On refined volatility smile expansion in the Heston model

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Heston Model

- **Dynamics**

\[ dS_t = S_t \sqrt{V_t} \, dW_t, \quad S_0 = 1, \]
\[ dV_t = (a + bV_t) \, dt + c \sqrt{V_t} \, dZ_t, \quad V_0 = v_0 > 0, \]

- **Correlated Brownian motions**

\[ d\langle W, Z \rangle_t = \rho \, dt, \quad \rho \in [-1, 1] \]

- **Parameters**

\[ a \geq 0, \quad b \leq 0, \quad c > 0 \]
Density and smile asymptotics

- Consider a fixed maturity $T > 0$.
- $D_T :=$ density of $S_T$.
- How heavy are the tails?
  
  $$D_T(x) \sim ? \quad (x \rightarrow 0, \infty)$$

- Implied Black-Scholes volatility ($k = \log K$ is the log-strike)
  
  $$\sigma_{BS}^2(k, T) \sim ? \quad (k \rightarrow \pm \infty)$$
Known results

- Drăgulescu, Yakovenko (2002): Stationary variance regime. Leading growth order of distribution function of $S_T$, by (non-rigorous) saddle-point argument
- Gulisashvili-Stein (2009): Precise density asymptotics for uncorrelated Heston model
Main results (right tail), SG et al. 2010

- Density asymptotics for $x \to \infty$

$$D_T(x) = A_1 x^{-A_3} e^{A_2 \sqrt{\log x}} (\log x)^{-3/4+a/c^2} (1 + O((\log x)^{-1/2}))$$

- Implied volatility for $k = \log K \to \infty$

$$\sigma_{BS} (k, T) \sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{\varphi(k)}{k^{1/2}}\right)$$

($\varphi$ arbitrary function tending to $\infty$)
Interpretation of smile expansion

- Implied volatility for $k = \log K \to \infty$

$$\sigma_{BS} (k, T) \sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O \left( \frac{\varphi(k)}{k^{1/2}} \right)$$

- $\beta_1$ does not depend on $\sqrt{v_0}$
- $\beta_2$ depends linearly on $\sqrt{v_0}$
- Changes of $\sqrt{v_0}$ have second-order effects
- Increase $\sqrt{v_0}$: parallel shift, slope not affected
- Changes in mean-reversion level $\bar{v} = -a/b$ seen only in $\beta_3$
General remarks

- Constants depend on: critical moment, critical slope, critical curvature
- Critical moment etc. defined in a model-free manner
- Closed form of Fourier (Mellin) transform not needed
- Work only with affine principles (Riccati equations)
Lee’s moment formula (2004)

- Model-free result
- Relates critical moment to implied volatility

\[ s^* := \sup \{ s : E[S_T^s] < \infty \} \]

\[ s^* =: \frac{1}{2\beta_1^2} + \frac{\beta_1^2}{8} + \frac{1}{2} \]

\[ \limsup_{k \to \infty} \frac{\sigma_{BS}(k, T) \sqrt{T}}{\sqrt{k}} = \beta_1 \]

- Refinements by Benaim, Friz (2008), Gulisashvili (2009)
Heston Model: Mgf of log-spot $X_t$

- Moment generating function

$$E[e^{sX_t}] = \exp(\phi(s, t) + \nu_0 \psi(s, t))$$

- Riccati equations

$$\partial_t \phi = F(s, \psi), \quad \phi(0) = 0,$$
$$\partial_t \psi = R(s, \psi), \quad \psi(0) = 0$$

$$F(s, \nu) = a \nu,$$
$$R(s, \nu) = \frac{1}{2}(s^2 - s) + \frac{1}{2}c^2 \nu^2 + b \nu + s \rho c \nu$$

- Explicit solution possible, but cumbersome expression
Moment explosion

- Critical moment for time $T$

  $$s^* := \sup \{ s \geq 1 : E[S^s_T] < \infty \}$$

- Explosion time for moment of order $s$

  $$T^*(s) = \sup \{ t \geq 0 : E[S^s_t] < \infty \}$$

- Critical slope, critical curvature:

  $$\sigma := -\partial_s T^*|_{s^*} \geq 0 \quad \text{and} \quad \kappa := \partial_s^2 T^*|_{s^*}$$
Explicit Explosion time for the Heston model

- Explosion time for moment of order $s$

$$T^*(s) = \frac{2}{\sqrt{-\Delta(s)}} \left( \arctan \frac{\sqrt{-\Delta(s)}}{s\rho c + b} + \pi \right),$$

$$\Delta(s) := (s\rho c + b)^2 - c^2 (s^2 - s)$$

- Critical moment $s^*$: Find numerically from

$$T^*(s^*) = T.$$
Mellin (Fourier) inversion

- Mellin transform of spot: \( M(u) = E[e^{(u-1)X_T}] \)
- Analytic in a complex strip
- Density of \( S_T \) by Mellin inversion:
  \[
  D_T(x) = \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} x^{-u} M(u) \, du.
  \]
- Valid for contour in analyticity strip of the Mellin transform
- Justification: exponential decay of \( M(u) \) at \( \pm i\infty \).
Mellin transform analytic in a strip

\[ u_- < \Re(u) < u^* = s^* + 1 \]

Leading order of density for \( x \to \infty \)

\[ x^{-u^*-\varepsilon} \ll D_T(x) \ll x^{-u^*+\varepsilon}, \]

depends on \textit{location} of singularity

Refinement: lower order factors depend on \textit{type} of singularity
Saddle point method

- Recall:
  \[ D_T(x) = \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} x^{-u} M(u) du \]

- Shift contour to the right, close to the singularity.
- Let it pass through a saddle point of the integrand.
- For large \( x \), the integral is concentrated around the saddle.
- Local expansion of integrand yields expansion of whole integral.
- (Laplace, Riemann, Debye...)
New integration contour

Contour runs through saddle point $\hat{u} = \hat{u}(x)$

Moves to the right as $x \to \infty$
The surface $|x^{-u}M(u)|$
Asymptotics of $\psi$ and $\phi$ near critical moment

- Recall $M(u) = \exp(\phi(u - 1, t) + \nu_0\psi(u - 1, t))$
- For $u \to u^*$ we have (with $\beta := \sqrt{2\nu_0/c\sqrt{\sigma}}$

\[
\psi(u - 1, T) = \frac{\beta^2}{u^* - u} + \text{const} + O(u^* - u),
\]

\[
\phi(u - 1, T) = \frac{2a}{c^2} \log \frac{1}{u^* - u} + \text{const} + O(u^* - u)
\]

- Found from Riccati equations
Saddle point method

- Finding the saddle point: \( 0 = \text{derivative of integrand} \)
- Use only first order expansion:

\[
0 = \frac{\partial}{\partial u} x^{-u} \exp \left( \frac{\beta^2}{u^* - u} \right)
\]

- Approximate saddle point at

\[
\hat{u}(x) = u^* - \beta / \sqrt{\log x}
\]
Contour depends on $x$:

$$u = \hat{u}(x) + iy, \quad -\infty < y < \infty$$

Divide contour into three parts:

$$|y| < (\log x)^{-\alpha} \quad \text{(central part)},$$

upper tail, lower tail (symmetric)

Uniform local expansion at saddle point $\Rightarrow$ need large $\alpha$

Tails negligible $\Rightarrow$ need small $\alpha$

Can take $\frac{2}{3} < \alpha < \frac{3}{4}$
Recall Mellin transform

\[ M(u) = \exp(\phi(u - 1, t) + v_0 \psi(u - 1, t)) \]

Determine singular expansions of \( \phi \) and \( \psi \) from Riccati equations

Abbreviation \( L := \log x \)

Local expansion of the integrand:

\[ x^{-u} M(u) = Cx^{-u^*} \exp \left( 2\beta L^{1/2} + \frac{a}{c^2} \log L - \beta^{-1} L^{3/2} y^2 + o(1) \right) \]
Gaussian integral

\[
\int_{-L^{-\alpha}}^{L^{-\alpha}} \exp(-\beta^{-1} L^{3/2} y^2) dy
\]

\[
= \beta^{1/2} L^{-3/4} \int_{-\beta^{-1/2} L^{3/4-\alpha}}^{\beta^{-1/2} L^{3/4-\alpha}} \exp(-w^2) dw
\]

\[
\sim \beta^{1/2} L^{-3/4} \int_{-\infty}^{\infty} \exp(-w^2) dw = \sqrt{\pi} \beta^{1/2} L^{-3/4}
\]
Tail estimate

- Finding saddle point + local expansion fairly routine
- Problem: Verify concentration
- Needs some insight into behaviour of function away from saddle point
- Show exponential decay by ODE comparison
Result of saddle point method

- Density asymptotics for $x \to \infty$

$$D_T(x) = A_1 x^{-A_3} e^{A_2 \sqrt{\log x}} (\log x)^{-3/4+a/c^2} \left(1+O((\log x)^{-1/2})\right)$$

- Constants in terms of critical moment and critical slope:

$$A_3 = u^* = s^* + 1 \quad \text{and} \quad A_2 = 2 \frac{\sqrt{2v_0}}{c \sqrt{\sigma}}$$

- Easily extended to full asymptotic expansion
Explicit expression for constant factor

From closed form of $\phi$ and $\psi$:

$$A_1 = \frac{1}{2\sqrt{\pi}} (2v_0)^{1/4} - a/c^2 \quad c^{2a/c^2 - 1/2} \quad e^{-a/c^2 - 1/4}$$

$$\times \exp \left( -v_0 \left( \frac{b + s^* \rho c}{c^2} + \frac{\kappa}{c^2 \sigma^2} \right) - \frac{aT}{c^2} (b + c\rho s^*) \right)$$

$$\times \left( \frac{2b^2 + 2bc\rho s^* + c^2s^*(1 - (1 - \rho^2)s^*)}{c^2s^*(s^* - 1) \sinh \frac{1}{2} \sqrt{b^2 + 2bc\rho s^* + c^2s^*(1 - (1 - \rho^2)s^*)}} \right)^{2a/c^2}$$
Call prices and Smile asymptotics

Gulisashvili (2009): Assumes that density of spot varies regularly at infinity

\[ D_T(x) = x^{-\gamma} h(x), \]

\( h \) varies slowly at infinity, \( \gamma > 2 \)

- Expansions of call prices and implied volatility
- Similarly for left tail
Smile asymptotics

- Implied volatility for log-strike $k \to \infty$

\[
\sigma_{BS}(k, T) \sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O \left( \frac{\varphi(k)}{k^{1/2}} \right)
\]

- Constants

\[
\beta_1 = \sqrt{2} \left( \sqrt{A_3 - 1} - \sqrt{A_3 - 2} \right),
\]
\[
\beta_2 = \frac{A_2}{\sqrt{2}} \left( \frac{1}{\sqrt{A_3 - 2}} - \frac{1}{\sqrt{A_3 - 1}} \right),
\]
\[
\beta_3 = \frac{1}{\sqrt{2}} \left( \frac{1}{4} - \frac{a}{c^2} \right) \left( \frac{1}{\sqrt{A_3 - 1}} - \frac{1}{\sqrt{A_3 - 2}} \right)
\]
Call price for strike $K \to \infty$

$$C(K) = \frac{A_1}{(-A_3 + 1)(-A_3 + 2)} K^{A_3 + 2} e^{A_2 \sqrt{\log K}} (\log K)^{-\frac{3}{4} + \frac{a}{c^2}}$$

$$\times \left( 1 + O \left( (\log K)^{-\frac{1}{4}} \right) \right)$$
Figure: Implied variance $\sigma(k, 1)^2$ in terms of log-strikes compared to the first order (dashed) and third order (dotted) approximations.

A. Gulisashvili: *Asymptotic formulas with error estimates for call pricing functions and the implied volatility at extreme strikes*, 2009

