Ambit Processes: Aspects of Theory and Applications

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Ambit fields and processes

Figure: Ambit framework
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Synopsis

- Ambit fields and processes
- Brownian semistationary processes (BSS processes)
- Multipower variations
- Turbulence
- Forwards and spots in energy markets
**Ambit fields**

\[
Y_t(x) = \mu + \int_{A_t(x)} g(\xi, s; t, x) M(d\xi, ds) \\
+ \int_{D_t(x)} q(\xi, s; t, x) a_s(\xi) d\xi ds.
\]

Here \(A_t(x)\), and \(D_t(x)\) are termed *ambit sets*, \(g\) and \(q\) are deterministic (matrix) functions, \(\sigma \geq 0\) and \(a\) are stochastic fields, and \(M\) is a random measure on \(\mathbb{R}^d \times \mathbb{R}\). Typically, \(M\) will be constructed from a *Lévy basis* \(L\).

*Example:* \(M(d\xi, ds) = \sigma_s(\xi) L(d\xi, ds)\) where the random field \(\sigma\) is referred to as the *intermittency* or *volatility*. This may itself be modeled as a positive ambit field.

**Ambit processes**

\[X_{\theta} = Y_{t(\theta)}(x(\theta))\]

where \(\tau(\theta) = (x(\theta), t(\theta))\) is a curve in space-time.
Lévy basis

A Lévy basis $L$ is an independently scattered random measure (on $\mathbb{R}^d$) whose values are infinitely divisible.

$L$ is said to be homogeneous if for any (bounded) Borel set the law of $L(B)$ is the same as the law of $L(B + z)$ for all $z \in \mathbb{R}^d$.

White noise is the special type of Lévy basis $L$ for which $L$ is Gaussian and homogeneous, with mean 0 and variance $E\left\{L(B)^2\right\} = \text{leb}(B)$. 
Stationary regimes

To model fields and processes that are *stationary in time and space*, the damping functions $g$ and $q$ are chosen to have the form

$$
g (\xi, s; t, x) = g (x - \xi, t - s)
$$

$$
q (\xi, s; t, x) = q (x - \xi, t - s)
$$

and the ambit sets are taken to be *homogeneous* and nonanticipative, i.e. $A_t (x)$ is of the form $A_t (x) = A + (x, t)$ where $A$ only involves negative time coordinates, and similarly for $D_t (x)$. Furthermore, in case $M (d\xi, ds) = \sigma_s (\xi) L (d\xi, ds)$, $\{\sigma_s (\xi)\}$ and $\{a_s (\xi)\}$ are assumed to be stationary fields, and the Lévy basis $L$ is homogeneous.
If only *stationarity in time* is required then one may choose

\[ g (\zeta, s; t, x) = g (\zeta, t - s; x) \]

\[ q (\zeta, s; t, x) = q (\zeta, t - s; x) \]

and the compensator \( n \) of the Lévy basis can be taken to satisfy

\[ n (dy; d\zeta, ds) = \nu (dy; d\zeta) ds. \]
Turbulence

Most extensive data sets on turbulent velocities only provide the time series of the main component of the velocity vector (i.e. the component in the main direction of the fluid flow) at a single location in space.

The turbulence modelling framework then particularises to the class of $\mathcal{BSS}$ models (Brownian semistationary processes). We discuss this class next, returning to turbulence settings later.
**BSS processes**

The class of *Brownian semistationary (BSS)* processes is the subclass of the ambit processes corresponding to a degenerate space component (null-spatial case) and having the form

\[
Y_t = \int_{-\infty}^{t} g(t - s) \sigma_s W(ds) + \int_{-\infty}^{t} q(t - s) a_s ds
\]

where \( W \) is Brownian motion on \( \mathbb{R} \), \( \sigma \) and \( a \) are c.d.l.g processes and \( g \) and \( q \) are deterministic continuous memory function on \( \mathbb{R} \), with \( g(t) = q(t) = 0 \) for \( t \leq 0 \).

When \( \sigma \) and \( a \) are stationary, as will be assumed throughout this talk, then so is \( Y \).

It is sometimes convenient to indicate the formula for \( Y \) as

\[
Y = g * \sigma \bullet W + q * a \bullet \text{leb}.
\]
We consider the \( \text{BSS} \) processes to be the natural analogue, in stationarity related settings, of the class \( \text{BSM} \) of Brownian semimartingales.

\[
Y_t = \int_0^t \sigma_s \, dW_s + \int_0^t a_s \, ds.
\]

Note  The \( \text{BSS} \) processes

\[
Y_t = \int_{-\infty}^{t} g(t - s) \sigma_s \, dW(ds) + \int_{-\infty}^{t} q(t - s) a_s \, ds
\]

are, in general, not semimartingales
Important example

Suppose $Y = g * \sigma \bullet W$ with $g(t) = t^{\nu-1} e^{-\lambda t}$.

$\frac{1}{2} < \nu < 1$ nonSM
$\nu = 1$ SM
$1 < \nu < \frac{3}{2}$ nonSM
A key object of interest, whether for \( \mathcal{BSM} \) or \( \mathcal{BSS} \) processes, is the *integrated squared volatility*

\[
\sigma_t^2 = \int_0^t \sigma_s^2 \, ds
\]

for any \( t \in \mathbb{R} \).

*Realised multipower variations* (RMPVs) of \( Y \) can be used to estimate elements of the main terms in \( Y \), i.e. \( Y = g \ast \sigma \ast W \). In particular they can be used to draw inference on \( \sigma_t^2 \) or on the small scale behaviour of the damping function \( g \).

However, because of the nonsemimartingale character of \( \mathcal{BSS} \) processes the probabilistic limit theory of RMPVs for \( \mathcal{BSS} \) processes is decisively different from that for \( \mathcal{BSM} \) processes.

*Next:* Brief review of the theory of RMPVs for \( \mathcal{BSS} \) processes.
Multipower Variation

In the following the process $Y$ is assumed to be observed at time points $t_i = i \Delta_n$ with $i = 0, \ldots, \lfloor t/\Delta_n \rfloor$ and $\Delta_n \to 0$. A realised multipower variation of a stochastic process $Y$ is defined as an object of the type

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor-k+1} \prod_{j=1}^{k} |\Delta_{i+j-1}^n Y|^{p_j}$$

where $\Delta_i^n Y = Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}$ and $p_1, \ldots, p_k \geq 0$. 
For the case where $Y \in BSM$, i.e. $Y = \sigma \cdot W + a \cdot leb$, it was established in [BNGJPS07] that

$$n^{p_+/2-1} \sum_{i=1}^{[t/\Delta_n]-k+1} \prod_{j=1}^{k} |\Delta_{i+j-1}^n Y|^{p_j} \xrightarrow{ucp} \mu_{p_1} \cdots \mu_{p_k} \int_0^t |\sigma_s|^{p_+} ds$$

where $p_+ = \sum_{j=1}^k p_j$ and $\mu_p = E[|u|^p]$, $u \sim N(0,1)$.
Moreover, under a regularity condition on the volatility process $\sigma$, there is an associated stable central limit theorem:

$$
\sqrt{n}\left(n^{p^+}/2-1\right)^{[t/\Delta_n]-k+1} \sum_{i=1}^{k} \prod_{j=1}^{\Delta_i+j-1} Y_{i+j-1}^p - \mu_{p_1} \cdots \mu_{p_k} \int_0^t |\sigma_s|^{p^+} ds
$$

$$
\xrightarrow{st} \sqrt{C} \int_0^t |\sigma_s|^{p^+} dB_s
$$

where $B$ is another Brownian motion, defined on an extension of the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ and independent of $\mathcal{F}$, and $C$ is a known constant.
Normalised RMPVs

This was for $BSM$ processes. When we pass on to $BSS$ processes the situation changes significantly (recall that $BSS$ processes are generally not of $BSM$ type). In order to obtain similar limit results one has to look at normalised RMPV. ([BNSch09], [BNCP08], [BNCP10])

The techniques for obtaining the theorems now, among other things, involves establishing a $CLT$ for triangular arrays of Gaussian variables, using Malliavin calculus. (This $CLT$ is of some independent interest.)
Furthermore, while the focus for $BSM$, and more generally for Ito processes, has been on inference concerning the quadratic variation of the processes, and especially the $\sigma^2+$ component, in the $BSS$ setting the interest is not just in regard to $\sigma^2+$ but also concerns inference on the damping function $g$.

For inference on $g$ it is pertinent to study the behaviour of ratios of $RMPVs$ and, in fact, not only of 'classical' ($\Delta$ case) $RMPVs$ but also $RMPVs$ based on second order differences ($\diamond$ case) instead of first order differences (something that would make no difference in the $BSM$ case).
Normalised RMPVs

The *normalised RMPVs* of types $\Delta$ and $\diamondsuit$ are defined as

\[
\text{MPV}^\Delta(Y, p_1, \ldots, p_k)_t^n = \Delta_n(\tau^\Delta_n)^{-p^+} \sum_{i=1}^{k-1} \prod_{l=0}^{i} |\Delta^n_{i+l} Y|^{p_{l+1}}
\]

(1)

\[
\text{MPV}^{\diamondsuit}(Y, p_1, \ldots, p_k)_t^n = \Delta_n(\tau^{\diamondsuit}_n)^{-p^+} \sum_{i=2}^{k} \prod_{l=0}^{i-1} |\diamondsuit^n_{i+l} Y|^{p_{l+1}}
\]

(2)

where $\Delta^n_i Y = Y_{i\Delta_n} - Y_{(i-1)\Delta_n}$ and $\diamondsuit^n_i Y = X_{i\Delta_n} - 2Y_{(i-1)\Delta_n} + Y_{(i-2)\Delta_n}$, where $p_l \geq 0$ and $p^+ = \sum_{i=1}^{k} p$ and where $\tau^\Delta_n$ is a known function of $g$.

We have established LLNs and CLTs for inference on $\sigma^{2+}$. Except for the normalisation these results are similar in nature to those for $\text{BSM}$s but the proofs are very different. (All specifics omitted here.)
Inference on \( g \)

For inference on \( g \) it is (as already mentioned) pertinent to study *Realised Variation Ratios* (RVRs), i.e. ratios of *RMPVs*.

**Note** The relevant *Realised Variation Ratios* (RVRs), of types \( \Delta \) and \( \diamondsuit \), do not involve normalisation of *RMPVs*. Thus, in this sense, the limit theory for RVRs is ‘nonparametric’.
Inference on $g$. An example:

To illustrate, consider the key case where $g(t) = t^{\nu-1} e^{-\lambda t}$.

The interesting situations are where $\nu \in \left( \frac{1}{2}, 1 \right) \cup \left( 1, \frac{3}{2} \right)$. Using $\Delta$-$RMPVs$ it is only possible to establish a $CLT$ for $\nu$ in $\left( \frac{1}{2}, 1 \right)$. Passing to $\diamond$-$RMPVs$ this can be strengthened to $\nu \in \left( \frac{1}{2}, 1 \right) \cup \left( 1, \frac{5}{4} \right)$.

But the range $\nu \in \left[ \frac{5}{4}, 1 \right)$ is of particular interest in the context of turbulence, and that can be covered only by modifications for which the convergence rate is somewhat slower than the $\sqrt{n}$ rate that holds in the other intervals. (Details omitted.)
Turbulence

*Null-spatial case* (spatial dimension 0).

As mentioned earlier, most extensive data sets on turbulent velocities only provide the time series of the main component of the velocity vector (i.e. the component in the main direction of the fluid flow) at a single location in space. The turbulence modelling framework then particularises to the class of BSS models
Realised quadratic variation for the Brookhaven data set

Figure: $S_2$ Brookhaven data
Figure: $S_2$ Brookhaven data

Log–log–plot of the second order structure function for the Brookhaven data set. Red line has slope $2/3$. 
Detailed analysis of the data and physical insights have made it possible to determine specifications of the ingredients of the \textit{BSS} framework resulting in a closely realistic general model, describing the observations from a variety of extensive empirical and experimental recordings. The model does, in particular, reproduce the empirical traits just shown and also give a universal description of the distributions of the velocity differences. ([BNBSch04], [BNSch07a], [BNSch08a], [BNSch08b])

Here we omit the details, except for a brief discussion of the use of Realised Variation Ratios for the study of the small scale nature of $g$. 
Small scale nature of $g$ and Realised Variation Ratios

Example  \textit{Realised Bipower Variation/Realised Quadratic Variation: ($\Delta$ case)}

$$RVR_t^n = \frac{MPV^\Delta(Y, 1, 1)_t^n}{MPV^\Delta(Y, 2, 0)_t^n} = \frac{\frac{\pi}{2} \sum_{i=1}^{[t/\Delta_n]−k+1} |\Delta_i^n Y||\Delta_{i+1}^n Y|}{\sum_{i=1}^{[t/\Delta_n]−k+1} |\Delta_i^n Y|^2}.$$
$RVR^4_t(\delta) = 10000$
$\delta = 1$

Figure: RVR
Turbulence

*Tempo-spatial settings:*  \( d \times 1 \) case (spatial dimension \( d \))

We specify the \( d \)-dimensional velocity vector \( Y \) of a stationary turbulent fluid, at position \( x \) in \( \mathbb{R}^d \) and at time \( t \), by

\[
Y_t(x) = \mu + \int_{A^+(x,t)} g(t - s, x - \zeta) \sigma_s(\zeta) \, W(d\zeta, ds) \\
+ \int_{D^+(x,t)} q(t - s, x - \zeta) \sigma_s^2(\zeta) d\zeta ds.
\]

where \( W \) is Brownian white noise.

*Next*  Choice of ambit sets and volatility fields from physical theory and stylised features.
Figure: Sound cone
Choice of volatility field \([\text{[BNSch04]}]\)

**Arithmetic:**

\[
\sigma^2_t (x) = \int_{C^+ (x, t)} h(t - s, x - \bar{\xi}) L (d\bar{\xi}, ds)
\]

**Example: OU\(^\wedge\) fields**  Let \(L\) be a positive homogeneous Lévy basis and define

\[
\sigma^2_t (x) = \int_\wedge^+ (x, t) e^{-\lambda(t-s)} L (d\bar{\xi}, ds).
\]

For fixed \(x\) this determines a stationary Markovian process.
Choice of volatility field

*Geometric:*

\[
\sigma_t^2(x) = \exp \left\{ \int_{C+(x,t)} h(t-s, x-\xi) L(d\xi, ds) \right\}
\]

Example: **SI fields**

By suitable choice of the kernel \( h \) and the ambit set \( C \) we obtain a version of \( \sigma \) that constitutes a *continuous* analogue of *multiplicative cascade processes* in the description of turbulent energy-dissipation fields.
The SI specification allows explicit analytic calculations, in particular showing scaling relations of \textit{n-point correlators}. The correlators are defined by

\begin{equation}
c ( (x_1, t_1) m_1; \cdots , (x_n, t_n) m_n ) = \frac{m ( (x_1, t_1), m_1; \cdots , (x_n, t_n), m_n )}{m ( (x_1, t_1), m_1 ) \cdots m ( (x_n, t_n), m_n )}
\end{equation}

where

\begin{equation}
m ( (x_1, t_1) m_1; \cdots , (x_n, t_n) m_n ) = \mathbb{E} \left\{ Y_{t_1} (x_1)^{m_1} \cdots Y_{t_n} (x_n)^{m_n} \right\}.
\end{equation}

\textbf{Note}  Cancellation of factors
Figure: Ambit sets
The probabilistic limit behaviour of *normalised RQV* in this tempo-spatial \((1 + 1)\) setting is presently under study.

In particular, the limit in probability of the \(nRQV\) is typically an integral of the squared volatility field over the boundary of the ambit set \(A\), the measure on the boundary being determined by the nature of the damping function \(g\).
Figure: Hornsrev
Forwards and Spots in Energy Markets  joint work with Fred Espen Benth and Almut Veraart

Using the ambit framework we aim to model both the forward and the spot price processes directly and coherently, encompassing main stylised features.

Stylised features:

- Samuelson effect: This effect refers to the empirical trait, observed on forward prices in power markets, that when the time to maturity approaches 0 the volatility of the forward starts increasing and converges to the volatility of the spot eventually.
- High correlation between neighbouring contracts near maturity.
The forward price is modelled as an ambit field $f_t(u)$ that is stationary in time:

$$f_t(x) = \int_{A_t(x)} k(\bar{\xi}, t-s; x) \sigma_s(\bar{\xi}) L(d\bar{\xi}, ds)$$

where the 'space variable' $x$ is the time to maturity.

Three ingredients for specification:

- Deterministic damping function $k$
- Volatility field $\sigma$ ($\sigma_s(\bar{\xi})$ stationary in $s$)
- Family of ambit sets $A_t(x) = A_0(x) + (0, t)$
$T = t + u$

Figure: Forwards
Under weak conditions, the forward $f_t(u)$ will converge, as time to maturity tends to zero, to a process $s_t$ of the form

$$ s_t = \int_{-\infty}^{t} k^* (\xi, t - s) \sigma_s (\xi) L (d\xi, ds) . $$

This process is then taken as the model for the spot price.

Further, in this setup, we can indeed flexibly model

- The Samuelson effect
- High correlation between neighbouring contracts near maturity
Extension to include delivery periods

Forward price

\[ f_t(x, \tau) = \int_{A_t(x, \tau)} k(\xi, \chi, t - s; x, \tau) \sigma_s(\xi, \chi) L(d\xi, d\chi, ds) \]

where \( \tau \) denotes the period of delivery, starting at time \( t + x \).
References


Basse, A., Graversen, S.E. and Pedersen, J. (2010): Martingale-type processes indexed by $\mathbb{R}$. (Submitted.)


