Absolutely Continuous Compensators

Conference in Honor of Walter Schachermayer
Philip Protter
ORIE, Cornell

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Based on work with Svante Janson and Sokhna M’Baye
Reduced Form Models

• Let $\tau$ be the random time an event of interest happens
• We do not know the distribution of $\tau$
• We have a filtration $\mathbb{F}$ of observable events, and a probability measure $P$
• We let $N_t = 1 \{t \geq \tau\}$ and let $A = (A_t)_{t \geq 0}$ be its compensator; that is
  \[ N_t - A_t = \text{a martingale}. \]
• A common assumption is that $A$ is of the form $A_t = \int_0^t \lambda_s ds$
• This depends on both $\mathbb{F}$ and $P$
Examples from the Literature

- Eduardo Schwartz and Walter Torous, 1989: $\tau$ represents the time of prepayment of a mortgage
- Stanton, 1995: Extension of Schwartz and Torous (still mortgage prepayments)
- MHA Davis and Lischka, 1999: $\tau$ is the time of default of a convertible bond
- Hughston and Turnbull, 2001: Basic formal construction of the reduced form approach to Credit Risk
- Bakshi and Madan, 2002: Used in Catastrophe Loss models
- Ciochetti et al, 2003: $\tau$ is the default time of a commercial mortgage
Examples from the Literature, Continued

- Dassios and Jang, 2003: $\tau$ is the time of a catastrophic event, in reinsurance models
- Leif Andersen and Buffum, 2004: $\tau$ is the default time in convertible bond models
- Jarrow, Lando, and Yu, 2005: $\tau$ is the default time in commercial paper models
- Christopoulos, Jarrow and Yildirim, 2008: $\tau$ is the time a commercial mortgage loan is delinquent
- Chava and Jarrow, 2008: $\tau$ is the default time of a Loan Commitment, or Credit Line
- Jarrow, 2010: Catastrophe bonds
Structural Versus Reduced Form Models in Credit Risk (Merton, 1973)

- We begin with a filtered space $(\Omega, \mathcal{H}, P, \mathbb{H})$ where $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$
- Let $X$ be a Markov process on $(\Omega, \mathcal{H}, P, \mathbb{H})$ given by
  \[
  dX_t = 1 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t \mu(s, X_s) ds
  \]
- In a structural model we assume we observe $G = (\sigma(X_s; 0 \leq s \leq t))_{t \geq 0}$ and so $G \subset \mathbb{H}$
- Default occurs when the firm’s value $X$ crosses below a given threshold level process $L = (L_t)_{t \geq 0}$
- If $L$ is constant, then the default time is $\tau = \inf\{t > 0 : X_t \leq L\}$, and $\tau$ is a predictable time for $G$ and $\mathbb{H}$
Two objections to the Structural Model Approach

- It is assumed that the coefficients $\sigma$ and $\mu$ in the diffusion equation are knowable.
- It is also assumed the level crossing that leads to default is knowable.
- The default time is a predictable stopping time.
The Reduced Form Approach (Jarrow, Turnbull, Duffie, Lando, Jeanblanc...)

- We assume that a stopping time $\tau$ is given, which is a default time
- We assume that $\tau$ is a totally inaccessible time
- This means that $M_t = 1\{t \geq \tau\} - A_t = \text{a martingale}$
- $A$ is adapted, continuous, and non decreasing
- Usually it is **implicitly assumed** that $A$ is of the form
  \[
  A_t = \int_0^t \lambda_s ds,
  \]
  where $\lambda$ is the instantaneous likelihood of the arrival of $\tau$
The Hybrid Approach (Giesecke, Goldberg, ...) 

- We assume the structural approach, but instead of a level crossing time as a default time, we replace it with a random curve.
- This can make the stopping time totally inaccessible, and of the form found in the reduced form approach.
- Giesecke has also pointed out that the increasing process $A$ need no longer have absolutely continuous paths.
The Filtration Shrinkage Approach (Çetin, Jarrow, Protter, Yildirim)

- $\tau$ can be the time of default for the structural approach
- One does not know the structural approach, so one models this by shrinking the filtration to the presumed level of observable events
- The result is that $\tau$ becomes totally inaccessible, and one recovers the reduced form approach
- **Advantage:** This relates the structural and reduced form approaches which facilitate empirical methods to estimate $\tau$
- Motivates studying compensators of stopping times and their behavior under filtration shrinkage
When does the compensator $A$ have absolutely continuous paths?

- **Ethier-Kurtz Criterion:** $A_0 = 0$ and suppose for $s \leq t$

  $$E\{A_t - A_s \mid \mathcal{G}_s\} \leq K(t - s)$$

  then $A$ is of the form $A_t = \int_0^t \lambda_s ds$

- **Yan Zeng, PhD Thesis, Cornell, 2006:** There exists an increasing process $D_t$ with $dD_t \ll dt$ a.s. and

  $$E\{A_t - A_s \mid \mathcal{G}_s\} \leq E\{D_t - D_s \mid \mathcal{G}_t\},$$

  then $A$ is of the form $A_t = \int_0^t \lambda_s ds$
Shrinkage Result; M. Jacobsen, 2005

- Suppose $1_{\{t \geq \tau\}} - \int_0^t \lambda_s ds$ is a martingale in $\mathcal{H}$
- Suppose also $\tau$ is a stopping time in $\mathcal{G}$ where $\mathcal{G} \subset \mathcal{H}$. Then
  \[
  1_{\{t \geq \tau\}} - \int_0^t \circ \lambda_s ds
  \]
  is a martingale in $\mathcal{G}$

where $\circ \lambda$ denotes the optional projection of the process $\lambda$ onto the filtration $\mathcal{G}$
Is there a general condition such that all stopping times have absolutely continuous compensators?

- Let $X$ be a strong Markov process; suppose it also a Hunt process.
- (Çinlar and Jacod, 1981) On a space $(\Omega, \mathcal{F}, \mathbb{F}, P^x)$, up to a change of time and space, if $X$ is a semimartingale we have the representation

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t c(X_s)dW_s$$

$$+ \int_0^t \int_{\mathbb{R}} k(X_s-z)1_{\{|k(X_s-z)| \leq 1\}}[n(ds, dz) - ds\nu(dz)]$$

$$+ \int_0^t \int_{\mathbb{R}} k(X_s-z)1_{\{|k(X_s-z)| > 1\}} n(ds, dz)$$
Lévy system of a Hunt process

- For a Hunt process semimartingale $X$ with measure $P^{\mu}$ a Lévy system $(K, H)$ where $K$ is a kernel on $\mathbb{R}$ and $H$ is a continuous additive functional of $X$, satisfies the following relationship:

$$E^{\mu} \left( \sum_{0<s\leq t} f(X_{s^-}, X_s) 1 \{ X_{s^-} \neq X_s \} \right)$$

$$= E^{\mu} \left( \int_{0}^{t} dH_s \int_{\mathbb{R}} K(X_{s^-}, dy) f(X_s, y) \right)$$

- For $X$ a strong Markov process as in the Činlar-Jacod theorem, we can take the continuous additive functional $H$ to be $H_t = t$
In a “natural” Markovian space, all compensators of stopping times have absolutely continuous paths

**Theorem:** Let $\mathbb{F}$ be the natural (completed) filtration of a Hunt process $X$ on a space $(\Omega, \mathcal{F}, P^{\mu})$ and let $(K, H)$ be a Lévy system for $X$. If $dH_t \ll dt$ then for any totally inaccessible stopping time $\tau$ the compensator of $\tau$ has absolutely continuous paths a.s. That is, there exists an adapted process $\lambda$ such that

$$1_{\{t \geq \tau\}} - \int_0^t \lambda_s ds$$

is an $\mathbb{F}$ martingale. (1)

Moreover if $dH_t$ is not equivalent to $dt$, then there exists a stopping time $\nu$ such that (1) does not hold.
Jumping Filtrations

- Jacod and Skorohod define a **jumping filtration** $\mathbb{F}$ to be a filtration such that there exists a sequence of stopping times $(T_n)_{n=0,1,...}$ increasing to $\infty$ a.s. with $T_0 = 0$ and such that for all $n \in \mathbb{N}$, $t > 0$, the $\sigma$-fields $\mathcal{F}_t$ and $\mathcal{F}_{T_n}$ coincide on \{ $T_n \leq t < T_{n+1}$ \}
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- **Theorem:** Let $N = (N_t)_{t \geq 0}$ be a point process without explosions that generates a quasi-left continuous jumping filtration, and suppose there exists a process $(\lambda_s)_{s \geq 0}$ such that

$$N_t - \int_0^t \lambda_s \, ds = \text{a martingale.} \quad (2)$$

Let $\mathbb{D} = (\mathcal{D}_t)_{t \geq 0}$ be the (automatically right continuous) filtration generated by $N$ and completed in the usual way. Then for any $\mathbb{D}$ totally inaccessible stopping time $R$ we have that the compensator of $1_{\{t \geq R\}}$ has absolutely continuous paths, a.s.
Increasing Processes

- **Theorem:** $Z$ is an increasing process; suppose there exists $\lambda$ such that

$$Z_t - \int_0^t \lambda_s ds = a \text{ martingale}$$
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Let $R$ be a stopping time such that $P(\Delta Z_R > 0 \cap \{R < \infty\}) = P(R < \infty)$; then $R$ too has an absolutely continuous compensator; that is, there exists a process $\mu$ such that

$$1_{\{t \geq R\}} - \int_0^t \mu_s ds = \text{a martingale}$$
Increasing Processes

• **Theorem:** \( Z \) is an increasing process; suppose there exists \( \lambda \) such that

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1\{t \geq R\} - \int_0^t \mu_s ds = \text{a martingale}
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• **Consequence:** If \( N \) is a Poisson process with parameter \( \lambda \), and \( R \) is a totally inaccessible stopping time on the minimal space generated by \( N \), then the compensator of \( R \) has absolutely continuous paths.
**Filtration Shrinkage and Compensators**

- **Dellacherie’s Theorem:** Let $R$ be a nonnegative random variable with $P(R = 0) = 0$, $P(R > t) > 0$ for each $t > 0$. Let $\mathcal{F}_t = \sigma(t \land R)$. Let $F$ denote the law of $R$. Then the compensator $A = (A_t)_{t \geq 0}$ of the process $1_{\{R \geq t\}}$ is given by

$$A_t = \int_0^t \frac{1}{1 - F(u-)} dF(u).$$

If $F$ is continuous, then $A$ is continuous, $R$ is totally inaccessible, and $A_t = -\ln(1 - F(R \land t))$. 
Filtration Shrinkage and Compensators

- **Dellacherie’s Theorem:** Let \( R \) be a nonnegative random variable with \( P(R = 0) = 0, P(R > t) > 0 \) for each \( t > 0 \). Let \( \mathcal{F}_t = \sigma(t \wedge R) \). Let \( F \) denote the law of \( R \). Then the compensator \( A = (A_t)_{t \geq 0} \) of the process \( 1_{\{R \geq t\}} \) is given by

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A_t = \int_0^t \frac{1}{1 - F(u-)} dF(u).
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If \( F \) is continuous, then \( A \) is continuous, \( R \) is totally inaccessible, and \( A_t = -\ln(1 - F(R \wedge t)) \).

- We know by Jacobsen’s theorem, that once a compensator is absolutely continuous, it still is in any smaller filtration.
• It is *a priori* possible that a stopping time $R$ has a singular compensator in a filtration $\mathcal{F}$, but an absolutely continuous compensator in a smaller filtration
• It is a priori possible that a stopping time $R$ has a singular compensator in a filtration $\mathcal{H}$, but an absolutely continuous compensator in a smaller filtration

• **Conjecture:** If a stopping time $R$ has an absolutely continuous law, then it has an absolutely continuous compensator in any filtration rendering it totally inaccessible.
• It is *a priori* possible that a stopping time $R$ has a singular compensator in a filtration $\mathbb{H}$, but an absolutely continuous compensator in a smaller filtration.

• **Conjecture:** If a stopping time $R$ has an absolutely continuous law, then it has an absolutely continuous compensator in any filtration rendering it totally inaccessible.

• **This conjecture is false.** A stopping time can be constructed with Brownian local time at zero as its compensator. In its minimal filtration, the compensator is absolutely continuous with respect to $t \mapsto E(L_t)$, which is absolutely continuous with respect to $dt$. 
Equivalent Probabilities

- Let $\tau$ be a stopping time on a space $(\Omega, \mathcal{F}, P, \mathbb{F})$ and suppose it has an absolutely continuous compensator; that is,

\[
M_t = 1\{t \geq \tau\} - \int_0^t \lambda_s ds = \text{a martingale}
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- Let $Q$ be equivalent to $P$, a situation which often arises in Mathematical Finance, with risk neutral measures; let $Z = \frac{dQ}{dP}$ and $Z_t = E\left\{\frac{dQ}{dP} \middle| \mathcal{F}_t\right\}$
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- Then $\tau$ has an absolutely continuous compensator, given by the relation

$$1\{t \geq \tau\} - \int_0^t \lambda_s ds - \int_0^t \frac{1}{Z_s} d\langle Z, M \rangle_s = \text{a martingale}$$
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• **Note:** Since $[M, M]_t = 1\{t \geq \tau\}$ we have that $\langle M, M \rangle_t = \int_0^t \lambda_s ds$, and the result follows by the Kunita-Watanabe inequality.
Initial Enlargement

• Again, let $\tau$ be a stopping time on a space $(\Omega, \mathcal{F}, P, \mathbb{F})$ and suppose it has an absolutely continuous compensator; that is,

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- Suppose we expand $\mathbb{F}$ by adding a random variable $L$, with law $\eta(dx)$, to $\mathcal{F}_0$ and $\mathcal{F}_t$ for all $t > 0$. 

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Suppose we expand $\mathbb{F}$ by adding a random variable $L$, with law $\eta(dx)$, to $\mathcal{F}_0$ and $\mathcal{F}_t$ for all $t > 0$.

Let $Q_t(\omega, dx)$ be the conditional distribution of $L$ given $\mathcal{F}_t$, and suppose further that $Q_t(\omega, ds) \ll \eta(dx)$ and we write $Q_t(\omega, dx) = q^*_{t} \eta_t(dx)$
Initial Enlargement

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• Suppose we expand \( \mathbb{F} \) by adding a random variable \( L \), with law \( \eta(dx) \), to \( \mathcal{F}_0 \) and \( \mathcal{F}_t \) for all \( t > 0 \).

• Let \( Q_t(\omega, dx) \) be the conditional distribution of \( L \) given \( \mathcal{F}_t \), and suppose further that \( Q_t(\omega, ds) \ll \eta(dx) \) and we write \( Q_t(\omega, dx) = q^x_t \eta_t(dx) \)

• We write

\[
\langle q^x, M \rangle_t = \int_0^t k^x_s q^x_s d\langle M, M \rangle_s
\]
The compensator of $\tau$ under the enlarged filtration $\mathbb{G}$ given by $G_t = \mathcal{F}_t \vee \sigma(t \wedge T)$ is

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• Again, note that $\langle M, M \rangle_t = \int_0^t \lambda_s ds$, so that the compensator is absolutely continuous
Progressive Expansion of Filtrations

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• We enlarge the filtration $\mathbb{F}$ with $L$ such that the new filtration, $\mathbb{G}$ makes $L$ a stopping time; the method of expansion is called **progressive expansion**. We call the enlarged filtration $\mathbb{G}$
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• Then $\tau$ has an absolutely continuous compensator in $\mathbb{G}$ as well.
Analogous Results for the Entire Space

- We will say that on a space \((\Omega, \mathcal{G}, P, \mathcal{G})\) that a probability \(Q\) has **Property AC** if under \(Q\), all totally inaccessible stopping times have absolutely continuous compensators.

- A class of examples with **Property AC** are strong Markov spaces, where the Lévy system of the Markov process is itself absolutely continuous.

- Theorem: Suppose that \((\Omega, \mathcal{G}, P, \mathcal{G}, X)\) is a given system, and that there exists a probability \(Q^*\) equivalent to \(P\) such that \(Q^*\) has **Property AC**. Then if \(Q\) is the set of all probability measures equivalent to \(P\), we have that **Property AC** holds under any \(Q \in Q\).

- This last theorem is especially useful for applications in Finance.
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- This last theorem is especially useful for applications in Finance.
• **Theorem:** Under initial expansion, we have an analogous result. Expand \( \mathcal{G} \) by adding a random variable \( L \) initially to obtain \( \mathcal{H} \). If there exists \( Q^* \in \mathcal{Q} \) with Property AC under \( \mathcal{G} \), then \( Q^* \) has Property AC in \( \mathcal{H} \), and so all \( Q \in \mathcal{Q} \).
• **Theorem**: Under initial expansion, we have an analogous result. Expand $\mathcal{G}$ by adding a random variable $L$ initially to obtain $\mathcal{H}$. If there exists $Q^* \in \mathcal{Q}$ with Property AC under $\mathcal{G}$, then $Q^*$ has Property AC in $\mathcal{H}$, and so all $Q \in \mathcal{Q}$.

• **Theorem**: Let $L$ be a positive random variable and progressively expand $\mathcal{G}$ with $L$ to get a filtration $\mathcal{J}$. If $Q^* \in \mathcal{Q}$ has Property AC for $\mathcal{G}$, then it also does for $\mathcal{J}$ as long as we restrict ourselves to totally inaccessible stopping times in $\mathcal{G}$. Moreover this is true for any $Q \in \mathcal{Q}$. 
• **Theorem:** Under initial expansion, we have an analogous result. Expand $\mathcal{G}$ by adding a random variable $L$ initially to obtain $\mathcal{H}$. If there exists $Q^* \in \mathcal{Q}$ with Property AC under $\mathcal{G}$, then $Q^*$ has Property AC in $\mathcal{H}$, and so all $Q \in \mathcal{Q}$.

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• In general, whether this extends to all of $\mathcal{J}$ depends on the nature of the compensator of $L$.
Thank you