Optimal investment with high-watermark performance fee

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based on joint work with

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Objective

- build and analyze a model of optimal investment and consumption where the investment opportunity is represented by a hedge-fund using the "two-and-twenty rule"
- analyze the impact of the high-watermark fee on the investor
Previous work on hedge-funds and high-watermarks

All existing work analyzes the impact/incentive of the high-watermark fees on fund managers

▶ extensive finance literature
  ▶ Goetzmann, Ingersoll and Ross, Journal of Finance 2003
  ▶ Panagea and Westerfield, Journal of Finance 2009
  ▶ Agarwal, Daniel and Naik Journal of Finance, forthcoming
  ▶ Aragon and Qian, preprint 2007

▶ recently studied in mathematical finance
  ▶ Guasoni and Obloj, preprint 2009
The model: investment opportunities

An investor with investment opportunities

- the hedge fund with fund share value process $F$, given exogenously
- the money market paying interest rate zero

Observation: since the money market pays zero rate, the investor leaves all the wealth $X_t$ with the hedge-fund manager (so we call it one investment opportunity)

- some is invested in the fund: $\theta_t$ at time $t$
- the rest (the money market) sits with the manager: $X_t - \theta_t$
The model: dynamic trading strategies

The investor makes continuous-time investments and withdrawals from the fund, which amounts to choosing the predictable $\theta_t$.

Evolution of the total wealth for a trading strategy

- without high-watermark fee

$$dX_t = \theta_t \frac{dF_t}{F_t}, \quad X_0 = x$$

- with high-watermark proportional fee $\lambda > 0$

$$\begin{cases} 
  dX_t = \theta_t \frac{dF_t}{F_t} - \lambda dM_t, & X_0 = x \\
  M_t = \max_{0 \leq s \leq t} (X_s \lor m) 
\end{cases}$$

High-watermark of the investor

$$M_t = \max_{0 \leq s \leq t} (X_s \lor m).$$

Observation: can be also interpreted as taxes on gains, paid right when gains are realized (pointed out by Paolo Guasoni)
Path-wise solutions of the state equation

(same as Guasoni and Obloj)
Denote by $I_t$ the **paper profits** from investing in the fund

$$I_t = \int_0^t \theta_u \frac{dF_u}{F_u}$$

Then

$$X_t = x + I_t - \frac{\lambda}{\lambda + 1} \max_{0 \leq s \leq t} [I_s - (m - x)]^+$$

The high-watermark of the investor is

$$M_t = m + \frac{1}{\lambda + 1} \max_{0 \leq s \leq t} [I_s - (m - x)]^+$$

Observations:

- the fee $\lambda$ can exceed 100% and the investor can still make a profit
- the high-watermark is measured **before** the fee is paid
Connection to the Skorohod map

Denote by $Y = M - X$ the distance from paying fees. Then $Y$ satisfies the equation:

$$\begin{cases} dY_t = -\theta_t \frac{dF_t}{F_t} + (1 + \lambda) dM_t \\ Y_0 = m, \end{cases}$$

where $Y \geq 0$ and

$$\int_0^t \mathbb{1}_{\{Y_s \neq 0\}} dM_s = 0, \quad (\forall) \ t \geq 0.$$

Skorohod map

$$l. = \int_0^\cdot \theta_u \frac{dF_u}{F_u} \to (Y, M) \approx (X, M).$$

Remark: $Y$ will be chosen as state in more general models.
The model: investment and consumption

The investor chooses

- have $\theta_t$ in the fund at time $t$
- consume at a rate $\gamma_t$

Observation: the high-watermark of the investor should take into account his accumulated consumption

$$M_t = \max_{0 \leq s \leq t} \left( \left\{ X_s + \int_0^s \gamma_u du \right\} \lor m \right)$$

The evolution of the wealth is

$$\begin{cases} 
  dX_t = \theta_t \frac{dF_t}{F_t} - \gamma_t dt - \lambda dM_t, & X_0 = x \\
  M_t = \max_{0 \leq s \leq t} \left( \left\{ X_s + \int_0^s \gamma_u du \right\} \lor m \right) 
\end{cases}$$
consumption is a part of the running-max, as opposed to the literature on draw-dawn constraints

- Grossman and Zhou
- Cvitanic and Karatzas
- Elie and Touzi
- Roche

we still have a similar path-wise representation for the wealth in terms of the "paper profit" \( l_t \) and the accumulated consumption

\[
C_t = \int_0^t \gamma_u du.
\]
Optimal investment and consumption

Admissible strategies

\[ \mathcal{A}(x, m) = \{ (\theta, \gamma) : X > 0 \} \].

Can represent investment and consumption strategies in terms of proportions

\[ c = \gamma / X, \quad \pi = \theta. \]

Observation:
- no closed form path-wise solutions for \( X \) in terms of \( (\pi, c) \)
  (unless \( c = 0 \))
Maximize discounted utility from consumption on infinite horizon

\[ \mathcal{A}(x, m) \ni (\theta, \gamma) \rightarrow \arg\max \mathbb{E} \left[ \int_{0}^{\infty} e^{-\beta t} U(\gamma_t) \, dt \right]. \]

Where \( U : (0, \infty) \rightarrow \mathbb{R} \) is the CRRA utility

\[ U(x) = \frac{x^{1-p}}{1-p}, \quad p > 0. \]

Finally, choose a geometric Brownian-Motion model for the fund share price

\[ \frac{dF_t}{F_t} = \alpha dt + \sigma dW_t. \]
Dynamic programming: state processes

First guess: state \((X, M, C)\).
High-watermark fees are paid only when \(X + C = M\) so we can actually choose as only states \(X\) and \(N = M - C\).
The state process \((X, N)\) is a two-dimensional controlled diffusion with reflection on \(\{X = N\}\).

\[
\begin{align*}
  dX_t &= (\theta_t \alpha - \gamma_t)dt + \theta_t \sigma dW_t - \lambda dM_t, \quad X_0 = x \\
  dN_t &= -\gamma_t dt + dM_t, \quad N_0 = m
\end{align*}
\]

Recall we have path-wise solutions in terms of \((\theta, \gamma)\).
Objective: expect to find the value function \(v(x, m)\) as a solution of the HJB, and find the (feed-back) optimal controls.
Dynamic programming equation

Use Itô and write formally the HJB

$$\sup_{\gamma \geq 0, \theta} \left\{ -\beta v + U(\gamma) + (\alpha \theta - \gamma) v_x + \frac{1}{2} \sigma^2 \theta^2 v_{xx} - \gamma v_m \right\} = 0$$

for $m > x > 0$ and the boundary condition

$$-\lambda v_x(x, x) + v_m(x, x) = 0.$$  

(Formal) optimal controls

$$\hat{\theta}(x, m) = -\frac{\alpha}{\sigma^2} \frac{v_x(x, m)}{v_{xx}(x, m)}$$

$$\hat{\gamma}(x, m) = I(v_x(x, m) + v_m(x, m))$$
HJB cont’d

Denote by $\tilde{U}(y) = \frac{p}{1-p} y^{\frac{p-1}{p}}$, $y > 0$ the dual function of the utility. The HJB becomes

$$-\beta v + \tilde{U}(v_x + v_m) - \frac{1}{2} \frac{\alpha^2}{\sigma^2} \frac{v_x^2}{v_{xx}} = 0, \quad m > x > 0$$

plus the boundary condition

$$-\lambda v_x(x, x) + v_m(x, x) = 0.$$  

Observation:

- if there were no $v_m$ term in the HJB, we could solve it closed-form as in Roche or Elie-Touzi using the (dual) change of variable $y = v_x(x, m)$
- no closed-from solutions in our case (even for power utility)
Reduction to one-dimension

Since we are using power utility

\[ U(x) = \frac{x^{1-p}}{1-p}, \quad p > 0 \]

we can reduce to one-dimension

\[ v(x, m) = x^{1-p} v(1, \frac{m}{x}) \]

and

\[ v(x, m) = m^{1-p} v(\frac{x}{m}, 1) \]

- first is nicer economically (since for \( \lambda = 0 \) we get a constant function \( v(1, \frac{m}{x}) \))
- the second gives a nicer ODE (works very well if there is a closed form solution, see Roche)

There is no closed form solution, so we can choose either one-dimensional reduction.
Reduction to one-dimension cont’d

We decide to denote \( z = \frac{m}{x} \geq 1 \) and

\[
v(x, m) = x^{1-p} u(z).
\]

Use

\[
\begin{align*}
v_m(x, m) &= u'(z) \cdot x^{-p}, \\
v_x(x, m) &= \left((1-p)u(z) - zu'(z)\right) \cdot x^{-p}, \\
v_{xx}(x, m) &= \left(-p(1-p)u(z) + 2pzu'(z) + z^2 u''(z)\right) \cdot x^{-1-p},
\end{align*}
\]

to get the reduced HJB

\[
-\beta u + \bar{U}((1-p)u - (z-1)u') - \frac{1}{2} \frac{\alpha^2}{\sigma^2} \frac{(1-p)u - zu')^2}{-p(1-p)u + 2pzu' + z^2 u''} = 0
\]

for \( z > 1 \) with boundary condition

\[
-\lambda(1-p)u(1) + (1 + \lambda)u'(1) = 0
\]
(Formal) optimal proportions

\[ \hat{\pi}(z) = \frac{\alpha}{p\sigma^2} \cdot \frac{(1 - p)u - zu'}{(1 - p)u - 2zu' - \frac{1}{p}z^2u''}, \]

\[ \hat{c}(z) = \left( v_x + v_m \right)^{-\frac{1}{p}} \frac{x}{x} = \left( (1 - p)u - (z-1)u' \right)^{-\frac{1}{p}} \]

Optimal controls

\[ \hat{\theta}(x, m) = x\hat{\pi}(z), \quad \hat{\gamma}(x, m) = x\hat{c}(x, m) \]

Objective: solve the HJB analytically and then do verification
Solution of the HJB for $\lambda = 0$

This is the classical Merton problem. The optimal investment proportion is given by

$$\pi_0 \triangleq \frac{\alpha}{p\sigma^2},$$

while the value function equals

$$v_0(x, m) = \frac{1}{1 - p} c_0^{-p} x^{1-p}, \quad 0 < x \leq m,$$

where

$$c_0 \triangleq \frac{\beta}{p} - \frac{1}{2} \frac{1 - p}{p^2} \cdot \frac{\alpha^2}{\sigma^2}$$

is the optimal consumption proportion. It follows that the one-dimensional value function is constant

$$u_0(z) = \frac{1}{1 - p} c_0^{-p}, \quad z \geq 1.$$
Solution of the HJB for $\lambda > 0$

If $\lambda > 0$ we expect that (additional boundary condition)

$$\lim_{z \to \infty} u(z) = u_0.$$ 

(For very large high-watermark, the investor gets almost the Merton expected utility)

**Theorem 1** The HJB has a smooth solution.

Idea of solving the HJB:
- find a viscosity solution using Perron’s method
- show that the viscosity solution is $C^2$

Avoid the Dynamic Programming Principle.
**Theorem 2** The closed loop equation

\[
\begin{aligned}
&dX_t = \hat{\theta}(X_t, N_t) \frac{dF_t}{F_t} - \hat{\gamma}(X_t, N_t) dt - \lambda dM_t, \quad X_0 = x \\
&M_t = \max_{0 \leq s \leq t} \left( \left\{ X_s + \int_0^s \hat{\gamma}(X_t, N_t) du \right\} \lor m \right)
\end{aligned}
\]

where

\[
N_t = M_t - \int_0^s \hat{\gamma}(X_t, N_t) du
\]

has a unique strong solution \(0 < \hat{X} \leq \hat{N}\).

Idea of proof: use the path-wise representation together with the Itô-Picard theory.

**Theorem 3** The controls \(\hat{\theta}(\hat{X}_t, \hat{N}_t)\) and \(\hat{\gamma}(\hat{X}_t, \hat{N}_t)\) are optimal.

Idea of proof: uniform integrability. Has to be done separately for \(p < 1\) and \(p > 1\).
The impact of fees

Certainty equivalent return defined by

\[ \tilde{u}_0(\tilde{\alpha}(z)) = u(z) \]

all other parameters being equal. Can be solved as

\[ \tilde{\alpha}^2(z) = 2\sigma^2 \frac{p^2}{1-p} \left( \frac{\beta}{p} - \left( (1-p)u(z) \right)^{-\frac{1}{p}} \right), \quad z \geq 1. \]

The relative size of the certainty equivalent excess return is therefore

\[ \frac{\tilde{\alpha}(z)}{\alpha} = \sqrt{2\sigma p} \left( \frac{\beta}{p} - \left( (1-p)u(z) \right)^{-\frac{1}{p}} \right)^{\frac{1}{2}}, \quad z \geq 1. \]
Certainty equivalent initial wealth:

\[ v_0(\tilde{x}) = v(x, n) = x^{1-p} u(z) \]

all other parameters being equal. Can be solved as.

\[ \tilde{x}(z) = x \cdot \left( \frac{u(z)}{u_0} \right)^{\frac{1}{1-p}} = x \cdot ((1 - p)c_0 u(z))^{\frac{1}{1-p}}, \quad z \geq 1. \]

The quantity

\[ \frac{\tilde{x}(z)}{x} = \left( \frac{u(z)}{u_0} \right)^{\frac{1}{1-p}} = ((1 - p)c_0 u(z))^{\frac{1}{1-p}}, \quad z \geq 1, \]

is the relative certainty equivalent wealth.
Figure: Parameters: $p = 3$, $\beta = 5\%$, $\alpha = 10\%$, $\sigma = 30\%$, $\lambda = 20\%$
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Conclusions

Point of view of Finance:
- model optimal investment with high-watermark fees from the point of view of the investor
- analyze the impact of the fees

Point of Mathematics:
- we are controlling a two-dimensional diffusion
- solve the problem using direct dynamic programming: first find a smooth solution of the HJB and then do verification
Work in progress and future work

with Gerard Brunick and Karel Janeček

- presence of (multiple and correlated) traded stocks, interest rates and hurdles: can still be modeled as a two-dimensional diffusion problem using $X$ and $Y = M - X$ as state processes (reduced to one-dimension by scaling)
- analytic approximations when $\lambda$ is small
- more than one fund: genuinely multi-dimensional problem with reflection
- stochastic volatility, jumps, etc
The Most Important Thing

Happy Birthday to Walter!