^k}^{t_i^k}\|\dot w(\tau)\|_{H^1(\Omega)}\mathrm{d}\tau + \beta W_k\int_{t_{i-1}^k}^{t_i^k}\|\dot w(\tau)\|_{H^1(\Omega)}\mathrm{d}\tau\\ &\quad + M\int_{t_{i-1}^k}^{t_i^k}\|\dot f(\tau)\|_{2,\Omega}\mathrm{d}\tau + F\int_{t_{i-1}^k}^{t_i^k}\|\dot w(\tau)\|_{H^1(\Omega)}\mathrm{d}\tau + (L+CM^p)\int_{t_{i-1}^k}^{t_i^k}a_3(\tau)\mathrm{d}\tau\,,\end{aligned} \tag{3.4.7}$$

where $L = \mathcal{H}^1(\Lambda)$, C is a positive constant independent of k, and

$$W_k := \frac{1}{2}\sup_{j=1,\dots,k}\|w_j^k - w_{j-1}^k\|_{H^1(\Omega)} \qquad \text{and} \qquad F := \sup_{t\in[0,T]}\|f(t)\|_{2,\Omega}\,.$$

Adding to both sides of (3.4.7) the term $\frac{\varepsilon}{2}\frac{(s_\varepsilon^{k,i-1}-s_\varepsilon^{k,i-2})^2}{\tau_k}$ and iterating the previous argument, we get

$$\begin{aligned}&\mathcal{E}_m(t_i^k, s_\varepsilon^{k,i}) + \frac{\varepsilon}{2}\sum_{j=1}^{i}\frac{(s_\varepsilon^{k,j}-s_\varepsilon^{k,j-1})^2}{\tau_k}\\ &\qquad\le \mathcal{E}_m(0,s_0) + (\beta M + \beta W_k + F)\int_0^T\|\dot w(t)\|_{H^1(\Omega)}\,\mathrm{d}t\\ &\qquad + M\int_0^T\|\dot f(t)\|_{2,\Omega}\,\mathrm{d}t + (L+CM^p)\int_0^T a_3(t)\,\mathrm{d}t\,.\end{aligned} \tag{3.4.8}$$

By (3.3.1), $F < +\infty$ and $W_k \to 0$ as $k \to +\infty$, so (3.4.5) and (3.4.8) imply (3.4.1), and the proof is thus concluded. □

For every k and every $\varepsilon > 0$, let us define the piecewise constant interpolations $\bar t_k(t) := t_i^k$ and $\bar s_\varepsilon^k(t) := s_i^k$ for $t \in (t_{i-1}^k, t_i^k]$, and the piecewise affine interpolation function

$$s_\varepsilon^k(t) := s_\varepsilon^{k,i-1} + \frac{s_\varepsilon^{k,i}-s_\varepsilon^{k,i-1}}{\tau_k}(t-t_{i-1}^k) \qquad \text{for } t\in(t_{i-1}^k, t_i^k]\,.$$

The next proposition is the equivalent of the Griffith's criterion in the discrete setting.

Proposition 3.4.3. *For every $k\in\mathbb{N}$, every $\varepsilon>0$, and every $t\in[0,\mathcal{T}(\bar s_\varepsilon^k))$ we have:*

(a) $\dot s_\varepsilon^k(t) \ge 0$*;*

(b) $\mathfrak{G}^+(\bar t_k(t), \bar s_\varepsilon^k(t)) - \kappa - \varepsilon\dot s_\varepsilon^k(t) \le 0$*;*

(c) $(\mathfrak{G}^+(\bar t_k(t), \bar s_\varepsilon^k(t)) - \kappa - \varepsilon\dot s_\varepsilon^k(t))\,\dot s_\varepsilon^k(t) = 0$*.*

Proof. Property (a) follows immediately from the definition of s^k_ε.

Let us prove (b). Fix $t \in (t^k_{i-1}, t^k_i]$ such that $t < \mathcal{T}(\bar{s}^k_\varepsilon)$. By construction, for every $\sigma \geq s^{k,i-1}_\varepsilon$ we have

$$\mathcal{E}_m(t^k_i, s^{k,i}_\varepsilon) + \frac{\varepsilon}{2}\,\frac{(s^{k,i}_\varepsilon - s^{k,i-1}_\varepsilon)^2}{\tau_k} \leq \mathcal{E}_m(t^k_i, \sigma) + \frac{\varepsilon}{2}\,\frac{(\sigma - s^{k,i-1}_\varepsilon)^2}{\tau_k}\,. \tag{3.4.9}$$

If $\sigma > s^{k,i}_\varepsilon$, dividing (3.4.9) by $\sigma - s^{k,i}_\varepsilon$, we obtain

$$\frac{\mathcal{E}_m(t^k_i, s^{k,i}_\varepsilon) - \mathcal{E}_m(t^k_i, \sigma)}{\sigma - s^{k,i}_\varepsilon} - \frac{\varepsilon}{2\tau_k}\,\frac{(\sigma - s^{k,i-1}_\varepsilon)^2 - (s^{k,i}_\varepsilon - s^{k,i-1}_\varepsilon)^2}{\sigma - s^{k,i}_\varepsilon} \leq 0\,,$$

so, passing to the limit as $\sigma \searrow s^{k,i}_\varepsilon$ and taking into account Theorem 3.2.2, we get (b).

If $\dot{s}^k_\varepsilon(t) = 0$, then (c) is clearly satisfied. Otherwise, $s^{k,i}_\varepsilon > s^{k,i-1}_\varepsilon$, hence we can consider (3.4.9) with $\sigma \in (s^{k,i-1}_\varepsilon, s^{k,i}_\varepsilon)$. Dividing by $\sigma - s^{k,i}_\varepsilon$ and passing to the limit as $\sigma \nearrow s^{k,i}_\varepsilon$, from Theorem 3.2.3 it follows that

$$\mathfrak{G}^-(t^k_i, \bar{s}^k_\varepsilon(t)) - \kappa - \varepsilon\dot{s}^k_\varepsilon(t) \geq 0\,. \tag{3.4.10}$$

Thanks to point (a) of Proposition 3.2.10 and to the previous step, we deduce that

$$\mathfrak{G}^+(t^k_i, \bar{s}^k_\varepsilon(t)) = \mathfrak{G}^-(t^k_i, \bar{s}^k_\varepsilon(t))\,,$$

hence (c) holds. □

3.5 Viscous evolution

This section is devoted to the proof of Theorem 3.3.4. For every $\varepsilon > 0$, we pass to the limit as $k \to +\infty$, in order to find a viscous evolution.

Let us prove the following compactness result.

Proposition 3.5.1. *For every $\varepsilon > 0$, there exists $s_\varepsilon \in H^1([0,T])$ such that*

(a) *up to a subsequence, $s^k_\varepsilon \rightharpoonup s_\varepsilon$ weakly in $H^1([0,T])$ and $s^k_\varepsilon, \bar{s}^k_\varepsilon \to s_\varepsilon$ uniformly in $[0,T]$;*

(b) *s_ε is monotone non-decreasing;*

(c) *$s_\varepsilon(0) = s_0$;*

(d) *$\varepsilon\|\dot{s}_\varepsilon\|^2_2$ is uniformly bounded with respect to $\varepsilon > 0$.*

Proof. Proposition 3.4.2 implies that $\varepsilon\|\dot{s}^k_\varepsilon\|^2_2$ is uniformly bounded with respect to $k \in \mathbb{N}$ and $\varepsilon > 0$, thus the sequence $(s^k_\varepsilon)_k$ is bounded in $H^1([0,T])$. Therefore, for every $\varepsilon > 0$ there exists $s_\varepsilon \in H^1([0,T])$ such that, up to a subsequence, $s^k_\varepsilon \rightharpoonup s_\varepsilon$ weakly in $H^1([0,T])$. In particular, by (3.4.1) and by the lower semicontinuity of the L^2-norm, property (d) holds.

Applying the Ascoli-Arzelà theorem, up to a further subsequence we can assume that $s_\varepsilon^k \to s_\varepsilon$ uniformly in $[0,T]$ as $k \to +\infty$. Since, by (3.4.1),

$$|s_\varepsilon^k(t) - \bar{s}_\varepsilon^k(t)| \le \left| \frac{s_\varepsilon^{k,i} - s_\varepsilon^{k,i-1}}{\tau_k} (t - t_{i-1}^k) \right| + |s_\varepsilon^{k,i} - s_\varepsilon^{k,i-1}| \le C\sqrt{\tau_k}$$

for some $C > 0$, we deduce that $\bar{s}_\varepsilon^k \to s_\varepsilon$ uniformly in $[0,T]$, hence (a) is proved.

Since, by construction, $s_\varepsilon^k(0) = s_0$ for every k, it follows that $s_\varepsilon(0) = s_0$. Finally, from the monotonicity of $\bar{s}_\varepsilon^k$ and the uniform convergence proved in (a), we deduce that s_ε is monotone non-decreasing. $\square$

We are now ready to prove Theorem 3.3.4

Proof of Theorem 3.3.4. Fix $\varepsilon > 0$. Let us prove that $s_\varepsilon \in H^1([0,T])$ found in Proposition 3.5.1 is a viscous evolution for the cohesive crack growth with $s_\varepsilon(0) = s_0$.

Since $s_\varepsilon \in H^1([0,T])$, its derivative $\dot{s}_\varepsilon$ exists a.e. in $[0,T]$ and is nonnegative by monotonicity (see (b) of Proposition 3.5.1).

To prove properties (2) and (3) of Definition 3.3.3, in view of Remark 2.2.30 we have to distinguish between two possibilities:

$$\mathcal{T}(s_\varepsilon) = \lim_k \mathcal{T}(\bar{s}_\varepsilon^k) \qquad \text{or} \qquad \mathcal{T}(s_\varepsilon) < \limsup_k \mathcal{T}(\bar{s}_\varepsilon^k)\,. \tag{3.5.1}$$

Let us consider the first case. By properties (a) of Proposition 3.2.10 and (b) of Proposition 3.4.3, for every $\psi \in L^2([0,T])$ with $\psi \ge 0$ we have

$$\int_0^{\mathcal{T}(\bar{s}_\varepsilon^k)} (\varepsilon \dot{s}_\varepsilon^k(t) + \kappa - \mathfrak{G}^-(\bar{t}_k(t), \bar{s}_\varepsilon^k(t)))\, \psi(t)\, \mathrm{d}t \ge 0\,. \tag{3.5.2}$$

By the weak convergence $s_\varepsilon^k \rightharpoonup s_\varepsilon$ in $H^1([0,T])$, taking the lim sup as $k \to +\infty$ in (3.5.2) we get

$$\int_0^{\mathcal{T}(s_\varepsilon)} (\varepsilon \dot{s}_\varepsilon(t) + \kappa)\, \psi(t)\, \mathrm{d}t - \liminf_k \int_0^T \mathfrak{G}^-(\bar{t}_k(t), \bar{s}_\varepsilon^k(t))\, \psi(t)\, \mathbf{1}_{[0,\mathcal{T}(\bar{s}_\varepsilon^k))}(t)\, \mathrm{d}t \ge 0\,. \tag{3.5.3}$$

By Proposition 3.2.10,

$$\mathfrak{G}^-(\bar{t}_k(t), \bar{s}_\varepsilon^k(t))\, \psi(t)\, \mathbf{1}_{[0,\mathcal{T}(\bar{s}_\varepsilon^k))}(t) \ge 0 \qquad \text{for a.e. } t \in [0,T]\,.$$

Therefore, applying Fatou's lemma to the last term in (3.5.3), taking into account (a) of Proposition 3.5.1, the convergence $\bar{t}_k(t) \to t$ for every $t \in [0,T]$, and the lower semicontinuity of $\mathfrak{G}^-$, we deduce that

$$\int_0^{\mathcal{T}(s_\varepsilon)} (\varepsilon \dot{s}_\varepsilon(t) + \kappa - \mathfrak{G}^-(t, s_\varepsilon(t)))\, \psi(t)\, \mathrm{d}t \ge 0\,. \tag{3.5.4}$$

Inequality (3.5.4) holds for every $\psi \in L^2([0,T])$, $\psi \ge 0$, hence we have proved property (2) of Definition 3.3.3.

In order to prove condition (3), we first notice that, thanks to the bound (3.4.5), to the definition of $\mathfrak{G}^+$ (see (3.2.2) and (3.2.7)), and to the hypotheses (2.3.1), (3.1.2), and (3.3.1), there exists $C > 0$ such that

$$\mathfrak{G}^+(\bar{t}_k(t), \bar{s}^k_\varepsilon(t)) \leq C \tag{3.5.5}$$

uniformly with respect to $k \in \mathbb{N}$, $\varepsilon > 0$, and $t \in [0, \mathcal{T}(\bar{s}^k_\varepsilon))$.

Integrating (c) of Proposition 3.4.3 over the interval $[0, \mathcal{T}(\bar{s}^k_\varepsilon))$, we obtain

$$\int_0^{\mathcal{T}(\bar{s}^k_\varepsilon)} (\mathfrak{G}^+(\bar{t}_k(t), \bar{s}^k_\varepsilon(t)) - \kappa - \varepsilon \dot{\bar{s}}^k_\varepsilon(t))\, \dot{\bar{s}}^k_\varepsilon(t)\, \mathrm{d}t = 0\,. \tag{3.5.6}$$

Passing to the lim sup in (3.5.6) as $k \to +\infty$, by Proposition 3.5.1 and the lower semicontinuity of the L^2-norm, we get

$$\begin{aligned}
0 &= \limsup_k \int_0^{\mathcal{T}(\bar{s}^k_\varepsilon)} (\mathfrak{G}^+(\bar{t}_k(t), \bar{s}^k_\varepsilon(t)) - \kappa - \varepsilon \dot{\bar{s}}^k_\varepsilon(t)) \dot{\bar{s}}^k_\varepsilon(t) \mathrm{d}t \\
&\leq \limsup_k \int_0^{\mathcal{T}(\bar{s}^k_\varepsilon)} \mathfrak{G}^+(\bar{t}_k(t), \bar{s}^k_\varepsilon(t)) \dot{\bar{s}}^k_\varepsilon(t) \mathrm{d}t - \int_0^{\mathcal{T}(s_\varepsilon)} \dot{s}_\varepsilon(t) \mathrm{d}t - \varepsilon \liminf_k \|\dot{\bar{s}}^k_\varepsilon \mathbf{1}_{[0,\mathcal{T}(\bar{s}^k_\varepsilon))}\|_2^2 \\
&\leq \limsup_k \int_0^{T} \mathfrak{G}^+(\bar{t}_k(t), \bar{s}^k_\varepsilon(t)) \dot{\bar{s}}^k_\varepsilon(t)\, \mathbf{1}_{[0,\mathcal{T}(\bar{s}^k_\varepsilon))}(t) \mathrm{d}t - \int_0^{\mathcal{T}(s_\varepsilon)} (\kappa + \varepsilon \dot{s}_\varepsilon(t))\, \dot{s}_\varepsilon(t) \mathrm{d}t\,.
\end{aligned} \tag{3.5.7}$$

By property (a) of Proposition 3.4.3, we can continue the chain of inequalities (3.5.7), obtaining

$$0 \leq \limsup_k \int_0^T F_k(t)\, \dot{\bar{s}}^k_\varepsilon(t)\, \mathrm{d}t - \int_0^{T(s_\varepsilon)} (\kappa + \varepsilon \dot{s}_\varepsilon(t))\, \dot{s}_\varepsilon(t)\, \mathrm{d}t\,, \tag{3.5.8}$$

where we have set

$$F_k(t) := \sup_{h \geq k} \mathfrak{G}^+(\bar{t}_h(t), \bar{s}^h_\varepsilon(t))\, \mathbf{1}_{[0,\mathcal{T}(\bar{s}^h_\varepsilon))}(t)$$

for every $t \in [0, T]$ and every $k \in \mathbb{N}$.

By definition, $F_k(t)$ converges pointwise to

$$F(t) := \limsup_k \mathfrak{G}^+(\bar{t}_k(t), \bar{s}^k_\varepsilon(t))\, \mathbf{1}_{[0,\mathcal{T}(\bar{s}^k_\varepsilon))}(t) = \limsup_k \mathfrak{G}^+(\bar{t}_k(t), \bar{s}^k_\varepsilon(t))\, \mathbf{1}_{[0,\mathcal{T}(s_\varepsilon))}(t)\,.$$

By estimate (3.5.5) and the dominated convergence theorem, $F_k \to F$ strongly in $L^2([0,T])$. Therefore, by Proposition 3.5.1, (3.5.8) becomes

$$\int_0^{\mathcal{T}(s_\varepsilon)} (F(t) - \kappa - \varepsilon \dot{s}_\varepsilon(t))\, \dot{s}_\varepsilon(t)\, \mathrm{d}t \geq 0\,.$$

Finally, by Proposition 3.2.10, we deduce that $F(t) \leq \mathfrak{G}^+(t, s_\varepsilon(t))\, \mathbf{1}_{[0,\mathcal{T}(s_\varepsilon))}(t)$, hence, thanks to the nonnegativity of $\dot{s}_\varepsilon$, we obtain

$$\int_0^{\mathcal{T}(s_\varepsilon)} (\mathfrak{G}^+(t, s_\varepsilon(t)) - \kappa - \varepsilon \dot{s}_\varepsilon(t))\, \dot{s}_\varepsilon(t)\, \mathrm{d}t \geq 0\,. \tag{3.5.9}$$

With the same argument, we can prove that (3.5.9) holds on every $I \subseteq [0, \mathcal{T}(s_\varepsilon))$ measurable. This implies property (3) of Definition 3.3.3.

For the second case in (3.5.1), we can assume, up to a further subsequence, that $\mathcal{T}(s_\varepsilon) < \mathcal{T}(\bar{s}_\varepsilon^k)$ for every k. Therefore, we just have to replace $\mathcal{T}(\bar{s}_\varepsilon^k)$ with $\mathcal{T}(s_\varepsilon)$ in (3.5.2) and (3.5.6) and repeat the previous arguments. This concludes the proof of the theorem. □

3.6 The quasi-static evolution

We now pass to the limit as the parameter ε tends to zero. This allows us to prove the existence of a quasi-static evolution of the cohesive crack growth problem in the sense of Definition 3.3.5.

In order to prove the properties of Definition 3.3.5, we need the following technical lemma.

Lemma 3.6.1. *Let $z, z_k : [0,T] \to \mathbb{R}$ be non-decreasing monotone functions such that $z_k(t) \to z(t)$ for every $t \in [0,T]$. Let z be continuous at $\hat{t} \in [0,T]$. Then, for every $t_k \to \hat{t}$ in $[0,T]$ it is $z_k(t_k) \to z(\hat{t})$.*

Proof. Fix $\eta > 0$. By continuity, there exists $\delta > 0$ such that $|z(\hat{t}) - z(t)| < \eta$ for every $|t - \hat{t}| < 2\delta$, $t \in [0,T]$.

Since $t_k \to \hat{t}$, there exists $\bar{k} \in \mathbb{N}$ such that $|t_k - \hat{t}| < \delta$ for every $k \geq \bar{k}$, so that

$$|z(t_k) - z(\hat{t})| < \eta$$

for every $k \geq \bar{k}$. By monotonicity, $z(\hat{t} - \delta) \leq z(t_k) \leq z(\hat{t} + \delta)$ for every $k \geq \bar{k}$.

Pointwise convergence implies that, up to a redefinition of $\bar{k}$,

$$|z_k(\hat{t} - \delta) - z(\hat{t} - \delta)| < \eta \qquad \text{and} \qquad |z_k(\hat{t} + \delta) - z(\hat{t} + \delta)| < \eta$$

for every $k \geq \bar{k}$.

By continuity of z and the choice of δ, we have $|z(\hat{t}) - z(\hat{t} \pm \delta)| < \eta$. Then, by monotonicity and the above inequalities, we get

$$z(\hat{t}) - 2\eta < z(\hat{t} - \delta) - \eta < z_k(\hat{t} - \delta) \leq z_k(t_k) \leq z_k(\hat{t} + \delta) < z(\hat{t} + \delta) + \eta < z(\hat{t}) + 2\eta$$

for $k \geq \bar{k}$. Being $\eta > 0$ arbitrary, the thesis follows. □

We are now ready to prove Theorem 3.3.6.

Proof of Theorem 3.3.6. Let $\varepsilon_k \searrow 0$ and let s_{ε_k} be a sequence of viscous evolutions for the cohesive crack growth problem. Since s_{ε_k} are monotone non-decreasing and uniformly bounded in time, by Helly's theorem there exists $s \in BV([0,T])$ monotone non-decreasing such that, up to a subsequence, $s_{\varepsilon_k} \to s$ pointwise in $[0,T]$. Let us prove that s is a quasi-static evolution of the cohesive crack growth problem with $s(0) = s_0$.

Since $s_{\varepsilon_k}(0) = s_0$, of course $s(0) = s_0$. We already know that s is monotone non-decreasing, thus it remains to prove that s satisfies the weak Griffith's principle, that is, properties (1), (2), and (3) of Definition 3.3.5.

Let us prove condition (1). We argue as in the proof of Theorem 3.3.4. By Remark 2.2.30, we distinguish between the two possibilities

$$\mathcal{T}(s) = \lim_k \mathcal{T}(s_{\varepsilon_k}) \qquad \text{or} \qquad \mathcal{T}(s) < \limsup_k \mathcal{T}(s_{\varepsilon_k}) . \tag{3.6.1}$$

In the first case, by property (2) of Definition 3.3.3 we have, for every $\psi \in L^2([0,T])$ with $\psi \geq 0$,

$$\int_0^{\mathcal{T}(s_{\varepsilon_k})} (\kappa + \varepsilon_k \dot{s}_{\varepsilon_k}(t) - \mathfrak{G}^-(t, s_{\varepsilon_k}(t)))\, \psi(t)\, \mathrm{d}t \geq 0 . \tag{3.6.2}$$

Thanks to (d) of Proposition 3.5.1, we deduce that $\varepsilon_k \dot{s}_{\varepsilon_k} \to 0$ in $L^2([0,T])$ as $k \to +\infty$. Therefore, passing to the lim sup as $k \to +\infty$ in (3.6.2), we get

$$\begin{aligned} 0 &\leq \limsup_k \int_0^{\mathcal{T}(s_{\varepsilon_k})} (\kappa + \varepsilon_k \dot{s}_{\varepsilon_k}(t) - \mathfrak{G}^-(t, s_{\varepsilon_k}(t)))\, \psi(t)\, \mathrm{d}t \\ &= \int_0^{\mathcal{T}(s)} \kappa \psi(t)\, \mathrm{d}t - \liminf_k \int_0^T \mathfrak{G}^-(t, s_{\varepsilon_k}(t))\, \psi(t)\, \mathbf{1}_{[0,\mathcal{T}(s_{\varepsilon_k}))}(t)\, \mathrm{d}t . \end{aligned} \tag{3.6.3}$$

Applying Fatou's lemma to (3.6.3), taking into account the lower semicontinuity of $\mathfrak{G}^-$ and the convergence $\mathcal{T}(s_{\varepsilon_k}) \to \mathcal{T}(s)$, we obtain

$$\int_0^{\mathcal{T}(s)} (\kappa - \mathfrak{G}^-(t, s(t)))\, \psi(t)\, \mathrm{d}t \geq 0$$

for every $\psi \in L^2([0,T])$ with $\psi \geq 0$, hence

$$\mathfrak{G}^-(t, s(t)) \leq \kappa \qquad \text{for a.e. } t \in [0, \mathcal{T}(s)) . \tag{3.6.4}$$

In particular, (3.6.4) is true for every $t \in [0, \mathcal{T}(s)) \backslash J(s)$.

For the second case of (3.6.1), we may assume, up to a subsequence, that $\mathcal{T}(s) < \mathcal{T}(s_{\varepsilon_k})$ for every k. Then, we have to replace $\mathcal{T}(s_{\varepsilon_k})$ with $\mathcal{T}(s)$ in (3.6.2) and repeat the previous argument. Thus, property (1) of Definition 3.3.5 holds.

We now prove property (2). Let $t \in [0, \mathcal{T}(s)) \cap J(s)$ be a jump point of s. Since $s_{\varepsilon_k} \to s$ pointwise, we may suppose that $t < \mathcal{T}(s_{\varepsilon_k})$. By the monotonicity of s, $s(t^-) < s(t^+)$. For every $s(t^-) \leq a < b \leq s(t^+)$, there exist two sequences $t_k^a, t_k^b \to t$ such that $s_{\varepsilon_k}(t_k^a) = a$ and $s_{\varepsilon_k}(t_k^b) = b$ for every $k \in \mathbb{N}$. For every $\psi \in L^2([s_0, L])$ with $\psi \geq 0$, we have, by (3) of Definition 3.3.3,

$$\int_{t_k^a}^{t_k^b} (\mathfrak{G}^+(\tau, s_{\varepsilon_k}(\tau)) - \kappa - \varepsilon_k \dot{s}_{\varepsilon_k}(\tau))\, \psi(s_{\varepsilon_k}(\tau))\, \dot{s}_{\varepsilon_k}(\tau)\, \mathrm{d}\tau \geq 0 . \tag{3.6.5}$$

Since $\dot{s}_{\varepsilon_k} \geq 0$ a.e. in $[0,T]$, from (3.6.5) we deduce that

$$\int_{t_k^a}^{t_k^b} (\mathfrak{G}^+(\tau, s_{\varepsilon_k}(\tau)) - \kappa)\, \psi(s_{\varepsilon_k}(\tau))\, \dot{s}_{\varepsilon_k}(\tau)\, \mathrm{d}\tau \geq 0 . \tag{3.6.6}$$

We perform a change of variable setting $\sigma := s_{\varepsilon_k}(\tau)$ and

$$\hat{t}_k(\sigma) := \min\{\tau \in [t_k^a, t_k^b] : s_{\varepsilon_k}(\tau) = \sigma\},$$

so that (3.6.6) becomes

$$\int_a^b (\mathfrak{G}^+(\hat{t}_k(\sigma), \sigma) - \kappa)\, \psi(\sigma)\, \mathrm{d}\sigma \geq 0\,. \tag{3.6.7}$$

Passing to the lim sup in (3.6.7) as $k \to +\infty$, applying Fatou's lemma and recalling Proposition 3.2.10, we get

$$\int_a^b (\mathfrak{G}^+(t, \sigma) - \kappa)\, \psi(\sigma)\, \mathrm{d}\sigma \geq 0\,. \tag{3.6.8}$$

Since (3.6.8) holds for every $\psi \in L^2([s_0, L])$, $\psi \geq 0$, and every $a < b$ in $[s(t^-), s(t^+)]$, then

$$\mathfrak{G}^+(t, \sigma) \geq \kappa \qquad \text{for every } \sigma \in [s(t^-), s(t^+)]\,.$$

It remains to prove property (3) of Definition 3.3.5. Let $t \in [0, \mathcal{T}(s))$ be such that $\mathfrak{G}^+(t, s(t)) < 1$. By the previous step, $t \notin J(s)$. Let us prove that s is constant in a neighborhood of t. To this end, we first prove that there exists $\delta > 0$ such that, for k large enough,

$$\mathfrak{G}^+(\tau, s_{\varepsilon_k}(\tau)) < \kappa \qquad \text{for every } \tau \in (t - \delta, t + \delta)\,. \tag{3.6.9}$$

Assume by contradiction that this is not the case. From the pointwise convergence $s_{\varepsilon_k} \to s$, we deduce that, for k large enough, $t \in [0, \mathcal{T}(s_{\varepsilon_k}))$. Therefore, we may assume that there exist a subsequence $\varepsilon_{k_h} \searrow 0$ and a sequence $\delta_h \searrow 0$ such that (3.6.9) is not satisfied in the interval $(t - \delta_h, t + \delta_h)$, i.e., we can find $t_h \in (t - \delta_h, t + \delta_h)$ such that, for every h,

$$\mathfrak{G}^+(t_h, s_{\varepsilon_{k_h}}(t_h)) \geq \kappa\,. \tag{3.6.10}$$

Since $t_h \to t$ and $t \notin J(s)$, by Lemma 3.6.1 we have $s_{\varepsilon_{k_h}}(t_h) \to s(t)$ as $h \to +\infty$. By the upper semicontinuity of $\mathfrak{G}^+$ we get, passing to the lim sup in (3.6.10) as $h \to +\infty$, $\mathfrak{G}^+(t, s(t)) \geq \kappa$, which is a contradiction.

Combining (3.6.9) and properties (1) and (3) of Definition 3.3.3, we deduce that, for k large enough, $\dot{s}_{\varepsilon_k}(\tau) = 0$ for every $\tau \in (t - \delta, t + \delta)$, thus s_{ε_k} is constant in this interval. Since $s_{\varepsilon_k} \to s$ pointwise in $[0, T]$ as $k \to +\infty$, we get that s is constant in the same interval. Therefore, s is differentiable in t and $\dot{s}(t) = 0$. This concludes the proof of the theorem. □

We conclude this section with a remark on the energy balance.

Remark 3.6.2. At this stage, we do not have any energy balance. This is due to the fact that we can not ensure that along a quasi-static evolution $s \in BV([0, T])$ the generalized energy release rates $\mathfrak{G}^+$ and $\mathfrak{G}^-$ coincide.

We give the hypotheses on the energy functional (3.3.3) which guarantee, applying the abstract results in [48], the existence of a *special* quasi-static evolution satisfying an energy balance and a more restrictive Griffith's criterion. Let $\mathbb{C}$ be $C^{1,1}$, Λ be a simple $C^{3,1}$ curve, and let $\varphi \in C^{1,1}([0,T]\times\mathbb{R}^2;\mathbb{R})$ be such that (3.1.2) and (3.3.2) hold with $p=2$. Moreover, let $f \in C^{1,1}([0,T]\times\overline{\Omega};\mathbb{R}^2)$ and $w \in C^{1,1}([0,T];H^1(\Omega;\mathbb{R}^2))$. Then, with the arguments used in [48, Sections 3.1, 3.2], it is possible to show that for every $t\in(0,T)$ and every $s\in(0,L)$ there exists the left derivative $\partial_t^-\mathcal{E}_m$ of the reduced energy with respect to time. In particular,

$$\partial_t^-\mathcal{E}_m(t,s) = \min\{H(t,s,u) : u \in A(t,s) \text{ is a minimizer of } \mathcal{E}(t,s,w(t))\},$$

where we have set

$$H(t,s,u) := \int_\Omega \mathbb{C}\mathrm{E}u\cdot\mathrm{E}\dot{w}(t)\,\mathrm{d}x - \int_\Omega \dot{f}(t)\cdot u\,\mathrm{d}x - \int_\Omega f(t)\cdot\dot{w}(t)\,\mathrm{d}x + \int_{\Gamma_s} D_t\varphi(t,[u])\,\mathrm{d}\mathcal{H}^1 .$$

Applying the results in [48, Section 5.2], we can also prove that for every $s_0\in(0,L)$ there exists a quasi-static evolution $s\in BV([0,T])$ for the cohesive crack growth problem with $s(0)=s_0$, which satisfies a refined Griffith's criterion: condition (1) in Definition 3.3.5 is replaced by

(1') for every $t\in[0,\mathcal{T}(s))\setminus J(s)$:

$$\mathfrak{G}^+(t,s(t)) \le \kappa .$$

Moreover, we have the following energy balance:
for every $t\in(0,\mathcal{T}(s))$

$$\begin{aligned}
&\mathcal{E}_m(t,s(t)) + \kappa s(t^-) - \kappa s(0^+) + \int_{s_0}^{s(0^+)} \mathfrak{G}^+(0,\sigma)\,\mathrm{d}\sigma + \int_{s(t^-)}^{s(t)} \mathfrak{G}^+(t,\sigma)\,\mathrm{d}\sigma \\
&\qquad + \sum_{\tau\in(0,t)\cap J(s)} \left(\kappa s(\tau^-) - \kappa s(\tau^+) + \int_{s(\tau^-)}^{s(\tau^+)} \mathfrak{G}^+(\tau,\sigma)\,\mathrm{d}\sigma\right) \\
&\qquad = \mathcal{E}_m(0,s_0) + \int_0^t \partial_t^-\mathcal{E}_m(\tau,s(\tau))\,\mathrm{d}\tau .
\end{aligned}$$

In [48], such an evolution is called *special local energetic solution.*

3.7 The case of many curves

In this section we address the study of the evolution of multiple non-interacting cracks.

We assume that the fractures grow along a prescribed number of pairwise disjoint simple $C^{2,1}$-curves $\Lambda_1,\dots,\Lambda_M$ with $\mathcal{H}^1(\Lambda_l) =: L_l$. The assumptions on every Λ_l are the same of Section 3.1. For $l=1,\dots,M$, we denote by $\lambda_l\colon[0,L_l]\to\mathbb{R}^2$ the arc-length parametrization of the l-th curve Λ_l and by ν_l, τ_l the unit normal and unit tangent vectors to Λ_l, respectively.

Let us set $\Xi := [0, L_1] \times \ldots \times [0, L_M] \subseteq \mathbb{R}^M$. For every $s = (s_1, \ldots, s_M) \in \Xi$, we set

$$\Gamma_s := \Gamma^1_{s_1} \cup \ldots \cup \Gamma^M_{s_M} \qquad \text{and} \qquad \Omega_s := \Omega \setminus \Gamma_s \,,$$

where $\Gamma^l_{s_l} \subseteq \Lambda_l$ is as in (3.1.1). Then, the set of admissible fractures is given by

$$\{\Gamma_s : \, s \in \Xi\} \,. \tag{3.7.1}$$

In this setting, we generalize the activation threshold considered in the energy (3.1.7) with the norm defined by

$$|s|_1 := \sum_{l=1}^{M} |s_l| \qquad \text{for every } s \in \mathbb{R}^M \,.$$

Therefore, for every $t \in [0, T]$, $s \in \Xi$, and $u \in H^1(\Omega_s; \mathbb{R}^2)$, the total energy of the system is

$$\mathcal{E}(t, s, u) := \frac{1}{2} \int_{\Omega_s} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}u \, \mathrm{d}x - \int_{\Omega_s} g(t, x, u) \, \mathrm{d}x + \int_{\Gamma_s} \varphi(t, [u]) \, \mathrm{d}\mathcal{H}^1 + |s|_1 \,,$$

where $\mathbb{C}$, φ, and g have the usual hypotheses stated in Section 3.1 and 3.3, and, for simplicity, $\kappa = 1$. Given the Dirichlet boundary datum $w \in H^1(\Omega; \mathbb{R}^2)$, we define $\mathcal{A}(s, w)$ and the reduced energy $\mathcal{E}_m(t, s, w)$ as in (3.1.9) and in (3.1.12), respectively.

We now show how to extend the results of Section 3.2 to this setting. In particular, we are interested in the analogous of the energy release rates. For $l = 1, \ldots, M$, let us define $\Xi_l := [0, L_1] \times \ldots \times [0, L_{l-1}] \times (0, L_l) \times [0, L_{l+1}] \times \ldots \times [0, L_M]$. Let $l = 1, \ldots, M$ and $s \in \Xi_l$ be fixed. By hypothesis, there exists $\eta > 0$ such that the curve Λ_l is the graph of a $C^{2,1}$-function ψ^l_s on $(\lambda^1_l(s_l) - \eta, \lambda^1_l(s_l) + \eta)$, where λ^1_l is the first component of $\lambda_l = (\lambda^1_l, \lambda^2_l)$. We may also assume that $d(\lambda_l(s_l), \Lambda_h) \geq 2\eta$ for every $h \neq l$. Given $\delta \in \mathbb{R}$ such that $s_l + \delta \in [0, L_l]$ and a cut-off function $\vartheta \in C^\infty_c(\mathrm{B}_{\eta/2}(0))$ with $\vartheta = 1$ in $\overline{\mathrm{B}}_{\eta/3}(0)$, we define, as in (3.2.1), $F^l_{s,\delta} \colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F^l_{s,\delta}(x) := x + \begin{pmatrix} (\lambda^1_l(s_l + \delta) - \lambda^1_l(s_l))\vartheta(\lambda_l(s_l) - x) \\ \psi^l_s(x_1 + (\lambda^1_l(s_l + \delta) - \lambda^1_l(s_l))\vartheta(\lambda_l(s_l) - x)) - \psi^l_s(x_1) \end{pmatrix} \tag{3.7.2}$$

if $x = (x_1, x_2) \in \mathrm{B}_{\eta/2}(\lambda_l(s_l))$, while $F^l_{s,\delta}(x) := x$ for $x \in \mathbb{R}^2 \setminus \mathrm{B}_{\eta/2}(\lambda_l(s_l))$.

Lemma 2.2.24 holds also in this context for every $l = 1, \ldots, M$ setting

$$\rho^l_s(x) := \partial_\delta(F^l_{s,\delta}(x))|_{\delta=0} = (\lambda^1_l)'(s)\vartheta(\lambda_l(s) - x) \begin{pmatrix} 1 \\ (\psi^l_s)'(x_1) \end{pmatrix} .$$

Similar to (3.2.2), for $l = 1, \ldots, M$, $t \in [0, T]$, $s \in \Xi_l$, $w \in H^1(\Omega; \mathbb{R}^2)$, and

$u \in \mathcal{A}(s,w)$, we set

$$\begin{aligned}
G_l(t,u,\vartheta) :=& -\frac{1}{2}\int_{\Omega_s} (D\mathbb{C}\,\rho_s^l)\nabla u\cdot\nabla u\,\mathrm{d}x \\
&- \int_{\Omega_s} \mathbb{C}\nabla\big((\nabla\rho_s^l - \mathrm{div}\rho_s^l\,\mathrm{I})u\big)\cdot\nabla u\,\mathrm{d}x \\
&+ \int_{\Omega_s} \mathbb{C}(\nabla u\nabla\rho_s^l)\cdot\nabla u\,\mathrm{d}x - \frac{1}{2}\int_{\Omega_s}\mathbb{C}\nabla u\cdot\nabla u\,\mathrm{div}\rho_s^l\,\mathrm{d}x \\
&+ \int_{\Omega_s} D_\xi g(t,x,u)\cdot\big[(\nabla\rho_s^l - \mathrm{div}\rho_s^l\,\mathrm{I})u - \nabla u\,\rho_s^l\big]\,\mathrm{d}x \\
&- \int_{\Gamma_s} D_\xi\varphi(t,[u])\cdot\big((\nabla\rho_s^l - \mathrm{div}\rho_s^l\,\mathrm{I})u\big)\,\mathrm{d}\mathcal{H}^1 \\
&- \int_{\Gamma_s} \varphi(t,[u])\,\nu\otimes\tau\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cdot\nabla\rho_s^l\,\mathrm{d}\mathcal{H}^1 \\
&- \int_{\Gamma_s} \varphi(t,[u])\,\mathrm{div}\rho_s^l\,\mathrm{d}\mathcal{H}^1\,,
\end{aligned} \tag{3.7.3}$$

where ϑ is as in (3.7.2) and $D\mathbb{C}\,\rho_s^l$ is as in (3.2.3).

Moreover, we define

$$\partial_{s,l}^{+}\mathcal{E}_m(t,s,w) := \lim_{\delta\searrow 0}\frac{\mathcal{E}_m(t,s+\delta e_l,w) - \mathcal{E}_m(t,s,w)}{\delta}\,, \tag{3.7.4}$$

$$\partial_{s,l}^{-}\mathcal{E}_m(t,s,w) := \lim_{\delta\nearrow 0}\frac{\mathcal{E}_m(t,s+\delta e_l,w) - \mathcal{E}_m(t,s,w)}{\delta}\,, \tag{3.7.5}$$

where $\{e_1,\dots,e_M\}$ is the canonical basis of $\mathbb{R}^M$. With the same techniques used in Theorems 3.2.2, 3.2.3, and in Proposition 3.2.10, we can prove that the limits in (3.7.4) and (3.7.5) exist and have explicit formulas similar to (3.2.6) and (3.2.8).

Theorem 3.7.1. *For every $t\in[0,T]$, every $l=1,\dots,M$, every $s\in\Xi_l$, and every $w\in H^1(\Omega;\mathbb{R}^2)$, the limits in* (3.7.4) *and* (3.7.5) *exist and*

$$\begin{aligned}
\partial_{s,l}^{+}\mathcal{E}_m(t,s,w) &= 1 - \mathfrak{G}_l^{+}(t,s,w)\,,\\
\partial_{s,l}^{-}\mathcal{E}_m(t,s,w) &= 1 - \mathfrak{G}_l^{-}(t,s,w)\,,
\end{aligned} \tag{3.7.6}$$

where we have set

$$\begin{aligned}
&\mathfrak{G}_l^{+}(t,s,w) := \max\{G_l(t,u_s,\vartheta) :\ u_s\in\mathcal{A}(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\},\\
&\mathfrak{G}_l^{-}(t,s,w) := \min\{G_l(t,u_s,\vartheta) :\ u_s\in\mathcal{A}(s,w) \text{ is a minimizer of } \mathcal{E}(t,s,\cdot)\}
\end{aligned} \tag{3.7.7}$$

for a given cut-off function ϑ as in (3.7.2)*. In particular, $\mathfrak{G}_l^{+}$ and $\mathfrak{G}_l^{-}$ do not depend on the choice of ϑ.*

Moreover, $\mathfrak{G}_l^{+},\mathfrak{G}_l^{-}\colon[0,T]\times\Xi_l\times H^1(\Omega;\mathbb{R}^2)\to[0,+\infty)$ are upper and lower semicontinuous on $[0,T]\times\mathrm{int}(\Xi)\times H^1(\Omega;\mathbb{R}^2)$, respectively.

Remark 3.7.2. The functions $\mathfrak{G}_l^+$ and $\mathfrak{G}_l^-$ introduced in Theorem 3.7.1 can be interpreted as partial energy release rates, in the sense that they characterize the partial derivatives with respect to the variable $s_l \in [0, L_l]$ of the reduced energy $\mathcal{E}_m$.

Also in this setting, the notion of quasi-static evolution will be related to the properties of $\mathfrak{G}_l^\pm$, see Theorems 3.7.6 and 3.8.1.

We now deal with the construction of a quasi-static evolution. As in Section 3.3, we replace g with the power spent by the body forces $f \in AC([0,T]; L^2(\Omega;\mathbb{R}^2))$. Given a boundary datum $w \in AC([0,T]; H^1(\Omega;\mathbb{R}^2))$, we redefine the reduced energy $\mathcal{E}_m \colon [0,T] \times \Xi \to \mathbb{R}$ and the energy release rates $\mathfrak{G}_l^\pm \colon [0,T] \times \Xi_l \to [0,+\infty)$ by

$$\mathcal{E}_m(t,s) := \mathcal{E}_m(t,s,w(t)) \qquad \text{and} \qquad \mathfrak{G}_l^\pm(t,s) := \mathfrak{G}_l^\pm(t,s,w(t)) \,.$$

We notice again that $\mathcal{E}_m$ is continuous on $[0,T] \times \operatorname{int}(\Xi)$, while, for every $l = 1, \dots, M$, $\mathfrak{G}_l^+$ and $\mathfrak{G}_l^-$ are upper and lower semicontinuous, respectively.

For every $k \in \mathbb{N}$, we consider a time discretization $\{t_i^k\}_{i=0}^k$ of the form $t_i^k := i\tau_k$, where $\tau_k := T/k$. Fixed $\varepsilon > 0$, we define recursively $s_\varepsilon^{k,i} \in \Xi$: $s_\varepsilon^{k,0} := s_0 \in \operatorname{int}(\Xi)$, the initial condition, and, for $i \geq 1$, we set $s_\varepsilon^{k,i}$ to be a solution of the incremental minimum problem

$$\min \left\{ \mathcal{E}_m(t_i^k, s) + \frac{\varepsilon}{2} \frac{|s - s_\varepsilon^{k,i-1}|_2^2}{\tau_k} \, : \, s \in \Xi,\, s_l \geq (s_\varepsilon^{k,i-1})_l \text{ for } l = 1, \dots, M \right\}, \tag{3.7.8}$$

where

$$|s|_2 := \left(\sum_{l=1}^M s_l^2 \right)^{1/2} \qquad \text{for every } s \in \mathbb{R}^M \,.$$

The proof of existence of solution to (3.7.8) is similar to the proof of Proposition 3.4.1.

We introduce the interpolation functions: for every $t \in (t_{i-1}^k, t_i^k]$ we set

$$\begin{gathered} \bar{t}_k(t) := t_i^k \,, \\ \bar{s}_{\varepsilon,l}^k(t) := (s_\varepsilon^{k,i})_l \,, \quad \bar{s}_\varepsilon^k(t) := (\bar{s}_{\varepsilon,1}^k(t), \dots, \bar{s}_{\varepsilon,M}^k(t)) \,, \\ s_{\varepsilon,l}^k(t) := (s_\varepsilon^{k,i-1})_l + \frac{(s_\varepsilon^{k,i})_l - (s_\varepsilon^{k,i-1})_l}{\tau_k} (t - t_{i-1}^k) \,, \quad s_\varepsilon^k(t) := (s_{\varepsilon,1}^k(t), \dots, s_{\varepsilon,M}^k(t)) \,. \end{gathered}$$

In particular, as in Proposition 3.4.2, we get

$$\varepsilon \int_0^T |\dot{s}_\varepsilon^k(t)|_2^2 \, \mathrm{d}t \leq C \tag{3.7.9}$$

uniformly in ε and k, where $\dot{s}_\varepsilon^k(t) := (\dot{s}_{\varepsilon,1}^k(t), \dots, \dot{s}_{\varepsilon,M}^k(t))$.

As in Proposition 3.4.3, we have a discrete Griffith's criterion.

Proposition 3.7.3. *For every $\varepsilon > 0$, every $k \in \mathbb{N}$, every $l = 1, \dots, M$, and every $t \in [0, \mathcal{T}((\bar{s}_\varepsilon^k)_l))$ we have*

(a) $\dot{s}_{\varepsilon,l}^k(t) \geq 0$*;*

(*b*) $\mathfrak{G}_l^+(\bar{t}_k(t), \bar{s}_\varepsilon^k(t)) - 1 - \varepsilon\, \dot{s}_{\varepsilon,l}^k(t) \leq 0$;

(*c*) $(\mathfrak{G}_l^+(\bar{t}_k(t), \bar{s}_\varepsilon^k(t)) - 1 - \varepsilon\, \dot{s}_{\varepsilon,l}^k(t))\, \dot{s}_{\varepsilon,l}^k(t) = 0$.

Proof. It is sufficient to repeat the argument of Proposition 3.4.3 componentwise. □

We define the *failure time* and the *jump set* for a vector valued function whose components are monotone non-decreasing.

Definition 3.7.4. Let $a, b_1, \ldots, b_M > 0$ and let $s_l : [0, a] \to [0, b_m]$ be a monotone non-decreasing function for $l = 1, \ldots, M$. Let $s = (s_1, \ldots, s_M)$. We define

- the *failure time* of s as
$$\mathcal{T}(s) := \min_{l=1,\ldots,M} \mathcal{T}(s_l)\,,$$
where $\mathcal{T}(s_l)$ is as in Definition 2.2.29;
- the *jump set* of s as
$$J(s) := \bigcup_{l=1}^{M} J(s_l)\,,$$

We can now pass to the limit as $k \to +\infty$. As in Proposition 3.5.1, for fixed $\varepsilon > 0$, we find $s_\varepsilon \in H^1([0,T])$ such that, up to a subsequence, s_ε^k converges to s_ε weakly in $H^1([0,T])$ and uniformly in $[0,T]$. Moreover, $\bar{s}_\varepsilon^k \to s_\varepsilon$ uniformly. By (3.7.9) and the lower semicontinuity of the L^2-norm, there exists $C > 0$ such that for every $\varepsilon > 0$

$$\varepsilon \int_0^T |\dot{s}_\varepsilon(t)|_2^2 \,\mathrm{d}t \leq C\,, \tag{3.7.10}$$

where $\dot{s}_\varepsilon(t) := (\dot{s}_\varepsilon^1(t), \ldots, \dot{s}_\varepsilon^M(t))$.

The map $t \mapsto s_\varepsilon(t)$ is a viscous evolution with $s_\varepsilon(0) = s_0$, see Definition 3.3.3. Indeed, taking into account Proposition 3.2.10, the following result holds.

Proposition 3.7.5. *For every $\varepsilon > 0$, every $l = 1, \ldots, M$, and a.e. $t \in [0, \mathcal{T}(s_\varepsilon))$:*

(*a*) $\dot{s}_\varepsilon^l(t) \geq 0$;

(*b*) $\mathfrak{G}_l^-(t, s_\varepsilon(t)) - 1 - \varepsilon \dot{s}_\varepsilon^l(t) \leq 0$;

(*c*) $(\mathfrak{G}_l^+(t, s_\varepsilon(t)) - 1 - \varepsilon \dot{s}_\varepsilon^l(t))\, \dot{s}_\varepsilon^l(t) \geq 0$.

Proof. Argue componentwise as in Theorem 3.3.4. □

As in the proof of Theorem 3.3.6, there exist a subsequence $\varepsilon_k \to 0$ and a function $s \in BV([0,T], \Xi)$ such that $s_{\varepsilon_k} \to s$ pointwise. Moreover, every component s_m is monotone non-decreasing in $[0,T]$.

Repeating componentwise the argument of Theorem 3.3.6, we can prove a Griffith's criterion in the continuity points of s.

Theorem 3.7.6. *The following facts hold:*

(a) s_l *is monotone non-decreasing for every* $l = 1, \ldots, M$*;*

(b) *for every* $l = 1, \ldots, M$ *and every* $t \in [0, \mathcal{T}(s)) \setminus J(s)$, $\mathfrak{G}_l^-(t, s(t)) \leq 1$*;*

(c) *if* $t \in [0, \mathcal{T}(s)) \setminus J_s$ *and* $\mathfrak{G}_l^+(t, s(t)) < 1$ *for some* $m = 1, \ldots, M$*, then* s_l *is differentiable in* t *and* $\dot{s}_l(t) = 0$.

However, in this setting it is difficult to state the properties of $\mathfrak{G}_l^\pm$ in the jump points: in particular, we do not have the equivalent to condition (2) of Definition 3.3.5. Therefore, following the steps of [47, 52, 58], we define a reparametrization that shall give some information on the behavior of the cracks at the jump points.

3.8 Parametrized solutions

We perform a change of variable which transforms the lengths in absolutely continuous functions. Roughly speaking, this is done by a parametrization of time on the jump points of the viscous solution s_ε.

For $\varepsilon > 0$ and $t \in [0, T]$, we set

$$\sigma_\varepsilon(t) := t + |s_\varepsilon(t)|_1 - |s_0|_1 = t + \sum_{m=1}^{M} (s_\varepsilon^l(t) - s_0^l). \tag{3.8.1}$$

Thanks to the properties of s_ε, see Proposition 3.7.5, σ_ε is strictly increasing, continuous, and $\dot{\sigma}_\varepsilon(t) \geq 1$ for every $\varepsilon > 0$ and a.e. $t \in [0, T]$, hence we can find its inverse $\sigma \mapsto \tilde{t}_\varepsilon(\sigma)$ for $0 \leq \sigma \leq \mathcal{S}_\varepsilon := \sigma_\varepsilon(T)$. We deduce that $\tilde{t}_\varepsilon$ is strictly increasing, continuous, and $0 < \tilde{t}'_\varepsilon(\sigma) \leq 1$ for every $\varepsilon > 0$ and a.e. $\sigma \in [0, \mathcal{S}_\varepsilon]$ (here, the symbol $'$ denotes the derivative with respect to σ).

For $l = 1, \ldots, M$ and $\sigma \in [0, \mathcal{S}_\varepsilon]$, we set

$$\tilde{s}_\varepsilon^l(\sigma) := s_\varepsilon^l(\tilde{t}_\varepsilon(\sigma)), \quad \tilde{s}_\varepsilon(\sigma) := (\tilde{s}_\varepsilon^1(\sigma), \ldots, \tilde{s}_\varepsilon^M(\sigma)),$$
$$\tilde{s}'_\varepsilon(\sigma) := ((\tilde{s}_\varepsilon^1)'(\sigma), \ldots, (\tilde{s}_\varepsilon^M)'(\sigma)).$$

By (3.8.1), we have $\sigma = \tilde{t}_\varepsilon(\sigma) + |\tilde{s}_\varepsilon(\sigma)|_1 - |s_0|_1$. Deriving this relation, we obtain

$$\tilde{t}'_\varepsilon(\sigma) + |\tilde{s}'_\varepsilon(\sigma)|_1 = 1 \tag{3.8.2}$$

for every $\varepsilon > 0$ and a.e. $\sigma \in [0, \mathcal{S}_\varepsilon]$. By (3.8.2) and the monotonicity of $\tilde{s}_\varepsilon^l$, we get $0 \leq (\tilde{s}_\varepsilon^l)'(\sigma) \leq 1$ for every $\varepsilon > 0$, every $l = 1, \ldots, M$, and a.e. $\sigma \in [0, \mathcal{S}_\varepsilon]$. Moreover, in view of (3.8.2), $\tilde{t}_\varepsilon$ and $\tilde{s}_\varepsilon$ are Lipschitz functions.

We define $\widetilde{\mathfrak{G}}_{l,\varepsilon}^\pm(\sigma) := \mathfrak{G}_l^\pm(\tilde{t}_\varepsilon(\sigma), \tilde{s}_\varepsilon(\sigma))$ for $\sigma \in [0, \mathcal{T}(\tilde{s}_\varepsilon))$ and $\bar{\mathcal{S}} := \sup_{\varepsilon > 0} \mathcal{S}_\varepsilon$, which is bounded by a constant depending on T and on the lengths L_l. Since in the limit $\varepsilon \searrow 0$ it will be useful to deal with functions defined on the same interval, we extend the functions $\tilde{t}_\varepsilon$, $\tilde{s}_\varepsilon$, $\tilde{t}'_\varepsilon$, and $\tilde{s}'_\varepsilon$ on $(\mathcal{S}_\varepsilon, \bar{\mathcal{S}}]$ by $\tilde{t}_\varepsilon(\sigma) := \tilde{t}_\varepsilon(\mathcal{S}_\varepsilon)$, $\tilde{s}_\varepsilon(\sigma) := \tilde{s}_\varepsilon(\mathcal{S}_\varepsilon)$, $\tilde{t}'_\varepsilon(\sigma) := 0$, and $\tilde{s}'_\varepsilon(\sigma) := 0$. In the sequel, we will also need $\widetilde{\mathcal{T}}(\tilde{s}_\varepsilon) := \min\{S_\varepsilon, \mathcal{T}(\tilde{s}_\varepsilon)\}$.

Recalling that $\tilde{t}_\varepsilon'(\sigma) > 0$ on $[0, \mathcal{S}_\varepsilon]$, the Griffith's criterion stated in Proposition 3.7.5 reads in the new variables as

$$(\tilde{s}_\varepsilon^l)'(\sigma) \geq 0\,, \tag{3.8.3}$$

$$\widetilde{\mathfrak{G}}_{l,\varepsilon}^-(\sigma)\,\tilde{t}_\varepsilon'(\sigma) - \tilde{t}_\varepsilon'(\sigma) - \varepsilon(\tilde{s}_\varepsilon^l)'(\sigma) \leq 0\,, \tag{3.8.4}$$

$$\big(\widetilde{\mathfrak{G}}_{l,\varepsilon}^+(\sigma)\,\tilde{t}_\varepsilon'(\sigma) - \tilde{t}_\varepsilon'(\sigma) - \varepsilon(\tilde{s}_\varepsilon^l)'(\sigma)\big)\,(\tilde{s}_\varepsilon^l)'(\sigma) \geq 0 \tag{3.8.5}$$

for every l, every ε, and a.e. $\sigma \in [0, \widetilde{\mathcal{T}}(\tilde{s}_\varepsilon))$.

We now pass to the limit along a subsquence $\varepsilon_k \searrow 0$. The sequences $\tilde{t}_{\varepsilon_k}, \tilde{s}_{\varepsilon_k}$ are bounded in $W^{1,\infty}([0,\bar{\mathcal{S}}])$ and in $W^{1,\infty}([0,\bar{\mathcal{S}}];\mathbb{R}^M)$, respectively. Therefore, up to a further subsequence, we have that $\tilde{t}_{\varepsilon_k}$ (resp. $\tilde{s}_{\varepsilon_k}$) converge weakly* in $W^{1,\infty}([0,\bar{\mathcal{S}}])$ (resp. in $W^{1,\infty}([0,\bar{\mathcal{S}}];\mathbb{R}^M)$) to some functions $\tilde{t}$ (resp. $\tilde{s}$). We can also assume that $\mathcal{S}_{\varepsilon_k} \to \mathcal{S}$ and $\tilde{t} \in W^{1,\infty}([0,\mathcal{S}])$ (resp. $,\tilde{s} \in W^{1,\infty}([0,\mathcal{S}];\mathbb{R}^M)$). In particular, writing (3.8.2) in an integral form and passing to the limit, we deduce that for a.e. $\sigma \in [0,\mathcal{S}]$

$$\tilde{t}'(\sigma) + |\tilde{s}'(\sigma)|_1 = 1\,. \tag{3.8.6}$$

We set $\widetilde{\mathcal{T}}(\tilde{s}) := \min\{\mathcal{S}, \mathcal{T}(\tilde{s})\}$ and, for $l = 1\ldots, M$ and $\sigma \in [0, \widetilde{\mathcal{T}}(\tilde{s}))$,

$$\widetilde{\mathfrak{G}}_l^\pm(\sigma) := \mathfrak{G}_l^\pm(\tilde{t}(\sigma), \tilde{s}(\sigma))\,.$$

As in Remark 2.2.30, we have

$$\widetilde{\mathcal{T}}(\tilde{s}) \leq \liminf_k\, \widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k})\,. \tag{3.8.7}$$

Finally, we observe that (3.7.10) gives

$$\begin{aligned}\varepsilon_k \int_0^{\mathcal{S}_{\varepsilon_k}} |\tilde{s}_{\varepsilon_k}'(\sigma)|_2^2\,\mathrm{d}\sigma &= \varepsilon_k \int_0^{\mathcal{S}_{\varepsilon_k}} |\dot{s}_{\varepsilon_k}(\tilde{t}_{\varepsilon_k}(\sigma))|_2^2 (\tilde{t}_{\varepsilon_k}')^2(\sigma)\,\mathrm{d}\sigma \\ &\leq \varepsilon_k \int_0^{\mathcal{S}_{\varepsilon_k}} |\dot{s}_{\varepsilon_k}(\tilde{t}_{\varepsilon_k}(\sigma))|_2^2\,\tilde{t}_{\varepsilon_k}'(\sigma)\,\mathrm{d}\sigma = \varepsilon_k \int_0^T |\dot{s}_{\varepsilon_k}(t)|_2^2\,\mathrm{d}t \leq C\end{aligned} \tag{3.8.8}$$

uniformly in k. Therefore, $\varepsilon_k \tilde{s}_{\varepsilon_k}' \mathbf{1}_{[0,\mathcal{S}_{\varepsilon_k}]} \to 0$ in $L^2([0,\bar{\mathcal{S}}];\mathbb{R}^M)$.

Passing to the limit as $k \to +\infty$, we are now able to show that the parametrized solution $\tilde{s}$ satisfies a Griffith's criterion involving also the jump points of $\tilde{s}$. This is the aim of the following theorem.

Theorem 3.8.1. *The Lipschitz continuous functions $\tilde{t}$ and $\tilde{s}$ satisfy for a.e. $\sigma \in [0, \widetilde{\mathcal{T}}(\tilde{s}))$:*

(a) $\tilde{t}'(\sigma) \geq 0$ *and* $\tilde{s}_l'(\sigma) \geq 0$ *for* $l = 1, \ldots, M$*;*

(b) *if* $\tilde{t}'(\sigma) > 0$*, then* $\widetilde{\mathfrak{G}}_l^-(\sigma) \leq 1$ *for* $l = 1, \ldots, M$*;*

(c) *if* $\tilde{t}'(\sigma) > 0$ *and* $\tilde{s}_l'(\sigma) > 0$ *for some* $l \in \{1, \ldots, M\}$*, then* $\widetilde{\mathfrak{G}}_l^+(\sigma) \geq 1$*;*

(d) *if* $\tilde{t}'(\sigma) = 0$*, then there exists* $l \in \{1, \dots, M\}$ *such that* $\tilde{s}'_l(\sigma) > 0$*. Moreover,* $\widetilde{\mathfrak{G}}_l^+(\sigma) \geq 1$ *for such* l*.*

Proof. By the monotonicity of $\tilde{t}$ and $\tilde{s}$, we have $\tilde{t}'(\sigma) \geq 0$ and $\tilde{s}'_m(\sigma) \geq 0$ for every m and a.e. $\sigma \in [0, \mathcal{S}]$. Moreover, by (3.8.6) they can not be simultaneously zero.

As in the proofs of Theorems 3.3.4 and 3.3.6, we have to distinguish between two possibilities:

$$\widetilde{\mathcal{T}}(\tilde{s}) = \lim_k \widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k}) \qquad \text{or} \qquad \widetilde{\mathcal{T}}(\tilde{s}) < \limsup_k \widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k}) . \tag{3.8.9}$$

Let us consider the first case. Let us fix $l = 1, \dots, M$ and $\psi \in L^2([0, \bar{\mathcal{S}}])$ with $\psi \geq 0$. Thanks to (3.8.4), for every k we have

$$\int_0^{\widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k})} (\tilde{t}'_{\varepsilon_k}(\sigma) - \widetilde{\mathfrak{G}}^-_{l,\varepsilon_k}(\sigma)\tilde{t}'_{\varepsilon_k}(\sigma) + \varepsilon_k(\tilde{s}^l_{\varepsilon_k})'(\sigma))\, \psi(\sigma)\, \mathrm{d}\sigma \geq 0 , \tag{3.8.10}$$

where ε_k is the subsequence previously fixed.

Since $\widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k}) \to \widetilde{\mathcal{T}}(\tilde{s})$, $\tilde{t}'_{\varepsilon_k}$ converges to $\tilde{t}'$ weakly* in $L^\infty([0, \bar{\mathcal{S}}])$, and $\varepsilon_k \tilde{s}'_{\varepsilon_k} \mathbf{1}_{[0, S_{\varepsilon_k}]} \to 0$ in $L^2([0, \bar{\mathcal{S}}]; \mathbb{R}^M)$, passing to the lim sup in (3.8.10) as $k \to +\infty$ we get

$$\begin{aligned} 0 &\leq \limsup_k \int_0^{\widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k})} (\tilde{t}'_{\varepsilon_k}(\sigma) - \widetilde{\mathfrak{G}}^-_{l,\varepsilon_k}(\sigma)\tilde{t}'_{\varepsilon_k}(\sigma) + \varepsilon_k(\tilde{s}^l_{\varepsilon_k})'(\sigma))\, \psi(\sigma)\, \mathrm{d}\sigma \\ &= \int_0^{\widetilde{\mathcal{T}}(\tilde{s})} \tilde{t}'(\sigma)\, \psi(\sigma)\, \mathrm{d}\sigma - \liminf_k \int_0^{\bar{\mathcal{S}}} \widetilde{\mathfrak{G}}^-_{l,\varepsilon_k}(\sigma)\tilde{t}'_{\varepsilon_k}(\sigma)\, \mathbf{1}_{[0, \widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k}))}\, \psi(\sigma)\, \mathrm{d}\sigma . \end{aligned} \tag{3.8.11}$$

By the monotonicity of $\tilde{t}_\varepsilon$, we can continue the chain of inequalities in (3.8.11)

$$0 \leq \int_0^{\widetilde{\mathcal{T}}(\tilde{s})} \tilde{t}'(\sigma)\, \psi(\sigma)\, \mathrm{d}\sigma - \liminf_k \int_0^{\bar{\mathcal{S}}} F_k(\sigma)\, \tilde{t}'_{\varepsilon_k}(\sigma)\, \psi(\sigma)\, \mathrm{d}\sigma , \tag{3.8.12}$$

where we have set

$$F_k(\sigma) := \inf_{h \geq k} \widetilde{\mathfrak{G}}^-_{l,\varepsilon_h}(\sigma)\, \mathbf{1}_{[0, \widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_h}))}(\sigma) .$$

The sequence F_k is uniformly bounded and converges pointwise to

$$F(\sigma) := \liminf_k \widetilde{\mathfrak{G}}^-_{l,\varepsilon_k}(\sigma)\, \mathbf{1}_{[0, \widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k}))}(\sigma) = \liminf_k \widetilde{\mathfrak{G}}^-_{l,\varepsilon_k}(\sigma)\, \mathbf{1}_{[0, \widetilde{\mathcal{T}}(\tilde{s}))}(\sigma) .$$

Therefore, applying the dominated convergence theorem, we get $F_k \to F$ in L^2 and

$$\int_0^{\widetilde{\mathcal{T}}(\tilde{s})} (\tilde{t}'(\sigma) - F(\sigma)\, \tilde{t}'(\sigma))\, \psi(\sigma)\, \mathrm{d}\sigma \geq 0 . \tag{3.8.13}$$

By Proposition 3.2.10, we deduce that $F(\sigma) \geq \widetilde{\mathfrak{G}}^-_l(\sigma)$. Hence, in view of property (a), (3.8.13) becomes

$$\int_0^{\widetilde{\mathcal{T}}(\tilde{s})} (\tilde{t}'(\sigma) - \widetilde{\mathfrak{G}}^-_l(\sigma)\, \tilde{t}'(\sigma))\, \psi(\sigma)\, \mathrm{d}\sigma \geq 0 ,$$

which proves (b) by the arbitrariness of ψ.

For the second case of (3.8.9), we may assume, up to a subsequence, that $\widetilde{\mathcal{T}}(\tilde{s}) < \widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k})$, hence it is sufficient to replace $\widetilde{\mathcal{T}}(\tilde{s}_{\varepsilon_k})$ with $\widetilde{\mathcal{T}}(\tilde{s})$ in (3.8.10) and repeat the previous argument. Thus property (b) is proved.

We notice that if (a), (b) and (3.8.6) hold, then (c) and (d) are equivalent to the following property:
if $\widetilde{\mathfrak{G}}_l^+(\bar{\sigma}) < 1$ for some m and some $\bar{\sigma} \in [0, \widetilde{\mathcal{T}}(\tilde{s}))$, then $\tilde{s}_l$ is locally constant around $\bar{\sigma}$.
Let us assume that $\widetilde{\mathfrak{G}}_l^+(\bar{\sigma}) < 1$. Then, arguing as in the proof of Theorem 3.3.6, there exist $\bar{k} \in \mathbb{N}$ and $\delta > 0$ such that $\widetilde{\mathfrak{G}}_{l,\varepsilon_k}^+(\sigma) < 1$ for every $\sigma \in (\bar{\sigma} - \delta, \bar{\sigma} + \delta)$ and every $k \geq \bar{k}$. From (3.8.5) we deduce that $\tilde{s}_{\varepsilon_k}^l$ is constant in $(\bar{\sigma} - \delta, \bar{\sigma} + \delta)$. Since $\tilde{s}_{\varepsilon_k}^l$ converges to $\tilde{s}_l$ weakly* in $W^{1,\infty}([0, \bar{\mathcal{S}}])$, we get that $\tilde{s}_l$ is locally constant around $\bar{\sigma}$, and this concludes the proof of the theorem. □

Remark 3.8.2. As usual in these cases, since the reduced energy $\mathcal{E}_m$ is continuous only on $[0, T] \times \mathrm{int}(\Xi)$ and, as a consequence, $\mathfrak{G}_l^{\pm}$ are not upper and lower semicontinuous on the whole $[0, T] \times \Xi$, the evolution we have described is meaningful up to the failure time $\widetilde{\mathcal{T}}(\tilde{s})$.

Chapter 4

Lower semicontinuity result for a free discontinuity functional with a boundary term

4.1 Introduction and setting of the problem

In this last chapter we are interested in the lower semicontinuity of free discontinuity functionals of the form

$$\mathcal{F}(u) := \int_{S_u \setminus \Sigma} \psi(x, \nu_u)\, \mathrm{d}\mathcal{H}^{n-1} + \int_{\Sigma} g(x, u^+, u^-)\, \mathrm{d}\mathcal{H}^{n-1}\,, \tag{4.1.1}$$

defined on $GSBV^p(\Omega; \mathbb{R}^m)$, for $p \in (1, +\infty)$ and Ω an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary. In (4.1.1), (Σ, ν_Σ) is a prescribed orientable Lipschitz manifold of dimension $n-1$ and Lipschitz constant L (see Definition 1.1.5) with $\Sigma \subseteq \overline{\Omega}$ and

$$\mathcal{H}^{n-1}(\Sigma) < +\infty\,, \qquad \mathcal{H}^{n-1}(\overline{\Sigma} \setminus \Sigma) = 0\,, \qquad \mathcal{H}^{n-1}((\overline{\Sigma \cap \Omega}) \cap \partial\Omega) = 0\,, \tag{4.1.2}$$

while $u^\pm$ are the traces of u on Σ, defined according to the orientation of ν_Σ (see Remark 1.2.10). To give a precise definition of $\mathcal{F}$ when $\Sigma \cap \partial\Omega \neq \emptyset$, the function u is extended to 0 out of Ω, so that the traces u^+ and u^- are well defined $\mathcal{H}^{n-1}$-a.e. on Σ .

Remark 4.1.1. When $\Sigma = \partial\Omega$, the functional (4.1.1) reduces to

$$\mathcal{F}(u) := \int_{S_u} \psi(x, \nu_u)\, \mathrm{d}\mathcal{H}^{n-1} + \int_{\partial\Omega} g(x, u)\, \mathrm{d}\mathcal{H}^{n-1}\,. \tag{4.1.3}$$

Remark 4.1.2. Let us set

$$\mathcal{N}^\pm := \{x \in \Sigma \cap \partial\Omega :\, \nu_\Sigma(x) = \pm\nu_\Omega(x)\}\,. \tag{4.1.4}$$

In view of our convention on the traces $u^{\pm}$ on $\Sigma \cap \partial\Omega$, it is not restrictive to assume that

$$\begin{aligned} &\text{if } x \in \mathcal{N}^+, \text{ then } g(x,s,t) = g(x,s,0) \text{ for every } s,t \in \mathbb{R}^m, \\ &\text{if } x \in \mathcal{N}^-, \text{ then } g(x,s,t) = g(x,0,t) \text{ for every } s,t \in \mathbb{R}^m. \end{aligned} \tag{4.1.5}$$

In Theorem 4.2.1 we prove that $\mathcal{F}$ is lower semicontinuous with respect to the weak convergence in $GSBV^p(\Omega;\mathbb{R}^m)$ under the following set of assumptions:

(H1) $\psi\colon \overline{\Omega} \times \mathbb{R}^n \to [0,+\infty)$ is continuous;

(H2) there exist $0 < c_1 \le c_2$ such that

$$c_1|\nu| \le \psi(x,\nu) \le c_2|\nu|$$

for every $(x,\nu) \in \overline{\Omega} \times \mathbb{R}^n$;

(H3) $\psi(x,\cdot)$ is a norm on $\mathbb{R}^n$ for every $x \in \overline{\Omega}$;

(H4) $g\colon \Sigma \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is a Borel function;

(H5) $g(\cdot,0,0) \in L^1(\Sigma)$;

(H6) $g(x,\cdot,\cdot)$ is lower semicontinuous for every $x \in \Sigma$;

(H7) for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ and for every $s,t,s',t' \in \mathbb{R}^m$

$$g(x,s,t) \le g(x,s',t) + \psi(x,\nu_\Sigma(x))\,, \tag{4.1.6}$$

$$g(x,s,t) \le g(x,s,t') + \psi(x,\nu_\Sigma(x))\,. \tag{4.1.7}$$

Remark 4.1.3. We notice that the functional $\mathcal{F}$ defined by (4.1.1) takes finite values on $GSBV^p(\Omega;\mathbb{R}^m)$ as a consequence of (H1)-(H7).

Remark 4.1.4. If $\Sigma = \partial\Omega$ and $g\colon \partial\Omega \times \mathbb{R}^m \to \mathbb{R}$ satisfies

$$\begin{gathered} s \mapsto g(x,s) \text{ is lower semicontinuous for every } x \in \Sigma, \\ g(x,s) \le g(x,t) + \psi(x,\nu_\Omega(x)) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Sigma \text{ and every } s,t \in \mathbb{R}^m, \end{gathered}$$

then $(x,s,t) \mapsto g(x,s+t)$ fulfills (H6) and (H7).

Remark 4.1.5. The inequalities (4.1.6) and (4.1.7) in (H7) are equivalent to

$$\operatorname{osc} g(x,\cdot,t) \le \psi(x,\nu_\Sigma(x)) \qquad \text{and} \qquad \operatorname{osc} g(x,s,\cdot) \le \psi(x,\nu_\Sigma(x))\,,$$

where for every function $\gamma\colon \mathbb{R}^m \to \mathbb{R}$

$$\operatorname{osc}\gamma := \sup_{s,t\in\mathbb{R}^m} |\gamma(s)-\gamma(t)| = \sup_{s\in\mathbb{R}^m} \gamma(s) - \inf_{s\in\mathbb{R}^m} \gamma(s)\,.$$

The proof of the lower semicontinuity theorem is divided into three steps. By the blow-up technique introduced in [14, 35, 36] we first prove that

$$\mathcal{F}(u) \leq \liminf_k \mathcal{F}(u_k) \tag{4.1.8}$$

whenever u_k converges to u pointwise and $u_k, u \in BV(\Omega; \mathrm{N})$ for some finite subset N of $\mathbb{R}^m$ (see Theorem 4.2.4). In Theorem 4.2.7 we extend (4.1.8) by approximation to functions belonging to $SBV^p(\Omega; \mathbb{R}^m)$. The third step is a truncation argument, which allows us to conclude in the general case $u \in GSBV^p(\Omega; \mathbb{R}^m)$. In Theorem 4.2.8 we show that condition (H7) is also necessary for the lower semicontinuity of the functional $\mathcal{F}$ in $GSBV^p(\Omega; \mathbb{R}^m)$, provided that g satisfies the following properties:

(H8) there exists $a \in L^1(\Sigma)^+$ such that $g(x,s,t) \geq -a(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ and every $s, t \in \mathbb{R}^m$;

(H9) $g(\cdot, s, t) \in L^1(\Sigma)$ for every $s, t \in \mathbb{R}^m$.

Finally, in Section 4.3 we prove a relaxation result for a functional $\mathcal{F}$ of the form (4.1.1), i.e., we give an integral representation formula for $sc^-\mathcal{F}$, defined as the greatest sequentially weakly lower semicontinuous functional on $GSBV^p(\Omega; \mathbb{R}^m)$ which is less than or equal to $\mathcal{F}$. In (4.1.1) we still assume that ψ satisfies (H1)-(H3). As for g, instead of (H4)-(H7), we set for every $(x,s,t) \in \Sigma \times \mathbb{R}^m \times \mathbb{R}^m$

$$\begin{aligned} g_{12}(x,s,t) &:= \min\Big\{ g_1(x,s,t), \inf_{\tau \in \mathbb{R}^m} g_1(x,s,\tau) + \psi(x, \nu_\Sigma(x)) \Big\}, \\ g_1(x,s,t) &:= \min\Big\{ g(x,s,t), \inf_{\sigma \in \mathbb{R}^m} g(x,\sigma,t) + \psi(x, \nu_\Sigma(x)) \Big\} \end{aligned}$$

and we suppose that

(A1) g is Borel measurable;

(A2) $g(x, \cdot, \cdot)$ is continuous on $\mathbb{R}^m \times \mathbb{R}^m$ for every $x \in \Sigma$;

(A3) there exists $a \in L^1(\Sigma)^+$ such that $g(x,s,t) \geq -a(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ and every $s, t \in \mathbb{R}^m$;

(A4) for every $M > 0$ there exists $a_M \in L^1(\Sigma)$ such that $g(x,s,t) \leq a_M(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ and every $s, t \in \mathbb{R}^m$ with $|s|, |t| \leq M$;

(A5) $g_{12}(x, \cdot, \cdot)$ is continuous on $\mathbb{R}^m \times \mathbb{R}^m$ for every $x \in \Sigma$.

Remark 4.1.6. Note that (A5) is not a consequence of (A2). Indeed, there are easy examples where g_1 and g_{12} are not even lower semicontinuous. However, if $g(x, \cdot, \cdot)$ is uniformly continuous on $\mathbb{R}^m \times \mathbb{R}^m$ for every $x \in \Sigma$, then the functions $g_1(x, \cdot, \cdot)$ and $g_{12}(x, \cdot, \cdot)$ are uniformly continuous on $\mathbb{R}^m \times \mathbb{R}^m$ for every $x \in \Sigma$.

In Theorem 4.3.3 we show that

$$sc^{-}\mathcal{F}(u)=\int_{S_u\setminus\Sigma}\psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\Sigma}g_{12}(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

Therefore, the relaxed functional $sc^{-}\mathcal{F}$ is again of the form (4.1.1) and the density g_{12} on Σ is a Carathéodory function which satisfies (H4)-(H7). The mechanical interpretation of this result is that, if the potential g of the surface force is too strong, it is energetically more convenient to create a new crack near the surface Σ.

We conclude the chapter with a relaxation result for the functional $\mathcal{G}\colon L^q(\Omega;\mathbb{R}^m)\to\overline{\mathbb{R}}$, $q\in(1,+\infty)$, defined by

$$\mathcal{G}(u):=\int_{\Omega}W(x,\nabla u)\mathrm{d}x+\int_{\Omega}f(x,u)\mathrm{d}x+\int_{S_u\setminus\Sigma}\psi(x,\nu_u)\mathrm{d}\mathcal{H}^{n-1}+\int_{\Sigma}g(x,u^+,u^-)\mathrm{d}\mathcal{H}^{n-1}$$

for $u\in GSBV^p(\Omega;\mathbb{R}^m)\cap L^q(\Omega;\mathbb{R}^m)$ and $\mathcal{G}(u)=+\infty$ otherwise in $L^q(\Omega;\mathbb{R}^m)$. More precisely, we characterize the functional $sc^{-}\mathcal{G}$, defined this time as the greatest lower semicontinuous functional in $L^q(\Omega;\mathbb{R}^m)$ which is less than or equal to $\mathcal{G}$. We assume that $W(x,\xi)$ is quasiconvex and has a p-growth with respect to ξ, and that $f(x,s)$ has a q-growth with respect to s. In Theorem 4.3.5 we prove that

$$sc^{-}\mathcal{G}(u)=\int_{\Omega}W(x,\nabla u)\mathrm{d}x+\int_{\Omega}f(x,u)\mathrm{d}x+\int_{S_u\setminus\Sigma}\psi(x,\nu_u)\mathrm{d}\mathcal{H}^{n-1}+\int_{\Sigma}g_{12}(x,u^+,u^-)\mathrm{d}\mathcal{H}^{n-1}$$

if $u\in GSBV^p(\Omega;\mathbb{R}^m)\cap L^q(\Omega;\mathbb{R}^m)$, and $sc^{-}\mathcal{G}(u)=+\infty$ otherwise in $L^q(\Omega;\mathbb{R}^m)$.

The results reported in this chapter are contained in the paper [4] in collaboration with G. Dal Maso and R. Toader.

4.2 Lower semicontinuity

This section is devoted to the proof of the following lower semicontinuity result.

Theorem 4.2.1. *Let $p\in(1,+\infty)$ and $\mathcal{F}$ be defined as in* (4.1.1) *with ψ and g satisfying (H1)-(H7). Then $\mathcal{F}$ is lower semicontinuous with respect to the weak convergence in $GSBV^p(\Omega;\mathbb{R}^m)$.*

As already mentioned, the strategy of the proof of Theorem 4.2.1 is the following: by the blow-up technique developed in [35, 36] we first prove the lower semicontinuity property for functions belonging to $BV(\Omega;\mathrm{N})$ for some finite set $\mathrm{N}\subseteq\mathbb{R}^m$. Then we extend this result to $SBV^p(\Omega;\mathbb{R}^m)$ by approximation and, finally, to $GSBV^p(\Omega;\mathbb{R}^m)$ by a simple truncation argument.

The following lemma shows that, in order to prove Theorem 4.2.1, it is not restrictive to assume that g is a nonnegative Carathéodory function satisfying (H5) and (H7).

Lemma 4.2.2. *There exists a sequence* $g_\lambda\colon \Sigma\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ *of nonnegative Carathéodory functions satisfying (H5) and (H7) such that* $g_\lambda(x,\cdot,\cdot)$ *is Lipschitz continuous with Lipschitz constant* λ*, and, setting*

$$\mathcal{F}_\lambda(u) := \int_{S_u\setminus\Sigma}\psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g_\lambda(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}$$

for every $u\in GSBV^p(\Omega;\mathbb{R}^m)$*, the following property holds: if* $u_k,u\in GSBV^p(\Omega;\mathbb{R}^m)$ *satisfy*

$$\mathcal{F}_\lambda(u)\le\liminf_k \mathcal{F}_\lambda(u_k)\quad \textit{for every } \lambda\,,$$

then

$$\mathcal{F}(u)\le\liminf_k \mathcal{F}(u_k)\,. \tag{4.2.1}$$

Proof. For every $(x,s,t)\in\Sigma\times\mathbb{R}^m\times\mathbb{R}^m$ and every $\lambda\in\mathbb{N}$ let

$$g_\lambda(x,s,t) := \inf_{\sigma,\tau\in\mathbb{R}^m}\{g(x,\sigma,\tau)-g(x,0,0)+2c_2+\lambda|(s,t)-(\sigma,\tau)|\}\,, \tag{4.2.2}$$

where c_2 is the constant in (H2). Let us prove that g_λ is a Carathéodory function. For every $s,t\in\mathbb{R}^m$ and every $c\in\mathbb{R}$, we have that

$$\begin{aligned}
&\{x\in\Sigma:\ g_\lambda(x,s,t)<c\}\\
&=\{x\in\Sigma:\ \exists\,\sigma,\tau\in\mathbb{R}^m \text{ such that } g(x,\sigma,\tau)-g(x,0,0)+2c_2+\lambda|(s,t)-(\sigma,\tau)|<c\}\\
&=\Pi_\Sigma\big(\{(x,\sigma,\tau)\in\Sigma\times\mathbb{R}^m\times\mathbb{R}^m: g(x,\sigma,\tau)-g(x,0,0)+2c_2+\lambda|(s,t)-(\sigma,\tau)|<c\}\big)\,,
\end{aligned}$$

where $\Pi_\Sigma\colon\Sigma\times\mathbb{R}^m\times\mathbb{R}^m\to\Sigma$ denotes the projection onto Σ. Since g is Borel, applying the projection theorem (see, e.g., [21, Proposition 8.4.4]), we get that the set $\{x\in\Sigma:\ g_\lambda(x,s,t)<c\}$ is $\mathcal{H}^{n-1}$-measurable. Hence $g_\lambda(\cdot,s,t)$ is $\mathcal{H}^{n-1}$-measurable for every $s,t\in\mathbb{R}^m$. It is easy to see that for $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma$ the function $g_\lambda(x,\cdot,\cdot)$ is Lipschitz continuous with Lipschitz constant λ, thus g_λ is a Carathéodory function.

By (H2) and (H7) for g we have that g_λ is nonnegative and satisfies (H7). The inequalities $0\le g_\lambda(x,0,0)\le 4c_2$ imply that $g_\lambda(\cdot,0,0)\in L^1(\Sigma)$. Since $g(x,\cdot,\cdot)$ is lower semicontinuous and g_λ is the Yosida approximation of $g(x,\cdot,\cdot)-g(x,0,0)+2c_2$, we have that $g_\lambda(x,s,t)\nearrow g(x,s,t)-g(x,0,0)+2c_2$ for every $(x,s,t)\in\Sigma\times\mathbb{R}^m\times\mathbb{R}^m$ (see for instance [16, Section 1.3]).

Let u_k,u be as in the statement of the lemma. Then, by definition of g_λ and $\mathcal{F}_\lambda$,

$$\mathcal{F}_\lambda(u)\le\liminf_k\mathcal{F}_\lambda(u_k)\le\liminf_k\mathcal{F}(u_k)-\int_\Sigma g(x,0,0)\,\mathrm{d}\mathcal{H}^{n-1}+2c_2\,\mathcal{H}^{n-1}(\Sigma)\,. \tag{4.2.3}$$

By the monotone convergence theorem, we get that

$$\lim_\lambda\mathcal{F}_\lambda(u)=\mathcal{F}(u)-\int_\Sigma g(x,0,0)\,\mathrm{d}\mathcal{H}^{n-1}+2c_2\,\mathcal{H}^{n-1}(\Sigma)\,.$$

The previous equality, together with (4.2.3), implies (4.2.1). □

In the sequel, we will also need the following technical lemma, where $\mathrm{R}^C_{\rho,\xi}(x)$ is defined as in (1.1.2).

Lemma 4.2.3. *Let $g\colon \Sigma\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ be a Carathéodory function satisfying properties (H5) and (H7). Then, for every $C>0$ and for every compact subset K of $\mathbb{R}^m\times\mathbb{R}^m$ we have that*

$$\lim_{\rho\searrow 0}\frac{1}{\rho^{n-1}}\int_{\Sigma\cap\mathrm{R}^C_{\rho,\xi}(x)}\sup_{(s,t)\in K}|g(y,s,t)-g(x,s,t)|\,\mathrm{d}\mathcal{H}^{n-1}(y)=0 \tag{4.2.4}$$

for $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma$ and every $\xi\in\mathbb{S}^{n-1}$.

Proof. For every $(x,\delta)\in\Sigma\times(0,+\infty)$ we set

$$\omega(x,\delta):=\sup_{\substack{(s,t),(\sigma,\tau)\in K\\ |(s,t)-(\sigma,\tau)|\le\delta}}|g(x,s,t)-g(x,\sigma,\tau)|\,. \tag{4.2.5}$$

Then $\omega(x,\delta)\to 0$ as $\delta\searrow 0$ for every $x\in\Sigma$ such that $g(x,\cdot,\cdot)$ is continuous on $\mathbb{R}^m\times\mathbb{R}^m$. Moreover, by properties (H2) and (H7), we have that $\omega(\cdot,\delta)\in L^1(\Sigma)$ for every $\delta>0$.

Fix a sequence $\delta_k\searrow 0$. For every $k\in\mathbb{N}$, let $(s^k_1,t^k_1),\dots,(s^k_{l_k},t^k_{l_k})\in K$ satisfy

$$K\subseteq\bigcup_{i=1}^{l_k}\mathrm{B}_{\delta_k}(s^k_i,t^k_i)\,,$$

where, in this proof, $\mathrm{B}_r(s,t)$ denotes the open ball in $\mathbb{R}^m\times\mathbb{R}^m$ of radius r and center (s,t).

Fix $x\in\Sigma$ with the following properties: x is a Lebesgue point of $\omega(\cdot,\delta_k)$ and of $g(\cdot,s^k_i,t^k_i)$ for every k and every $i=1,\dots,l_k$, $\omega(x,\delta_k)\to 0$ as $k\to+\infty$, and $\nu_\Sigma(x)$ is normal to Σ at x. Note that these properties are satisfied by $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma$.

Finally, fix $k\in\mathbb{N}$. For every $(s,t)\in K$, let $j_s\in\{1,\dots,l_k\}$ be such that $|(s,t)-(s^k_{j_s},t^k_{j_s})|<\delta_k$. Then, for $\mathcal{H}^{n-1}$-a.e. $y\in\Sigma$ we have that

$$\begin{aligned}|g(y,s,t)&-g(x,s,t)|\\ &\le|g(y,s,t)-g(y,s^k_{j_s},t^k_{j_s})|+|g(y,s^k_{j_s},t^k_{j_s})-g(x,s^k_{j_s},t^k_{j_s})|\\ &\quad+|g(x,s^k_{j_s},t^k_{j_s})-g(x,s,t)|\\ &\le\omega(y,\delta_k)+\sup_{i=1,\dots,l_k}|g(y,s^k_i,t^k_i)-g(x,s^k_i,t^k_i)|+\omega(x,\delta_k)\,.\end{aligned} \tag{4.2.6}$$

Inequality (4.2.6) implies that, for every $\xi\in\mathbb{S}^{n-1}$ and every $C>0$,

$$\begin{aligned}\frac{1}{\rho^{n-1}}&\int_{\Sigma\cap\mathrm{R}^C_{\rho,\xi}(x)}\sup_{(s,t)\in K}|g(y,s,t)-g(x,s,t)|\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &\le\frac{1}{\rho^{n-1}}\int_{\Sigma\cap\mathrm{R}^C_{\rho,\xi}(x)}(\omega(y,\delta_k)+\omega(x,\delta_k))\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &\quad+\sum_{i=1}^{l_k}\frac{1}{\rho^{n-1}}\int_{\Sigma\cap\mathrm{R}^C_{\rho,\xi}(x)}|g(y,s^k_i,t^k_i)-g(x,s^k_i,t^k_i)|\,\mathrm{d}\mathcal{H}^{n-1}(y)\,.\end{aligned} \tag{4.2.7}$$

Since, by assumption, $x \in \Sigma$ is a Lebesgue point of $\omega(\cdot, \delta_k)$ and of $g(\cdot, s_i^k, t_i^k)$, passing to the lim sup as $\rho \searrow 0$ in (4.2.7) we obtain that for every $k \in \mathbb{N}$

$$\begin{aligned}\limsup_{\rho \searrow 0} \frac{1}{\rho^{n-1}} \int_{\Sigma \cap \mathrm{R}^C_{\rho,\xi}(x)} \sup_{(s,t)\in K} |g(y,s,t) - g(x,s,t)| \, \mathrm{d}\mathcal{H}^{n-1}(y) \\ \leq 2\mathcal{H}^{n-1}(T_x(\Sigma) \cap \mathrm{R}^C_{1,\xi}(0)) \, \omega(x, \delta_k) \, ,\end{aligned} \tag{4.2.8}$$

where $T_x(\Sigma)$ is the tangent space defined in (1.1.3). Passing to the limit as $k \to +\infty$ in (4.2.8) we get (4.2.4). □

Let us introduce some notation which will be useful in the sequel. Let N be a finite subset of $\mathbb{R}^m$, U an open subset of Ω such that $\{x \in \Omega : \, d(x, \Sigma \cup \partial\Omega) < \eta\} \subseteq U$ for some $\eta > 0$, and let Ω' be a bounded smooth open subset of $\mathbb{R}^n$ such that $\Omega \subset\subset \Omega'$. For every $u \in BV(U; \mathrm{N}) := \{v \in BV(U; \mathbb{R}^m) : \, v(x) \in \mathrm{N} \text{ for } \mathcal{L}^n\text{-a.e. } x \in U\}$, its extension to 0 on $\Omega' \setminus \Omega$ is still denoted by u. We notice that $U' := (\Omega' \setminus \Omega) \cup U$ is open and that this extension belongs to $BV(U'; \mathrm{N}')$, where $\mathrm{N}' := \mathrm{N} \cup \{0\}$. For every $B \in \mathcal{B}(U')$ we set

$$\mathcal{F}_U(u, B) := \int_{U \cap S_u \cap B \setminus \Sigma} \psi(x, \nu_u) \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\Sigma \cap B} g(x, u^+, u^-) \, \mathrm{d}\mathcal{H}^{n-1} \, , \tag{4.2.9}$$

where, in the second integral, $u^\pm$ denote the traces on the two faces of Σ of u, according to Remark 1.2.10.

Since ψ and g satisfy (H2), (H5), and (H7), we have that $\mathcal{F}_U(u, \cdot)$ is a measure defined on $\mathcal{B}(U')$. If, in addition, $\mathcal{H}^{n-1}(S_u) < +\infty$, in view of (4.1.2) $\mathcal{F}_U(u, \cdot)$ belongs to $\mathcal{M}_b(U')$ (this is always the case if $u \in BV(U; \mathrm{N})$ for some finite set $\mathrm{N} \subseteq \mathbb{R}^m$). Finally, we notice that if g is nonnegative, then $\mathcal{F}_U(u, \cdot)$ is nonnegative.

We are now ready to state the lower semicontinuity result on $BV(U; \mathrm{N})$.

Theorem 4.2.4. *Let ψ and g be functions satisfying (H1)-(H7). Assume in addition that g is a nonnegative Carathéodory function. Let* N *be a finite subset of $\mathbb{R}^m$ and let U be an open subset of Ω such that $\{x \in \Omega : \, d(x, \Sigma \cup \partial\Omega) < \eta\} \subseteq U$ for some $\eta > 0$. Then*

$$\mathcal{F}_U(u, U \cup \Sigma) \leq \liminf_k \mathcal{F}_U(u_k, U \cup \Sigma)$$

for every $u_k, u \in BV(U; \mathrm{N})$ such that u_k converges to u pointwise $\mathcal{L}^n$-a.e. in U.

In order to prove Theorem 4.2.4, we need the following blow-up lemma.

Lemma 4.2.5. *Let ψ, g, N, U, η, u_k, and u be as in Theorem 4.2.4 and let $U' := (\Omega' \setminus \Omega) \cup U$. For every $x \in \Sigma$, let $\xi(x) \in \mathbb{S}^{n-1}$ be as in Definition 1.1.5. Assume that $\mathcal{F}_U(u_k, U \cup \Sigma)$ is bounded and that $\mathcal{F}_U(u_k, \cdot) \rightharpoonup \mu$ weakly* in $\mathcal{M}_b(U')$ for some $\mu \in \mathcal{M}_b(U')$. Then*

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\lfloor\Sigma}(x) \geq g(x, u^+(x), u^-(x)) \tag{4.2.10}$$

for $\mathcal{H}^{n-1}$*-a.e.* $x \in \Sigma$*, and*

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\lfloor(S_u \setminus \Sigma)}(x) \geq \psi(x, \nu_u(x)) \tag{4.2.11}$$

for $\mathcal{H}^{n-1}$*-a.e.* $x \in S_u \setminus \Sigma$*.*

Proof. Let us perform the blow-up on Σ. Let $L > 0$ be the Lipschitz constant of Σ and $\Lambda := L\sqrt{n}$. Let $x_0 \in \Sigma$ be such that $\nu_\Sigma(x_0)$ is normal to Σ at x_0 and (H7) holds. We introduce the simplified notation $\mathrm{R}_\rho(x_0) := \mathrm{R}^{\Lambda}_{\rho,\xi(x_0)}(x_0)$, $\mathrm{R}_\rho := \mathrm{R}^{\Lambda}_{\rho,\xi(x_0)}(0)$, and $\mathrm{R}^{\pm}_\rho(x_0) := \{y \in \mathrm{R}_\rho(x_0) : (y - x_0)\cdot \nu_\Sigma(x_0) \gtrless 0\}$, where $\mathrm{R}^{C}_{\rho,\xi}(x)$ is defined in (1.1.2). We assume in addition that x_0 satisfies the following conditions:

$$x_0 \notin (\overline{\Sigma \cap \Omega}) \cap \partial\Omega \tag{4.2.12}$$

$$\lim_{\rho \searrow 0} \frac{1}{\rho^{n-1}} \int_{\Sigma \cap \mathrm{R}_\rho(x_0)} |\nu_\Sigma(x) - \nu_\Sigma(x_0)| \, \mathrm{d}\mathcal{H}^{n-1}(x) = 0\,, \tag{4.2.13}$$

$$\lim_{\rho \searrow 0} \frac{1}{\rho^{n}} \int_{\mathrm{R}^{\pm}_\rho(x_0)} |u(x) - u^{\pm}(x_0)| \, \mathrm{d}x = 0\,, \tag{4.2.14}$$

$$\text{there exists } \lim_{\rho \searrow 0} \frac{\mu(\mathrm{R}_\rho(x_0))}{\mathcal{H}^{n-1}\lfloor\Sigma\,(\mathrm{R}_\rho(x_0))} = \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\lfloor\Sigma}(x_0)\,, \tag{4.2.15}$$

$$\lim_{\rho \searrow 0} \frac{1}{\rho^{n-1}} \int_{\Sigma \cap \mathrm{R}_\rho(x_0)} \sup_{s,t \in T} |g(x,s,t) - g(x_0,s,t)| \, \mathrm{d}\mathcal{H}^{n-1}(x) = 0\,. \tag{4.2.16}$$

We notice that conditions (4.2.12)-(4.2.16) are satisfied for $\mathcal{H}^{n-1}$-a.e. $x_0 \in \Sigma$ as a consequence of the properties of the traces of BV functions, of hypotheses (4.1.2), of Lemma 4.2.3, and of a generalized version of Besicovitch differentiation theorem (see [62] and [34, Sections 1.2.1-1.2.2]).

Since $\nu_\Sigma(x_0)$ is normal to Σ at x_0, we have that

$$\lim_{\rho \searrow 0} \frac{\mathcal{H}^{n-1}\lfloor\Sigma\,(\mathrm{R}_\rho(x_0))}{\rho^{n-1}} = \mathcal{H}^{n-1}(T_{x_0}(\Sigma) \cap \mathrm{R}_1)\,, \tag{4.2.17}$$

where $T_{x_0}(\Sigma)$ is the tangent space defined in (1.1.3) and, according to the notation introduced above, $\mathrm{R}_1 = \mathrm{R}^{\Lambda}_{1,\xi(x_0)}(0)$. Let

$$\gamma(x_0) := \lim_{\rho \searrow 0} \frac{\mu(\mathrm{R}_\rho(x_0))}{\rho^{n-1}}\,. \tag{4.2.18}$$

From (4.2.15) and (4.2.17) we get that the limit in (4.2.18) exists and

$$\gamma(x_0) = \mathcal{H}^{n-1}(T_{x_0}(\Sigma) \cap \mathrm{R}_1) \lim_{\rho \searrow 0} \frac{\mu(\mathrm{R}_\rho(x_0))}{\mathcal{H}^{n-1}\lfloor\Sigma\,(\mathrm{R}_\rho(x_0))}\,. \tag{4.2.19}$$

Using the definition (4.2.18), we shall first express $\gamma(x_0)$ as limit of suitable rescalings of the functional $\mathcal{F}_U$. Then we shall estimate $\gamma(x_0)$ from below using g, and finally we shall deduce (4.2.10) thanks to (4.2.19).

By the weak*-convergence of $\mathcal{F}_U(u_k,\cdot)$ to μ, we have that

$$\mathcal{F}_U(u_k,\mathrm{R}_\rho(x_0)) \to \mu(\mathrm{R}_\rho(x_0)) \tag{4.2.20}$$

for every $\rho > 0$ out of an at most countable set. Thus, we can fix a sequence $\rho_j \searrow 0$ such that $\Omega \cap \mathrm{R}_{\rho_j}(x_0) \subseteq U$, (4.2.20) holds for every ρ_j, and

$$\lim_j \frac{\mu(\mathrm{R}_{\rho_j}(x_0))}{\rho_j^{n-1}} = \gamma(x_0)\,. \tag{4.2.21}$$

Since Σ is a Lipschitz manifold with Lipschitz constant L, for j sufficiently large the function φ_{x_0} of Definition 1.1.4 is well-defined and Lipschitz continuous on the $(n-1)$-dimensional cube $\mathrm{Q}^{n-1}_{\rho_j,\xi(x_0)}(x_0)$, with Lipschitz constant L. Let $\tilde{x} := x_0 - (x_0\cdot\xi(x_0))\xi(x_0)$ be the center of $\mathrm{Q}^{n-1}_{\rho_j,\xi(x_0)}(x_0)$. Then, for every $y \in \mathrm{Q}^{n-1}_{\rho_j,\xi(x_0)}(x_0)$ we have that

$$|\varphi_{x_0}(y) - \varphi_{x_0}(\tilde{x})| \le L|y-\tilde{x}| \le \frac{\Lambda}{2}\rho_j\,. \tag{4.2.22}$$

In view of the definition of the rectangle $\mathrm{R}_{\rho_j}(x_0)$, inequality (4.2.22) implies that

$$\mathrm{R}_{\rho_j}(x_0) \cap \Sigma = \{y + \varphi_{x_0}(y)\xi(x_0) : y \in \mathrm{Q}^{n-1}_{\rho_j,\xi(x_0)}(x_0)\}\,.$$

We define

$$A^{\rho_j}_{\pm} := \{y + t\xi(x_0) : y \in \mathrm{Q}^{n-1}_{\rho_j,\xi(x_0)}(x_0),\, |t - x_0\cdot\xi(x_0)| < \Lambda\rho_j,\, t \gtrless \varphi_{x_0}(y)\}\,. \tag{4.2.23}$$

It is easy to see that $A^{\rho_j}_+$ and $A^{\rho_j}_-$ are connected, have Lipschitz boundaries, and that $\nu_\Sigma(x)$ points towards $A^{\rho_j}_+$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathrm{R}_{\rho_j}(x_0) \cap \Sigma$. Moreover, thanks to (4.2.12), it is not restrictive to assume that if $x_0 \in \Sigma \cap \partial\Omega$, then $A^{\rho_j}_+ = \mathrm{R}_{\rho_j}(x_0) \cap \Omega$ and $A^{\rho_j}_- = \mathrm{R}_{\rho_j}(x_0) \setminus \overline{\Omega}$, or viceversa, according to the orientation of $\nu_\Omega(x_0)$ with respect to $\nu_\Sigma(x_0)$. Conversely, if $x_0 \in \Sigma \setminus \partial\Omega$, we assume that $\mathrm{R}_{\rho_j}(x_0) \subseteq \Omega$.

It is now convenient to rescale $\mathcal{F}_U$ to the rectangle R_1 and, consequently, to define the corresponding rescaled sets and functions: let $\Omega_j := \{y \in \mathbb{R}^n : x_0 + \rho_j y \in \Omega\}$, $\Sigma_j := \{y \in \mathbb{R}^n : x_0 + \rho_j y \in \Sigma\}$,

$$A^{\pm}_j := \{y \in \mathbb{R}^n : x_0 + \rho_j y \in A^{\rho_j}_{\pm}\}\,, \tag{4.2.24}$$

and $u^j_k(y) := u_k(x_0 + \rho_j y)$ for $y \in \mathrm{R}_1$, noticing that $u^j_k(y) = 0$ for $y \in \mathrm{R}_1 \setminus \Omega_j$. By the change of variables $x = x_0 + \rho_j y$ with $y \in \mathrm{R}_1$ we have

$$\begin{aligned}\frac{\mathcal{F}_U(u_k,\mathrm{R}_{\rho_j}(x_0))}{\rho_j^{n-1}} &= \int_{\Omega_j\cap S_{u^j_k}\cap \mathrm{R}_1\setminus\Sigma_j} \psi(x_0+\rho_j y, \nu_{u^j_k}(y))\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &\quad + \int_{\Sigma_j\cap\mathrm{R}_1} g(x_0+\rho_j y, (u^j_k)^+(y), (u^j_k)^-(y))\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &= \mathcal{F}^{\rho_j}(u^j_k,\mathrm{R}_1)\,,\end{aligned} \tag{4.2.25}$$

where

$$\mathcal{F}^{\rho_j}(v,B) := \int_{\Omega_j\cap S_v\cap B\setminus\Sigma_j} \psi(x_0+\rho_j y,\nu_v(y))\,\mathrm{d}\mathcal{H}^{n-1}(y) + \int_{\Sigma_j\cap B} g(x_0+\rho_j y, v^+(y), v^-(y))\,\mathrm{d}\mathcal{H}^{n-1}(y)$$

for every $j\in\mathbb{N}$, every $v\in BV(\mathrm{R}_1;\mathrm{N}')$, and every $B\in\mathcal{B}(\mathrm{R}_1)$.

Let us introduce $u^j(y) := u(x_0+\rho_j y)$ and

$$u^{x_0}(y) := \begin{cases} u^+(x_0) & \text{if } y\in\mathrm{R}_1^+, \\ u^-(x_0) & \text{if } y\in\mathrm{R}_1^-, \end{cases} \tag{4.2.26}$$

where we have set $\mathrm{R}_1^\pm := \{y\in\mathrm{R}_1 : y\cdot\nu_\Sigma(x_0) \gtrless 0\}$. By hypothesis, $u_k^j\to u^j$ in $L^1(\mathrm{R}_1;\mathrm{N}')$ as $k\to+\infty$ and, by (4.2.14), $u^j\to u^{x_0}$ in $L^1(\mathrm{R}_1;\mathrm{N}')$. Therefore, we can find a sequence $k_j\nearrow+\infty$ such that $u^j_{k_j}\to u^{x_0}$ in $L^1(\mathrm{R}_1;\mathrm{N}')$ as $j\to+\infty$ and, by (4.2.20) and (4.2.25),

$$\left|\mathcal{F}^{\rho_j}(u^j_{k_j},\mathrm{R}_1) - \frac{\mu(\mathrm{R}_{\rho_j}(x_0))}{\rho_j^{n-1}}\right| < \frac{1}{j}. \tag{4.2.27}$$

By (4.2.21) and (4.2.27) we get that

$$\gamma(x_0) = \lim_j \mathcal{F}^{\rho_j}(u^j_{k_j},\mathrm{R}_1). \tag{4.2.28}$$

Besides $\mathcal{F}^{\rho_j}(v,B)$, it is convenient to consider also the functional $\mathcal{F}^{\rho_j}_{x_0}(v,B)$ defined by "freezing" the value of the first argument of ψ and g at x_0:

$$\mathcal{F}^{\rho_j}_{x_0}(v,B) := \int_{\Omega_j\cap S_v\cap B\setminus\Sigma_j} \psi(x_0,\nu_v)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Sigma_j\cap B} g(x_0,v^+,v^-)\,\mathrm{d}\mathcal{H}^{n-1}$$

for every $j\in\mathbb{N}$, every $v\in BV(\mathrm{R}_1;\mathrm{N}')$, and every $B\in\mathcal{B}(\mathrm{R}_1)$.

Equalities (4.2.16) and (4.2.28), together with the uniform continuity of ψ on $\overline{\Omega}\times\mathbb{S}^{n-1}$, imply that

$$\gamma(x_0) = \lim_j \mathcal{F}^{\rho_j}_{x_0}(u^j_{k_j},\mathrm{R}_1). \tag{4.2.29}$$

The next step of the proof is to show that, in order to give an estimate of $\gamma(x_0)$ in terms of g, we can restrict ourselves to functions which are equal to $u^+(x_0)$ or $u^-(x_0)$ near $\partial\mathrm{R}_1$. To this end, let us define, for every $j\in\mathbb{N}$, the functions

$$u^{x_0}_j(y) := \begin{cases} u^+(x_0) & \text{if } y\in A_j^+, \\ u^-(x_0) & \text{if } y\in A_j^-, \end{cases}$$

where $A_j^\pm$ are introduced in (4.2.24). The difference between this definition and (4.2.26) is that in (4.2.26) the interface is flat and coincides with $T_{x_0}(\Sigma)\cap\mathrm{R}_1$, while here the interface is the rescaled version Σ_j of Σ. It is clear that $u^{x_0}_j\in BV(\mathrm{R}_1;\mathrm{N}')$ and $u^{x_0}_j\to u^{x_0}$ in $L^1(\mathrm{R}_1;\mathrm{N}')$ as $j\to+\infty$.

Given $\varepsilon > 0$, we now modify the functions $u^j_{k_j}$ near $\partial \mathrm{R}_1$ in order to obtain new functions v_j in $BV(\mathrm{R}_1;\mathrm{N}')$ such that $v_j \to u^{x_0}$ in $L^1(\mathrm{R}_1;\mathrm{N}')$, $v_j = u^{x_0}_j$ in a neighborhood of $\partial \mathrm{R}_1$, and

$$\limsup_j \mathcal{F}^{\rho_j}_{x_0}(v_j,\mathrm{R}_1) \le \lim_j \mathcal{F}^{\rho_j}_{x_0}(u^j_{k_j},\mathrm{R}_1) + \varepsilon = \gamma(x_0) + \varepsilon\,. \tag{4.2.30}$$

This will be done following the lines of an interpolation argument proposed in [7, Lemma 4.4].

To this aim, we consider the distance function $d\colon \mathrm{N}' \times \mathrm{N}' \to \{0,1\}$ defined by $d(i,j) := 1$ for $i,j \in \mathrm{N}'$ with $i \neq j$ and $d(i,i) := 0$. Let us fix $0 < r_1 < r_2 < 1$ and a function $\varphi \in C^\infty(\mathrm{R}_1)$ such that $0 \le \varphi \le 1$, $\varphi = 1$ in $\mathrm{R}_1 \setminus \overline{\mathrm{R}}_{r_2}$, and $\varphi = 0$ in $\overline{\mathrm{R}}_{r_1}$. By Sard Lemma and Coarea Formula, for every j we can find $t_j \in (0,1)$ such that

$$\partial\{\varphi < t_j\} = \{\varphi = t_j\} \text{ is } C^\infty\,, \tag{4.2.31}$$

$$\mathcal{H}^{n-1}(\{\varphi = t_j\}) < +\infty\,, \tag{4.2.32}$$

$$\mathcal{H}^{n-1}(S_{u^j_{k_j}} \cap \{\varphi = t_j\}) = \mathcal{H}^{n-1}(\Sigma_j \cap \{\varphi = t_j\}) = 0\,, \tag{4.2.33}$$

$$\begin{aligned} \int_{\{\varphi = t_j\}\cap \mathrm{R}_{r_2}\setminus \mathrm{R}_{r_1}} d(u^{x_0}_j, u^j_{k_j})\,\mathrm{d}\mathcal{H}^{n-1} &\le \int_{\mathrm{R}_{r_2}\setminus \mathrm{R}_{r_1}} d(u^{x_0}_j, u^j_{k_j})\,|\nabla\varphi|\,\mathrm{d}x \\ &\le C\,\mathcal{L}^n(\{u^{x_0}_j \neq u^j_{k_j}\} \cap \mathrm{R}_{r_2}\setminus \mathrm{R}_{r_1})\,, \end{aligned} \tag{4.2.34}$$

where $C := \|\nabla\varphi\|_\infty$. For such a t_j we set

$$v^{r_1,r_2}_j(x) := \begin{cases} u^j_{k_j}(x) & \text{if } \varphi(x) < t_j\,, \\ u^{x_0}_j(x) & \text{if } \varphi(x) \ge t_j\,. \end{cases}$$

Then $v^{r_1,r_2}_j \in BV(\mathrm{R}_1;\mathrm{N}')$, $v^{r_1,r_2}_j = u^{x_0}_j$ in $\mathrm{R}_1 \setminus \mathrm{R}_{r_2}$, $v^{r_1,r_2}_j = u^j_{k_j}$ in R_{r_1}, and $v^{r_1,r_2}_j \to u^{x_0}$ in $L^1(\mathrm{R}_1;\mathrm{N}')$ as $j \to +\infty$. By (4.2.32), $\mathcal{F}^{\rho_j}_{x_0}(v^{r_1,r_2}_j,\cdot)$ is a nonnegative bounded Radon measure on R_1. Thus, to estimate $\mathcal{F}^{\rho_j}_{x_0}(v^{r_1,r_2}_j,\mathrm{R}_1)$, we integrate separately on the sets $\{\varphi < t_j\}$ and $\{\varphi > t_j\}$, and on the interface $\{\varphi = t_j\}$. Taking into account (H2) and (4.2.31)-(4.2.34), we get that

$$\begin{aligned} &\mathcal{F}^{\rho_j}_{x_0}(v^{r_1,r_2}_j,\mathrm{R}_1) \le \mathcal{F}^{\rho_j}_{x_0}(u^j_{k_j},\mathrm{R}_{r_2}) + \mathcal{F}^{\rho_j}_{x_0}(u^{x_0}_j,\mathrm{R}_1\setminus\mathrm{R}_{r_1}) + c_2\int_{\{\varphi = t_j\}\cap \mathrm{R}_{r_2}\setminus \mathrm{R}_{r_1}} d(u^{x_0}_j, u^j_{k_j})\,\mathrm{d}\mathcal{H}^{n-1} \\ &\le \mathcal{F}^{\rho_j}_{x_0}(u^j_{k_j},\mathrm{R}_1) + \mathcal{F}^{\rho_j}_{x_0}(u^{x_0}_j,\mathrm{R}_1\setminus\mathrm{R}_{r_1}) + c_2 C\,\mathcal{L}^n(\{u^{x_0}_j \neq u^j_{k_j}\} \cap \mathrm{R}_{r_2}\setminus \mathrm{R}_{r_1})\,. \end{aligned} \tag{4.2.35}$$

Since $u^j_{k_j}, u^{x_0}_j$ converge to u^{x_0} in $L^1(\mathrm{R}_1;\mathrm{N}')$, passing to the $\limsup$ as $j \to +\infty$ in (4.2.35) we deduce that

$$\limsup_j \mathcal{F}^{\rho_j}_{x_0}(v^{r_1,r_2}_j,\mathrm{R}_1) \le \limsup_j \big(\mathcal{F}^{\rho_j}_{x_0}(u^j_{k_j},\mathrm{R}_1) + \mathcal{F}^{\rho_j}_{x_0}(u^{x_0}_j,\mathrm{R}_1 \setminus \mathrm{R}_{r_1})\big)\,. \tag{4.2.36}$$

Obviously, $u_j^{x_0}$ does not have jump points in $\mathrm{R}_1 \setminus \Sigma_j$. Hence, recalling that (Σ, ν_Σ) is an orientable Lipschitz manifold, we have that

$$\begin{aligned}\mathcal{F}_{x_0}^{\rho_j}(u_j^{x_0}, \mathrm{R}_1 \setminus \mathrm{R}_{r_1}) &= \int_{\Sigma_j \cap \mathrm{R}_1 \setminus \mathrm{R}_{r_1}} g(x_0, (u_j^{x_0})^+, (u_j^{x_0})^-)\, \mathrm{d}\mathcal{H}^{n-1} \\ &= g(x_0, u^+(x_0), u^-(x_0))\, \mathcal{H}^{n-1}(\Sigma_j \cap \mathrm{R}_1 \setminus \mathrm{R}_{r_1})\,.\end{aligned} \tag{4.2.37}$$

Since $\nu_\Sigma(x_0)$ is normal to Σ at x_0, $\mathcal{H}^{n-1}(\Sigma_j \cap \mathrm{R}_1 \setminus \mathrm{R}_{r_1}) \to \mathcal{H}^{n-1}(T_{x_0}(\Sigma) \cap \mathrm{R}_1 \setminus \mathrm{R}_{r_1})$ as $j \to +\infty$. Therefore, given $\varepsilon > 0$, we can choose $0 < r_1 < r_2 < 1$ such that

$$g(x_0, u^+(x_0), u^-(x_0)) \lim_j \mathcal{H}^{n-1}(\Sigma_j \cap \mathrm{R}_1 \setminus \mathrm{R}_{r_1}) < \varepsilon\,,$$

and set $v_j := v_j^{r_1, r_2}$. By (4.2.36) and (4.2.37), we get (4.2.30).

We now study the behavior of v_j and $\mathcal{F}_{x_0}^{\rho_j}(v_j, \cdot)$ on the interface between the sets $\{v_j = u_j^{x_0}\}$ and $\{v_j \neq u_j^{x_0}\}$. To this aim, we define, for every j,

$$E_j^\pm := A_j^\pm \cap \{v_j \neq u_j^{x_0}\}\,.$$

Since $v_j, u_j^{x_0} \in BV(\mathrm{R}_1; \mathrm{N}')$ and $v_j = u_j^{x_0}$ in a neighborhood of $\partial \mathrm{R}_1$, the sets $E_j^\pm$ have finite perimeter and $E_j^\pm \subset\subset \mathrm{R}_1$. We set also

$$\mathrm{t}(E_j^\pm) := \{y \in \partial A_j^\pm : \widetilde{\mathbf{1}_{E_j^\pm}}(y) = 1\}\,,$$

where $\widetilde{\mathbf{1}_{E_j^\pm}}$ is defined in (1.2.7).

By the definitions of $A_j^\pm$, of $E_j^\pm$, and of $\mathrm{t}(E_j^\pm)$, for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma_j \setminus \mathrm{t}(E_j^\pm)$ we have that

$$\begin{aligned}\lim_{r \searrow 0} \frac{1}{r^n} \int_{\mathrm{B}_r(x) \cap A_j^\pm} |v_j(y) - u^\pm(x_0)|\, \mathrm{d}y &= \lim_{r \searrow 0} \frac{1}{r^n} \int_{\mathrm{B}_r(x) \cap E_j^\pm} |v_j(y) - u^\pm(x_0)|\, \mathrm{d}y \\ \leq c \lim_{r \searrow 0} \frac{\mathcal{L}^n(\mathrm{B}_r(x) \cap E_j^\pm)}{r^n} &= c \lim_{r \searrow 0} \frac{1}{r^n} \int_{\mathrm{B}_r(x) \cap A_j^\pm} \mathbf{1}_{E_j^\pm}(y)\, \mathrm{d}y = 0\,,\end{aligned} \tag{4.2.38}$$

where $c := 2 \max\{|s| : s \in \mathrm{N}\}$. Equality (4.2.38) implies that for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma_j \setminus \mathrm{t}(E_j^\pm)$ the traces $v_j^\pm(x)$ on the two sides of Σ_j are equal to $u^\pm(x_0)$, respectively.

We now prove that

$$\int_{\partial^* E_j^\pm \setminus \mathrm{t}(E_j^\pm)} \nu_{E_j^\pm}\, \mathrm{d}\mathcal{H}^{n-1} = \mp \int_{\mathrm{t}(E_j^\pm)} \nu_{\Sigma_j}\, \mathrm{d}\mathcal{H}^{n-1}\,. \tag{4.2.39}$$

By Lemma 1.2.14 and by the definition of v_j, we have that, up to an $\mathcal{H}^{n-1}$-negligible set,

$$\mathrm{t}(E_j^\pm) = \Sigma_j \cap \partial^* E_j^\pm\,. \tag{4.2.40}$$

Since $E_j^{\pm}$ have finite perimeter, we get that

$$\int_{\partial^* E_j^{\pm}} \nu_{E_j^{\pm}} \, \mathrm{d}\mathcal{H}^{n-1} = (D\mathbf{1}_{E_j^{\pm}})(\mathbb{R}^n) = 0\,, \tag{4.2.41}$$

where $\nu_{E_j^{\pm}}$ are the inner unit normals to $E_j^{\pm}$. By the definitions of $A_{\pm}^{\rho_j}$ and of $A_j^{\pm}$ given in (4.2.23)-(4.2.24), by Definition 1.1.5, and by the equality (4.2.40), for j large enough $\nu_{E_j^{\pm}} = \pm\nu_{\Sigma_j}$ $\mathcal{H}^{n-1}$-a.e. on $\mathrm{t}(E_j^{\pm})$. Hence, by (4.2.41) we have that

$$\begin{aligned} 0 &= \int_{\partial^* E_j^{\pm}} \nu_{E_j^{\pm}} \, \mathrm{d}\mathcal{H}^{n-1} = \int_{\partial^* E_j^{\pm} \cap \Sigma_j} \nu_{E_j^{\pm}} \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\partial^* E_j^{\pm} \setminus \Sigma_j} \nu_{E_j^{\pm}} \, \mathrm{d}\mathcal{H}^{n-1} \\ &= \pm \int_{\mathrm{t}(E_j^{\pm})} \nu_{\Sigma_j} \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\partial^* E_j^{\pm} \setminus \mathrm{t}(E_j^{\pm})} \nu_{E_j^{\pm}} \, \mathrm{d}\mathcal{H}^{n-1}\,, \end{aligned}$$

which implies (4.2.39).

From (4.2.13) we obtain that

$$\begin{aligned} \lim_j \int_{\mathrm{t}(E_j^{\pm})} &|\nu_{\Sigma_j}(y) - \nu_{\Sigma}(x_0)| \, \mathrm{d}\mathcal{H}^{n-1}(y) \\ &\le \lim_j \int_{\Sigma_j \cap \mathrm{R}_1} |\nu_{\Sigma_j}(y) - \nu_{\Sigma}(x_0)| \, \mathrm{d}\mathcal{H}^{n-1}(y) = 0\,. \end{aligned} \tag{4.2.42}$$

Therefore, thanks to the continuity of ψ, to hypothesis (H3), and to equalities (4.2.39) and (4.2.42), we get that

$$\lim_j \left| \psi\Big(x_0, \int_{\mathrm{t}(E_j^{\pm})} \nu_{\Sigma_j}(y) \, \mathrm{d}\mathcal{H}^{n-1}(y)\Big) - \psi(x_0, \nu_{\Sigma}(x_0)) \, \mathcal{H}^{n-1}(\mathrm{t}(E_j^{\pm})) \right| = 0\,, \tag{4.2.43}$$

and, by Jensen inequality, for every j it holds

$$\begin{aligned} \psi\Big(x_0, \int_{\mathrm{t}(E_j^{\pm})} \nu_{\Sigma_j}(y) \, \mathrm{d}\mathcal{H}^{n-1}(y)\Big) &= \psi\Big(x_0, \int_{\partial^* E_j^{\pm} \setminus \mathrm{t}(E_j^{\pm})} \nu_{E_j^{\pm}}(y) \, \mathrm{d}\mathcal{H}^{n-1}(y)\Big) \\ \le \int_{\partial^* E_j^{\pm} \setminus \mathrm{t}(E_j^{\pm})} \psi(x_0, \nu_{E_j^{\pm}}(y)) \, \mathrm{d}\mathcal{H}^{n-1}(y) &= \int_{A_j^{\pm} \cap \partial^* E_j^{\pm}} \psi(x_0, \nu_{E_j^{\pm}}(y)) \, \mathrm{d}\mathcal{H}^{n-1}(y)\,, \end{aligned} \tag{4.2.44}$$

where in the last step we have used the equality $\partial^* E_j^{\pm} \setminus \mathrm{t}(E_j^{\pm}) = A_j^{\pm} \cap \partial^* E_j^{\pm}$.

We are now ready to estimate from below $\gamma(x_0)$ in terms of $g(x_0, u^+(x_0), u^-(x_0))$ and then to conclude the blow-up argument on Σ. Recalling inequality (4.2.38) and the inclusions $\partial^* E_j^{\pm} \setminus \Sigma_j \subseteq S_{v_j} \cap A_j^{\pm} \subseteq S_{v_j} \cap \mathrm{R}_1 \setminus \Sigma_j$, we can write (4.2.30) as

$$\begin{aligned}\gamma(x_0) + \varepsilon \geq \limsup_j \Big(& \int_{\Omega_j \cap A_j^+ \cap \partial^* E_j^+} \psi(x_0, \nu_{E_j^+}(y))\, \mathrm{d}\mathcal{H}^{n-1}(y) \\ & + \int_{\Omega_j \cap A_j^+ \cap (S_{v_j} \setminus \partial^* E_j^+) \cap \mathrm{R}_1} \psi(x_0, \nu_{v_j}(y))\, \mathrm{d}\mathcal{H}^{n-1}(y) + \int_{\Omega_j \cap A_j^- \cap \partial^* E_j^-} \psi(x_0, \nu_{E_j^-}(y))\, \mathrm{d}\mathcal{H}^{n-1}(y) \\ & + \int_{\Omega_j \cap A_j^- \cap (S_{v_j} \setminus \partial^* E_j^-) \cap \mathrm{R}_1} \psi(x_0, \nu_{v_j}(y))\, \mathrm{d}\mathcal{H}^{n-1}(y) + \int_{\mathrm{t}(E_j^+) \cup (\mathrm{t} E_j^-)} g(x_0, v_j^+(y), v_j^-(y))\, \mathrm{d}\mathcal{H}^{n-1}(y) \\ & + g(x_0, u^+(x_0), u^-(x_0))\, \mathcal{H}^{n-1}((\Sigma_j \setminus (\mathrm{t}(E_j^+) \cup \mathrm{t}(E_j^-)) \cap \mathrm{R}_1) \Big).\end{aligned} \tag{4.2.45}$$

Taking into account (4.2.38)-(4.2.44) and splitting the set $\mathrm{t}(E_j^+) \cup \mathrm{t}(E_j^-)$ into the union of the pairwise disjoint sets $\mathrm{t}(E_j^+) \setminus \mathrm{t}(E_j^-)$, $\mathrm{t}(E_j^-) \setminus \mathrm{t}(E_j^+)$, and $\mathrm{t}(E_j^+) \cap \mathrm{t}(E_j^-)$, from (4.2.45) we obtain

$$\begin{aligned}\gamma(x_0) + \varepsilon \geq \limsup_j \Big(& \psi(x_0, \nu_\Sigma(x_0))(\mathcal{H}^{n-1}(\mathrm{t}(E_j^+)) + \mathcal{H}^{n-1}(\mathrm{t}(E_j^-))) \\ & + \int_{\mathrm{t}(E_j^+) \setminus \mathrm{t}(E_j^-)} g(x_0, v_j^+(y), u^-(x_0)) \mathrm{d}\mathcal{H}^{n-1}(y) + \int_{\mathrm{t}(E_j^-) \setminus \mathrm{t}(E_j^+)} g(x_0, u^+(x_0), v_j^-(y)) \mathrm{d}\mathcal{H}^{n-1}(y) \\ & + \int_{\mathrm{t}(E_j^+) \cap \mathrm{t}(E_j^-)} g(x_0, v_j^+(y), v_j^-(y)) \mathrm{d}\mathcal{H}^{n-1}(y) \\ & + g(x_0, u^+(x_0), u^-(x_0)) \mathcal{H}^{n-1}((\Sigma_j \setminus (\mathrm{t}(E_j^+) \cup \mathrm{t}(E_j^-)) \cap \mathrm{R}_1) \Big) \\ = \limsup_j \Big(& \psi(x_0, \nu_\Sigma(x_0)) \big(\mathcal{H}^{n-1}(\mathrm{t}(E_j^+) \setminus \mathrm{t}(E_j^-)) + \mathcal{H}^{n-1}(\mathrm{t}(E_j^-) \setminus \mathrm{t}(E_j^+)) \\ & + 2\mathcal{H}^{n-1}(\mathrm{t}(E_j^+) \cap \mathrm{t}(E_j^-))\big) + \int_{\mathrm{t}(E_j^+) \setminus \mathrm{t}(E_j^-)} g(x_0, v_j^+(y), u^-(x_0)) \mathrm{d}\mathcal{H}^{n-1}(y) \\ & + \int_{\mathrm{t}(E_j^-) \setminus \mathrm{t}(E_j^+)} g(x_0, u^+(x_0), v_j^-(y)) \mathrm{d}\mathcal{H}^{n-1}(y) + \int_{\mathrm{t}(E_j^+) \cap \mathrm{t}(E_j^-)} g(x_0, v_j^+(y), v_j^-(y)) \mathrm{d}\mathcal{H}^{n-1}(y) \\ & + g(x_0, u^+(x_0), u^-(x_0))\, \mathcal{H}^{n-1}((\Sigma_j \setminus (\mathrm{t}(E_j^+) \cup \mathrm{t}(E_j^-))) \cap \mathrm{R}_1) \Big).\end{aligned}$$

Using (H7) in the previous inequality we get

$$\begin{aligned}\gamma(x_0) + \varepsilon &\geq g(x_0, u^+(x_0), u^-(x_0)) \limsup_j \mathcal{H}^{n-1}(\Sigma_j \cap \mathrm{R}_1) \\ &= g(x_0, u^+(x_0), u^-(x_0))\, \mathcal{H}^{n-1}(T_{x_0}(\Sigma) \cap \mathrm{R}_1),\end{aligned} \tag{4.2.46}$$

where in the last equality we have used the fact that $\nu_\Sigma(x_0)$ is normal to Σ at x_0. Passing to the limit in (4.2.46) as $\varepsilon \searrow 0$ we get

$$\gamma(x_0) \geq g(x_0, u^+(x_0), u^-(x_0))\, \mathcal{H}^{n-1}(T_{x_0}(\Sigma) \cap \mathrm{R}_1)$$

for $\mathcal{H}^{n-1}$-a.e. $x_0 \in \Sigma$. In view of (4.2.19) we have (4.2.10).

Let us define the functional

$$\Psi_U(v) := \int_{U\cap S_v} \psi(x,\nu_v)\,\mathrm{d}\mathcal{H}^{n-1}$$

for every $v \in BV(U;\mathrm{N})$, and its localized version

$$\Psi_U(v,B) := \int_{U\cap S_v\cap B} \psi(x,\nu_v)\,\mathrm{d}\mathcal{H}^{n-1}$$

for every $v \in BV(U;\mathrm{N})$ and every $B \in \mathcal{B}(U')$, where we recall that $U' = (\Omega'\setminus\Omega)\cup U$. We already know that Ψ_U is lower semicontinuous in $BV(U;\mathrm{N})$ with respect to the pointwise convergence (see [6, 8]). Now we show, using the blow-up technique, that (4.2.11) holds for $\mathcal{H}^{n-1}$-a.e. $x \in S_u\setminus\Sigma$. Indeed, let $x \in S_u\setminus\Sigma$ be such that

$$x \notin \overline{\Sigma}\,, \tag{4.2.47}$$

$$\text{there exists the approximate unit normal vector } \nu_u(x) \text{ to } S_u \text{ at } x, \tag{4.2.48}$$

$$\lim_{\rho\searrow 0}\frac{1}{\rho^{n-1}}\int_{S_u\cap \mathrm{B}_\rho(x)} |\nu_u(y)-\nu_u(x)|\,\mathrm{d}\mathcal{H}^{n-1}(y) = 0\,, \tag{4.2.49}$$

$$\text{there exists } \lim_{\rho\searrow 0}\frac{\mu(\mathrm{B}_\rho(x))}{\mathcal{H}^{n-1}(\mathrm{B}_\rho(x)\cap(S_u\setminus\Sigma))} = \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\lfloor(S_u\setminus\Sigma)}(x)\,. \tag{4.2.50}$$

We notice that properties (4.2.47)-(4.2.50) are satisfied by $\mathcal{H}^{n-1}$-a.e. $x \in S_u\setminus\Sigma$ as a consequence of hypotheses (4.1.2), of well-known properties of BV functions, and of the Besicovitch differentiation theorem.

Let $\rho_j \searrow 0$ be such that, for every $j \in \mathbb{N}$, $\mathrm{B}_{\rho_j}(x) \subseteq U\setminus\Sigma$ and $\mathcal{F}_U(u_k,\mathrm{B}_{\rho_j}(x)) \to \mu(\mathrm{B}_{\rho_j}(x))$ as $k \to +\infty$. Then, in view of the continuity of ψ, of the definition (4.2.9) of $\mathcal{F}_U$, and of conditions (4.2.47) and (4.2.49), we have

$$\begin{aligned}
\lim_{\rho\searrow 0}\frac{\mu(\mathrm{B}_\rho(x))}{\mathcal{H}^{n-1}(\mathrm{B}_\rho(x)\cap(S_u\setminus\Sigma))} &= \lim_j \frac{\mu(\mathrm{B}_{\rho_j}(x))}{\mathcal{H}^{n-1}(\mathrm{B}_{\rho_j}(x)\cap S_u)}\\
&= \lim_j\lim_k \frac{\mathcal{F}_U(u_k,\mathrm{B}_{\rho_j}(x))}{\mathcal{H}^{n-1}(\mathrm{B}_{\rho_j}(x)\cap S_u)} = \lim_j\lim_k \frac{\Psi_U(u_k,\mathrm{B}_{\rho_j}(x))}{\mathcal{H}^{n-1}(\mathrm{B}_{\rho_j}(x)\cap S_u)}\\
&\geq \lim_j \frac{\Psi_U(u,\mathrm{B}_{\rho_j}(x))}{\mathcal{H}^{n-1}(\mathrm{B}_{\rho_j}(x)\cap S_u)} = \psi(x,\nu_u(x))\,.
\end{aligned} \tag{4.2.51}$$

Since (4.2.50) holds, the previous inequality implies (4.2.11). This concludes the proof of the lemma. □

We are now ready to prove Theorem 4.2.4.

Proof of Theorem 4.2.4. Let ψ, g, N, U, η, u_k, u be as in the statement of the theorem, and let $U' := (\Omega'\setminus\Omega)\cup U$, as in Lemma 4.2.5.

Assume that

$$\liminf_k \mathcal{F}_U(u_k, U\cup\Sigma) < +\infty. \tag{4.2.52}$$

Up to a subsequence, we may suppose that the lim inf in (4.2.52) is a limit and that there exists $M > 0$ such that $\mathcal{F}_U(u_k, U\cup\Sigma) \le M$. Then the sequence of nonnegative measures $\mathcal{F}_U(u_k,\cdot)$ is bounded in $\mathcal{M}_b(U')$. Therefore, there exists a nonnegative measure $\mu \in \mathcal{M}_b(U')$ such that, up to a subsequence, $\mathcal{F}_U(u_k,\cdot) \rightharpoonup \mu$ weakly* in $\mathcal{M}_b(U')$.

Applying Lemma 4.2.5 and recalling the definition (4.2.9) of $\mathcal{F}_U$, we get that

$$\begin{aligned}\mathcal{F}_U(u, U\cup\Sigma) = \int_{U\cap S_u\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} \le \mu(U\cup\Sigma) \le \mu(U') \\ \le \liminf_k \mathcal{F}_U(u_k, U') = \liminf_k \mathcal{F}_U(u_k, U\cup\Sigma)\,,\end{aligned}$$

and the proof is thus concluded. □

Remark 4.2.6. We notice that, if we assume g to be symmetric on $\mathbb{R}^m\times\mathbb{R}^m$, that is, $g(x,s,t) = g(x,t,s)$ for $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma$ and every $s,t\in\mathbb{R}^m$, then the orientability property given in Definition 1.1.5 is not needed to prove Theorem 4.2.4 and Lemma 4.2.5: indeed in this case it is enough to assume Σ to be a Lipschitz manifold of dimension $n-1$.

In the following theorem we prove the lower semicontinuity of the functional $\mathcal{F}$ with respect to the weak convergence in $SBV^p(\Omega;\mathbb{R}^m)$, $p\in(1,+\infty)$.

Theorem 4.2.7. *Let $p\in(1,+\infty)$. Let ψ and g satisfy (H1)-(H7). Then the functional $\mathcal{F}$ is lower semicontinuous with respect to the weak convergence in $SBV^p(\Omega;\mathbb{R}^m)$.*

Proof. Through this proof, the superscript j, with $1\le j\le m$, stands for the j-th component of a vector in $\mathbb{R}^m$.

Thanks to Lemma 4.2.2 we restrict our attention to the case of a nonnegative Carathéodory function g.

We apply the approximation argument of [6, Theorem 3.3]. Let $u_k, u \in SBV^p(\Omega;\mathbb{R}^m)$ be such that u_k converges to u weakly in $SBV^p(\Omega;\mathbb{R}^m)$. By Definition 1.2.7, we have that

$$\sup_k \|u_k\|_\infty < +\infty\,, \qquad \sup_k \|\nabla u_k\|_p < +\infty\,, \qquad \sup_k \mathcal{H}^{n-1}(S_{u_k}) < +\infty\,. \tag{4.2.53}$$

Hence, for the sake of simplicity, we may assume that u_k takes values in $(0,1)^m$ for every k. Moreover, thanks to (4.2.53) and to hypotheses (4.1.2), (H2), (H5), and (H7), we have

$$\liminf_k \mathcal{F}(u_k) < +\infty\,. \tag{4.2.54}$$

By the second inequality in (4.2.53) for every $l\in\mathbb{N}$, $l\ge1$, we can find an open subset A_l of Ω such that

$$\bigcup_{k\in\mathbb{N}} S_{u_k}\cup S_u \subseteq A_l\,, \qquad \sup_k \int_{A_l} |\nabla u_k|\,\mathrm{d}x < 2^{-l}\,,$$

and $\{x \in \Omega : d(x, \Sigma \cup \partial\Omega) < \eta_l\} \subseteq A_l$ for some $\eta_l > 0$. We also set $B_{k,l} := A_l \setminus S_{u_k}$.

Let us fix $l \in \mathbb{N}$. By the Coarea Formula, for every $k \in \mathbb{N}$, every $i = 1, \ldots, l$, and every $j = 1, \ldots, m$, we can find $\xi^j_{i,k}$ such that

$$\xi^j_{i,k} \in \Big(\frac{i-1}{l}, \frac{2i-1}{2l}\Big], \tag{4.2.55}$$

$$\{x \in \Omega : u^j_k(x) > \xi^j_{i,k}\} \text{ is of finite perimeter,} \tag{4.2.56}$$

$$\mathcal{L}^n(\{x \in \Omega : u^j_k(x) = \xi^j_{i,k}\}) = 0\,, \tag{4.2.57}$$

$$\begin{aligned} &\mathcal{H}^{n-1}(B_{k,l} \cap \partial^*\{x \in \Omega : u^j_k(x) > \xi^j_{i,k}\}) \\ &\quad \le 2l \int_{\frac{i-1}{l}}^{\frac{i}{l}} \mathcal{H}^{n-1}(B_{k,l} \cap \partial^*\{x \in \Omega : u^j_k(x) > t\})\,\mathrm{d}t \le 2l|Du^l_k|(B_{k,l})\,. \end{aligned} \tag{4.2.58}$$

We set also $\xi^j_{0,k} := 0$ and $\xi^j_{l+1,k} := 1$.

We denote by $\mathcal{S}$ the family of functions $\sigma \colon \{1, \ldots, m\} \to \{0, \ldots, l\}$. For every $\sigma \in \mathcal{S}$ we define $\eta^j_\sigma := \sigma(j)/l$ and

$$\begin{aligned} &Q_{\sigma,k} := \{s \in \mathbb{R}^m : \xi^j_{\sigma(j),k} < s^j < \xi^j_{\sigma(j)+1,k} \text{ for } j = 1, \ldots, m\}\,, \\ &E_{\sigma,k} := \{x \in \Omega : u_k(x) \in Q_{\sigma,k}\}\,. \end{aligned} \tag{4.2.59}$$

We notice that $\eta_\sigma \in \overline{Q}_{\sigma,k}$ and the sets $\{E_{\sigma,k}\}_{\sigma \in \mathcal{S}}$ are pairwise disjoint and of finite perimeter by (4.2.56).

For every k we define a piecewise constant function v_k by

$$v_k(x) := \begin{cases} \eta_\sigma & \text{if } x \in E_{\sigma,k} \text{ for some } \sigma \in \mathcal{S}, \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.2.60}$$

If we set $\mathrm{N} := \{\eta_\sigma\}_{\sigma \in \mathcal{S}}$, from (4.2.56) we infer $v_k \in BV(\Omega; \mathrm{N})$. Moreover, by construction of η_σ and of v_k, we have that $\|u_k - v_k\|_{\infty,\Omega} \le 2m/l$ and $\|u^\pm_k - v^\pm_k\|_{\infty,\Sigma} \le 2m/l$.

We now estimate $\mathcal{F}_{A_l}(v_k, A_l \cup \Sigma)$. Since $A_l \setminus B_{k,l} \subseteq S_{u_k}$, we get

$$\begin{aligned} &\int_{A_l \cap S_{v_k} \setminus (B_{k,l} \cup \Sigma)} \psi(x, \nu_{v_k}) \mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g(x, v^+_k, v^-_k) \mathrm{d}\mathcal{H}^{n-1} \\ &\le \int_{S_{u_k} \setminus \Sigma} \psi(x, \nu_{u_k}) \mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g(x, u^+_k, u^-_k) \mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma \omega\Big(x, \frac{2\sqrt{2}m}{l}\Big) \mathrm{d}\mathcal{H}^{n-1}\,, \end{aligned} \tag{4.2.61}$$

where ω is a modulus of continuity defined as in (4.2.5) with $K = [0,1]^m \times [0,1]^m$. We recall that $\omega(\cdot, \delta) \to 0$ in $L^1(\Sigma)$ as $\delta \to 0$. By (H2) and (4.2.58), on the set $B_{k,l}$

we have

$$\begin{aligned}\int_{S_{v_k}\cap B_{k,l}\setminus\Sigma} \psi(x,\nu_{v_k})\,\mathrm{d}\mathcal{H}^{n-1} &\le c_2\,\mathcal{H}^{n-1}(B_{k,l}\cap S_{v_k}) \le c_2\,\mathcal{H}^{n-1}\Big(B_{k,l}\cap \bigcup_{\sigma\in\mathcal{S}}\partial^* E_{\sigma,k}\Big)\\ &\le c_2\sum_{j=1}^{m}\sum_{i=1}^{l}\mathcal{H}^{n-1}(B_{k,l}\cap\partial^*\{x\in\Omega:\, u_k^j(x)>\xi_{i,k}^j\}) \qquad (4.2.62)\\ &\le 2c_2 l\sum_{j=1}^{m}|Du_k^j|(B_{k,l}) \le 2c_2 ml|Du_k|(B_{k,l}) \le Cl2^{1-l}\end{aligned}$$

for some $C>0$ independent of l. Summing up (4.2.61) and (4.2.62) and recalling definition (4.2.9) of $\mathcal{F}_{A_l}$, we obtain

$$\begin{aligned}\mathcal{F}_{A_l}(v_k, A_l\cup\Sigma) &= \int_{A_l\cap S_{v_k}\setminus\Sigma}\psi(x,\nu_{v_k})\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g(x,v_k^+,v_k^-)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\le \int_{S_{u_k}\setminus\Sigma}\psi(x,\nu_{u_k})\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g(x,u_k^+,u_k^-)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\quad+\int_\Sigma \omega\Big(x,\frac{2\sqrt{2}m}{l}\Big)\,\mathrm{d}\mathcal{H}^{n-1} + Cl2^{1-l} \qquad (4.2.63)\\ &= \mathcal{F}(u_k) + \int_\Sigma \omega\Big(x,\frac{2\sqrt{2}m}{l}\Big)\,\mathrm{d}\mathcal{H}^{n-1} + Cl2^{1-l}\,.\end{aligned}$$

Assumptions (4.1.2) and (H2), together with inequalities (4.2.54) and (4.2.63), imply that

$$\sup_k \mathcal{H}^{n-1}(S_{v_k}\cap A_l) < +\infty\,.$$

Hence v_k satisfies the hypotheses of the compactness Theorem 1.2.8 in $SBV(A_l;\mathbb{R}^m)$: there exists $w_l\in SBV(A_l;\mathbb{R}^m)$ such that, up to a subsequence, $v_k\to w_l$ pointwise $\mathcal{L}^n$-a.e. in A_l. Moreover, $w_l\in BV(A_l;\mathrm{N})$. Thus, we are in a position to apply Theorem 4.2.4 on A_l:

$$\begin{aligned}\mathcal{F}_{A_l}(w_l, A_l\cup\Sigma) &\le \liminf_k\, \mathcal{F}_{A_l}(v_k, A_l\cup\Sigma)\\ &\le \liminf_k\, \mathcal{F}(u_k) + \int_\Sigma \omega\Big(x,\frac{2\sqrt{2}m}{l}\Big)\,\mathrm{d}\mathcal{H}^{n-1} + Cl2^{1-l}\,. \qquad (4.2.64)\end{aligned}$$

Since $u_k\to u$ and $v_k\to w_l$ pointwise $\mathcal{L}^n$-a.e. in A_l, we have that

$$\|w_l-u\|_{\infty,A_l}\le 2m/l \qquad\text{and}\qquad \|w_l^\pm-u^\pm\|_{\infty,\Sigma}\le 2m/l\,. \qquad (4.2.65)$$

In addition, for every $\sigma\in\mathcal{S}$ there exists $E_{\sigma,l}$ of finite perimeter such that

$$w_l = \sum_{\sigma\in\mathcal{S}}\eta_\sigma\,\mathbf{1}_{E_{\sigma,l}}\,.$$

Up to a subsequence, we may assume that

$$\xi^j_{i,k} \to \xi^j_i \in \Big[\frac{i-1}{l}, \frac{2i-1}{2l}\Big]\,. \tag{4.2.66}$$

We define the cube

$$Q_\sigma := \{s \in \mathbb{R}^m : \, \xi^j_{\sigma(j)} < s^j < \xi^j_{\sigma(j)+1}\}\,.$$

Recalling the pointwise convergences of u_k to u and of v_k to w_l in A_l, from (4.2.55), (4.2.59), (4.2.60), and (4.2.66) we easily deduce that $\mathcal{L}^n(E_{\sigma,l} \setminus u^{-1}(\overline{Q}_\sigma)) = 0$. Thus, up to a negligible set, we have

$$E_{\sigma,l} \subseteq A_l \cap u^{-1}(\overline{Q}_\sigma)\,. \tag{4.2.67}$$

We now pass to the limit as $l \to +\infty$. For every $\varepsilon > 0$, let $l_0 \in \mathbb{N}$ be such that $\mathrm{diam}(Q_\sigma) < \varepsilon/3$ for every $\sigma \in \mathcal{S}$ and every $l \geq l_0$. Then, for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$ such that $|u^+(x) - u^-(x)| > \varepsilon$ we have that the sets

$$\{y \in \Omega : \, |u(y) - u^+(x)| < \varepsilon/3\} \qquad \text{and} \qquad \{y \in \Omega : \, |u(y) - u^-(x)| < \varepsilon/3\}$$

have density $1/2$ at x. Therefore, from (4.2.67) we deduce that, up to an $\mathcal{H}^{n-1}$-negligible set,

$$S_\varepsilon := \{x \in S_u : \, |u^+(x) - u^-(x)| > \varepsilon\} \subseteq A_l \cap S_{w_l}\,. \tag{4.2.68}$$

In view of (4.2.68), we have that $\nu_u = \pm\nu_{w_l}$ $\mathcal{H}^{n-1}$-a.e. in S_ε for every $l \geq l_0$, and, by (4.2.65), $\|w_l^\pm - u^\pm\|_{\infty,\Sigma} \to 0$ as $l \to +\infty$. Thus, recalling (4.2.64) and applying Fatou Lemma, we get

$$\begin{aligned}
&\int_{S_\varepsilon \setminus \Sigma} \psi(x, \nu_u)\, \mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g(x, u^+, u^-)\, \mathrm{d}\mathcal{H}^{n-1} \\
&\qquad \leq \liminf_l \int_{S_\varepsilon \setminus \Sigma} \psi(x, \nu_{w_l})\, \mathrm{d}\mathcal{H}^{n-1} + \liminf_l \int_\Sigma g(x, w_l^+, w_l^-)\, \mathrm{d}\mathcal{H}^{n-1} \\
&\qquad \leq \liminf_l \mathcal{F}_{A_l}(w_l, A_l \cup \Sigma) \leq \liminf_k \mathcal{F}(u_k)\,.
\end{aligned} \tag{4.2.69}$$

Since $S_\varepsilon \nearrow S_u$, we conclude the proof of the theorem by passing to the limit in (4.2.69) as $\varepsilon \searrow 0$. $\square$

We now conclude with the proof of Theorem 4.2.1.

Proof of Theorem 4.2.1. Let us assume that g is a nonnegative Carathéodory function such that, for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$, $g(x, \cdot, \cdot)$ is Lipschitz continuous with Lipschitz constant $\lambda > 0$. Let $u_k, u \in GSBV^p(\Omega; \mathbb{R}^m)$ be such that u_k converges to u weakly in $GSBV^p(\Omega; \mathbb{R}^m)$ and $\liminf_k \mathcal{F}(u_k) < +\infty$.

By Proposition 1.2.9, for every $h,k\in\mathbb{N}$ we have that

$$T_h(u_k):=(T_h(u_k^1),\dots,T_h(u_k^m))\in SBV^p(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)\,.$$

By definition of T_h and by the weak convergence of u_k in $GSBV^p(\Omega;\mathbb{R}^m)$, for every h the sequences $\{T_h(u_k)\}_k$ and $\{\nabla(T_h(u_k))\}_k$ are bounded in $L^\infty(\Omega;\mathbb{R}^m)$ and in $L^p(\Omega;\mathbb{M}^{m\times n})$, respectively. Moreover, $S_{T_h(u_k)}\subseteq S_{u_k}$ for every $h,k\in\mathbb{N}$. Therefore, by the compactness Theorem 1.2.8, we deduce that $T_h(u_k)$ converges to $T_h(u)$ weakly in $SBV^p(\Omega;\mathbb{R}^m)$ as $k\to+\infty$.

Let $h\in\mathbb{N}$ be fixed. We now construct a new function $g_h\colon\Sigma\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ such that $0\le g_h\le g$ and g_h satisfies (H4)-(H7). For every $x\in\Sigma$ and every $s,t\in\mathbb{R}^m$ we set

$$g_h(x,s,t):=\begin{cases} g(x,s,t) & \text{if } |s|,|t|<h\,,\\ \inf\limits_{\sigma\in\mathbb{R}^m} g(x,\sigma,t) & \text{if } |s|\ge h, |t|<h\,,\\ \inf\limits_{\tau\in\mathbb{R}^m} g(x,s,\tau) & \text{if } |s|<h, |t|\ge h\,,\\ \inf\limits_{\sigma,\tau\in\mathbb{R}^m} g(x,\sigma,\tau) & \text{if } |s|,|t|\ge h\,. \end{cases}$$

It is clear that $0\le g_h\le g$. Let us prove that g_h satisfies properties (H4)-(H7). By construction, g_h is a Borel function and $g_h(\cdot,0,0)=g(\cdot,0,0)\in L^1(\Sigma)$, hence (H4) and (H5) hold.

To prove (H6) we consider two sequences $s_j,t_j\in\mathbb{R}^m$ converging to s and t, respectively. By definition of g_h and by the continuity of $g(x,\cdot,\cdot)$, there is only one non-trivial alternative:

$$|s_j|,|s|\ge h \text{ and } |t_j|,|t|<h\,.$$

In this case, by the Lipschitz continuity of $g(x,\cdot,\cdot)$ we have that for $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma$ and every $\tau,\tau'\in\mathbb{R}^m$

$$\Big|\inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,\tau)-\inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,\tau')\Big|\le\lambda|\tau-\tau'|\,,$$

which implies that

$$\lim_j g_h(x,s_j,t_j)=\lim_j\inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,t_j)=\inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,t)=g_h(x,s,t)\,.$$

This concludes the proof of (H6).

To prove (4.1.6), we fix $s,s',t\in\mathbb{R}^m$ and distinguish between the cases $|t|<h$ and $|t|\ge h$. If $|t|<h$, since g satisfies (H7) we have that, for $\mathcal{H}^{n-1}$-a.e. $x\in\Sigma$,

$$\begin{aligned} g_h(x,s,t)\le g(x,s,t)&\le\inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,t)+\psi(x,\nu_\Sigma(x))\\ &\le g_h(x,s',t)+\psi(x,\nu_\Sigma(x))\,. \end{aligned}\tag{4.2.70}$$

Otherwise, if $|t|\ge h$,

$$\begin{aligned} g_h(x,s,t)&\le\inf_{\tau\in\mathbb{R}^m} g(x,s,\tau)\le\inf_{\sigma,\tau\in\mathbb{R}^m} g(x,\sigma,\tau)+\psi(x,\nu_\Sigma(x))\\ &\le g_h(x,s',t)+\psi(x,\nu_\Sigma(x))\,. \end{aligned}\tag{4.2.71}$$

Thanks to (4.2.70) and (4.2.71), we get that g_h satisfies (4.1.6). Inequality (4.1.7) can be proved in the same way. Therefore, g_h fulfills property (H7).

Finally, it is easy to see that for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$, every $s, t \in \mathbb{R}^m$, and every $h \in \mathbb{N}$

$$g_h(x, s, t) = g_h(x, T_h(s), T_h(t)) . \tag{4.2.72}$$

Let us define the functional $\mathcal{F}_h \colon GSBV^p(\Omega; \mathbb{R}^m) \to \mathbb{R}$ by

$$\mathcal{F}_h(v) := \int_{S_v \setminus \Sigma} \psi(x, \nu_v) \, \mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g_h(x, v^+, v^-) \, \mathrm{d}\mathcal{H}^{n-1} .$$

Since ψ and g_h satisfy (H1)-(H7), we can apply Theorem 4.2.7 to $\mathcal{F}_h$. Hence, in view of the weak convergence of $T_h(u_k)$ to $T_h(u)$ in $SBV^p(\Omega; \mathbb{R}^m)$, we get that

$$\mathcal{F}_h(T_h(u)) \leq \liminf_k \mathcal{F}_h(T_h(u_k)) . \tag{4.2.73}$$

As a consequence of (4.2.72), of the inclusion $S_{T_h(u_k)} \subseteq S_{u_k}$, and of the inequality $g_h \leq g$, we have that $\mathcal{F}_h(T_h(u_k)) \leq \mathcal{F}(u_k)$ for every h, k. Thus, from (4.2.73) we deduce that

$$\int_{S_{T_h(u)} \setminus \Sigma} \psi(x, \nu_u) \, \mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g_h(x, u^+, u^-) \, \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_k \mathcal{F}(u_k) . \tag{4.2.74}$$

Since $g_h(\cdot, u^+, u^-) \nearrow g(\cdot, u^+, u^-)$ pointwise $\mathcal{H}^{n-1}$-a.e. in Σ and $S_{T_h(u)} \nearrow S_u$, passing to the limit in (4.2.74) as $h \to +\infty$ we obtain

$$\mathcal{F}(u) \leq \liminf_k \mathcal{F}(u_k) ,$$

which concludes the proof of the theorem in the particular case of a nonnegative Carathéodory function g with $g(x, \cdot, \cdot)$ Lipschitz continuous with Lipschitz constant λ. The general case follows by Lemma 4.2.2. □

We now show that condition (H7) is also necessary for the lower semicontinuity of the functional $\mathcal{F}$.

Theorem 4.2.8. *Let ψ satisfy (H1)-(H3) and let g be a Carathéodory function such that* (4.1.5)*, (H8), and (H9) hold. Let $\mathcal{F} \colon GSBV^p(\Omega; \mathbb{R}^m) \to \overline{\mathbb{R}}$ be the functional defined in* (4.1.1)*. Assume that $\mathcal{F}$ is lower semicontinuous with respect to the weak convergence in $GSBV^p(\Omega; \mathbb{R}^m)$. Then ψ and g fulfill property (H7).*

Proof. Let $L > 0$ be the Lipschitz constant of Σ and $\Lambda := L\sqrt{n}$. Let us prove that g satisfies the inequality (4.1.6) on $\Sigma \cap \Omega$. Let $x_0 \in \Sigma \cap \Omega$ be such that $\nu_\Sigma(x_0)$ is normal to Σ at x_0, and let $\xi(x_0) \in \mathbb{S}^{n-1}$ be as in Definition 1.1.5. As in the proof of Lemma 4.2.5, we set $\mathrm{R}_\rho(x_0) := \mathrm{R}^\Lambda_{\rho, \xi(x_0)}(x_0)$, where $\mathrm{R}^C_{\rho,\xi}(x)$ is defined in (1.1.2). In particular, for ρ sufficiently small we may suppose that $\mathrm{R}_\rho(x_0) \subseteq \Omega$, that $\mathcal{H}^{n-1}(\Sigma \cap$

$\partial \mathrm{R}_\rho(x_0)) = 0$, that the function φ_{x_0} of Definition 1.1.4 is well-defined and Lipschitz continuous on the $(n-1)$-dimensional cube $\mathrm{Q}^{n-1}_{\rho,\xi(x_0)}(x_0)$, and that

$$\mathrm{R}_\rho(x_0) \cap \Sigma = \{y + \varphi_{x_0}(y)\xi(x_0) : y \in \mathrm{Q}^{n-1}_{\rho,\xi(x_0)}(x_0)\}.$$

We assume in addition that x_0 satisfies the following conditions:

$$x_0 \text{ is a Lebesgue point for } g(\cdot,\sigma,\tau) \text{ for every } \sigma,\tau \in \mathbb{Q}^m, \tag{4.2.75}$$

$$g(x_0,\cdot,\cdot) \text{ is continuous on } \mathbb{R}^m \times \mathbb{R}^m, \tag{4.2.76}$$

$$\lim_{\rho \searrow 0} \frac{1}{\rho^{n-1}} \int_{\mathrm{R}_\rho(x_0)\cap\Sigma} |\nu_\Sigma(x) - \nu_\Sigma(x_0)| \, \mathrm{d}\mathcal{H}^{n-1}(x) = 0. \tag{4.2.77}$$

We notice that properties (4.2.75)-(4.2.77) are satisfied for $\mathcal{H}^{n-1}$-a.e. $x_0 \in \Sigma \cap \Omega$.

We define the sets $A^\rho_\pm$ as in (4.2.23). For every $k \in \mathbb{N}$ we set

$$\begin{aligned} \Sigma_k &:= (\mathrm{R}_\rho(x_0) \cap \Sigma) + \tfrac{1}{k}\xi(x_0) \\ &= \Big\{ y + \big(\varphi_{x_0}(y) + \tfrac{1}{k}\big)\xi(x_0) : y \in \mathrm{Q}^{n-1}_{\rho,\xi(x_0)}(x_0) \Big\}, \end{aligned} \tag{4.2.78}$$

and

$$A^{\rho,k}_+ := \Big\{ y + t\xi(x_0) : y \in \mathrm{Q}^{n-1}_{\rho,\xi(x_0)}(x_0),\ \varphi_{x_0}(y) + \tfrac{1}{k} < t < x_0 \cdot \xi(x_0) + \Lambda\rho \Big\}. \tag{4.2.79}$$

It is easy to see that for k large enough we have $\Sigma_k, A^{\rho,k}_+ \subseteq A^\rho_+$ and

$$\nu_{\Sigma_k}(x) = \pm \nu_\Sigma\big(x - \tfrac{1}{k}\xi(x_0)\big) \tag{4.2.80}$$

for $\mathcal{H}^{n-1}$ a.e. $x \in \Sigma_k$.

Let us fix three distinct points $s, s', t \in \mathbb{Q}^m \setminus \{0\}$. We introduce the functions

$$u_k(x) := \begin{cases} s' & \text{if } x \in A^\rho_+ \setminus A^{\rho,k}_+, \\ s & \text{if } x \in A^{\rho,k}_+, \\ t & \text{if } x \in A^\rho_-, \\ 0 & \text{if } x \in \Omega \setminus \mathrm{R}_\rho(x_0), \end{cases} \qquad u(x) := \begin{cases} s & \text{if } x \in A^\rho_+, \\ t & \text{if } x \in A^\rho_-, \\ 0 & \text{if } x \in \Omega \setminus \mathrm{R}_\rho(x_0). \end{cases} \tag{4.2.81}$$

It is clear that $u_k, u \in GSBV^p(\Omega;\mathbb{R}^m)$ and that u_k converges to u weakly in $GSBV^p(\Omega;\mathbb{R}^m)$ as $k \to +\infty$. Moreover,

$$S_{u_k} = \partial \mathrm{R}_\rho(x_0) \cup \Sigma_k \cup (\Sigma \cap \mathrm{R}_\rho(x_0)) \qquad \text{and} \qquad S_u = \partial \mathrm{R}_\rho(x_0) \cup (\Sigma \cap \mathrm{R}_\rho(x_0)). \tag{4.2.82}$$

Thanks to the lower semicontinuity of the functional $\mathcal{F}$, to hypothesis (H3), and to (4.2.80)-(4.2.82), we have that

$$\begin{aligned} &\int_{\partial \mathrm{R}_\rho(x_0)} \psi(x, \nu_{R_\rho(x_0)}) \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\mathrm{R}_\rho(x_0)\cap\Sigma} g(x,s,t) \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\Sigma \setminus \mathrm{R}_\rho(x_0)} g(x,0,0) \, \mathrm{d}\mathcal{H}^{n-1} \\ &= \mathcal{F}(u) \le \liminf_k \mathcal{F}(u_k) = \liminf_k \int_{\Sigma_k} \psi\big(x, \nu_\Sigma\big(x - \tfrac{1}{k}\xi(x_0)\big)\big) \, \mathrm{d}\mathcal{H}^{n-1} \\ &+ \int_{\partial \mathrm{R}_\rho(x_0)} \psi(x, \nu_{R_\rho(x_0)}) \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\mathrm{R}_\rho(x_0)\cap\Sigma} g(x,s',t) \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\Sigma \setminus \mathrm{R}_\rho(x_0)} g(x,0,0) \, \mathrm{d}\mathcal{H}^{n-1}. \end{aligned} \tag{4.2.83}$$

Therefore, by the change of coordinates $y = x - \frac{1}{k}\xi(x_0)$ and taking into account the uniform continuity of ψ on $\overline{\Omega} \times \mathbb{S}^{n-1}$, from (4.2.83) we get

$$\begin{aligned} &\int_{\mathrm{R}_\rho(x_0)\cap\Sigma} g(x,s,t)\,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad\le \liminf_k \int_{\mathrm{R}_\rho(x_0)\cap\Sigma} \psi\big(x + \tfrac{1}{k}\xi(x_0), \nu_\Sigma(x)\big)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\mathrm{R}_\rho(x_0)\cap\Sigma} g(x,s',t)\,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad= \int_{\mathrm{R}_\rho(x_0)\cap\Sigma} \psi(x,\nu_\Sigma(x))\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\mathrm{R}_\rho(x_0)\cap\Sigma} g(x,s',t)\,\mathrm{d}\mathcal{H}^{n-1}\,. \end{aligned} \tag{4.2.84}$$

Dividing (4.2.84) by ρ^{n-1} and passing to the limit as $\rho \searrow 0$, thanks to properties (4.2.75)-(4.2.77) and to (H1), we obtain that

$$g(x_0,s,t) \le g(x_0,s',t) + \psi(x_0,\nu_\Sigma(x_0)) \tag{4.2.85}$$

for every triple of distinct points $s, s', t \in \mathbb{Q}^m \setminus \{0\}$. By density and by (4.2.76), we conclude that g satisfies (4.1.6) for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma \cap \Omega$. To prove the same result for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma \cap \partial\Omega$, we use a similar argument and take into account (4.1.5). The proof of (4.1.7) is analogous. □

We conclude this section with an existence result whose proof follows directly from Theorems 1.2.13 and 4.2.1. Let $W\colon \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ satisfy (1.2.8) and (1.2.9), and let $f\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function such that

$$a_3|s|^q - b_3(x) \le f(x,s) \le a_4|s|^q + b_4(x) \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}^m \tag{4.2.86}$$

for some $1 < q < +\infty$, $0 < a_3 \le a_4$, and $b_3, b_4 \in L^1(\Omega)$.

We define the functional $\mathcal{G}\colon L^q(\Omega;\mathbb{R}^m) \to \overline{\mathbb{R}}$ by

$$\mathcal{G}(u) := \int_\Omega W(x,\nabla u)\,\mathrm{d}x + \int_\Omega f(x,u)\,\mathrm{d}x + \int_{S_u\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} \tag{4.2.87}$$

for every $u \in GSBV^p(\Omega;\mathbb{R}^m) \cap L^q(\Omega;\mathbb{R}^m)$, and $\mathcal{G}(u) := +\infty$ otherwise in $L^q(\Omega;\mathbb{R}^m)$.

In the following theorem we state an existence result for the minimum problem

$$\min\{\mathcal{G}(u) : u \in L^q(\Omega;\mathbb{R}^m)\}\,. \tag{4.2.88}$$

Theorem 4.2.9. *Let ψ and g satisfy (H1)-(H7). Let $W\colon \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ satisfy* (1.2.8) *and* (1.2.9)*, and let $f\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function such that* (4.2.86) *holds. Then the minimum problem* (4.2.88) *admits a solution.*

Proof. The proof is based on the direct method of the calculus of variations. Let $u_k \in L^q(\Omega;\mathbb{R}^m)$ be a minimizing sequence for (4.2.88). Then $u_k \in GSBV^p(\Omega;\mathbb{R}^m)$ and, by hypotheses (1.2.8), (1.2.9), (4.1.2), (H1)-(H7), and (4.2.86), we have that $\|u_k\|_q$,

$\|\nabla u_k\|_p$, and $\mathcal{H}^{n-1}(S_{u_k})$ are bounded uniformly with respect to k. By the compactness Theorem 1.2.12, there exists $u \in GSBV^p(\Omega;\mathbb{R}^m) \cap L^q(\Omega;\mathbb{R}^m)$ such that, up to a subsequence, u_k converges to u weakly in $GSBV^p(\Omega;\mathbb{R}^m)$.

Applying Theorems 1.2.13 and 4.2.1, and the Fatou Lemma, we get that

$$\mathcal{G}(u) \leq \liminf_k \mathcal{G}(u_k)\,,$$

thus u is a solution of (4.2.88). □

4.3 Relaxation result

In this section we give a relaxation result for functionals of the form (4.1.1) in $GSBV^p(\Omega;\mathbb{R}^m)$, $p \in (1,+\infty)$.

Let us recall the setting of the problem. Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, let (Σ, ν_Σ) be an orientable Lipschitz manifold of dimension $n-1$ and Lipschitz constant L with $\Sigma \subseteq \overline{\Omega}$ and such that (4.1.2) holds. We consider a function $\psi\colon \overline{\Omega} \times \mathbb{R}^n \to [0,+\infty)$ satisfying properties (H1)-(H3), and a function $g\colon \Sigma \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ such that (A1)-(A4) hold (we refer to Section 4.1).

In the statement and in the proof of Theorem 4.3.3 we will use the following functions

$$g_1(x,s,t) := \min\left\{g(x,s,t),\ \inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,t) + \psi(x,\nu_\Sigma(x))\right\}, \tag{4.3.1}$$

$$g_2(x,s,t) := \min\left\{g(x,s,t),\ \inf_{\tau\in\mathbb{R}^m} g(x,s,\tau) + \psi(x,\nu_\Sigma(x))\right\}, \tag{4.3.2}$$

$$g_{12}(x,s,t) := \min\left\{g_1(x,s,t),\ \inf_{\tau\in\mathbb{R}^m} g_1(x,s,\tau) + \psi(x,\nu_\Sigma(x))\right\}, \tag{4.3.3}$$

$$g_{21}(x,s,t) := \min\left\{g_2(x,s,t),\ \inf_{\sigma\in\mathbb{R}^m} g_2(x,\sigma,t) + \psi(x,\nu_\Sigma(x))\right\}. \tag{4.3.4}$$

We will prove in Lemma 4.3.2 that $g_{12} = g_{21}$. In Theorem 4.3.3 we need g_{12} to satisfy the additional hypothesis (A5) stated in Section 4.1.

Remark 4.3.1. If (4.1.5) holds, it is easy to see that for every $s,t \in \mathbb{R}^m$

$$\begin{array}{ll}
g_1(x,s,t) = g_1(x,s,0) \quad \text{and} \quad g_2(x,s,t) = g(x,s,0) & \text{if } x \in \mathcal{N}^+\,,\\
g_1(x,s,t) = g(x,0,t) \quad \text{and} \quad g_2(x,s,t) = g_2(x,0,t) & \text{if } x \in \mathcal{N}^-\,,\\
g_{12}(x,s,t) = g_{21}(x,s,t) = g_1(x,s,0) & \text{if } x \in \mathcal{N}^+\,,\\
g_{12}(x,s,t) = g_{21}(x,s,t) = g_2(x,0,t) & \text{if } x \in \mathcal{N}^-\,,
\end{array}$$

where $\mathcal{N}^\pm$ are as in (4.1.4).

In the following lemma we discuss some properties of the functions introduced in (4.3.1)-(4.3.4).

Lemma 4.3.2. *Assume (A1)-(A4). Then the functions* $g_1, g_2, g_{12}, g_{21} \colon \Sigma \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ *are Borel measurable and satisfy the inequalities*

$$g_1,\, g_2,\, g_{12},\, g_{21} \geq -a\,, \tag{4.3.5}$$

$$g_1,\, g_2,\, g_{12},\, g_{21} \leq g\,. \tag{4.3.6}$$

Moreover, for every $x \in \Sigma$ *they are upper semicontinuous with respect to* (s,t)*. Finally,* g_{12} *and* g_{21} *fulfill property (H7) and*

$$g_{12}(x,s,t) = g_{21}(x,s,t) = \sup_{\gamma \in \Gamma_g} \gamma(x,s,t)\,, \tag{4.3.7}$$

where Γ_g *is the set of all functions* $\gamma \colon \Sigma \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ *satisfying (H7) and such that* $\gamma \leq g$*.*

Proof. Since for every $x \in \Sigma$ the function $g(x,\cdot,\cdot)$ is upper semicontinuous on $\mathbb{R}^m \times \mathbb{R}^m$, we have that

$$g_1(x,s,t) := \min\Big\{ g(x,s,t),\ \inf_{\sigma \in \mathbb{Q}^m} g(x,\sigma,t) + \psi(x,\nu_\Sigma(x)) \Big\}\,.$$

Since g is also Borel measurable, this implies that g_1 is Borel measurable and, for every $x \in \Sigma$, $g_1(x,\cdot,\cdot)$ is upper semicontinuous. The same argument applies to g_2, g_{12}, g_{21}. The inequalities (4.3.5) and (4.3.6) follow immediately from (A3) and (4.3.1)-(4.3.4).

Let us prove that g_{12} fulfills property (H7). By definitions (4.3.1) and (4.3.3), it is easy to see that g_1 satisfies (4.1.6) and g_{12} satisfies (4.1.7). Therefore, for every $x \in \Sigma$ and every $s, s', t \in \mathbb{R}^m$, the following inequalities hold:

$$g_{12}(x,s,t) \leq g_1(x,s,t) \leq g_1(x,s',t) + \psi(x,\nu_\Sigma(x))\,, \tag{4.3.8}$$

$$g_{12}(x,s,t) \leq \inf_{\tau \in \mathbb{R}^m} g_1(x,s,\tau) + \psi(x,\nu_\Sigma(x)) \leq \inf_{\tau \in \mathbb{R}^m} g_1(x,s',\tau) + 2\psi(x,\nu_\Sigma(x)). \tag{4.3.9}$$

From (4.3.3), (4.3.8), and (4.3.9), we infer that g_{12} satisfies (4.1.6), which completes the proof of (H7). A similar argument can be used for g_{21}.

We now prove (4.3.7). To this end, we first check that

$$g_1(x,s,t) = \sup_{\gamma \in \Gamma_g^1} \gamma(x,s,t)\,, \tag{4.3.10}$$

where Γ_g^1 is the set of all functions $\gamma \colon \Sigma \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ satisfying (4.1.6) for every $x \in \Sigma$ and such that $\gamma \leq g$. Let $G_1(x,s,t)$ be the right-hand side of (4.3.10). Since g_1 satisfies (4.1.6) and $g_1 \leq g$, we have that $g_1 \leq G_1$. Conversely, let $\gamma \in \Gamma_g^1$. Then

$$\gamma(x,s,t) \leq \inf_{\sigma \in \mathbb{R}^m} \gamma(x,\sigma,t) + \psi(x,\nu_\Sigma(x)) \leq \inf_{\sigma \in \mathbb{R}^m} g(x,\sigma,t) + \psi(x,\nu_\Sigma(x))$$

for every $x \in \Sigma$ and every $s, t \in \mathbb{R}^m$. Since $\gamma \leq g$, the previous inequality implies that $\gamma \leq g_1$. Taking the supremum for $\gamma \in \Gamma_g^1$, we deduce that $G_1 \leq g_1$. Since the opposite inequality has already been proved, we have that (4.3.10) holds. With the same argument it is possible to show that

$$g_2(x,s,t) = \sup_{\gamma\in\Gamma_g^2} \gamma(x,s,t)\,, \tag{4.3.11}$$

where Γ_g^2 is the set of all functions $\gamma\colon \Sigma\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ satisfying (4.1.7) for every $x\in\Sigma$ and such that $\gamma\le g$.

Since g_{12} satisfies (H7) and $g_{12}\le g$, we have that

$$g_{12}(x,s,t) \le \sup_{\gamma\in\Gamma_g} \gamma(x,s,t)\,. \tag{4.3.12}$$

For the converse inequality, let us fix $\gamma\in\Gamma_g$ and let $G(x,s,t)$ be the right-hand side of (4.3.12). Then, in view of (4.3.10), we have that $\gamma\le g_1$ and

$$\gamma(x,s,t) \le \inf_{\tau\in\mathbb{R}^m} \gamma(x,s,\tau)+\psi(x,\nu_\Sigma(x)) \le \inf_{\tau\in\mathbb{R}^m} g_1(x,s,\tau)+\psi(x,\nu_\Sigma(x)) \tag{4.3.13}$$

for every $x\in\Sigma$ and every $s,t\in\mathbb{R}^m$. In view of (4.3.13) we get that $\gamma\le g_{12}$. Thus, $G\le g_{12}$, which, together with (4.3.12), gives $g_{12}=G$. In the same way, using (4.3.11), we can show that $g_{21}=G$, and this concludes the proof of the lemma. □

Given $p\in(1,+\infty)$, we define the functional $\mathcal{F}\colon GSBV^p(\Omega;\mathbb{R}^m)\to\overline{\mathbb{R}}$ as in (4.1.1) and the functional $sc^-\mathcal{F}\colon GSBV^p(\Omega;\mathbb{R}^m)\to\overline{\mathbb{R}}$ as the greatest sequentially lower semicontinuous functional on $GSBV^p(\Omega;\mathbb{R}^m)$ which is less than or equal to $\mathcal{F}$. We are now ready to state the main theorem of this section.

Theorem 4.3.3. *Let ψ and g satisfy (H1)-(H3), (A1)-(A5), and* (4.1.5). *Then we have*

$$sc^-\mathcal{F}(u) = \int_{S_u\setminus\Sigma}\psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g_{12}(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} \tag{4.3.14}$$

for every $u\in GSBV^p(\Omega;\mathbb{R}^m)$.

For what follows, it is convenient to define the functionals $\mathcal{F}_{12},\mathcal{F}_1\colon GSBV^p(\Omega;\mathbb{R}^m)\to\overline{\mathbb{R}}$ by

$$\mathcal{F}_{12}(u) := \int_{S_u\setminus\Sigma}\psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g_{12}(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{4.3.15}$$

$$\mathcal{F}_{1}(u) := \int_{S_u\setminus\Sigma}\psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g_{1}(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{4.3.16}$$

for every $u\in GSBV^p(\Omega;\mathbb{R}^m)$. The functional $\mathcal{F}_1$ is "intermediate" between $\mathcal{F}$ and $\mathcal{F}_{12}$ and will be used in the proof of Theorem 4.3.3.

In order to prove Theorem 4.3.3 we need the following approximation lemma.

Lemma 4.3.4. *Let $r \in [1, +\infty)$. Then for every $u \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ and every $\varepsilon > 0$ there exists $v \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ such that*

$$\|v - u\|_{r,\Omega} < \varepsilon\,, \qquad \|\nabla v - \nabla u\|_{p,\Omega} < \varepsilon\,, \tag{4.3.17}$$

$$\mathcal{H}^{n-1}(S_v) < \mathcal{H}^{n-1}(S_u) + 4\mathcal{H}^{n-1}(\Sigma) + \varepsilon\,, \tag{4.3.18}$$

$$\mathcal{F}(v) < \mathcal{F}_{12}(u) + \varepsilon\,. \tag{4.3.19}$$

Proof. Let us set $\Sigma' := \big(\Sigma \setminus (\overline{\Sigma \cap \Omega})\big) \cup (\Sigma \cap \Omega)$. In view of hypotheses (4.1.2), we have

$$\mathcal{H}^{n-1}(\Sigma \setminus \Sigma') = 0\,. \tag{4.3.20}$$

Moreover, Σ' is open in the relative topology of Σ.

By Definition 1.1.5 and by Lindelöff theorem, there exists a sequence of points $x_i \in \Sigma'$ and corresponding $(n-1)$-dimensional rectangles Δ_{x_i}, intervals I_{x_i}, vectors $\xi(x_i) \in \mathbb{S}^{n-1}$, and Lipschitz functions $\varphi_{x_i} \colon \Delta_{x_i} \to \mathrm{I}_{x_i}$ such that the following conditions hold, where, for simplicity of notation, we have set $V_i := \{y + t\xi(x_i) : y \in \Delta_{x_i},\, t \in \mathrm{I}_{x_i}\}$:

$$V_i \cap \Sigma = \{y + \varphi_{x_i}(y)\xi(x_i) : y \in \Delta_{x_i}\}\,, \tag{4.3.21}$$

$$\nu_\Sigma(x) \cdot \xi(x_i) \text{ has constant sign for } x \in V_i \cap \Sigma\,, \tag{4.3.22}$$

$$\Sigma' \subseteq \bigcup_{i \in \mathbb{N}} \{y + \varphi_{x_i}(y)\xi(x_i) : y \in \Delta_{x_i}\}\,, \tag{4.3.23}$$

$$V_i \cap \Sigma \subset\subset \Omega \quad \text{or} \quad V_i \cap \Sigma \subset\subset \Sigma' \cap \partial\Omega\,. \tag{4.3.24}$$

As in the proof of Lemma 4.2.5, we define

$$A_i^{\pm} := \{y + t\xi(x_i) : y \in \Delta_{x_i},\, t \in \mathrm{I}_{x_i},\, t \gtrless \varphi_{x_i}(y)\}\,. \tag{4.3.25}$$

Therefore, for every $i \in \mathbb{N}$, Σ splits the set V_i into two disjoint connected open subsets A_i^+ and A_i^-, with $\nu_\Sigma(x)$ pointing towards A_i^+ for $\mathcal{H}^{n-1}$-a.e. $x \in V_i \cap \Sigma$.

Let u be as in the statement of the lemma. We set

$$B_{g_1} := \big\{x \in \Sigma' : g_1(x, u^+(x), u^-(x)) > \inf_{\tau \in \mathbb{R}^m} g_1(x, u^+(x), \tau) + \psi(x, \nu_\Sigma(x))\big\}\,, \tag{4.3.26}$$

where g_1 is defined in (4.3.1). Clearly, B_{g_1} is an $\mathcal{H}^{n-1}$-measurable subset of Σ'. By Remark 4.3.1, we have that $B_{g_1} \cap \partial\Omega \subseteq \mathcal{N}^-$, where the set $\mathcal{N}^-$ is defined in (4.1.4). This implies that $\nu_\Sigma(x) = -\nu_\Omega(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in B_{g_1} \cap \partial\Omega$. Therefore, from (4.3.24) and (4.3.25) we deduce that

$$\begin{aligned} &\mathcal{H}^{n-1}(V_i \cap B_{g_1} \cap \partial\Omega) > 0 \\ &\qquad \Longrightarrow \quad \nu_\Sigma = -\nu_\Omega \ \mathcal{H}^{n-1}\text{-a.e. in } V_i \cap \Sigma \text{ and } A_i^- \subseteq \Omega. \end{aligned} \tag{4.3.27}$$

Moreover, by (4.3.3), (4.3.20), and (4.3.26), for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ we have

$$g_{12}(x, u^+(x), u^-(x)) = \begin{cases} g_1(x, u^+(x), u^-(x)) & \text{if } x \in \Sigma \setminus B_{g_1}, \\ \inf\limits_{\tau \in \mathbb{R}^m} g_1(x, u^+(x), \tau) + \psi(x, \nu_\Sigma(x)) & \text{if } x \in B_{g_1}. \end{cases} \tag{4.3.28}$$

Given $\varepsilon > 0$, our first aim is to construct a new function $w \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ such that

$$\|w-u\|_{r,\Omega} < \frac{\varepsilon}{2}\,, \qquad \|\nabla w - \nabla u\|_{p,\Omega} < \frac{\varepsilon}{2}\,, \tag{4.3.29}$$

$$\mathcal{H}^{n-1}(S_w) < \mathcal{H}^{n-1}(S_u) + 2\mathcal{H}^{n-1}(\Sigma) + \frac{\varepsilon}{2}\,, \tag{4.3.30}$$

$$\mathcal{F}_1(w) < \mathcal{F}_{12}(u) + \frac{\varepsilon}{2}\,, \tag{4.3.31}$$

where $\mathcal{F}_1$ is defined in (4.3.16). Roughly speaking, the idea of the proof of (4.3.29)-(4.3.31) is to construct a sort of copy of the "bad" set B_{g_1} inside Ω near Σ. This modified version of B_{g_1} will be part of the jump set of the new function w which will be constructed in such a way that $w = u$ far from B_{g_1}, $w^+ = u^+$ on Σ, and $g_1(x, w^+(x), w^-(x))$ is close to $\inf_{\tau\in\mathbb{R}^m} g_1(x, u^+(x), \tau)$ for $x \in B_{g_1}$.

We now start our construction. Let us fix an auxiliary parameter $\delta > 0$ which will be chosen at the end of the proof in order to get (4.3.29)-(4.3.31) and (4.3.17)-(4.3.19). Given an enumeration $\{q_j\}_{j\in\mathbb{N}}$ of $\mathbb{Q}^m$, for every j we define

$$B^j_{g_1} := \bigl\{x \in B_{g_1} : g_1(x, u^+(x), q_j) < \inf_{\tau\in\mathbb{R}^m} g_1(x, u^+(x), \tau) + \delta\bigr\} \setminus \bigcup_{l=1}^{j-1} B^l_{g_1}\,. \tag{4.3.32}$$

The sets $\{B^j_{g_1}\}_{j\in\mathbb{N}}$ are pairwise disjoint $\mathcal{H}^{n-1}$-measurable subsets of Σ' such that $B_{g_1} = \bigcup_j B^j_{g_1}$. By taking suitable intersections with the sets V_i and their complements, it is not restrictive to assume that for every j there exists $i_j \in \mathbb{N}$ such that $B^j_{g_1} \subseteq V_{i_j}$ and $B^j_{g_1} \cap V_l = \emptyset$ for $l \neq i_j$.

The next step is to approximate B_{g_1} with the union of a finite number of relatively open subsets of Σ'. Let us set $M := \|u\|_{\infty,\Omega}$. Since $\{B^j_{g_1}\}_{j\in\mathbb{N}}$ are pairwise disjoint and $\mathcal{H}^{n-1}(\Sigma) < +\infty$, we can find $H \in \mathbb{N}$ such that

$$\mathcal{H}^{n-1}\Bigl(\bigcup_{j>H} B^j_{g_1}\Bigr) < \delta\,, \qquad \int_{\bigcup_{j>H} B^j_{g_1}} a \, d\mathcal{H}^{n-1} < \delta\,, \qquad \int_{\bigcup_{j>H} B^j_{g_1}} a_M \, d\mathcal{H}^{n-1} < \delta\,. \tag{4.3.33}$$

where a, $a_M \in L^1(\Sigma)$ have been defined in (A3) and (A4), respectively.

For every $j \in \{1, \ldots, H\}$, we choose a compact set $K_j \subseteq \Sigma'$ such that $K_j \subseteq B^j_{g_1}$ and

$$\mathcal{H}^{n-1}(B^j_{g_1} \setminus K_j) < \frac{\delta}{2^j}\,, \qquad \int_{B^j_{g_1}\setminus K_j} a \, d\mathcal{H}^{n-1} < \frac{\delta}{2^j}\,, \qquad \int_{B^j_{g_1}\setminus K_j} a_M \, d\mathcal{H}^{n-1} < \frac{\delta}{2^j}\,. \tag{4.3.34}$$

Let us set $\widetilde{M} := \max\{M, |q_1|, \ldots, |q_H|\}$, where q_j are associated to each $B^j_{g_1}$ through definition (4.3.32). Since $\mathcal{H}^{n-1}\lfloor\Sigma$ is a bounded Radon measure, we can find a family $\{U_j\}_{j=1}^H$ of relatively open subsets of Σ such that the following conditions hold, where $\partial_\Sigma U_j$ denotes the boundary of U_j in the relative topology of Σ:

$$K_j \subseteq U_j \subset\subset V_{i_j} \cap \Sigma\,, \quad \overline{U}_l \cap \overline{U}_j = \emptyset \text{ for } l \neq j, \tag{4.3.35}$$

$$\mathcal{H}^{n-1}(U_j \setminus K_j) < \frac{\delta}{2^j}\,, \qquad \int_{U_j\setminus K_j} a \,\mathrm{d}\mathcal{H}^{n-1} < \frac{\delta}{2^j}\,, \qquad \int_{U_j\setminus K_j} a_{\widetilde{M}} \,\mathrm{d}\mathcal{H}^{n-1} < \frac{\delta}{2^j} \tag{4.3.36}$$

$$\mathcal{H}^{n-2}(\partial_\Sigma U_j) < +\infty\,, \tag{4.3.37}$$

where $a_{\widetilde{M}} \in L^1(\Sigma)$ has been defined in (A4).

We now move each U_j inside $\Omega \setminus \Sigma$ by translation. Let $j \in \{1, \dots, H\}$ be fixed. Thanks to (4.3.24), (4.3.25), (4.3.27), and (4.3.35), we may choose $\eta_j > 0$ such that

$$U_j - \zeta\xi(x_{i_j}) \subseteq A^-_{i_j} \quad \text{and} \quad U_j - \zeta\xi(x_{i_j}) \subset\subset V_{i_j} \cap \Omega \quad \text{for every } \zeta \in (0, \eta_j]\,. \tag{4.3.38}$$

Moreover, by the uniform continuity of ψ on $\overline{\Omega} \times \mathbb{S}^{n-1}$ and by (4.3.37), we may assume that:

$$\sup_{x\in U_j} |\psi(x - \eta_j\xi(x_{i_j}), \nu_\Sigma(x)) - \psi(x, \nu_\Sigma(x))| \leq \frac{\delta}{2^j}\,, \tag{4.3.39}$$

$$2^j\eta_j \in \left(0, \delta \,\min\left\{1, \frac{1}{|q_j|^r}, \frac{1}{\mathcal{H}^{n-2}(\partial_\Sigma U_j)}\right\}\right)\,. \tag{4.3.40}$$

We denote by C_j the open "cylinders"

$$C_j := \bigcup_{\zeta\in(0,\eta_j)} (U_j - \zeta\xi(x_{i_j}))\,. \tag{4.3.41}$$

By possibly changing η_j, by (4.3.38) we may assume that

$$C_j \subseteq A^-_{i_j} \cap \Omega\,, \tag{4.3.42}$$

$$\{U_j - \eta_j\xi(x_{i_j})\}_{j=1}^H \text{ are pairwise disjoint,} \tag{4.3.43}$$

$$\{C_j\}_{j=1}^H \text{ are pairwise disjoint,} \tag{4.3.44}$$

$$\|u\|_{r,C_j} < \frac{\delta}{2^j} \quad \text{and} \quad \|\nabla u\|_{p,C_j} < \frac{\delta}{2^j}\,. \tag{4.3.45}$$

Moreover, if

$$L_j := \bigcup_{\zeta\in(0,\eta_j)} (\partial U_j - \zeta\xi(x_{i_j}))$$

is the lateral surface of the cylinder C_j, by (4.3.40) we have that

$$\mathcal{H}^{n-1}\Big(\bigcup_{j=1}^H L_j\Big) \leq \sum_{j=1}^H \eta_j\, \mathcal{H}^{n-2}(\partial_\Sigma U_j) < \sum_{j=1}^H \frac{\delta}{2^j} < \delta\,. \tag{4.3.46}$$

Note that the trasversality condition $\nu_\Sigma(x)\cdot\xi(x_{i_j})>0$ for $\mathcal{H}^{n-1}$-a.e. $x\in U_j$ implies that

$$\partial C_j = U_j \cup (U_j - \eta_j\xi(x_{i_j})) \cup L_j\,. \tag{4.3.47}$$

We are now ready to define the function $w\in SBV^p(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$ satisfying inequalities (4.3.29)-(4.3.31). For every $x\in\Omega$, we set

$$w(x) := \begin{cases} q_j & \text{if } x\in C_j \text{ for some } j\in\{1,\dots,H\},\\ u(x) & \text{if } x\in\Omega\setminus\displaystyle\bigcup_{j=1}^H C_j\,. \end{cases} \tag{4.3.48}$$

By definition, $\|w\|_{\infty,\Omega}=\widetilde{M}$, $\nabla w\in L^p(\Omega;\mathbb{M}^{m\times n})$, and

$$S_w \subseteq S_u\cup\Sigma\cup\bigcup_{j=1}^H \big(L_j\cup(U_j-\eta_j\xi(x_{i_j}))\big)\,, \tag{4.3.49}$$

thus, by (4.3.35) and (4.3.46), we get that

$$\mathcal{H}^{n-1}(S_w) < \mathcal{H}^{n-1}(S_u) + 2\mathcal{H}^{n-1}(\Sigma)+\delta\,. \tag{4.3.50}$$

Estimate (4.3.50) implies that $\mathcal{H}^{n-1}(S_w)<+\infty$, hence $w\in SBV^p(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$. Thanks to (4.3.40), (4.3.41), (4.3.45), and (4.3.48), we have that, for some $c_r>0$ independent of δ,

$$\begin{aligned} \|w-u\|_{r,\Omega} &= \sum_{j=1}^H \|q_j-u\|_{r,C_j} \le \sum_{j=1}^H |q_j|\,(\mathcal{L}^n(C_j))^{1/r} + \|u\|_{r,C_j} \\ &< \sum_{j=1}^H |q_j|\,\eta_j^{1/r}\,(\mathcal{H}^{n-1}(U_j))^{1/r}+\delta \\ &\le \sum_{j=1}^H \Big(\frac{\delta}{2^j}\Big)^{1/r} (\mathcal{H}^{n-1}(\Sigma))^{1/r}+\delta < c_r\delta^{1/r}+\delta\,, \end{aligned} \tag{4.3.51}$$

and

$$\|\nabla w-\nabla u\|_{p,\Omega} = \sum_{j=1}^H \|\nabla u\|_{p,C_j} < \delta\,. \tag{4.3.52}$$

We now have to estimate $\mathcal{F}_1(w)$ in terms of $\overline{\mathcal{F}}(u)$. Let us start with the the jump term in $\mathcal{F}_1(w)$. Since $U_j\subseteq\Sigma$ for every $j=1,\dots,H$, for $\mathcal{H}^{n-1}$-a.e. $x\in U_j-\eta_j\xi(x_{i_j})$ we have

$$\nu_{C_j}(x) = \nu_\Sigma(x+\eta_j\xi(x_{i_j}))\,,$$

which implies that

$$\nu_w(x) = \pm\nu_\Sigma(x+\eta_j\xi(x_{i_j})) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x\in S_w\cap(U_j-\eta_j\xi(x_{i_j}))\,. \tag{4.3.53}$$

Moreover, it is clear that

$$\begin{aligned} \nu_w(x) &= \pm\nu_{C_j}(x) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in S_w \cap L_j, \\ \nu_w(x) &= \pm\nu_u(x) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in S_w \cap S_u. \end{aligned} \tag{4.3.54}$$

Therefore, thanks to (H3), (4.3.49), (4.3.53), and (4.3.54), we deduce that

$$\begin{aligned} \int_{S_w\setminus\Sigma} \psi(x,\nu_w)\,\mathrm{d}\mathcal{H}^{n-1} \leq \int_{S_u\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\bigcup_{j=1}^H L_j} \psi(x,\nu_{C_j})\,\mathrm{d}\mathcal{H}^{n-1} \\ + \sum_{j=1}^{H} \int_{U_j-\eta_j\xi(x_{i_j})} \psi(x,\nu_\Sigma(x+\eta_j\xi(x_{i_j})))\,\mathrm{d}\mathcal{H}^{n-1}\,. \end{aligned} \tag{4.3.55}$$

Hypothesis (H2) on ψ and inequality (4.3.46) imply that

$$\int_{\bigcup_{j=1}^H L_j} \psi(x,\nu_{C_j})\,\mathrm{d}\mathcal{H}^{n-1} \leq c_2\mathcal{H}^{n-1}\Big(\bigcup_{j=1}^H L_j\Big) < c_2\delta\,. \tag{4.3.56}$$

By (4.3.35), (4.3.39), and by the change of variables $y = x+\eta_j\xi(x_{i_j})$ in the last term of (4.3.55), we obtain that

$$\begin{aligned} \sum_{j=1}^{H}\int_{U_j-\eta_j\xi(x_{i_j})} \psi(x,\nu_\Sigma(x+\eta_j\xi(x_{i_j})))\mathrm{d}\mathcal{H}^{n-1}(x) \\ = \sum_{j=1}^{H}\int_{U_j} \psi(y-\eta_j\xi(x_{i_j}),\nu_\Sigma(y))\mathrm{d}\mathcal{H}^{n-1}(y) \\ \leq \sum_{j=1}^{H}\int_{U_j} \psi(y,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1} + \delta\mathcal{H}^{n-1}(\Sigma)\,. \end{aligned} \tag{4.3.57}$$

By (4.3.35), we can split the sum in the right-hand side of (4.3.57) in the following way:

$$\sum_{j=1}^{H}\int_{U_j} \psi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1} = \sum_{j=1}^{H}\int_{K_j} \psi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1} + \sum_{j=1}^{H}\int_{U_j\setminus K_j} \psi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{4.3.58}$$

In view of (H2) and of (4.3.36) and recalling that the sets K_j are pairwise disjoint and contained in B_{g_1}, (4.3.58) becomes

$$\begin{aligned} \sum_{j=1}^{H}\int_{U_j} \psi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1} &\leq \int_{\bigcup_{j=1}^H K_j} \psi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1} + c_2\,\delta \\ &\leq \int_{B_{g_1}} \psi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1} + c_2\,\delta\,. \end{aligned} \tag{4.3.59}$$

Therefore, collecting inequalities (4.3.55)-(4.3.57), and (4.3.59), we get that the jump term in $\mathcal{F}_1(w)$ can be controlled from above by

$$\begin{aligned}\int_{S_w\setminus\Sigma}&\psi(x,\nu_w)\,\mathrm{d}\mathcal{H}^{n-1}\\&\le\int_{S_u\setminus\Sigma}\psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{B_{g_1}}\psi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1}+\delta\,(2c_2+\mathcal{H}^{n-1}(\Sigma))\,.\end{aligned}\tag{4.3.60}$$

Finally, we give an estimate of the integral over Σ of $\mathcal{F}_1(w)$. We first split it into the contribution on $\bigcup_{j=1}^H K_j$ and on $\Sigma\setminus\bigcup_{j=1}^H K_j$:

$$\begin{aligned}\int_\Sigma &g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1}\\&=\int_{\bigcup_{j=1}^H K_j}g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\Sigma\setminus\bigcup_{j=1}^H K_j}g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1}\,.\end{aligned}\tag{4.3.61}$$

We notice that by (4.3.25) and (4.3.42), for every $j=1,\dots,H$ and for $\mathcal{H}^{n-1}$-a.e. $x\in U_j$ the unit normal $\nu_\Sigma(x)$ to Σ at x points outside C_j. Thus, by (4.3.48), we have that $w^-(x)=q_j$ for $\mathcal{H}^{n-1}$-a.e. $x\in U_j$. Moreover, since $w=u$ in $\Omega\setminus\bigcup_{j=1}^H C_j$,

$$w^+=u^+\quad\text{for }\mathcal{H}^{n-1}\text{-a.e. }x\in\Sigma.\tag{4.3.62}$$

Therefore, recalling that the sets K_j are pairwise disjoint, we can write the first integral in the right-hand side of (4.3.61) as

$$\int_{\bigcup_{j=1}^H K_j}g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1}=\sum_{j=1}^H\int_{K_j}g_1(x,u^+,q_j)\,\mathrm{d}\mathcal{H}^{n-1}\,.\tag{4.3.63}$$

Taking into account definition (4.3.32) of the sets $B^j_{g_1}$, the inclusion $K_j\subseteq B^j_{g_1}$, and inequalities (4.3.5), (4.3.33), and (4.3.34), we can continue (4.3.63) in the following way:

$$\begin{aligned}\int_{\bigcup_{j=1}^H K_j}g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1}&\le\sum_{j=1}^H\int_{K_j}\inf_{\tau\in\mathbb{R}^m}g_1(x,u^+,\tau)\,\mathrm{d}\mathcal{H}^{n-1}+\delta\mathcal{H}^{n-1}(K_j)\\&\le\int_{\bigcup_{j=1}^H B^j_{g_1}}\inf_{\tau\in\mathbb{R}^m}g_1(x,u^+,\tau)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\bigcup_{j=1}^H B^j_{g_1}\setminus K_j}a\,\mathrm{d}\mathcal{H}^{n-1}+\delta\,\mathcal{H}^{n-1}(\Sigma)\\&\le\int_{B_{g_1}}\inf_{\tau\in\mathbb{R}^m}g_1(x,u^+,\tau)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\bigcup_{j>H}B^j_{g_1}}a\,\mathrm{d}\mathcal{H}^{n-1}+\delta(\mathcal{H}^{n-1}(\Sigma)+1)\\&\le\int_{B_{g_1}}\inf_{\tau\in\mathbb{R}^m}g_1(x,u^+,\tau)\,\mathrm{d}\mathcal{H}^{n-1}+\delta\,(\mathcal{H}^{n-1}(\Sigma)+2)\,.\end{aligned}\tag{4.3.64}$$

We now consider the last term in (4.3.61). By (4.3.35), (4.3.37), and (4.3.48), we have that $w^- = u^-$ $\mathcal{H}^{n-1}$-a.e. on $\Sigma \setminus \bigcup_{j=1}^H U_j$. Thus, by (4.3.62), we obtain

$$\begin{aligned}
&\int_{\Sigma\setminus\bigcup_{j=1}^H K_j} g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1} \\
&\qquad = \int_{\Sigma\setminus\bigcup_{j=1}^H U_j} g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\bigcup_{j=1}^H U_j\setminus K_j} g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1} \\
&\qquad = \int_{\Sigma\setminus\bigcup_{j=1}^H U_j} g_1(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\bigcup_{j=1}^H U_j\setminus K_j} g_1(x,u^+,q_j)\,\mathrm{d}\mathcal{H}^{n-1}\,.
\end{aligned} \tag{4.3.65}$$

In view of (A4), (4.3.5), (4.3.6), and of (4.3.33)-(4.3.36), inequality (4.3.65) becomes

$$\begin{aligned}
\int_{\Sigma\setminus\bigcup_{j=1}^H K_j} g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1} &\le \int_{\Sigma\setminus\bigcup_{j=1}^H U_j} g_1(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\bigcup_{j=1}^H U_j\setminus K_j} a_{\widetilde{M}}\,\mathrm{d}\mathcal{H}^{n-1} \\
&< \int_{\Sigma\setminus\bigcup_{j=1}^H K_j} g_1(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\bigcup_{j=1}^H U_j\setminus K_j} a\,\mathrm{d}\mathcal{H}^{n-1} + \delta \\
&< \int_{\Sigma\setminus\bigcup_{j=1}^H B^j_{g_1}} g_1(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\bigcup_{j=1}^H B^j_{g_1}\setminus K_j} a_M\,\mathrm{d}\mathcal{H}^{n-1} + 2\delta \\
&< \int_{\Sigma\setminus B_{g_1}} g_1(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\bigcup_{j>H} B^j_{g_1}} a_M\,\mathrm{d}\mathcal{H}^{n-1} + 3\delta \\
&< \int_{\Sigma\setminus B_{g_1}} g_1(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} + 4\delta\,.
\end{aligned} \tag{4.3.66}$$

Therefore, (4.3.61), (4.3.64), and (4.3.66) imply that

$$\begin{aligned}
\int_\Sigma g_1(x,w^+,w^-)\mathrm{d}\mathcal{H}^{n-1} &< \int_{\Sigma\setminus B_{g_1}} g_1(x,u^+,u^-)\mathrm{d}\mathcal{H}^{n-1} \\
&\quad + \int_{B_{g_1}} \inf_{\tau\in\mathbb{R}^m} g_1(x,u^+,\tau)\mathrm{d}\mathcal{H}^{n-1} + \delta\,(\mathcal{H}^{n-1}(\Sigma)+6)\,.
\end{aligned} \tag{4.3.67}$$

Collecting inequalities (4.3.60) and (4.3.67) and using (4.3.28) in the last equality, we obtain that, for some $c > 0$ independent of δ,

$$\begin{aligned}
\mathcal{F}_1(w) &= \int_{S_w\setminus\Sigma} \psi(x,\nu_w)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g_1(x,w^+,w^-)\,\mathrm{d}\mathcal{H}^{n-1} \\
&< \int_{S_u\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{B_{g_1}} \psi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad + \int_{\Sigma\setminus B_{g_1}} g_1(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{B_{g_1}} \inf_{\tau\in\mathbb{R}^m} g_1(x,u^+,\tau)\,\mathrm{d}\mathcal{H}^{n-1} + c\delta \\
&= \mathcal{F}_{12}(u) + c\delta\,.
\end{aligned} \tag{4.3.68}$$

Choosing $0 < \delta < \varepsilon/2$ such that $c\delta < \varepsilon/2$ and $c_r\delta^{1/r}+\delta < \varepsilon/2$ in estimates (4.3.50), (4.3.51), (4.3.52), and (4.3.68), we deduce (4.3.29)-(4.3.31).

If we repeat the above argument replacing u and B_{g_1} of (4.3.26) with the function w and the set

$$\begin{aligned} B_g := & \left\{x \in \Sigma' : g(x, w^+(x), w^-(x)) > \inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,w^-(x)) + \psi(x,\nu_\Sigma(x))\right\} \\ = & \left\{x \in \Sigma' : g(x, u^+(x), w^-(x)) > \inf_{\sigma\in\mathbb{R}^m} g(x,\sigma,w^-(x)) + \psi(x,\nu_\Sigma(x))\right\}, \end{aligned}$$

we are able to construct a new function $v \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ such that:

$$\begin{gathered} \|v-w\|_{r,\Omega} < \frac{\varepsilon}{2}, \qquad \|\nabla v - \nabla w\|_{p,\Omega} < \frac{\varepsilon}{2}, \\ \mathcal{H}^{n-1}(S_v) < \mathcal{H}^{n-1}(S_w) + 2\mathcal{H}^{n-1}(\Sigma) + \frac{\varepsilon}{2}, \\ \mathcal{F}(v) < \mathcal{F}_1(w) + \frac{\varepsilon}{2}. \end{gathered}$$

The previous inequalities, together with (4.3.29)-(4.3.31), imply that v satisfies (4.3.17)-(4.3.19). This concludes the proof of the lemma. $\square$

Proof of Theorem 4.3.3. By the hypotheses of the theorem and by Lemma 4.3.2, the functions ψ and g_{12} satisfy hypotheses (H1)-(H7). Hence, from Theorem 4.2.1 we deduce that the functional $\mathcal{F}_{12}$ defined in (4.3.15) is lower semicontinuous with respect to the weak convergence in $GSBV^p(\Omega;\mathbb{R}^m)$. Since $g_{12} \le g$, we have that $\mathcal{F}_{12} \le \mathcal{F}$. Thus, by definition of $sc^-\mathcal{F}$, we easily get that $\mathcal{F}_{12} \le sc^-\mathcal{F}$ on $GSBV^p(\Omega;\mathbb{R}^m)$. Therefore, we only need to show the converse inequality, that is,

$$sc^-\mathcal{F}(u) \le \mathcal{F}_{12}(u) \qquad \text{for every } u \in GSBV^p(\Omega;\mathbb{R}^m). \tag{4.3.69}$$

Let us first prove (4.3.69) for $u \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$. To this end, we need to construct a recovery sequence for u. Applying Lemma 4.3.4, we can find a sequence $v_k \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ such that v_k converges to u weakly in $GSBV^p(\Omega;\mathbb{R}^m)$ and

$$\mathcal{F}(v_k) < \mathcal{F}_{12}(u) + \frac{1}{k} \quad \text{for every } k. \tag{4.3.70}$$

Passing to the $\liminf$ as $k \to +\infty$ in (4.3.70) we get

$$sc^-\mathcal{F}(u) \le \liminf_k sc^-\mathcal{F}(v_k) \le \liminf_k \mathcal{F}(v_k) \le \mathcal{F}_{12}(u).$$

This concludes the proof of (4.3.69) for $u \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$.

Let us now consider $u \in GSBV^p(\Omega;\mathbb{R}^m)$. Given a function $\varphi \in C_c^\infty(\mathbb{R}^m;\mathbb{R}^m)$ with $\varphi(s) = s$ if $|s| \le 1$, we can approximate u in $GSBV^p(\Omega;\mathbb{R}^m)$ with the sequence $\varphi_k(u) \in SBV^p(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$, where we have set $\varphi_k(s) := k\varphi(s/k)$. Clearly, $\varphi_k(u)$ converges to u pointwise $\mathcal{L}^n$-a.e. in Ω and $\nabla\varphi_k(u) \to \nabla u$ in $L^p(\Omega;\mathbb{M}^{m\times n})$.

Moreover, $S_{\varphi_k(u)} \subseteq S_u$ for every k. Hence, by Definition 1.2.11, $\varphi_k(u)$ converges to u weakly in $GSBV^p(\Omega;\mathbb{R}^m)$ and

$$\limsup_k \int_{S_{\varphi_k(u)}\setminus\Sigma} \psi(x,\nu_{\varphi_k(u)})\,\mathrm{d}\mathcal{H}^{n-1} \le \int_{S_u\setminus\Sigma} \psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{4.3.71}$$

Recalling that $\varphi_k \in C_c^\infty(\mathbb{R}^m;\mathbb{R}^m)$, we have that $\varphi_k(u)^\pm = \varphi_k(u^\pm)$ $\mathcal{H}^{n-1}$-a.e. in Σ. Therefore, since g_{12} is a Carathéodory function, we get that

$$g_{12}(x,\varphi_k(u)^+(x),\varphi_k(u)^-(x)) \to g_{12}(x,u^+(x),u^-(x)) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x\in\Sigma\,.$$

Thanks to hypothesis (A4) and to inequalities (4.3.5) and (4.3.6) of Lemma 4.3.2, we can apply the dominated convergence theorem to deduce that

$$\lim_k \int_\Sigma g_{12}(x,\varphi_k(u)^+,\varphi_k(u)^-)\,\mathrm{d}\mathcal{H}^{n-1} = \int_\Sigma g_{12}(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{4.3.72}$$

Collecting (4.3.71) and (4.3.72), we get that

$$sc^-\mathcal{F}(u) \le \liminf_k\, sc^-\mathcal{F}(\varphi_k(u)) \le \limsup_k\, \mathcal{F}_{12}(\varphi_k(u)) \le \mathcal{F}_{12}(u)\,,$$

which concludes the proof of (4.3.69) in the general case.

□

We conclude this section with a generalization of Theorem 4.3.3 which takes into account also the presence of volume terms. Let $q\in(1,+\infty)$, let $W\colon \Omega\times\mathbb{M}^{m\times n}\to\mathbb{R}$ satisfy (1.2.8) and (1.2.9), and let $f\colon\Omega\times\mathbb{R}^m\to\mathbb{R}$ be a Carathéodory function such that (4.2.86) holds. We consider the functional $\mathcal{G}\colon L^q(\Omega;\mathbb{R}^m)\to\overline{\mathbb{R}}$ defined as in (4.2.87). With the same notation used before, $sc^-\mathcal{G}$ denotes the greatest sequentially lower semicontinuous functional on $L^q(\Omega;\mathbb{R}^m)$ which is less than or equal to $\mathcal{G}$. Moreover, we define

$$\mathcal{G}_{12}(u) := \int_\Omega W(x,\nabla u)\,\mathrm{d}x + \int_\Omega f(x,u)\,\mathrm{d}x + \int_{S_u\setminus\Sigma}\psi(x,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Sigma g_{12}(x,u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1}$$

for $u\in GSBV^p(\Omega;\mathbb{R}^m)\cap L^q(\Omega;\mathbb{R}^m)$. We extend $\mathcal{G}_{12}$ to $+\infty$ out of $GSBV^p(\Omega;\mathbb{R}^m)$.

Theorem 4.3.5. *Let ψ and g satisfy (H1)-(H3), (A1)-(A5), and* (4.1.5)*. Then the functionals $sc^-\mathcal{G}$ and $\mathcal{G}_{12}$ coincide on $L^q(\Omega;\mathbb{R}^m)$.*

Proof. By (4.3.6) of Lemma 4.3.2, $\mathcal{G}_{12}\le\mathcal{G}$. Recalling that g_{12} satisfies properties (H4)-(H7), from Theorems 1.2.13 and 4.2.1 and from the hypotheses on f we deduce that $\mathcal{G}_{12}$ is sequentially lower semicontinuous in $L^q(\Omega;\mathbb{R}^m)$. Thus $\mathcal{G}_{12}\le sc^-\mathcal{G}$. By Lemma 4.3.4 and by the hypotheses on the volume densities W and f, we get also the opposite inequality in $SBV^p(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$. The conclusion follows by the truncation argument used in the last part of the proof of Theorem 4.3.3. □

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