Differentiability problems in Banach spaces

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Expanded notes of a talk based on a nearly finished research monograph "Fréchet differentiability of Lipschitz functions and porous sets in Banach spaces" written jointly with Joram Lindenstrauss and Jaroslav Tišer with some results coming from a joint work with Giovanni Alberti and Marianna Csörnyei

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Derivatives

For vector valued functions there are two main versions of derivatives: Gâteaux (or weak) derivatives and Fréchet (or strong) derivatives. For a function \( f \) from a Banach space \( X \) into a Banach space \( Y \) the Gâteaux derivative at a point \( x_0 \in X \) is by definition a bounded linear operator \( T: X \rightarrow Y \) so that

\[
\lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t} = Tu
\]

The operator \( T \) is called the Fréchet derivative of \( f \) at \( x_0 \) if it is a Gâteaux derivative of \( f \) at \( x_0 \) and the limit above holds uniformly in \( u \) in the unit ball (or unit sphere) in \( X \).

Existence of derivatives

The first continuous nowhere differentiable \( f: \mathbb{R} \rightarrow \mathbb{R} \) was constructed by Bolzano about 1820 (unpublished), who however did not give a full proof. Around 1850, Riemann mentioned such an example, which was later found slightly incorrect. The first published example with a valid proof is by Weierstrass in 1875.

The first general result on existence of derivatives for functions \( f: \mathbb{R} \rightarrow \mathbb{R} \) was found by Lebesgue (around 1900). He proved that a monotone function \( f: \mathbb{R} \rightarrow \mathbb{R} \) is differentiable almost everywhere. As a consequence it follows that every Lipschitz function \( f: \mathbb{R} \rightarrow \mathbb{R} \) i.e., a function which satisfies

\[
|f(s) - f(t)| \leq C|s - t|
\]

for some constant \( C \) and every \( s, t \in \mathbb{R} \), has a derivative a.e.

Sharpness of Lebesgue’s result

Lebesgue’s result is sharp in the sense that for every \( A \subset \mathbb{R} \) of measure zero there is a Lipschitz (and monotone) function \( f: \mathbb{R} \rightarrow \mathbb{R} \) which fails to have a derivative at any point of \( A \).

A more precise result was proved by Zahorski in 1946.

Theorem. A set \( A \subset \mathbb{R} \) is a \( G_{\delta} \) set of Lebesgue measure zero if and only if there is a Lipschitz function \( f: \mathbb{R} \rightarrow \mathbb{R} \) which is differentiable exactly at points of \( \mathbb{R} \setminus A \).

Explanation. A set \( A \subset \mathbb{R} \) is

\( G_\delta \) if there are open sets \( G_i \) so that \( A = \bigcap_{i=1}^{\infty} G_i \),

\( G_{\delta,\sigma} \) if there are \( G_\delta \) sets \( G_i \) so that \( A = \bigcup_{i=1}^{\infty} G_i \).
Rademacher’s Theorem

Lebesgue’s theorem was extended to Lipschitz functions \( f : \mathbb{R}^n \to \mathbb{R} \) by Rademacher in 1919 who showed that in this case \( f \) is also differentiable a.e.

However, this result is not as sharp as Lebesgue’s: in \( \mathbb{R}^n, n \geq 2 \) there are sets of measure zero containing points of differentiability of all Lipschitz \( f : \mathbb{R}^2 \to \mathbb{R} \) (Preiss 1990).

Doré and Maleva (in preparation) found a compact set in \( \mathbb{R}^2 \) of Hausdorff dimension one that contains points of differentiability of all Lipschitz functions \( f : \mathbb{R}^2 \to \mathbb{R} \).

With Alberti and Csörnyei (in preparation) we proved that Rademacher’s theorem is sharp for maps from \( \mathbb{R}^2 \) to \( \mathbb{R} \).

Many higher dimensional results are also known, but the question of sharpness of Rademacher’s theorem for maps from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \) is still open.

Almost everywhere

Recall that in infinite dimensional spaces there is no Lebesgue measure. If we wish to extend Lebesgue’s theorem to infinite dimensional setting, we have to extend the notion of a.e. (almost everywhere, except for a null set) to such spaces.

So we have to define in a reasonable way a family of negligible sets on such spaces. They should form a proper \( \sigma \)-ideal of subsets of the given space \( X \), i.e., be closed under subsets and countable unions, and should not contain all subsets of \( X \).

It turns out that there are infinitely many non-equivalent natural \( \sigma \)-ideals, some of which are suitable for some differentiability questions.

Infinitely many dimensions

The notion of a Lipschitz function makes sense for functions \( f : M \to N \) between metric spaces,

\[
\text{dist}(f(x), f(y)) \leq \text{const} \cdot \text{dist}(x, y).
\]

This gives rise to the study of derivatives of Lipschitz functions between Banach spaces \( X \) and \( Y \).

If \( \dim(X) < \infty \) and \( f \) is Lipschitz, the two notions of derivative coincide. However, if \( \dim X = \infty \) easy examples show that there is a big difference between Gâteaux and Fréchet differentiability even for simple Lipschitz functions.

Further obstacles

It is easy to find nowhere differentiable Lipschitz maps \( f : \mathbb{R} \to Y, Y \) a Banach space. For example, for \( Y = c_0 \) (the space of sequences converging to zero with maximum norm),

\[
f(t) = \left( \frac{\sin(t)}{1}, \frac{\sin(2t)}{2}, \frac{\sin(3t)}{3}, \ldots \right)
\]

Spaces \( Y \) for which this pathology does not happen were characterized in various ways by many authors (including Walter Schachermayer). They are called spaces with the RNP (Radon-Nikodým property) and include all reflexive spaces.

There are more obstacles to Fréchet differentiability; e.g. the norm on \( \ell_1 \) is nowhere Fréchet differentiable. Spaces for which this behaviour does not happen are called Asplund spaces. Among separable spaces they are precisely those with separable dual.
Gâteaux differentiability

Starting from about 1970, the theorem of Lebesgue has been extended to Gâteaux differentiability, with various notions of negligible sets, independently by a number of authors: Mankiewicz, Christensen, Aronszajn, Phelps, . . .

**Theorem.** Every Lipschitz map from a separable Banach space $X$ into a space $Y$ with the RNP is Gâteaux differentiable almost everywhere.

The situation concerning the existence of Gâteaux derivatives is generally deemed to be quite satisfactory. However, once one goes a bit deeper, fundamental questions remain unanswered.

Fréchet differentiability

The unsolved questions about Gâteaux differentiability pale in comparison with those concerning Fréchet differentiability. The only general positive result is

**Theorem (Preiss 1990).** Every real-valued Lipschitz function on an Asplund space has points of Fréchet differentiability.

However, no “almost everywhere” result is known. In other words, we do not whether every countable collection of real-valued Lipschitz functions on an Asplund space has a common point of Fréchet differentiability.

We do not even know whether three Lipschitz functions on a Hilbert space have a common point of Fréchet differentiability.

Γ-null sets

Lindenstrauss and Preiss (2003) defined a new $\sigma$-ideal of negligible sets: A set $N \subset X$ is $\gamma$-null if it is null on residually many infinite dimensional $C^1$ surfaces.

The differentiability result with these sets is rather curious.

**Theorem.** Every real-valued Lipschitz function on an Asplund space $X$ is Fréchet differentiable $\Gamma$-almost everywhere if and only if every set porous in $X$ is $\Gamma$-null.

Porous sets are special sets of Fréchet nondifferentiability: a set $E \subset X$ is porous if an only if the function $x \to \text{dist}(x, E)$ is Fréchet nondifferentiable at any point of $E$.

Notice that the class of porous sets is considerably smaller than the class of Fréchet nondifferentiability sets, already on $\mathbb{R}$.

Originally only asymptotically $c_0$ spaces were known to satisfy the assumption of this Theorem. We now know that it holds for spaces asymptotically smooth with modulus $o(t^n)$ for any $n$.

What about Hilbert spaces

In Hilbert spaces (as well as in all $\ell_p$, $1 < p < \infty$) there are porous sets that are not $\Gamma$-null. The best we can do is

**Theorem (Lindenstrauss, Preiss, Tišer).** Every pair of real-valued Lipschitz function on a Hilbert space has a common point of Fréchet differentiability.

**Theorem (Lindenstrauss, Preiss, Tišer).** Every collection of $n$ real-valued Lipschitz functions on $\ell_p$, $1 < p < \infty$, $p \geq n$ has a common point of Fréchet differentiability.

These results are special cases of a more general result.

**Theorem (Lindenstrauss, Preiss, Tišer).** If a Banach space $X$ has modulus of asymptotic smoothness $o(t^n \log^{n-1}(1/t))$ then every collection of $n$ real-valued Lipschitz functions on $X$ has a common point of Fréchet differentiability.
Why are 2 and 3 so different?

**Theorem (Preiss, Tišer).** There are three real-valued Lipschitz functions $f_1, f_2, f_3$ on a Hilbert space so that

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} = 0$$

at every point at which they are all Fréchet differentiable but not at every point at which they are all Gâteaux differentiable.

We call this phenomenon a (rather strong) failure of the multi-dimensional mean value estimate. In our positive results we can actually prove that the multi-dimensional mean value estimate holds. So the above is impossible with two functions.

**Asymptotic smoothness**

The modulus of asymptotic uniform smoothness of a Banach space $X$ is defined by

$$\bar{\rho}_X(t) = \sup_{\|x\|=1} \inf_{\dim(X/Y)<\infty} \sup_{\|y\| \leq t} \|x + y\| - 1, \quad t > 0.$$ 

The moduli of asymptotic uniform smoothness of $\ell_p$, $1 \leq p < \infty$, and $c_0$ are

$$\bar{\rho}_{\ell_p}(t) = (1 + t^p)^{1/p} - 1$$

$$\bar{\rho}_{c_0}(t) = \max(t - 1, 0)$$

Unlike the usual moduli of uniform smoothness which cannot go to zero faster that $t^2$, there is no limit on behaviour of these moduli as $t \downarrow 0$.

**ε-differentiability**

So called $\varepsilon$-Fréchet differentiability results for Lipschitz functions in asymptotically uniformly smooth spaces say that for every Lipschitz map $f$ of such a space $X$ into a finite dimensional space $Y$ and for every $\varepsilon > 0$ there are $x \in X$ and a bounded linear operator $T : X \to Y$ so that

$$\limsup_{y \to x} \frac{\|f(y) - f(x) - T(y - x)\|}{\|y - x\|} < \varepsilon.$$ 

For uniformly smooth spaces this was proved by Lindenstrauss and Preiss in 1996. To prove the general case we were joined by Bates, Johnson and Schechtman (1999).

This result is highly exceptional. It is the only general differentiability result for Lipschitz function proved in a situation when the multi-dimensional mean value estimates may fail.

**Mean value estimates**

Mean value estimates are inequalities controlling increment of the function with the help of its derivative. For Gâteaux derivatives they follow from any of the existing results.

**Theorem.** Let $f$ be a real-valued locally Lipschitz function on an open subset $G$ of a separable Banach space $X$ and let $a, b \in G$ be such that the straight segment $[a, b]$ is contained in $G$. Then for every $\varepsilon > 0$ there is a point $x \in G$ at which $f$ is Gâteaux differentiable and

$$f'(x)(b - a) > f(b) - f(a) - \varepsilon.$$ 

**Theorem.** On Asplund spaces this holds also for Fréchet derivatives.
Multi-dimensional mean value estimates

The multi-dimensional mean value estimates for vector-valued functions come from the Gauss-Green (divergence) theorem. Again, they hold for Gâteaux derivatives.

**Theorem (Lindenstrauss, Preiss, Tišer).** If $X$ has modulus of asymptotic smoothness of $(t^n \log^{n-1}(1/t))$ these estimates hold for Fréchet derivatives of maps $X \to \mathbb{R}^n$.

The constants $r$ and a number $\varepsilon$ are given a metric space $X$. The game is played by two players, (A) and (B).

Again, they hold for Gâteaux derivatives.

Theorem (Lindenstrauss, Preiss, Tišer). These estimates fail for Fréchet derivatives of maps $X \to \mathbb{R}^n$ provided that for some $q > n/(n-1)$, $X^*$ contains a normalized sequence such that for any real $c$ and natural $k$,

$$
\left\| \sum_{i=1}^{k} c_i x_i \right\| \leq \text{const} \cdot \left( \sum_{i=1}^{k} |c_i|^q \right)^{1/q}.
$$

The perturbation game

The game is played by two players, (A) and (B). They are given a metric space $(M, d)$, a function $f: M \to \mathbb{R}$ and a number $\varepsilon_0 > 0$.

- The game begins by (A) choosing a positive number $\eta_0$.
- Then (B) chooses a function $F_0: M \to \mathbb{R}$, a point $x_0 \in M$, and positive numbers $r_0, \varepsilon_1$.
- Then (A) chooses a positive number $\eta_1$.
- Then (B) chooses a function $F_1: M \to \mathbb{R}$, a point $x_1 \in M$, and positive numbers $r_1, \varepsilon_2$.
- Then (A) chooses a positive number $\eta_2$, etc.

The constants $r_k, \eta_k, \varepsilon_k$ limit (B)’s choice of the function $F_k$.

The rules are such that (A) has a “winning strategy”, i.e., one that leads to a sequence $(x_k)$ “converging” to a point at which $f + \sum_{k=0}^{\infty} F_k$ attains its minimum.

Variational arguments and differentiability

The following statement illustrates the role of perturbational variational principles in proving differentiability.

**Observation.** Suppose that $\Theta: X \to \mathbb{R}$ is Fréchet differentiable, $\psi: X \to \mathbb{R}$ is continuous and $f: X \to \mathbb{R}$ is Lipschitz and everywhere Gâteaux differentiable. Suppose further that at some $(x_0, u_0)$ the function $h: E \times X \to \mathbb{R}$

$$
h(x, u) = f'(x)(u) + \Theta(u) + \psi(x)
$$

attains its minimum. Then $f$ is Fréchet differentiable at $x_0$.

More delicate versions of the variational principle of Ekeland’s type are used to handle the situation when $f$ is not everywhere Gâteaux differentiable. We describe these principles as infinite two player games, which allows some unusual applications.
Porosity

A set $E$ in a Banach space $X$ is called porous if there is $0 < c < 1$ such that for every $x \in E$ and $\varepsilon > 0$ there is $y \in X$ so that $0 < \|y\| < \varepsilon$ and

$$B(x + y, c\|y\|) \cap E = \emptyset.$$ 

In our work, $\sigma$-porous sets (countable unions of porous sets) appear as sets of points with special type of Fréchet non-differentiability, namely that $f$ is differentiable but not regularly differentiable in some direction $u$.

A map $f : X \to Y$ is regularly differentiable at $x$ in the direction of $u \in X$ if it differentiable at $x$ in the direction some $u$ and for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\|f(x + z + tu) - f(x + z) - tf'(x; u)\| \leq \varepsilon(|t| + \|z\|)$$

whenever $t \in \mathbb{R}$, $z \in X$ and $|t| + \|z\| < \delta$.

$\Gamma_n$-null sets

A set $N \subset X$ is $\Gamma_n$-null if it is null on residually many $n$-dimensional $C^1$ surfaces.

The $\Gamma_1$ null sets are reasonably well understood.

**Theorem.** A separable Banach space $X$ has separable dual if and only if every set porous in $X$ is $\Gamma_1$-null.

Much less is known for $\Gamma_n$ sets, $n > 2$. Also the connection between $\Gamma_n$ sets for finite $n$ and $\Gamma$-null sets is not completely clear. But we have

**Theorem.** $G_{\delta\sigma}$ sets that are $\Gamma_n$ null for infinitely many $n$ are $\Gamma$-null.

Since Fréchet non-differentiability sets are $G_{\delta\sigma}$, this statement can be used, and has been used, to improve existing result on Fréchet differentiability $\Gamma$-almost everywhere.

A very optimistic differentiability conjecture

**Conjecture.** In every Asplund space $X$ there is a nontrivial notion of negligible sets such that for every locally Lipschitz map $f$ of an open subset $G \subset X$ into a Banach space $Y$ having the RNP:

- $f$ is Gâteaux differentiable almost everywhere in $G$.
- If $S \subset G$ is a set with negligible complement such that $f$ is Gâteaux differentiable at every point of $S$, then $\text{Lip}(f) = \sup_{x \in S} \|f'(x)\|$.
- If the set of Gâteaux derivatives of $f$ attained on some set $E \subset G$ is norm separable, then $f$ is Fréchet differentiable at almost every point of $E$.

In spite of having only little evidence for validity of this conjecture, we seem to be very far from disproving it and equally far from proving it.

One of many problems

**Problem.** Can a separable Hilbert space be written as

$$H = \bigcup_{i=1}^{\infty} P_i \cup \bigcup_{i=1}^{\infty} G_i$$

where $P_i$ are porous and $G_i$ are Gâteaux nondifferentiability sets of Lipschitz maps into RNP spaces (or of real-valued Lipschitz functions)? More generally, which spaces can be decomposed in this way?

The only known results are

- $\ell_1$ can be written in this way,
- spaces admitting a norm with modulus of asymptotic uniform smoothness $o(t^n)$ for every $n$ cannot be written in this way.