Hedging under arbitrage

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• Usually, there are several trading strategies at one’s disposal to obtain a given wealth at a specified time.
• Imagine an investor who wants to hold the stock $S_i$ with price $S_i(0)$ of a company in a year.
• Surely, she could just buy the stock today for a price $S_i(0)$.
• This might not be an “optimal strategy”, even under a classical no-arbitrage situation (“no free lunch with vanishing risk”).
• There can be other “strategies” which require less initial capital than $S_i(0)$ but enable her to hold the stock after one year.
• But how much initial capital does she need at least and how should she trade?
Two generic examples

- Reciprocal of the three-dimensional Bessel process (NFLVR):
  \[ d\tilde{S}(t) = -\tilde{S}^2(t)dW(t) \]

- Three-dimensional Bessel process:
  \[ dS(t) = \frac{1}{S(t)}dt + dW(t) \]
Strict local martingales

• A stochastic process $X(\cdot)$ is a *local martingale* if there exists a sequence of stopping times $(\tau_n)$ with $\lim_{n \to \infty} \tau_n = \infty$ such that $X^{\tau_n}(\cdot)$ is a martingale.

• Here, in our context, a local martingale is a nonnegative stochastic process $X(\cdot)$ which does not have a drift:

$$dX(t) = X(t)\text{something}dW(t).$$

• Strict local martingales (local martingales, which are not martingales) do only appear in continuous time.

• Nonnegative local martingales are supermartingales.
We assume a Markovian market model.

- Our time is finite: $T < \infty$. Interest rates are zero.
- The stocks $S(\cdot) = (S_1(\cdot), \ldots, S_d(\cdot))^T$ follow

$$dS_i(t) = S_i(t) \left( \mu_i(t, S(t))dt + \sum_{k=1}^{K} \sigma_{i,k}(t, S(t))dW_k(t) \right)$$

with some measurability and integrability conditions.
- Markovian
- but not necessarily complete ($K > d$ allowed).
- The covariance process is defined as

$$a_{i,j}(t, S(t)) := \sum_{k=1}^{K} \sigma_{i,k}(t, S(t))\sigma_{j,k}(t, S(t)).$$

- The underlying filtration is denoted by $\mathbb{F} = \{\mathcal{F}(t)\}_{0\leq t \leq T}$. 
An important guy: the market price of risk.

- A *market price of risk* is an \( \mathbb{R}^K \)-valued process \( \theta(\cdot) \) satisfying
  \[
  \mu(t, S(t)) = \sigma(t, S(t))\theta(t).
  \]
- We assume it exists and
  \[
  \int_0^T \|\theta(t)\|^2 dt < \infty.
  \]
- The market price of risk is not necessarily unique.
- We will always use a Markovian version of the form \( \theta(t, S(t)) \).
  (needs argument!)
Related is the stochastic discount factor.

- The *stochastic discount factor* corresponding to $\theta$ is denoted by
  \[
  Z^\theta(t) := \exp \left( - \int_0^t \theta^T(u, S(u)) dW(u) - \frac{1}{2} \int_0^t \|\theta(u, S(u))\|^2 du \right).
  \]
- It has dynamics
  \[
  dZ^\theta(t) = -\theta^T(t, S(t)) Z^\theta(t) dW(t).
  \]
- If $Z^\theta(\cdot)$ is a martingale, that is, if $E[Z^\theta(T)] = 1$, then it defines a risk-neutral measure $\mathbb{Q}$ with $d\mathbb{Q} = Z^\theta(T) d\mathbb{P}$.
- Otherwise, $Z^\theta(\cdot)$ is a strict local martingale and classical arbitrage is possible.
- From Itô’s rule, we have
  \[
  d \left( Z^\theta(t) S_i(t) \right) = Z^\theta(t) S_i(t) \sum_{k=1}^K (\sigma_{i,k}(t, S(t)) - \theta_k(t, S(t))) dW_k(t).
  \]
Everything an investor cares about: how and how much?

- We call \textit{trading strategy} the number of shares held by an investor: \( \eta(t) = (\eta_1(t), \ldots, \eta_d(t))^T \)
- We assume that \( \eta(\cdot) \) is progressively measurable with respect to \( \mathbb{F} \) and self-financing.
- The corresponding wealth process \( V^{v,\eta}(\cdot) \) for an investor with initial wealth \( V^{v,\eta}(0) = v \) has dynamics

\[
dV^{v,\eta}(t) = \sum_{i=1}^{d} \eta_i(t) dS_i(t).
\]
- We restrict ourselves to trading strategies which satisfy \( V^{1,\eta}(t) \geq 0 \)
The terminal payoff

• Let \( p : \mathbb{R}^d_+ \to [0, \infty) \) denote a measurable function.
• The investor wants to have the payoff \( p(S(T)) \) at time \( T \).
• For example,
  • market portfolio: \( \tilde{p}(s) = \sum_{i=1}^{d} s_i \)
  • money market: \( p^0(s) = 1 \)
  • stock: \( p^1(s) = s_1 \)
  • call: \( p^C(s) = (s_1 - L)^+ \) for some \( L \in \mathbb{R} \).
• We define a candidate for the hedging price as

\[
h^p(t, s) := \mathbb{E}^{t,s} \left[ \tilde{Z}^\theta(T)p(S(T)) \right],
\]

where \( \tilde{Z}^\theta(T) = Z^\theta(T)/Z^\theta(t) \) and \( S(t) = s \) under the expectation operator \( \mathbb{E}^{t,s} \).
Prerequisites

- We shall call \((t, s) \in [0, T] \times \mathbb{R}_+^d\) a point of support for \(S(\cdot)\) if there exists some \(\omega \in \Omega\) such that \(S(t, \omega) = s\).
- We have assumed Markovian stock price dynamics such that \(S(t)\) is \(\mathbb{R}^d\)-valued, unique and stays in the positive orthant and a square-integrable Markovian market price of risk \(\theta(t, S(t))\).
- We have defined

\[
h^p(t, s) := \mathbb{E}^{t, s}[\tilde{Z}^\theta(T)p(S(T))],
\]

where \(\tilde{Z}^\theta(T) = Z^\theta(T)/Z^\theta(t)\) and \(S(t) = s\) under the expectation operator \(\mathbb{E}^{t, s}\).
- In particular,

\[
h^p(T, s) := p(s).
\]
A first result: non path-dependent European claims

Assume that we have a contingent claim of the form $p(S(T)) \geq 0$ and that for all points of support $(t, s)$ for $S(\cdot)$ with $t \in [0, T)$ we have $h^p \in C^{1,2}(\mathcal{U}_{t,s})$ for some neighborhood $\mathcal{U}_{t,s}$ of $(t, s)$. Then, with $\eta^p_i(t, s) := D_i h^p(t, s)$ and $\nu^p := h^p(0, S(0))$, we get

$$V^{\nu^p, \eta^p}(t) = h^p(t, S(t)).$$

The strategy $\eta^p$ is optimal in the sense that for any $\tilde{\nu} > 0$ and for any strategy $\tilde{\eta}$ whose associated wealth process is nonnegative and satisfies $V^{\tilde{\nu}, \tilde{\eta}}(T) \geq p(S(T))$, we have $\tilde{\nu} \geq \nu^p$. Furthermore, $h^p$ solves the PDE

$$\frac{\partial}{\partial t} h^p(t, s) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} s_i s_j a_{i,j}(t, s) D^2_{i,j} h^p(t, s) = 0$$

at all points of support $(t, s)$ for $S(\cdot)$ with $t \in [0, T)$. 
The proof relies on Itô’s formula.

- Define the martingale $N^p(\cdot)$ as

$$N^p(t) := \mathbb{E}[Z^\theta(T)p(S(T))|\mathcal{F}(t)] = Z^\theta(t)h^p(t, S(t)).$$

- Use a localized version of Itô’s formula to get the dynamics of $N^p(\cdot)$. Since it is a martingale, its $dt$ term must disappear which yields the PDE.

- Then, another application of Itô’s formula yields

$$dh^p(t, S(t)) = \sum_{i=1}^{d} D_i h^p(t, S(t))dS_i(t) = dV^{v^p, \eta^p}(t).$$

- This yields directly $V^{v^p, \eta^p}(\cdot) \equiv h^p(\cdot, S(\cdot)).$
• Next, we prove optimality.

• Assume we have some initial wealth \( \tilde{v} > 0 \) and some strategy \( \tilde{\eta} \) with nonnegative associated wealth process such that \( V^{\tilde{v},\tilde{\eta}}(T) \geq p(S(T)) \) is satisfied.

• Then, \( Z^\theta(\cdot) V^{\tilde{v},\tilde{\eta}}(\cdot) \) is a supermartingale.

• This implies

\[
\tilde{v} \geq \mathbb{E}[Z^\theta(T) V^{\tilde{v},\tilde{\eta}}(T)] \geq \mathbb{E}[Z^\theta(T) p(S(T))] \\
= \mathbb{E}[Z^\theta(T) V^{v^p,\eta^p}(T)] = v^p
\]
Non-uniqueness of PDE

- Usually,

\[
\frac{\partial}{\partial t} v(t, s) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} s_i s_j a_{i,j}(t, s) D_{i,j}^2 v(t, s) = 0
\]

does not have a unique solution.

- However, if \( h^p \) is sufficiently differentiable, it can be characterized as the minimal nonnegative solution of the PDE.

- This follows as in the proof of optimality. If \( \tilde{h} \) is another nonnegative solution of the PDE with \( \tilde{h}(T, s) = p(s) \), then \( Z^\theta(\cdot)\tilde{h}(\cdot, S(\cdot)) \) is a supermartingale.
Corollary: Modified put-call parity

For any $L \in \mathbb{R}$ we have the modified put-call parity for the call- and put-options $(S_1(T) - L)^+$ and $(L - S_1(T))^+$, respectively, with strike price $L$:

$$
\mathbb{E}^{t,s} \left[ \tilde{Z}^\theta(T) (L - S_1(T))^+ \right] + h^{p_1}(t, s)
$$

$$
= \mathbb{E}^{t,s} \left[ \tilde{Z}^\theta(T) (S_1(T) - L)^+ \right] + L h^{p_0}(t, s),
$$

where $p_0(\cdot) \equiv 1$ denotes the payoff of one monetary unit and $p_1(s) = s_1$ the price of the first stock for all $s \in \mathbb{R}_+$. 
We shall call a function $f : [0, T] \times \mathbb{R}_+^d \to \mathbb{R}$ \textit{locally Lipschitz and bounded} on $\mathbb{R}_+^d$ if for all $s \in \mathbb{R}_+^d$ the function $t \to f(t, s)$ is right-continuous with left limits and for all $M > 0$ there exists some $C(M) < \infty$ such that for all $t \in [0, T]$.

$$\sup_{\frac{1}{M} \leq \|y\|, \|z\| \leq M} \frac{|f(t, y) - f(t, z)|}{\|y - z\|} + \sup_{\frac{1}{M} \leq \|y\| \leq M} |f(t, y)| \leq C(M).$$
Sufficient conditions for the differentiability of $h^p$.

(A1) The functions $\theta_k$ and $\sigma_{i,k}$ are for all $i = 1, \ldots, d$ and $k = 1, \ldots, K$ locally Lipschitz and bounded.

(A2) For all points of support $(t, s)$ for $S(\cdot)$ with $t \in [0, T)$ there exist some $C > 0$ and some neighborhood $\mathcal{U}$ of $(t, s)$ such that

$$\sum_{i=1}^{d} \sum_{j=1}^{d} a_{i,j}(u, y)\xi_i\xi_j \geq C\|\xi\|^2$$

for all $\xi \in \mathbb{R}^d$ and $(u, y) \in \mathcal{U}$.

(A3) The payoff function $p$ is chosen so that for all points of support $(t, s)$ for $S(\cdot)$ there exist some $C > 0$ and some neighborhood $\mathcal{U}$ of $(t, s)$ such that $h^p(u, y) \leq C$ for all $(u, y) \in \mathcal{U}$.

We will proceed in three steps to show that these conditions imply smoothness of $h^p$. 
Step 1: Stochastic flows

We define \( X^{t,s,z}(\cdot) := (S^{t,s}(\cdot), z\tilde{Z}_{\phi,t,s}(\cdot))^T \).

Take \((t, s) \in [0, T] \times \mathbb{R}^d\) a point of support for \( S(\cdot) \). Then under Assumption (A1) [locally Lipschitz and bounded] we have for all sequences \((t_k, s_k)_{k \in \mathbb{N}}\) with \( \lim_{k \to \infty} (t_k, s_k) = (t, s) \) that

\[
\lim_{k \to \infty} \sup_{u \in [t, T]} \| X^{t_k,s_k,1}(u) - X^{t,s,1}(u) \| = 0
\]

almost surely.

In particular, for \( K(\omega) \) sufficiently large we have that \( X^{t_k,s_k,1}(u, \omega) \) is strictly positive and \( \mathbb{R}^{d+1}_+ \)-valued for all \( k > K(\omega) \) and \( u \in [t, T] \).
Step 2: Schauder estimates

Fix a point \((t, s) \in [0, T) \times \mathbb{R}_+^d\) and a neighborhood \(U\) of \((t, s)\). Suppose Assumptions (A1) and (A2) [locally Lipschitz and bounded, non-degenerate \(a\)] hold.

Let \((f_k)_{k \in \mathbb{N}}\) denote a sequence of solutions of the Black-Scholes PDE on \(U\), uniformly bounded under the supremum norm on \(U\). If \(\lim_{k \to \infty} f_k(t, s) = f(t, s)\) on \(U\) for some function \(f : U \to \mathbb{R}\), then \(f\) solves also the PDE on some neighborhood \(\tilde{U}\) of \((t, s)\). In particular, \(f \in C^{1,2}(\tilde{U})\).

- Janson and Tysk (2006), Tysk and Ekström (2009)
- Interior Schauder estimates by Knerr (1980) together with Arzelà-Ascoli type of arguments
Step 3: Putting everything together

Under Assumptions (A1)-(A3) [locally Lipschitz and bounded, non-degenerate a, locally boundedness of \( h^p \)] there exists for all points of support \((t, s)\) for \( S(\cdot) \) with \( t \in [0, T) \) some neighborhood \( \mathcal{U} \) of \((t, s)\) such that the function \( h^p \) is in \( C^{1,2}(\mathcal{U}) \).

- Define \( \tilde{p}(s_1, \ldots, s_d, z) := zp(s_1, \ldots, s_d) \).
- Define \( \tilde{p}^M(\cdot) := \tilde{p}(\cdot)1_{\{\tilde{p}(\cdot) \leq M\}} \) for some \( M > 0 \)
- Approximate by sequence of continuous functions \( \tilde{p}^{M,m} \) such that \( \tilde{p}^{M,m} \leq 2M \) for all \( m \in \mathbb{N} \).
Proof (continuation)

- The corresponding expectations are defined as

\[ \tilde{h}^{p,M}(u, y) := \mathbb{E}^{u,y}[\tilde{p}^M(S_1(T), \ldots, S_d(T), \tilde{Z}^\theta(T))] \]

for all \((u, y) \in \tilde{U}\) for some neighborhood \(\tilde{U}\) of \((t, s)\) and equivalently \(\tilde{h}^{p,M,m}\).

- We have continuity of \(\tilde{h}^{p,M,m}\) for large \(m\) due to the bounded convergence theorem.

- A result from Jansen and Tysk (2006) yields that under Assumption (A2) [non-degenerate \(a\)] \(\tilde{h}^{p,M,m}\) is a solution of the PDE.

- Then, by Step 2 firstly, \(\tilde{h}^{p,M}\) and secondly, \(h^p\) also solve the PDE.
We can change the measure to compute $h^p$

- There exists not always an equivalent local martingale measure.
- However, after making some technical assumptions on the probability space and the filtration we can construct a new measure $\mathbb{Q}$ which corresponds to a “removal of the stock price drift”.
- Based on the work of Föllmer and Meyer and along the lines of Delbaen and Schachermayer.
Theorem: Under a new measure $\mathbb{Q}$ the drifts disappear.

There exists a measure $\mathbb{Q}$ such that $\mathbb{P} \ll \mathbb{Q}$. More precisely, for all nonnegative $\mathcal{F}(T)$-measurable random variables $Y$ we have

$$\mathbb{E}^\mathbb{P}[Z^\theta(T)Y] = \mathbb{E}^\mathbb{Q}\left[Y1_{\left\{ \frac{1}{Z^\theta(T)}>0 \right\}} \right].$$

Under this measure $\mathbb{Q}$, the stock price processes follow

$$dS_i(t) = S_i(t)\sum_{k=1}^{K} \sigma_{i,k}(t, S(t)) d\tilde{W}_k(t)$$

up to time $\tau^\theta := \inf\{t \in [0, T] : 1/Z^\theta(t) = 0\}$. Here,

$$\tilde{W}_k(t \wedge \tau^\theta) := W_k(t \wedge \tau^\theta) + \int_0^{t \wedge \tau^\theta} \theta_k(u, S(u))du$$

is a $K$-dimensional $\mathbb{Q}$-Brownian motion stopped at time $\tau^\theta$. 
What happens in between time 0 and time $T$: Bayes’ rule.

For all nonnegative $\mathcal{F}(T)$-measurable random variables $Y$ the representation

$$\mathbb{E}^Q \left[ Y \mathbf{1}_{\left\{ 1/Z^\theta(T) > 0 \right\}} \middle| \mathcal{F}(t) \right] = \mathbb{E}^P \left[ Z^\theta(T) Y \middle| \mathcal{F}(t) \right] \frac{1}{Z^\theta(t)} \mathbf{1}_{\left\{ 1/Z^\theta(t) > 0 \right\}}$$

holds $Q$-almost surely (and thus $P$-almost surely) for all $t \in [0, T]$. 
The class of Bessel processes with drift provides interesting arbitrage opportunities.

- We begin with defining an auxiliary stochastic process $X(\cdot)$ as
  \[ dX(t) = \left( \frac{1}{X(t)} - c \right) dt + dW(t) \]
  with $W(\cdot)$ denoting a Brownian motion and $c \geq 0$ a constant.
- $X(t)$ is for all $t \geq 0$ strictly positive since $X(\cdot)$ is a Bessel process under an equivalent measure.
- The stock price process is now defined via
  \[ dS(t) = \frac{1}{X(t)} dt + dW(t) = S(t) \left( \frac{1}{S^2(t) - S(t)ct} dt + \frac{1}{S(t)} dW(t) \right) \]
  with $S(0) = X(0) > 0$. 

After a change of measure, the Bessel process becomes Brownian motion.

- As a reminder:
  \[ dS(t) = \frac{1}{S(t) - ct} \, dt + dW(t). \]

- We have \( S(t) \geq X(t) > 0 \) for all \( t \geq 0 \).
- The market price of risk is \( \theta(t, s) = 1/(s - ct) \).
- Thus, the inverse stochastic discount factor \( 1/Z^\theta \) becomes zero exactly when \( S(t) \) hits \( ct \).
- Removing the drift with a change of measure as before makes \( S(\cdot) \) a Brownian motion (up to the first hitting time of zero by \( 1/Z^\theta(\cdot) \)) under \( \mathbb{Q} \).
The optimal strategy for getting one dollar at time $T$ can be explicitly computed.

- For $p(s) \equiv p^0(s) \equiv 1$ we get
  
  $$h^p_0(t, s) = \mathbb{E}^P \left[ \frac{Z^\theta(T)}{Z^\theta(t)} \cdot 1 \mid \mathcal{F}_t \right] \bigg|_{S(t)=s} = \mathbb{E}^Q[1_{\{1/Z^\theta(T)>0\}}\mid \mathcal{F}_t] \bigg|_{S(t)=s}$$  
  
  $$= \Phi \left( \frac{s - cT}{\sqrt{T-t}} \right) - \exp(2cs - 2c^2 t) \Phi \left( \frac{-s - cT + 2ct}{\sqrt{T-t}} \right).$$

- This yields the optimal strategy
  
  $$\eta^0(t, s) = \frac{2}{\sqrt{T-t}} \phi \left( \frac{s - cT}{\sqrt{T-t}} \right) - 2c \exp(2cs - 2c^2 t) \Phi \left( \frac{-s - cT + 2ct}{\sqrt{T-t}} \right).$$

- The hedging price $h^p$ satisfies on all points $\{s > ct\}$ the PDE
  
  $$\frac{\partial}{\partial t} h^p(t, s) + \frac{1}{2} D^2 h^p(t, s) = 0.$$
Conclusion

- No equivalent local martingale measure needed to find an optimal hedging strategy based upon the familiar delta hedge.
- Sufficient conditions are derived for the necessary differentiability of expectations indexed over the initial market configuration.
- The dynamics of stochastic processes under a non-equivalent measure and a generalized Bayes’ rule might be of interest themselves.
- We have computed some optimal trading strategies in standard examples for which so far only ad-hoc and not necessarily optimal strategies have been known.
Congratulations to Walter Schachermayer!
Strict local martingales II

Assume $X(\cdot)$ is a nonnegative local martingale:

$$dX(t) = X(t)\text{something}dW(t).$$

- We always have $\mathbb{E}[X(T)] \leq X(0)$.
- If $\mathbb{E}[X(T)] = X(0)$ then $X(\cdot)$ is a (true) martingale.
- If “something” behaves nice (for example is bounded) then $X(\cdot)$ is a martingale.
- If $\mathbb{E}[X(T)] < X(0)$ then $X(\cdot)$ is a strict local martingale.
Role of Markovian market price of risk

Let $M \geq 0$ be a random variable measurable with respect to $\mathcal{F}^S(T)$. Let $\nu(\cdot)$ denote any MPR and $\theta(\cdot, \cdot)$ a Markovian MPR. Then, with

$$M^\nu(t) := \mathbb{E} \left[ \frac{Z^\nu(T)}{Z^\nu(t)} M \mid \mathcal{F}_t \right] \quad \text{and} \quad M^\theta(t) := \mathbb{E} \left[ \frac{Z^\theta(T)}{Z^\theta(t)} M \mid \mathcal{F}_t \right]$$

for $t \in [0, T]$, we have $M^\nu(\cdot) \leq M^\theta(\cdot)$ almost surely.
Proof

- We define $c(\cdot) := \nu(\cdot) - \theta(\cdot, S(\cdot))$ and $c^n(\cdot) := c(\cdot) 1_{\{\|c(\cdot)\| \leq n\}}$
- Then,

$$\frac{Z^\nu(T)}{Z^\nu(t)} = \lim_{n \to \infty} \frac{Z^{c^n}(T)}{Z^{c^n}(t)} \cdot \exp \left( - \int_t^T \theta^T (dW(u) + c^n(u)du) - \frac{1}{2} \int_t^T \|\theta\|^2 du \right).$$
- Since $c^n(\cdot)$ is bounded, $Z^{c^n}(\cdot)$ is a martingale.
- Fatou’s lemma, Girsanov’s theorem and Bayes’ rule yield

$$M^\nu(t) \leq \lim inf_{n \to \infty} \mathbb{E}^{Q^n} \left[ \exp \left( - \int_t^T \theta^T dW^n(u) - \frac{1}{2} \int_t^T \|\theta\|^2 du \right) M \bigg| \mathcal{F}_t \right]$$
- Since $\sigma(\cdot, S(\cdot))c^n(\cdot) \equiv 0$ the process $S(\cdot)$ has the same dynamics under $Q^n$ as under $\mathbb{P}$. 
Open problem

The last result might be related to the “Markovian selection results”, as in Krylov (1973) and Ethier and Kurtz (1986). There, the existence of a Markovian solution for a martingale problem is studied. It is observed that a supremum over a set of expectations indexed by a family of distributions is attained and the maximizing distribution is a Markovian solution of the martingale problem.
Open problem

$h^p$ can be characterized as the minimal nonnegative solution of the Cauchy problem

\[
\frac{\partial}{\partial t} v(t, s) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} s_i s_j a_{i,j}(t, s) D_{i,j}^2 v(t, s) = 0
\]

\[
v(T, s) = p(s)
\]

Can an iterative method be constructed, which converges to the minimal solution of this PDE?
“Classical” Mathematical Finance I

- Reminder: \( dZ^\theta(t) = -\theta^T(t, S(t))Z^\theta(t)dW(t) \), where \( \theta \) denotes the market price of risk.
- Assume: \( Z^\theta(\cdot) \) is a true martingale.
- Then, there exists a risk-neutral measure \( Q \), under which \( S(\cdot) \) has dynamics

\[
dS_i(t) = S_i(t) \sum_{k=1}^{K} \sigma_{i,k}(t, S(t))dW_Q^k(t).
\]

- Then,

\[
h^p(t, s) = \mathbb{E}^{t, s} [\tilde{Z}^\theta(T)p(S(T))] = \mathbb{E}^{Q,t,s} [p(S(T))].
\]

- Below: Generalization to the situation where \( Z^\theta(\cdot) \) is a strict local martingale and risk-neutral measure \( Q \) does not exist.
“Classical” Mathematical Finance II

- If we assume that the number of stocks $d$ and the number of driving Brownian motions $K$ is equal, that is, $d = K$, and $\sigma$ has full rank, then the market is called complete.
- Then, by the Martingale Representation Theorem, there exists some strategy $\eta$ such that

$$V_{v,\eta}(T) = p(S(T))$$

for initial capital $v = h^p(0, S(0))$.
- That is, the contingent claim / payoff can be hedged.
- Often, one can use Itô’s rule to compute

$$\eta_i(t) = D_i h^p(t, S(t)),$$

which is called delta hedge.
“Classical” Mathematical Finance III

• Often, the hedging price $h^p$ needs to be computed numerically.
• Theory behind it: *Feynman-Kac Theorem*  
• It states that under some continuity and growth conditions on $a$ and $p$, any solution $v : [0, T] \times \mathbb{R}^d_+ \rightarrow \mathbb{R}$ of the Cauchy-Problem (*Black-Scholes PDE*)

$$
\frac{\partial}{\partial t} v(t, s) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} s_i s_j a_{i,j}(t, s) D_{i,j}^2 v(t, s) = 0
$$

$$
v(T, s) = p(s)
$$

with polynomial growth can be represented as

$$
v(t, s) = \mathbb{E}^Q_{t,s}[p(S(T))] = h^p(t, s),
$$

where $a(\cdot, \cdot) = \sigma(\cdot, \cdot)\sigma^T(\cdot, \cdot)$ and $S(\cdot)$ has $\mathcal{Q}$-dynamics

$$
dS_i(t) = S_i(t) \sum_{k=1}^{K} \sigma_{i,k}(t, S(t))dW^Q_k(t).
$$
Feynman-Kac does not always work.

- We have seen, as long as
  - some growth and continuity conditions on $\sigma$ and $p$ are satisfied,
  - the risk-neutral measure $Q$ exists,
  - $h^p$ is of polynomial growth,
  - the Black-Scholes equation has a solution
we know that the hedging price $h^p$ is a solution.

- Growth conditions are often not satisfied, for example

  $$d\tilde{S}(t) = -\tilde{S}^2(t)dW(t)$$

  with corresponding PDE

  $$\frac{\partial}{\partial t} v(t, s) + \frac{1}{2} s^4 D^2 v(t, s) = 0.$$ 

- Then, $v_1(t, s) = s$ and $v_2(t, s) = 2s\Phi\left(\frac{1}{s\sqrt{T-t}}\right) - s$ are solutions of polynomial growth, satisfying $v(T, s) = s$ and $v(t, 0) = 0.$
• Remember: We have assumed that there exists some $\theta$ which maps the volatility into the drift, that is $\sigma(\cdot, \cdot)\theta(\cdot, \cdot) = \mu(\cdot, \cdot)$.
• It can be shown that this assumption excludes “unbounded profit with bounded risk”.
• Thus “making (a considerable) something out of almost nothing” is not possible.
• However, it is still possible to “certainly make something more out of something”.
• The reason that the arbitrage is not scalable is due to the credit constraint (admissibility) $V^{1,\eta}(\cdot) \geq 0$. 
Digression: Problems of the no-arbitrage assumption.

- A typical market participant can statistically detect whether a market price of risk $\theta$ exists or does not exist.
- However, there exists no statistical test to decide whether $Z^\theta(\cdot)$ is a true martingale or not (whether arbitrage exists or does not exist).
- Instead of starting from the normative assumption of no arbitrage, *Stochastic Portfolio Theory* takes a descriptive approach.
- One goal is to find models which provide realistic dynamics of the market weights $S_i(\cdot)/(S_i(\cdot) + \ldots S_d(\cdot))$.
- These models tend to violate the no-arbitrage assumption.
Stationarity of the market weights.

Figure: Market weights against ranks on logarithmic scale, 1929 - 1999, from Fernholz, *Stochastic Portfolio Theory*, page 95.