Equivalent Measure Changes for Jump-Diffusions

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Analysis, Stochastics, and Applications
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Outline

1 Problem

2 Result

3 Applications
   CIR Short Rate Model
   Stochastic Volatility Model
1 Problem

2 Result

3 Applications

CIR Short Rate Model
Stochastic Volatility Model
Ingredients

- \( m, d \in \mathbb{N} \)
- State space (open or closed) \( E \subseteq \mathbb{R}^m \)
- Locally bounded measurable mappings
  \[
  b : E \to \mathbb{R}^{m \times 1}, \quad \sigma : E \to \mathbb{R}^{m \times d}
  \]
- Transition kernel \( \nu \) from \( E \) to \( \mathbb{R}^m \) such that
  \[
  x \mapsto \int_{\mathbb{R}^m} \|\xi\| \wedge \|\xi\|^2 \nu(x, d\xi)
  \]
  is locally bounded on \( E \)
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Special Semimartingale

- Filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\)
- Carrying \(d\)-dimensional Brownian motion \(W\), and
- Random measure \(\mu(dt, d\xi)\) associated to the jumps of \(\ldots\)
- \(\ldots\) the special (for simplicity) semimartingale \(X\) with canonical decomposition

\[
X_t = X_0 + \int_0^t b(X_s) \, ds + \int_0^t \sigma(X_s) \, dW_s \\
+ \int_0^t \int_{\mathbb{R}^m} \xi(\mu(ds, d\xi) - \nu(X_s, d\xi)) \, ds
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Density Process Heuristics I

- **Measurable mappings** . . .
  \[ \lambda : E \to \mathbb{R}^{d \times 1}, \quad \kappa : E \times \mathbb{R}^{m} \to (0, \infty) \]

- . . . such that the local martingale \( L \) is well defined:

  \[
  L_t = \int_0^t \lambda(X_s)^\top dW_s \\
  + \int_0^t \int_{\mathbb{R}^m} (\kappa(X_s, \xi) - 1) (\mu(ds, d\xi) - \nu(X_s, d\xi) ds)
  \]

- **Assume** its stochastic exponential

  \[
  \mathcal{E}_t(L) = \exp \left( L_t - \frac{1}{2} \int_0^t \|\lambda(X_s)\|^2 ds \\
  + \int_0^t \int_{\mathbb{R}^m} (\log \kappa(X_s, \xi) - \kappa(X_s, \xi) + 1) \mu(ds, d\xi) \right)
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  is a **true martingale**
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Heuristics II

- **Finite time horizon** $T$

- Define equivalent probability measure $Q \sim P$ on $\mathcal{F}_T$ by

$$\frac{dQ}{dP} = \mathcal{E}_T(L)$$

- Girsanov’s theorem implies that

$$\tilde{W}_t = W_t - \int_0^t \lambda(X_s) \, ds, \quad t \in [0, T]$$

is a $Q$-Brownian motion, and the compensator of $\mu(dt, d\xi)$ under $Q$ becomes

$$\tilde{\nu}(X_t, d\xi)dt = \kappa(X_t, \xi)\nu(X_t, d\xi)dt, \quad t \in [0, T].$$
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• Canonical decomposition of $X$ under $\mathbb{Q}$ reads

$$X_t = X_0 + \int_0^t \tilde{b}(X_s) \, ds + \int_0^t \sigma(X_s) \, d\tilde{W}_s + \int_0^t \int_{\mathbb{R}^m} \xi(\mu(ds, d\xi) - \tilde{\nu}(X_s, d\xi)) \, ds$$

• With modified drift function defined as

$$\tilde{b}(x) = b(x) + \sigma(x)\lambda(x) + \int_{\mathbb{R}^m} \xi(\kappa(x, \xi) - 1) \nu(x, d\xi).$$
Heuristics III

- Canonical decomposition of $X$ under $\mathbb{Q}$ reads

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Heuristics IV

• In other words: infinitesimal generator of $X$ under $\mathbb{Q}$ is

$$
\tilde{A}f(x) = \sum_{i=1}^{m} \tilde{b}_i(x) \frac{\partial f(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{m} (\sigma \sigma^\top)_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \int_{\mathbb{R}^m} \left( f(x + \xi) - f(x) - \sum_{i=1}^{m} \frac{\partial f(x)}{\partial x_i} \xi_i \right) \tilde{\nu}(x, d\xi)
$$

• Itô’s lemma implies: for any $f \in C^2_c(E)$,

$$
f(X_t) - f(X_0) - \int_0^t \tilde{A}f(X_s) \, ds
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is a $\mathbb{Q}$-martingale
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is a $\mathbb{Q}$-martingale
• QUESTION: when is $\mathcal{E}(L)$ a true martingale??

• EQUIVALENTLY: when is

$$\mathbb{E}[\mathcal{E}_T(L)] = 1$$

• Note: this does not depend on the filtration, but only on the law of $X$!
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1 Problem

2 Result

3 Applications
   CIR Short Rate Model
   Stochastic Volatility Model
Martingale Problem

- Canonical basis: $\Omega = \text{space of càdlàg paths in } E$, 

$$X_t(\omega) = \omega(t), \quad \mathcal{F}_t = \mathcal{F}^X_t$$

**Definition 2.1.**

A probability measure $Q$ on $(\Omega, \mathcal{F}^X)$ is a solution of the martingale problem for $\tilde{A}$ if for all $f \in C_c^2(E)$,

$$f(X_t) - f(X_0) - \int_0^t \tilde{A}f(X_s) \, ds$$

is a $Q$-martingale. The martingale problem for $\tilde{A}$ is well-posed if for every probability distribution $\eta$ on $E$ there exists a unique solution $Q$ with $Q \circ X_0^{-1} = \eta$. 
Martingale Problem

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Main Result

**Theorem 2.2.**
Assume that $x \mapsto \lambda(x)$ and

$$x \mapsto \int_{\mathbb{R}^m} (\kappa(x, \xi) \log \kappa(x, \xi) - \kappa(x, \xi) + 1) \nu(x, d\xi)$$

are locally bounded on $E$, and that the martingale problem for $\tilde{A}$ is well-posed. Then $\mathcal{E}(L)$ is a true martingale.
Proof I: Lépingle and Mémin [3]

- Localizing sequence of bounded stopping times $S_1 \leq S_2 \leq \cdots \uparrow \infty$ such that

\[
\Lambda_n := \frac{1}{2} \int_0^{S_n} \|\lambda(X_s)\|^2 \, ds \\
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is uniformly bounded

- Lépingle and Mémin [3, Théorème IV.3]:

\[
\mathcal{E}_{t \wedge S_n}(L) \text{ is a martingale}
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Proof II: Stopped Martingale Problem

- Girsanov’s theorem implies that for any $f \in C^2_c(E)$:

$$f(X_{t\wedge S_n}^S) - f(X_0) - \int_0^{t\wedge S_n} \tilde{A}f(X_{s\wedge S_n}^S) \, ds$$

is a $\mathcal{E}_{S_n}(L) \cdot \mathbb{P}$-martingale

- Uniqueness of the stopped martingale problem (Ethier and Kurtz [2, Theorem 4.6.1]) implies that

$$\mathcal{E}_{S_n}(L) \cdot \mathbb{P} = \mathbb{Q} \quad \text{on } \mathcal{F}_{S_n}^X$$

where $\mathbb{Q}$ is the solution of the martingale problem for $\tilde{A}$ with $\mathbb{Q} = \mathbb{P}$ on $\mathcal{F}_0^X$
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Proof III: Limit

- Monotone convergence theorem, and since \( \{ T < S_n \} \in \mathcal{F}^X_{T \wedge S_n} \):

\[
1 = \lim_{n \to \infty} Q[T < S_n] \\
= \lim_{n \to \infty} \mathbb{E}_P[\mathcal{E}_{T \wedge S_n}(L) 1_{\{T < S_n\}}] \\
= \lim_{n \to \infty} \mathbb{E}_P[\mathcal{E}_T(L) 1_{\{T < S_n\}}] \\
= \mathbb{E}_P[\mathcal{E}_T(L)]
\]
Equivalent Measure Changes for Jump-Diffusions

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Cox–Ingersoll–Ross (CIR) Model

- Model for short rate under $\mathbb{P}$: square root ("CIR") process
  \[ dX_t = (b + \beta X_t) \, dt + \sigma \sqrt{X_t} \, dW_t \]
- State space $E = (0, \infty)$
- Feller condition: 0 not attained iff $b \geq \sigma^2/2$
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Market Price of Risk Specification

- Aim: MPR specification that preserves affine structure:

\[
dX_t = \left( b^Q + \beta^Q X_t \right) dt + \sigma \sqrt{X_t} \left( dW_t + \frac{\ell + \lambda X_t}{\sigma \sqrt{X_t}} \right)
\]

- MPR parameters:

\[
\ell = b - b^Q, \quad \lambda = \beta - \beta^Q
\]

- Formal density process \( \mathcal{E} \left( -\frac{\ell + \lambda X}{\sigma \sqrt{X}} \right) \bullet W \)

- Novikov condition not satisfied, since

\[
\mathbb{E} \left[ e^{\frac{1}{2} \int_0^T \frac{1}{X_t} \, dt} \right] = \infty, \quad \mathbb{E} \left[ e^{\frac{1}{2} \int_0^T X_t \, dt} \right] = \infty
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for \( T \) large enough in general
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for \( T \) large enough in general
CFY Condition

- Assume that Feller condition is also satisfied for $b^Q$:
  \[ b^Q \geq \sigma^2/2 \]

- Then the martingale problem for
  \[ \tilde{A}f(x) = \left( b^Q + \beta^Q x \right) f'(x) + \frac{1}{2} \sigma^2 x f''(x) \]

  is well-posed in $E = (0, \infty)$

- CFY Theorem implies that $\mathcal{E} \left( -\frac{\ell + \lambda X}{\sigma \sqrt{X}} \bullet W \right)$ is a true martingale
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Stochastic Volatility

- Model for volatility: GARCH diffusion

\[ dX_t = (b + \beta X_t) \, dt + X_t \, dW_t^1 \]

- State space \( E = (0, \infty) \); that is, \( b \geq 0 \)

- Model for discounted S&P 500 index process:

\[ \frac{dS_t}{S_t} = X_t \left( \rho \, dW_t^1 + \sqrt{1 - \rho^2} \, dW_t^2 \right) \]

- Leverage effect: non-positive correlation \( \rho \leq 0 \) between

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- Model for volatility: GARCH diffusion
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Martingality of $S$

- Question: is $S$ a true martingale? (vital for pricing!)
- Write $S$ as stochastic exponential

$$S_t = S_0 \mathcal{E}_t \left( \lambda(X)^T \cdot W \right)$$

with

$$\lambda(x) = x \left( \frac{\rho}{\sqrt{1 - \rho^2}} \right)$$

- Novikov condition fails:

$$\mathbb{E} \left[ e^{\frac{1}{2} \int_0^T X_t^2 dt} \right] = \infty$$
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CFY Condition

- Apply auxiliary change of measure with density process

\[ \frac{S_t}{S_0} = \mathcal{E}_t \left( \lambda(X)^\top \cdot W \right) \]

- Formally, the generator of \( X \) becomes

\[ \tilde{A}f(x) = \left( b + \beta x + \rho x^2 \right) f'(x) + \frac{1}{2} xf''(x) \]

- Inspection shows: the martingale problem for \( \tilde{A} \) is well-posed in \( E = (0, \infty) \)

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