Error Estimates for Multinomial Approximations of American Options in a Class of Jump Diffusion Models

Yan Dolinsky

ETH Zurich

AnStAp 2010
15.07.2010

The setup

- $(\Omega, \mathcal{F}, P)$–complete probability space.
- $\{W(t) = (W_1(t), ..., W_d(t))\}_{t=0}^{\infty}$– standard d dimensional Brownian motion.
- $\{N(t)\}_{t=0}^{\infty}$– Poisson process with intensity λ and independent of W.
- $\{U^{(i)} = (U_1^{(i)}, ..., U_d^{(i)})\}_{i=1}^{\infty}$–sequence of i.i.d. random vectors with values in $(-1, \infty)^d$, independent from W and N. Assume that the random vector $U^{(1)}$ takes on a finite number of values and denote $u_j = EU_j^{(1)}$, $1 \leq j \leq d$.

The market

- $B(t) = B(0)\exp(rt)$.
- For $1 \leq i \leq d$,

$$S_i(t) = S_i(0)\exp(\mu_i t + \sum_{j=1}^{d} \sigma_{ij} W_j(t)) \prod_{j=1}^{N(t)} (1 + U_i^{(j)})$$

where $\sigma = (\sigma_{ij})_{1 \leq i,j \leq d}$ is a nonsingular matrix.

- Without loss of generality we assume that

$$\mu_i = r - \lambda - u_i - \frac{\sum_{j=1}^{d} \sigma_{ij}^2}{2}, \ 1 \leq i \leq d.$$

American options

- American option with the payoff process

$$Y(t) = F(S(t), t), \ 0 \le t \le T$$

  where $F : \mathbb{R}_+^d \times \mathbb{R}_+ \to \mathbb{R}_+$.
- The term

$$V = \sup_{\tau \in \mathcal{T}} E(\exp(-r\tau) Y(\tau))$$

  gives an arbitrage–free price for the American option.
- We assume that for some constant $L \ge 1$

$$|F(v, t) - F(\tilde{v}, s)| \le L \sum_{i=1}^{d} |v_i - \tilde{v}_i| + L(t - s)(1 + \sum_{i=1}^{d} |v_i|)$$

  for any $t \ge s \ge 0$ and $v, \tilde{v} \in \mathbb{R}^d$.

Construction of the discrete probability spaces

- Let $A \in M_{d+1}(\mathbb{R})$ be an orthogonal matrix such that it last column equals to $(\frac{1}{\sqrt{d+1}}, ..., \frac{1}{\sqrt{d+1}})$.
- Let $\Omega_\xi = \{1, 2, ..., d+1\}^\infty$ be the space of infinite sequences $\omega = (\omega_1, \omega_2, ...)$; $\omega_i \in \{1, 2, ..., d+1\}$ with the product probability $P^\xi = \{\frac{1}{d+1}, ..., \frac{1}{d+1}\}^\infty$.
- Define a sequence of i.i.d. random vectors $\xi^{(1)}, \xi^{(2)}, ...$ by

$$\xi^{(i)}(\omega) = \sqrt{d+1}(A_{\omega_i 1}, A_{\omega_i 2}, ..., A_{\omega_i d}), \ i \in \mathbb{N}.$$

We have $E\xi^{(1)} = 0$ and $E\xi_i^{(1)}\xi_j^{(1)} = \delta_{ij}$.

Discrete probability spaces

For any n we extend (Ω_ξ, P^ξ) to a probability space (Ω_n, P_n) such that it contains a three independent sequences of i.i.d. random vectors

- $\{\xi^{(k)}\}_{k=1}^{\infty}$–as above.
- $\{\tilde{U}^{(k)}\}_{k=1}^{\infty} \sim \{U^{(k)}\}_{k=1}^{\infty}$.
- $\{\rho^{n,k}\}_{k=1}^{n}$– sequence of Bernoulli random variables such that $P_n\{\rho^{n,1} = 1\} = 1 - \exp(-\lambda T/n)$.

Multinomial n–step market

- The market is active in the moments $0, \frac{T}{n}, \frac{2T}{n}, ..., T$.
- $B(t) = B(0)\exp(rt)$.
- $S_i^{(n)}(\frac{kT}{n}) = S_i(0)\exp(rkT/n)\prod_{m=1}^{k}(1+\sqrt{\frac{T}{n}}\sum_{j=1}^{d}\sigma_{ij}\xi_j^{(m)})$
$$\times \frac{\prod_{j=1}^{\tilde{N}^{n,k}}(1+\tilde{U}_i^{(j)})}{\left(1+(1-\exp(-\lambda T/n))u_i\right)^k}$$
where $\tilde{N}^{n,k} = \sum_{m=1}^{k}\rho^{n,m}$.
- $\mathcal{T}_n$–the set of all stopping times $\sigma : \Omega_n \to \{0, \frac{T}{n}, ..., T\}$ with respect to the filtration generated by $S^{(n)}$.

American options

- American option with the payoff process

$$Y^{(n)}\Big(\frac{kT}{n}\Big) = F\Big(S^{(n)}\Big(\frac{kT}{n}\Big), \frac{kT}{n}\Big), \ 0 \leq k \leq n.$$

- The term

$$V_n = \sup_{\tau \in \mathcal{T}_n} E_n(\exp(-r\tau) Y^{(n)}(\tau))$$

is an arbitrage–free price of the n–step market.

Dynamical programming algorithm for V_n

For any $0 \leq k \leq n$ define $F_k^{(n)} : \mathbb{R}_+^d \to \mathbb{R}_+$ by

$$F_n^{(n)}(x) = F(x, T)$$

and for $k < n$,

$$F_k^{(n)}(x) = \max\left(F(x, kT/n), EF_{k+1}^{(n)}(x_1\Gamma_1^{(n)}, ..., x_d\Gamma_d^{(n)})\right)$$

where for any $1 \leq i \leq d$

$$\Gamma_i^{(n)} = \exp(rT/n)\left(1 + \sqrt{\frac{T}{n}}\sum_{j=1}^{d}\sigma_{ij}\xi_j^{(1)}\right)\frac{1 + \tilde{U}_i^{(j)}\mathbb{I}_{\rho^{n,1}=1}}{1 + (1 - \exp(-\lambda T/n))u_i}.$$

Then

$$V_n = F_0^{(n)}(S(0)).$$

The main result

Theorem:

For any $\epsilon > 0$ there exists a constant C_ϵ such that for any n

$$|V - V_n| < C_\epsilon n^{\epsilon - \frac{1}{8}}.$$

If F is bounded then there exists a constant C such that for any n

$$|V - V_n| < C n^{-\frac{1}{8}}.$$

Preparations

Fix n. For $1 \leq i \leq d$ and $0 \leq k \leq n$ set

- $\beta_i = r - \frac{\sum_{j=1}^{d} \sigma_{ij}^2}{2}$.
- $N^{n,k} = \sum_{j=1}^{k} \mathbb{I}_{\{N(jT/n) - N((j-1)T/n) \geq 1\}}$.
- $S_i^{\mathcal{C},n}(kT/n) = S_i(0) \exp(\beta_i kT/n + \sum_{j=1}^{d} \sigma_{ij} W_j(kT/n)) \times \frac{\prod_{j=1}^{N^{n,k}} (1+U_i^{(j)})}{\left(1+(1-\exp(-\lambda T/n))u_i\right)^k}$.

First step

Set $Y^{\mathcal{C},n}(t) = F(t, S^{\mathcal{C},n}(t))$.
Let $\mathcal{T}^{(n)} \subset \mathcal{T}$ be the set of all stopping times (in the continuous model) with values in the set $\{0, \frac{T}{n}, ..., T\}$.
Define the map $\phi_n : \mathcal{T} \to \mathcal{T}^{(n)}$ by $\phi_n(\tau) = \frac{T}{n}\min\{k : \frac{kT}{n} \geq \tau\}$.
By using the regularity properties of F and the fact that $0 \leq \phi_n(\tau) - \tau \leq \frac{1}{n}$ we obtain that for

$$V_n^{\mathcal{C}} := \sup_{\tau \in \mathcal{T}^{(n)}} E(\exp(-r\tau) Y^{\mathcal{C},n}(\tau)),$$

$$|V_n^{\mathcal{C}} - V| \leq C_1 n^{-1/4}.$$

Second step

Fix n. For $1 \le i \le d$ and $0 \le k \le n$ set

- $S_i^{\mathcal{D},n}(kT/n) =$
  $S_i(0)\exp(\beta_i kT/n + \sqrt{T/n}\sum_{m=1}^{k}\sum_{j=1}^{d}\sigma_{ij}\xi_j^{(m)})$
  $$\times \frac{\prod_{j=1}^{\tilde{N}^{n,k}}(1+\tilde{U}_i^{(j)})}{\left(1+(1-\exp(-\lambda T/n))u_i\right)^k}.$$
- $Y^{\mathcal{D},n}(kT/n) = F(kT/n, S^{\mathcal{D},n}(kT/n))$.

By using the regularity properties of F we obtain that for

$$V_n^{\mathcal{D}} := \sup_{\tau\in\mathcal{T}_n} E(\exp(-r\tau)Y^{\mathcal{D},n}(\tau)),$$

$$|V_n^{\mathcal{D}} - V_n| \le C_2 n^{-1/4}.$$

Final step

We need to estimate

$$|V_n^{\mathcal{C}} - V_n^{\mathcal{D}}| := \Big| \sup_{\tau \in \mathcal{T}^{(n)}} E\Big(\exp(-r\tau) F(\tau, S^{\mathcal{C},n}(\tau))\Big)$$

$$- \sup_{\tau \in \mathcal{T}_n} E_n\Big(\exp(-r\tau) F(\tau, S^{\mathcal{D},n}(\tau))\Big)\Big|$$

where

$$S_i^{\mathcal{C},n}(kT/n) = S_i(0)\exp(\beta_i kT/n + \textstyle\sum_{j=1}^d \sigma_{ij} W_j(kT/n)) \times$$

$$\frac{\prod_{j=1}^{N^{n,k}} (1+U_i^{(j)})}{\Big(1+(1-\exp(-\lambda T/n))u_i\Big)^k}.$$

$$S_i^{\mathcal{D},n}(kT/n) = S_i(0)\exp(\beta_i kT/n + \textstyle\sum_{j=1}^d \sigma_{ij}\sqrt{T/n}\sum_{m=1}^k \xi_j^{(m)}) \times$$

$$\frac{\prod_{j=1}^{\tilde{N}^{n,k}} (1+\tilde{U}_i^{(j)})}{\Big(1+(1-\exp(-\lambda T/n))u_i\Big)^k}.$$

The main tool

Theorem:(Sakhanenko 2002). Let X, Y be a d–dimensional random vectors. Assume that $EX = EY$ and

$$E(X_i X_j) = E(Y_i Y_j) \ \ \forall i, j \in \{1, ..., d\}.$$

For any $z > 0$ and $n \in \mathbb{N}$ it is possible to construct a probability space which contains two sequence of i.i.d. random vectors $X^{(1)}, ..., X^{(n)}$, $Y^{(1)}, ..., Y^{(n)}$ such that $X^{(1)} \sim X$, $Y^{(1)} \sim Y$ and

$$P(\max_{1 \leq k \leq n} || \sum_{m=1}^{k} X^{(m)} - Y^{(m)} || > z) \leq \frac{\tilde{C} n}{z^3} E(||X||^3 + ||Y||^3).$$

Furthermore, for any k the random vectors $X^{(1)}, ..., X^{(k-1)}, Y^{(k)}, Y^{(k+1)}, ..., Y^{(n)}$, are independent.

The inequality $V_n^{\mathcal{D}} - V_n^{\mathcal{C}} < C_\epsilon n^{\epsilon - 1/8}$

From the Sakhanenko theorem it follows that we can construct a probability space which contains $\{W(kT/n)\}_{k=0}^n$, $\xi^{(1)}, \xi^{(2)}, ..., \xi^{(n)}$ such that

- $$P(\max_{1 \leq k \leq n} ||W(kT/n) - \sqrt{T/n} \sum_{m=1}^{k} \xi^{(m)}|| > n^{-1/8}) \leq \hat{C} n^{-1/8}.$$
- For any k the random vectors $\xi^{(1)}, ..., \xi^{(k-1)}, W(kT/n) - W((k-1)T/n), ..., W(T) - W((n-1)T/n)$ are independent.

We extend the probability space such that it will also contain two independent sequences of i.i.d. random vectors $\{U^{(k)}\}_{k=1}^\infty$ and $\{\rho^{n,k}\}_{k=1}^n$ which are independent of W and ξ.

Continue of the inequality $V_n^{\mathcal{D}} - V_n^{\mathcal{C}} < C_\epsilon n^{\epsilon-1/8}$

On the above probability space we have

$$V_n^{\mathcal{D}} - V_n^{\mathcal{C}} = \sup_{\tau\in\mathcal{T}(\xi,U,\rho)} E\Big(\exp(-r\tau)F(\tau,S^{\mathcal{D},n}(\tau))\Big) -$$

$$\sup_{\tau\in\mathcal{T}(W,U,\rho,\xi)} E\Big(\exp(-r\tau)F(\tau,S^{\mathcal{C},n}(\tau))\Big) \leq$$

$$\sup_{\mathcal{T}(\xi,U,\rho)} E\Big(\exp(-r\tau)F(\tau,S^{\mathcal{D},n}(\tau))\Big) -$$

$$\sup_{\tau\in\mathcal{T}(\xi,U,\rho)} E\Big(\exp(-r\tau)F(\tau,S^{\mathcal{C},n}(\tau))\Big) \leq$$

$$E \max_{0\leq k\leq n} |F(kT/n, S^{\mathcal{D},n}(kT/n)) - F(kT/n, S^{\mathcal{C},n}(kT/n))|$$

$$< C_\epsilon n^{\epsilon-1/8}.$$

The inequality $V_n^{\mathcal{C}} - V_n^{\mathcal{D}} < C_\epsilon n^{\epsilon - 1/8}$

We do it in a similar way.
Just change the roles of $\{W(kT/n)\}_{k=1}^n$ and $\{\xi^{(k)}\}_{k=1}^n$.

Weak convergence approach

It is well known that $S^{(n)} \Rightarrow S$ in distribution on the space $D[0, T]$ equipped with the Skorohod topology. By using the stability of optimal stopping values under weak convergence it can be proved that $\lim_{n\to\infty} V_n = V$. The main disadvantage of the weak convergence approach is that this machinery can not provide, in principle, speed of convergence estimates.

Strong approximation theorems

There are other strong approximation theorems (multidimensional case) which provide a better estimate of

$$\max_{1 \le k \le n} ||W(kT/n) - \sqrt{T/n} \sum_{m=1}^{k} \xi^{(m)}||$$

however the construction of the corresponding probability space does not respect filtrations in a very substantial way as they depend on the far away future.

Extension to path dependent payoffs

These results are remain valid for path dependent payoffs which satisfy strong regularity conditions.

For the Black–Scholes model (no Poisson process) the above results can be extended to path dependent options which satisfy Lipchitz type conditions.

Extension to game options

- Let $F, \Delta : \mathbb{R}_+^d \times \mathbb{R}_+ \to \mathbb{R}_+$.
- Consider a game option with the payoff function

$$H(\sigma, \tau) = F(S(\sigma \wedge \tau), \sigma \wedge \tau) + \mathbb{I}_{\sigma<\tau} \Delta(S(\sigma), \sigma).$$

- $V := \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} E\big(\exp(-r(\sigma \wedge \tau))H(\sigma, \tau)\big)$ is an arbitrage–free price for the Game option.

An interesting question is whether we can apply strong approximation theorems in order to approximate V by similar terms on the discrete probability spaces that were used for American options.

References

- Ya.Dolinsky and Yu.Kifer, "Binomial approximations for barrier options of Israeli style", Annals of Dynamic Games, vol. XI.
- B.Hu, J.Liang and L.Liang, "Optimal convergence rate of the binomial tree scheme for American options with jump diffusion and their free boundaries", SIAM Journal on Financial Mathematics. **1** (2010), 30–65.
- Yu.Kifer, "Error estimates for binomial approximations of game options", Ann. Appl. Probab. **16** (2006), 984-1033.
- D.Lamberton and L.C.G Rogers, "Optimal stopping and Embedding", J. Appl. Probab. **37** (2000), 1143–1148.

References

- H.He, "Convergence from discrete to continuous time contingent claim prices", Rev. Financial Stud. **3** (1990).
- Yu.Kifer, "Optimal stopping and strong approximation theorems", Stochastics **79** (2007), 253–273.
- S.Mulinacci, "American path–dependent options: analysis and approximations", Rend. Studi Econ. Quant. 2002 (2003), 93–120.
- R.Maller, D.Solomon and A.Szimayer, "A Multinomial Approximation of American Option Prices in a Levy Process Model", Math. Finance **16** (2006), 613–633.
- A.I Sakhanenko, "A New Way to Obtain Estimates in the Invariance Principle", High Dimensional Probability II, (2000) 221–243.