Optimal Portfolio Liquidation with Dynamic Coherent Risk

Andrey Selivanov\textsuperscript{1}  Mikhail Urusov\textsuperscript{2}

\textsuperscript{1}Moscow State University and Gazprom Export
\textsuperscript{2}Ulm University

Analysis, Stochastics, and Applications. A Conference in Honour of Walter Schachermayer – Vienna University, July 12–16, 2010
Outline

Optimal Portfolio Liquidation

Dynamic Risk

Main Result
Outline

Optimal Portfolio Liquidation

Dynamic Risk

Main Result
A trader sells \( x > 0 \) shares of a stock in an illiquid market. In selling the price falls from \( S_- \) to

\[
S_+ = S_- - \frac{1}{q} x.
\]

The trader gets the payout

\[
x \left( S_- - \frac{1}{2q} x \right)
\]

average price per share

instead of \( xS_- \)
How to sell optimally $X_0$ shares until time $N$?

$X_0$, $N$ are specified by a client, $X_0$ is very big

Time horizon is usually short

A strategy is a sequence $x = (x_i)_{i=0}^N$, where all $x_i \geq 0$ and $\sum_{i=0}^N x_i = X_0$

$x_i$ means the number of shares to sell at time $i$, $i = 0, \ldots, N$

$\mathcal{X}$ (resp., $\mathcal{X}_{\text{det}}$) denotes the set of adapted (resp., deterministic) strategies
Model for unaffected price
A random walk \((S_n)\) (short time horizon)

Model for price impact
A block-shaped limit order book with infinite resilience

Optimization problem
Minimize a certain dynamic coherent risk measure
Model for price impact

Linear permanent and temporary impacts with the coefficients $\gamma \geq 0$ resp. $\kappa > 0$

Selling $x_k \geq 0$ shares at times $k$, $k = 0, 1, \ldots$:

$$\tilde{S}_{n+} = \tilde{S}_{n-} - (\kappa + \gamma)x_n,$$

where $\tilde{S}_{n-} = S_n - \gamma \sum_{i=0}^{n-1} x_i$

Payout at time $n$:

$$x_n \left( \tilde{S}_{n-} - \frac{\kappa + \gamma}{2} x_n \right)$$

Cf. with Bertsimas and Lo (1998), Almgren and Chriss (2001)

LOB with finite resilience:
Notation $X_n := X_0 - \sum_{i=0}^{n-1} x_i$, $n = 1, \ldots, N + 1$, the number of shares remaining at hand at time $n−$. Note that $X_{N+1} = 0$

$(x_i) \leftrightarrow (X_i)$

Properties of strategies desirable for practitioners

(A) Dynamic consistency

(B) Presence of an intrinsic time horizon $N^\ast$ such that

- $N^\ast < N$ for small $X_0$,
- $N^\ast = N$ for large $X_0$,
- $N^\ast$ is increasing as a function of $X_0$

(C) Relative selling speed decreasing in the position size:

$$\frac{x_0}{X_0}$$ decreases as a function of $X_0$
Notation \( R_{N+} \) revenue from the liquidation

Almgren and Chriss (2001)

\[
- \mathbb{E} R_{N+} + \lambda \text{Var} R_{N+} \xrightarrow{\mathcal{X}_{\text{det}}} \min
\]

Optimal strategy is of the form

\[
\mathcal{X}_n = C_1 e^{-Kn} - C_2 e^{Kn} \quad (\ast)
\]

(A) + (B) − (C) −

Konishii and Makimoto (2001)

\[
- \mathbb{E} R_{N+} + \lambda \sqrt{\text{Var} R_{N+}} \xrightarrow{\mathcal{X}_{\text{det}}} \min
\]

Optimal strategy is again of the form \((\ast)\)

(A) − (B) − (C) +
It would be more interesting to optimize over $\mathcal{X}$ rather than over $\mathcal{X}_{\text{det}}$

Almgren and Lorenz (2007)

$$-E R_{N^+} + \lambda \text{Var} R_{N^+} \rightarrow \min_{\mathcal{X}}$$

(*) is no longer optimal

(A)–(C): ?

Schied, Schöneborn, and Tehranchi (2010) For $U(x) = -e^{-\alpha x}$,

$$E U(R_{N^+}) \rightarrow \max_{\mathcal{X}}$$

Optimal strategy is deterministic (cf. with Schied and Schöneborn (2009))

If $(S_n)$ is a Gaussian random walk, then the optimal strategy is the Almgren–Chriss one with $\lambda = \alpha / 2$

(A) + (B) — (C) —
Outline

Optimal Portfolio Liquidation

Dynamic Risk

Main Result
Static Risk

$(\Omega, \mathcal{F}, P)$

$R : \Omega \to \mathbb{R}$  P&L of a bank

How to measure risk of $R$?


Notation

$\rho(R)$ a law invariant coherent risk measure

$\tilde{\rho}(\text{Law } R) := \rho(R)$

E.g.

$\text{CV@R}_\lambda(R) = -E(R|R \leq q_\lambda(R))$

(modulo a technicality), where $q_\lambda(R)$ is $\lambda$-quantile of $R$
Dynamizing $\rho$

$$(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^N, P)$$

Cashflow $F = (F_n)_{n=0}^N$: an adapted process

$F_n$ means P&L of a bank at time $n$

Need to define dynamic risk $\rho(F)$

$$\rho(F) = (\rho_n(F))_{n=0}^N$$ an adapted process

$\rho_n(F) \equiv \rho(F_n, \ldots, F_N)$ means the risk of the remaining part $(F_n, \ldots, F_N)$ of the cashflow measured at time $n$

Define inductively:

$$\rho_N(F) = -F_N,$$

$$\rho_n(F) = -F_n + \rho(\text{Law}[-\rho_{n+1}(F) | \mathcal{F}_n]), \quad n = N - 1, \ldots, 0$$

Outline

Optimal Portfolio Liquidation

Dynamic Risk

Main Result
$X_0 > 0$ a large number of shares to sell until time $N$

$S_n = S_0 + \sum_{i=1}^{n} \xi_i$, where $(\xi_i)$ iid

$\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$, where $\mathcal{F}_0 = \text{triv}$

A strategy is an $(\mathcal{F}_n)$-adapted sequence $x = (x_i)_{i=0}^{N}$, where all $x_i \geq 0$ and $\sum_{i=0}^{N} x_i = X_0$

$\mathcal{X}$ (resp., $\mathcal{X}_{\text{det}}$) denotes the set of all (resp., deterministic) strategies

$(x_i) \leftrightarrow (X_i)$, where $X_n = X_0 - \sum_{i=0}^{n-1} x_i$
Problem Settings

Setting 1 For a strategy $x = (x_i)_{i=0}^N$ define the cashflow $F^x$ by

$$F^x_n = x_n \left( S_n - \gamma \sum_{i=0}^{n-1} x_i - \frac{\kappa + \gamma}{2} x_n \right), \quad n = 0, \ldots, N.$$ 

The problem: $\bar{\rho}_0(F^x) \longrightarrow \min$ over $x \in \mathcal{X}$

Setting 2 For a strategy $x$ define $G^x$ by $G^x_0 = 0$ and

$$G^x_n = x_{n-1} \left( S_{n-1} + \frac{\xi_n}{2} - \gamma \sum_{i=0}^{n-2} x_i - \frac{\kappa + \gamma}{2} x_{n-1} \right), \quad n = 1, \ldots, N+1.$$ 

The problem: $\bar{\rho}_0(G^x) \longrightarrow \min$ over $x \in \mathcal{X}$
Main Result

Standing assumption $0 < \tilde{\rho}(\text{Law } \xi) < \infty$

Set $a := \tilde{\rho}(\text{Law } \xi)/\kappa$, so $a > 0$

**Theorem** Optimal strategy is the same in both settings. Moreover, it is deterministic and given by the formulas

\[
x_i = \frac{X_0}{N^* + 1} + a \left( \frac{N^*}{2} - i \right), \quad i = 0, \ldots, N^*,
\]

\[
x_i = 0, \quad i = N^* + 1, \ldots, N,
\]

where

\[
N^* = N \land \left( \text{ceil} \frac{-1 + \sqrt{1 + 8X_0/a}}{2} - 1 \right)
\]

with ceil $y$ denoting the minimal integer $d$ such that $y \leq d$
Discussion

If we maximized over $\mathcal{X}_{\text{det}}$ rather than over $\mathcal{X}$, then the optimizer would be the same in both settings. This is not clear a priori when we maximize over $\mathcal{X}$

The proof consists of two parts: first we prove that optimizing over $\mathcal{X}$ does not do a better job, than optimizing over $\mathcal{X}_{\text{det}}$, and then perform just a deterministic optimization

Cf. with Alfonsi, Fruth, and Schied (2010), Schied, Schöneborn, and Tehranchi (2010), where the optimal strategies are also deterministic

Why is the optimal strategy deterministic?

Because here liquidity ($\kappa$) is deterministic

Cf. with Fruth, Schöneborn, and Urusov (2010), where stochastic liquidity leads to stochastic optimal strategies
Remarks

- (A) + (B) + (C) +
  (recall “+ − −” for the Almgren–Chriss strategy)

- \((X_n)\) parabola vs. \(X_n = C_1 e^{-Kn} - C_2 e^{Kn}\)
  (Almgren–Chriss is now a benchmark for practitioners)

- Setting \(N = \infty\) (time horizon is not specified by the client)
  we get a strategy with a purely intrinsic time horizon \(N^*\).
  Cf. with Almgren (2003), Schöneborn (2008)

- \(a \uparrow\) leads to a quicker liquidation in the beginning
  \(\implies\) reasonable dependence of the liquidation strategy on
  volatility risk \(\tilde{\rho}(\text{Law } \xi)\) and on liquidity risk \(\kappa\)
Thank you for your attention!
Possible Generalizations

- More general price impact?

  Optimal strategies are again deterministic

- Convex risk measure $\rho$?

  Optimal strategies are again deterministic, however, different in Settings 1 and 2

  Typically $(A) + (B) -$

  Also $(C) -$ in an example with entropic risk measure, which was worked out explicitly


