ON GROUPS OF HÖLDER DIFFEOMORPHISMS AND THEIR REGULARITY

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ABSTRACT. We study the set $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ of orientation preserving diffeomorphisms of $\mathbb{R}^d$ which differ from the identity by a Hölder $C_0^{n,\beta}$-mapping, where $n \in \mathbb{N}_{\ge 1}$ and $\beta \in (0,1]$. We show that $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ forms a group, but left translations in $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ are in general discontinuous. The groups $\mathcal{D}^{n,\beta-}(\mathbb{R}^d) := \bigcap_{\alpha<\beta} \mathcal{D}^{n,\alpha}(\mathbb{R}^d)$ (with its natural Fréchet topology) and $\mathcal{D}^{n,\beta+}(\mathbb{R}^d) := \bigcup_{\alpha>\beta} \mathcal{D}^{n,\alpha}(\mathbb{R}^d)$ (with its natural inductive locally convex topology) however are $C^{0,\omega}$ Lie groups for any *slowly vanishing* modulus of continuity ω. In particular, $\mathcal{D}^{n,\beta-}(\mathbb{R}^d)$ is a topological group and a so-called half-Lie group (with smooth right translations). We prove that the Hölder spaces $C_0^{n,\beta}$ are ODE closed, in the sense that pointwise time-dependent $C_0^{n,\beta}$-vector fields u have unique flows Φ in $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$. This includes, in particular, all Bochner integrable functions $u \in L^1([0,1], C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d))$. For the latter and $n \ge 2$, we show that the flow map $L^1([0,1], C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d)) \to C([0,1], \mathcal{D}^{n,\alpha}(\mathbb{R}^d))$, $u \mapsto \Phi$, is continuous (even $C^{0,\beta-\alpha}$), for every $\alpha < \beta$. As an application we prove that the corresponding Trouvé group $\mathcal{G}_{n,\beta}(\mathbb{R}^d)$ from image analysis coincides with the connected component of the identity of $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$.

1. Introduction

Let E be a Banach space of functions $f : \mathbb{R}^d \to \mathbb{R}^d$ which is continuously embedded in $C_0^1(\mathbb{R}^d, \mathbb{R}^d)$, i.e., C^1-mappings which vanish together with its first derivative at infinity. Let $u : [0,1]\times\mathbb{R}^d \to \mathbb{R}^d$ be a *pointwise time-dependent E-vector field*, i.e., $u(t,\cdot) \in E$ for all t, $u(\cdot,x)$ is measurable for all x, and $t \mapsto \|u(t,\cdot)\|_E$ is integrable. It is well-known that the corresponding pointwise flow

$$\Phi(t,x) = x + \int_0^t u(s,\Phi(s,x))\,ds, \quad x \in \mathbb{R}^d,\ t \in [0,1], \tag{1.1}$$

is a C^1-diffeomorphism of $\mathbb{R}^d$ at any t. The set of all diffeomorphisms $\Phi(1,\cdot)$ at time 1 which arise in this way form a group $\mathcal{G}_E$, which we call the *Trouvé group* of E, since this constructions is due to Trouvé [27]; details can be found in the book [28].

In general, not much is known about the Trouvé group. We are especially interested in precise regularity properties of its elements. This is intimately related to the question as to whether E is *ODE closed*, i.e., $\mathcal{G}_E \subseteq \mathrm{Id} + E$, and if not, what

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the *ODE hull* of E is. We define the ODE hull to be the intersection of all ODE closed spaces continuously embedded in $C^1_0(\mathbb{R}^d,\mathbb{R}^d)$ and continuously containing E, see Section 4.2; here it is reasonable to allow for locally convex spaces (mutatis mutandis) instead of just Banach spaces.

ODE closedness is closely related to stability and continuity or smoothness properties of composition of mappings. Indeed, it has been widely studied in the context of regular infinite dimensional Lie groups; cf. [10]. Our results are not covered by the general theory from [10]; Hölder (diffeomorphism) groups fail to be Lie groups.

In this paper we explore these questions in the case that E is a global Hölder space $C^{n,\beta}_0(\mathbb{R}^d,\mathbb{R}^d)$, $n\in\mathbb{N}_{\ge 1}$, $\beta\in(0,1]$. In search for an identification of the elements of the corresponding Trouvé group $\mathcal{G}_{n,\beta}(\mathbb{R}^d)$, it is natural to look at the set of orientation preserving diffeomorphisms of $\mathbb{R}^d$ which differ from the identity by a $C^{n,\beta}_0$-mapping, i.e.,

$$\mathcal{D}^{n,\beta}(\mathbb{R}^d) := \big\{\Phi\in \operatorname{Id}+C^{n,\beta}_0(\mathbb{R}^d,\mathbb{R}^d) : \det d\Phi(x)>0 \ \forall x\in\mathbb{R}^d\big\}.$$

We will show that $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ is a group with respect to composition, but it is not a topological group: left translations and inversion are in general not continuous. Left translations become continuous if the outer function is slightly more regular: $\phi\mapsto\psi\circ(\operatorname{Id}+\phi)$ is continuous from $C^{n,\alpha}_0(\mathbb{R}^d,\mathbb{R}^d)\to C^{n,\alpha}_0(\mathbb{R}^d,\mathbb{R}^d)$ if $\psi\in C^{n,\beta}_0(\mathbb{R}^d,\mathbb{R}^d)$ and $\alpha<\beta$ (the same holds if $\phi\in C^{n,1}_0(\mathbb{R}^d,\mathbb{R}^d)$ and $\psi\in C^{n+1,\beta}_0(\mathbb{R}^d,\mathbb{R}^d)$). Similarly, $\Phi\mapsto\Phi^{-1}$ is continuous from $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ to $\mathcal{D}^{n,\alpha}(\mathbb{R}^d)$ if $\alpha<\beta$. This motivates the definitions

$$\mathcal{D}^{n,\beta-}(\mathbb{R}^d) := \big\{\Phi\in \operatorname{Id}+C^{n,\beta-}_0(\mathbb{R}^d,\mathbb{R}^d) : \det d\Phi(x)>0 \ \forall x\in\mathbb{R}^d\big\},$$
$$\mathcal{D}^{n,\beta+}(\mathbb{R}^d) := \big\{\Phi\in \operatorname{Id}+C^{n,\beta+}_0(\mathbb{R}^d,\mathbb{R}^d) : \det d\Phi(x)>0 \ \forall x\in\mathbb{R}^d\big\},$$

where $C^{n,\beta-}_0 := \bigcap\{C^{n,\alpha}_0 : 0<\alpha<\beta\}$ and $C^{n,\beta+}_0 := \bigcup\{C^{n,\alpha}_0 : \beta<\alpha<1\}$, equipped with the natural projective, resp. inductive locally convex topology. We prove that $\mathcal{D}^{n,\beta\pm}(\mathbb{R}^d)$ are $C^{0,\omega}$ Lie groups (see Section 3.3) for every *slowly vanishing* modulus of continuity ω, i.e.,

$$\liminf_{t\downarrow 0}\frac{\omega(t)}{t^\gamma}>0 \quad \text{for all } \gamma>0.$$

This regularity cannot be improved; see Proposition 3.13. In particular, $\mathcal{D}^{n,\beta-}(\mathbb{R}^d)$ are topological groups (which remains open for $\mathcal{D}^{n,\beta+}(\mathbb{R}^d)$ since the underlying locally convex topology and the c^∞-topology fall apart in this case). The right translations are bounded affine linear (in the chart representation) and hence smooth. Consequently, $\mathcal{D}^{n,\beta-}(\mathbb{R}^d)$ are also *half-Lie groups* as defined in [20].

In the second part of the paper we study flows of time-dependent $C^{n,\beta}_0$-vector fields. Here we distinguish between:

(1) *Pointwise time-dependent $C^{n,\beta}_0$-vector fields*, i.e., mappings $u:[0,1]\times\mathbb{R}^d\to\mathbb{R}^d$ such that $u(t,\cdot)\in C^{n,\beta}_0(\mathbb{R}^d,\mathbb{R}^d)$ for all $t\in[0,1]$, $u(\cdot,x)$ is measurable for all $x\in\mathbb{R}^d$, and $t\mapsto\|u(t,\cdot)\|_{n,\beta}$ is integrable. (This corresponds to the notion defined at the beginning of the introduction.)
(2) *Strong time-dependent $C^{n,\beta}_0$-vector fields*, i.e., Bochner integrable functions $u\in L^1([0,1],C^{n,\beta}_0(\mathbb{R}^d,\mathbb{R}^d))$.

The latter notion involves strong measurability which entails that the image $u([0,1])$ is essentially separable; a non-trivial condition, since the Hölder spaces $C^{n,\beta}_0$ are non-separable. If u satisfies (2) then $u^\wedge$ satisfies (1), the converse is false. (For $f \in Z^{X\times Y}$ we consider $f^\vee \in (Z^Y)^X$ defined by $f^\vee(x)(y) = f(x,y)$, and with $g \in (Z^Y)^X$ we associate $g^\wedge \in Z^{X\times Y}$ with $g^\wedge(x,y) = g(x)(y)$.)

This deficiency has the effect that Carathéodory's solution theory for ODEs on Banach spaces which are Bochner integrable in time is not well suited for the Hölder space setting. Instead we work with pointwise estimates which has the additional benefit that our proofs only require the weaker assumptions in (1).

We show that, for all $n \in \mathbb{N}_{\ge 1}$, $\beta \in (0,1]$, pointwise time-dependent $C_0^{n,\beta}$-vector fields u have unique pointwise flows $\Phi \in C([0,1], \mathcal{D}^{n,\beta}(\mathbb{R}^d))$; in particular, $C_0^{n,\beta}$ is ODE closed (although composition in $\mathcal{D}^{n,\beta}$ is not continuous!). As a consequence, for $u \in L^1([0,1], C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d))$, the identity (1.1) lifts to an identity in $\mathcal{D}^{n,\alpha}(\mathbb{R}^d)$, for each $\alpha < \beta$ (see Theorem 5.3):

$$\Phi^\vee(t) = \mathrm{Id} + \int_0^t u(s) \circ \Phi^\vee(s)\, ds, \quad t \in [0,1].$$

Furthermore, we identify the corresponding Trouvé group:

$$\mathcal{G}_{n,\beta}(\mathbb{R}^d) = \mathcal{D}^{n,\beta}(\mathbb{R}^d)_0, \tag{1.2}$$

where $\mathcal{D}^{n,\beta}(\mathbb{R}^d)_0$ denotes the connected component of the identity in $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$. Thus there seems to be no natural topology on $\mathcal{G}_{n,\beta}(\mathbb{R}^d)$ which makes it a topological group. On the other hand, we also get

$$\mathcal{G}_{n,\beta-}(\mathbb{R}^d) = \mathcal{D}^{n,\beta-}(\mathbb{R}^d)_0, \quad \mathcal{G}_{n,\beta+}(\mathbb{R}^d) = \mathcal{D}^{n,\beta+}(\mathbb{R}^d)_0 \tag{1.3}$$

which endows $\mathcal{G}_{n,\beta\pm}(\mathbb{R}^d)$ with a $C^{0,\omega}$ Lie group structure, for every slowly vanishing modulus of continuity ω; on $\mathcal{G}_{n,\beta-}(\mathbb{R}^d)$ we also get a topological group structure and a half-Lie group structure. We wish to point out that our proof of (1.2) subsequently shows that the equality (1.2) also holds if in the definition of the Trouvé group $\mathcal{G}_{n,\beta}(\mathbb{R}^d)$ one restricts to pointwise time-dependent $C_0^{n,\beta}$-vector fields which are piecewise C^n in time; see Remark 4.8.

In the third part we investigate the continuity of the flow map $u \mapsto \Phi$. We find that as a mapping

$$L^1([0,1], C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d)) \to C([0,1], \mathcal{D}^{n,\alpha}(\mathbb{R}^d)) \tag{1.4}$$

the flow map is

- bounded, if $n \in \mathbb{N}_{\ge 1}$ and $0 < \alpha \le \beta \le 1$,
- continuous, even $C^{0,\beta-\alpha}$, if $n \in \mathbb{N}_{\ge 2}$ and $0 < \alpha < \beta \le 1$.

As a corollary we obtain that as a mapping

$$L^1([0,1], C_0^{n,\beta-}(\mathbb{R}^d,\mathbb{R}^d)) \to C([0,1], \mathcal{D}^{n,\beta-}(\mathbb{R}^d)) \tag{1.5}$$

the flow map is bounded for all $n \ge 1$ and continuous and $C^{0,\omega}$ if $n \ge 2$ (and arbitrary $\beta \in (0,1]$), for every slowly vanishing modulus of continuity ω.

In [2] similar results were obtained in the Sobolev case $E = H^s(\mathbb{R}^d,\mathbb{R}^d)$, for $s > d/2+1$. In particular, it was shown that H^s is ODE closed and that

$$\mathcal{G}_s(\mathbb{R}^d) = \mathcal{D}^s(\mathbb{R}^d)_0,$$

where $\mathcal{G}_s(\mathbb{R}^d)$ denotes the corresponding Trouvé group and $\mathcal{D}^s(\mathbb{R}^d)$ the group of orientation preserving diffeomorphisms of $\mathbb{R}^d$ which differ from the identity by a mapping in $H^s(\mathbb{R}^d,\mathbb{R}^d)$. The methods are quite different: thanks to the fact that

$D^s(\mathbb{R}^d)$ is a topological group (cf. [11]) Carathéodory's solution theory for ODEs on Banach spaces which are Bochner integrable in time is well suited for this setting.

The paper is structured as follows. We fix notation and present the main technical tools in Section 2. We also review some results on the composition of Hölder functions essentially due to [3]; since we need slightly altered versions we give proofs but relegate them to Appendix A. In Section 3 we investigate the groups $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$, $\mathcal{D}^{n,\beta-}(\mathbb{R}^d)$ and $\mathcal{D}^{n,\beta+}(\mathbb{R}^d)$. We prove ODE closedness of $C_0^{n,\beta}$ and the identities (1.2) and (1.3) in Section 4. In Section 5 we study the continuity of the flow maps (1.4) and (1.5).

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2. Definitions and preliminary results

2.1. **Hölder spaces.** Let $k \in \mathbb{N}$, $\alpha \in (0,1]$. Let E, F be Banach spaces and let $U \subseteq E$ be open. We consider the *global* Hölder space

$$C_b^{k,\alpha}(U,F) := \big\{ f \in C^k(U,F) : \|f\|_{k,\alpha} < \infty \big\},$$

where

$$\begin{aligned} \|f\|_{k,\alpha} &:= \max\{\|f\|_k, [f]_{k,\alpha}\}, \\ \|f\|_k &:= \sup\{\|f^{(l)}(x)\|_{L_l} : x \in U,\ 0 \le l \le k\}, \\ [f]_{k,\alpha} &:= \sup_{x,y \in U,\, x \ne y} \frac{\|f^{(k)}(x) - f^{(k)}(y)\|_{L_k}}{\|x-y\|^\alpha}. \end{aligned}$$

Here $f^{(l)} = d^l f : E \to L_l(E;F)$ is the Fréchet derivative of order l and $L_l(E;F)$ denotes the vector space of continuous l-linear mappings endowed with the operator norm $\|\cdot\|_{L_l}$.

We denote by $C_0^{k,\alpha}(E,F)$ the subspace of those mappings $f \in C_0^{k,\alpha}(E,F)$ that tend to 0 at infinity together with all their derivatives up to order k, i.e., for every $\epsilon > 0$ there is $r > 0$ such that $\|f^{(l)}(x)\|_{L_l} \le \epsilon$ if $\|x\| > r$ and $0 \le l \le k$.

All these spaces are Banach spaces.

Local Hölder spaces are denoted by $C^{k,\alpha}$, i.e., $f \in C^{k,\alpha}(U,F)$ if each $x \in U$ has a neighborhood V in U such that $f|_V \in C_b^{k,\alpha}(V,F)$.

Let us recall interpolation and inclusion relations for Hölder spaces. In the following $C_b^{n,0} := C_b^n$ and $\|\cdot\|_{n,0} := \|\cdot\|_n$.

Lemma 2.1 ([3, 3.1])**.** *Let $n \in \mathbb{N}$ and $0 \le \alpha < \beta < \gamma \le 1$ and set $\mu := \frac{\gamma-\beta}{\gamma-\alpha}$. Then*

$$\|f\|_{n,\beta} \le M_\alpha \|f\|_{n,\alpha}^\mu \|f\|_{n,\gamma}^{1-\mu}, \quad f \in C_b^{n,\gamma}(E,F),$$

where $M_0 := 2$ and $M_\alpha := 1$ for $\alpha > 0$.

Lemma 2.2 ([3, 3.7])**.** *Let $m, n \in \mathbb{N}$ and $\alpha, \beta \in [0,1]$ with $m + \alpha \le n + \beta$. Then $C_b^{n,\beta}(E,F) \subseteq C_b^{m,\alpha}(E,F)$ and*

$$\|f\|_{m,\alpha} \le 2\|f\|_{n,\beta}, \quad f \in C_b^{n,\beta}(E,F).$$

2.2. **The Bochner integral.** Cf. [4]. Let $I = [a, b] \subseteq \mathbb{R}$ be a closed interval (with the Lebesgue measure). Let E be a Banach space. A measurable function $f : I \to E$ is *Bochner integrable* if there is a sequence of integrable simple functions $s_n : I \to E$ such that $s_n \to f$ a.e. (i.e., f is *strongly measurable*) and $\int_a^b \|f - s_n\|\, dt \to 0$. In this case the *Bochner integral* is defined by

$$\int_a^b f\, dt = \lim_{n\to\infty} \int_a^b s_n\, dt.$$

By the Pettis measurability theorem, $f : I \to E$ is strongly measurable if and only if it is *weakly measurable* (i.e., $\lambda \circ f$ is measurable for all $\lambda \in E^*$) and *essentially separable valued* (i.e., $f(I \setminus N)$ is separable in E for some null set N). A strongly measurable function $f : I \to E$ is Bochner integrable if and only if $\int_a^b \|f\|\, dt < \infty$. Then the triangle inequality holds:

$$\Big\| \int_a^b f\, dt \Big\| \le \int_a^b \|f\|\, dt.$$

If $T : E \to F$ is a bounded linear operator into another Banach space F then $Tf : I \to F$ is Bochner integrable and

$$T \int_a^b f\, dt = \int_a^b Tf\, dt.$$

We will use the following version of the fundamental theorem of calculus.

Lemma 2.3. *If $f : [a, b] \to E$ is continuous, then*

$$\frac{d}{dt} \int_a^t f(s)\, ds = f(t),$$

for all $t \in I$. If $f : [a, b] \to E$ is C^1, then

$$f(b) - f(a) = \int_a^b f'(s)\, ds.$$

It is then straightforward to deduce a mean value inequality for C^1-mappings between Banach spaces.

2.3. **Carathéodory type ODEs.** Next we collect some results on Carathéodory type differential equations. Those are certain ODEs on Banach spaces whose right hand side is not continuous in time. We refer to [1] and to the appendix in [2].

Let E be a Banach space, $U \subseteq E$ some open subset and $I = [t_0, t_1]$ some real interval. We say that $f : I \times U \to E$ satisfies the *Carathéodory property* if:

(i) For every $t \in I$ the mapping $f(t, \cdot) : U \to E$ is continuous.
(ii) For every $x \in U$ the mapping $f(\cdot, x) : I \to E$ is strongly measurable.

Also the notion of solution of such an ODE is weakened: we say a continuous curve $\Phi : I \to U$ is a solution of the initial value problem

$$\partial_t x = f(t, x), \quad x(t_0) = x_0 \tag{2.1}$$

if and only if $s \mapsto f(s, \Phi(s))$ is Bochner integrable and

$$\Phi(t) = x_0 + \int_{t_0}^t f(s, \Phi(s))\, ds, \quad t \in I. \tag{2.2}$$

This already implies that $\Phi : I \to U$ is continuous. It is actually absolutely continuous in the sense that there exists a Bochner integrable $\gamma : I \to E$ such that $\Phi(t) = \Phi(t_0) + \int_{t_0}^t \gamma(s)\, ds$; in particular, Φ is differentiable a.e. and $\Phi' = \gamma$ a.e. (see [10, Lemma 1.28]).

The next theorem is the central existence and uniqueness result for Carathéodory type differential equations; it is taken from [2, Thm. A.2].

Theorem 2.4. *Let $I = [t_0, t_1]$ and let $f : I \times U \to E$ have the Carathéodory property. Let $B(x_0, \varepsilon) := \{x \in E : \|x - x_0\| < \epsilon\} \subseteq U$. In addition let m, l be positive locally integrable functions defined on I such that the estimates*

$$\|f(t, x_1) - f(t, x_2)\| \le l(t)\|x_1 - x_2\|$$
$$\|f(t, x)\| \le m(t)$$

are valid for almost all t and all $x, x_1, x_2 \in B(x_0, \varepsilon)$. Let δ be such that

$$\int_{t_0}^{t_0+\delta} m(s)\, ds < \varepsilon,$$

then (2.1) *has a unique solution $\phi : [t_0, t_0 + \delta] \to B(x_0, \varepsilon)$ in the sense of* (2.2)*.*

If the ODE is linear, we have global existence in time:

Theorem 2.5. *Let $I = [t_0, t_1]$. Let $A : I \to L(E)$ and $b : I \to E$ be Bochner integrable. Then for all $x_0 \in E$ there exists a unique solution on I of*

$$\partial_t x(t) = A(t) \cdot x(t) + b(t), \quad x(t_0) = x_0$$

in the sense of (2.2)*.*

2.4. **Composition in Hölder spaces.** Let us review some regularity results for the composition in Hölder spaces due to [3]. But in contrast to [3], we need the results for mappings $F : \mathbb{R}^d \to \mathbb{R}^d$ of the form $F = \mathrm{Id} + f$ where f is in some Hölder class; note that Id is unbounded and hence not a member of any $C_b^{n,\beta}(\mathbb{R}^d, \mathbb{R}^d)$. For this reason it is convenient to introduce the seminorm

$$[F]_n := \|F^{(n)}\|_0 = \sup_{x \in \mathbb{R}^d} \|F^{(n)}(x)\|_{L_n}.$$

If $F = \mathrm{Id} + f$ and $n \ge 1$, then

$$[F]_n \le 1 + [f]_n. \tag{2.3}$$

It is easy to adapt the proofs in [3] to our needs; they are outlined in Appendix A for completeness' sake.

Proposition 2.6 ([3, 4.2])**.** *Let E, F, G, H be Banach spaces and $U \subseteq E$ open. Let $m \in \mathbb{N}$, $\alpha \in (0, 1]$, and $b : F \times G \to H$ be a bilinear continuous mapping. Then $b_* : C_b^{m,\alpha}(U, F) \times C_b^{m,\alpha}(U, G) \to C_b^{m,\alpha}(U, H)$, defined by $b_*(f, g)(x) := b(f(x), g(x))$, is bilinear, continuous, and $\|b_*\| \le 2^{m+1}\|b\|$.*

The following theorem shows stability under composition and continuity of the right translation. We will denote by $f^\star$ the pull-back by $\mathrm{Id} + f$, i.e., $f^\star := (\mathrm{Id} + f)^*$.

Theorem 2.7 ([3, 6.2])**.** *Let $m \in \mathbb{N}_{\ge 1}$ and $\alpha \in (0, 1]$. Let $f \in C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$ and $g \in C_b^{m,\alpha}(\mathbb{R}^d, G)$ for some Banach space G. Then $g \circ (\mathrm{Id} + f) \in C_b^{m,\alpha}(\mathbb{R}^d, G)$ and there exists a constant $M = M(m) \ge 1$ such that*

$$\|g \circ (\mathrm{Id} + f)\|_{m,\alpha} \le M\|g\|_{m,\alpha}(1 + \|f\|_{m,\alpha})^{m+\alpha}. \tag{2.4}$$

In particular, for every fixed $f \in C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$*, the linear mapping*

$$f^\star : C_b^{m,\alpha}(\mathbb{R}^d, G) \to C_b^{m,\alpha}(\mathbb{R}^d, G), \quad g \mapsto f^\star(g) := g \circ (\mathrm{Id} + f)$$

is continuous.

Continuity of the left translation is the content of the following theorem. We denote by $B^{m,\alpha}(f,\delta) := \{g \in C_b^{m,\alpha} : \|f - g\|_{m,\alpha} < \delta\}$ the open ball with radius δ centered at f. By $g_\star$ we mean the push-forward by g precomposed with translation by Id, i.e., $g_\star := g_* \circ (\mathrm{Id} + \cdot\,)$.

Theorem 2.8 ([3, 6.2])**.** *Let* $m \in \mathbb{N}_{\geq 1}$ *and* $\alpha, \beta \in (0,1]$*,* $\alpha < \beta$*. Let* $g \in C_b^{m,\beta}(\mathbb{R}^d, G)$ *where* G *is some Banach space. Then, for every* $f_0 \in C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$*,* $R > 0$*, and* $f_1, f_2 \in B^{m,\alpha}(f_0, R)$*,*

$$\|g_\star(f_1) - g_\star(f_2)\|_{m,\alpha} \leq M \|g\|_{m,\beta} \|f_1 - f_2\|_{m,\alpha}^{\beta-\alpha}, \tag{2.5}$$

where $M = M(m, \|f_0\|_{m,\alpha}, R)$*. In particular,*

$$g_\star : C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d) \to C_b^{m,\alpha}(\mathbb{R}^d, G), \quad f \mapsto g_\star(f) := g \circ (\mathrm{Id} + f)$$

is continuous.

It follows that composition is even jointly continuous.

Corollary 2.9. *Let* $m \in \mathbb{N}_{\geq 1}$*,* $0 < \alpha < \beta \leq 1$ *and* G *some Banach space. Then, for all* $f_0 \in C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$*,* $g_0 \in C_b^{m,\beta}(\mathbb{R}^d, G)$*,* $R > 0$*,* $f_1, f_2 \in B^{m,\alpha}(f_0, R)$*, and* $g_1, g_2 \in B^{m,\beta}(g_0, R)$*,*

$$\|g_1 \circ (\mathrm{Id} + f_1) - g_2 \circ (\mathrm{Id} + f_2)\|_{m,\alpha} \leq M\big(\|g_1 - g_2\|_{m,\alpha} + \|f_1 - f_2\|_{m,\alpha}\big)^{\beta-\alpha}, \tag{2.6}$$

where $M = M(m, \|f_0\|_{m,\alpha}, \|g_0\|_{m,\beta}, R)$*. In particular,*

$$\mathrm{comp} : C_b^{m,\beta}(\mathbb{R}^d, G) \times C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d) \to C_b^{m,\alpha}(\mathbb{R}^d, G), \quad (g, f) \mapsto g \circ (\mathrm{Id} + f)$$

is continuous.

We will also need the following result on C^1 left translations.

Theorem 2.10 ([3, 6.7])**.** *Let* $m \in \mathbb{N}_{\geq 1}$ *and* $\alpha, \beta \in (0,1]$*,* $\alpha < \beta$*. Let* $g \in C_b^{m+1,\beta}(\mathbb{R}^d, G)$ *where* G *is some Banach space. Then* $g_\star : C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d) \to C_b^{m,\alpha}(\mathbb{R}^d, G)$ *is continuously differentiable.*

Together with Lemma 2.3, Theorem 2.10 implies Lipschitz continuity of the left translation in the following cases; but see also Theorem 2.14 below.

Corollary 2.11. *Let* $m \in \mathbb{N}_{\geq 1}$ *and* $0 < \alpha < \beta \leq 1$*. Let* $g \in C_b^{m+1,\beta}(\mathbb{R}^d, G)$*. Then* $g_\star : C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d) \to C_b^{m,\alpha}(\mathbb{R}^d, G)$ *satisfies for all* $f_1, f_2 \in C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$*,*

$$\|g_\star(f_1) - g_\star(f_2)\|_{m,\alpha} \leq M \|g\|_{m+1,\beta} (1 + \max_{i=1,2} \|f_i\|_{m,\alpha})^{m+1} \|f_1 - f_2\|_{m,\alpha}.$$

Remark 2.12. Let us stress the fact that left translation ceases to be continuous, resp. differentiable, if in Theorem 2.8, resp. Theorem 2.10, g is merely of class $C_b^{m,\alpha}$, resp. $C_b^{m+1,\alpha}$; see [3] and also Lemma 3.5.

We shall make frequent use of the Faà di Bruno formula for Banach spaces: Let E, F, G be Banach spaces, let $f : E \supseteq U \to F$ and $g : F \supseteq V \to G$ be k times

Fréchet differentiable, and assume $f(U) \subseteq V$. Then $g \circ f : U \to G$ is k times Fréchet differentiable, and for all $x \in U$,

$$d^k(g \circ f)(x) = \operatorname{sym} \sum_{l=1}^{k} \sum_{\gamma \in \Gamma(l,k)} c_\gamma g^{(l)}(f(x)) \Big(f^{(\gamma_1)}(x), \ldots, f^{(\gamma_l)}(x) \Big), \tag{2.7}$$

where $\Gamma(l,k) := \{\gamma \in \mathbb{N}_{>0}^l : |\gamma| = k\}$, $c_\gamma := \frac{k!}{l!\gamma!}$, and sym denotes symmetrization of multilinear mappings.

Faà di Bruno's formula applied to a function $h : U \to H$ of the form $h(x) = b(f(x), g(x))$, where f, g are k times Fréchet differentiable functions defined on a common domain $U \subseteq E$ and $b : F \times G \to H$ is a continuous bilinear map, gives

$$d^k h(x) = \operatorname{sym} \sum_{l=0}^{k} \binom{k}{l} b(f^{(l)}(x), g^{(k-l)}(x)). \tag{2.8}$$

This formula is of particular use when $h(x) = dg(x)(f(x))$, where $f, g : \mathbb{R}^d \to \mathbb{R}^d$, i.e. the bilinear map takes the form $b : L(\mathbb{R}^d, \mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$, $(l, x) \mapsto l(x)$.

Remark 2.13. Faà di Bruno's formula (2.7) implies that for $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to G$ both in C_0^k, we have $g \circ (\mathrm{Id} + f) \in C_0^k(\mathbb{R}^d, G)$. So the stated regularity results for the composition hold as well for $C_b^{m,\alpha}$, etc., replaced by $C_0^{m,\alpha}$, etc.

2.5. **Convenient calculus.** Occasionally, we shall use some tools from *convenient calculus* which extends differential calculus beyond Banach spaces; the main reference is [15], see also [9] and the three appendices in [16]. Let us briefly describe the concepts and results we will need.

Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called C^∞ if all derivatives exist and are continuous. It can be shown that the set $C^\infty(\mathbb{R}, E)$ of C^∞-curves in E does not depend on the locally convex topology of E, only on its associated bornology.

The *c^∞-topology* on E is the final topology with respect to $C^\infty(\mathbb{R}, E)$; equivalently it is the final topology with respect to all Lipschitz curves or all Mackey-convergent sequences in E. In general the c^∞-topology is finer than the given locally convex topology, and it is not a vector space topology; for Fréchet spaces the topologies coincide.

A locally convex vector space E is said to be a *convenient vector space* if it is Mackey-complete; equivalently, a curve $c : \mathbb{R} \to E$ is C^∞ if and only if $\lambda \circ c$ is C^∞ for all continuous (equivalently bounded) linear functionals λ on E.

Let E, F, and G be convenient vector spaces, and let $U \subseteq E$ be c^∞-open. A mapping $f : U \to F$ is called C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, U)$. For mappings on Fréchet spaces this notion of smoothness coincides with all other reasonable definitions. Multilinear mappings are C^∞ if and only if they are bounded. The space $C^\infty(U, F)$ with the initial structure with respect to all mappings $f \mapsto \lambda \circ f \circ c$, $c \in C^\infty(\mathbb{R}, E)$ and $\lambda \in E^*$, is again convenient. The exponential law holds: For c^∞-open $V \subseteq F$,

$$C^\infty(U, C^\infty(V, G)) \cong C^\infty(U \times V, G)$$

is a linear diffeomorphism of convenient vector spaces. A linear mapping $f : E \to C^\infty(V, G)$ is C^∞ (bounded) if and only if $\mathrm{ev}_v \circ f : E \to G$ is C^∞ for all $v \in V$.

There are, however, C^∞-mappings which are not continuous with respect to the underlying locally convex topology; clearly they are continuous for the c^∞-topology.

Beside the class C^∞ due to [7], [12], [13], convenient calculus was developed for the holomorphic class [21], the real analytic class [14], and all reasonable ultradifferentiable classes [16], [17], [19], [26].

For the classes $C^{k,\alpha}$ ($k \in \mathbb{N}$, $\alpha \in (0,1]$) it was established in a weaker sense (without general exponential law) by [8] (for $\alpha = 1$) and by [6], [5]. Let E, F be convenient vector spaces, and let $U \subseteq E$ be c^∞-open. A curve $c : \mathbb{R} \to F$ is *locally α-Hölder continuous*, we write $c \in C^{0,\alpha}(\mathbb{R}, F)$, if for each bounded interval $I \subseteq \mathbb{R}$,

$$\Big\{ \frac{c(t) - c(s)}{|t-s|^\alpha} : t, s \in I,\, t \neq s \Big\}$$

is bounded in F. A curve $c : \mathbb{R} \to F$ is $C^{k,\alpha}$, i.e., $c \in C^{k,\alpha}(\mathbb{R}, F)$, if all derivatives up to order k exist and are locally α-Hölder continuous. A mapping $f : U \to F$ between convenient vector spaces is called $C^{k,\alpha}$, if $f \circ c \in C^{k,\alpha}(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, U)$. If E and F are Banach spaces, then f is $C^{0,\alpha}$ in this sense if and only if it is in the sense of Section 2.1, i.e., $\|f(x) - f(x)\|/\|x - y\|^\alpha$ is locally bounded; see [5], [15, 12.7], or [18, Lemma], and note that this is a special case of Lemma 3.9 below.

2.6. **An application of convenient calculus.** We finish this section with a result which is not contained in [3]: if $\alpha = \beta$ in Theorem 2.10, resp. Corollary 2.11, the left translation $g_\star$ is still locally Lipschitz. Of course, Remark 2.13 applies to this theorem as well. In contrast to Corollary 2.11, we do not get an explicit bound for the Lipschitz constant.

Theorem 2.14. *Let $m \in \mathbb{N}_{\ge 1}$ and $0 < \alpha \le 1$. Let $g \in C_b^{m+1,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$. Then $g_\star : C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d) \to C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$ is locally Lipschitz.*

Proof. It suffices to check that $g_\star$ maps C^∞-curves to $C^{0,1}$-curves; cf. Section 2.5. That $t \mapsto f(t, \cdot)$ is C^∞ in $C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$ means, by [9, 4.1.19], that, for all $k \in \mathbb{N}$, $\|\partial_t^k f(t, \cdot)\|_{m,\alpha}$ is locally bounded in t.

Let $h(t, x) := g(x + f(t, x))$. Then, if $F := \mathrm{Id} + f$,

$$h(t,x) - h(s,x) = \int_s^t \partial_\tau h(\tau, x)\, d\tau = \int_s^t dg(F(\tau, x)) \partial_\tau f(\tau, x)\, d\tau$$

and, by (2.8),

$$\begin{aligned} d_x^k h(t,x) - d_x^k h(s,x) &= \int_s^t d_x^k \big(dg(F(\tau,x))\partial_\tau f(\tau,x)\big)\, d\tau \\ &= \operatorname{sym} \sum_{j=0}^k \binom{k}{j} \int_s^t d_x^j \big(dg(F(\tau,x))\big) \partial_\tau d_x^{k-j} f(\tau,x)\, d\tau. \end{aligned}$$

With Faà di Bruno's formula (2.7),

$$d_x^j \big(dg(F(\tau,x))\big) = \operatorname{sym} \sum_{l=1}^j \sum_{\gamma \in \Gamma(l,j)} c_\gamma g^{(l+1)}(F(\tau,x)) \big(d_x^{\gamma_1} F(\tau,x), \dots, d_x^{\gamma_l} F(\tau,x), \mathbb{1}\big)$$

it is easy to see that $t \mapsto h(t, \cdot)$ is locally Lipschitz into $C_b^m(\mathbb{R}^d, \mathbb{R}^d)$.

It remains to prove that $t \mapsto h(t, \cdot)$ is locally Lipschitz into $C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$. To this end we have to show that

$$\frac{[h(t,\cdot) - h(s,\cdot)]_{m,\alpha}}{t - s}$$

is locally bounded, i.e., for each bounded interval I, the set

$$\Big\{\frac{d_x^m h(t,x) - d_x^m h(t,y) - d_x^m h(s,x) + d_x^m h(s,y)}{\|x-y\|^\alpha |t-s|} : x \neq y \in \mathbb{R}^d, s \neq t \in I\Big\}$$

must be bounded. Without loss of generality we can assume that $\|x-y\| \le 1$ and thus $\|x-y\| \le \|x-y\|^\alpha$; if $\|x-y\| \ge 1$ then the result follows from the fact that $t \mapsto h(t,\cdot)$ is locally Lipschitz into $C_b^m(\mathbb{R}^d,\mathbb{R}^d)$. Let us define

$$A^{\gamma,i} = A^{\gamma,i}(x,y) := \big(d_x^{\gamma_1} F(\tau,x), \dots, d_x^{\gamma_i} F(\tau,x), d_x^{\gamma_{i+1}} F(\tau,y), \dots, d_x^{\gamma_l} F(\tau,y)\big)$$

and

$$B^{\gamma,h} := \begin{cases} \big(A^{\gamma,l}, \partial_t d_x^{m-j} f(\tau,x)\big) & \text{if } h = l+1, \\ \big(A^{\gamma,h}, \partial_t d_x^{m-j} f(\tau,y)\big) & \text{if } h \le l. \end{cases}$$

Then

$$\begin{aligned} &g^{(l+1)}(F(\tau,x))\big(B^{\gamma,l+1}\big) - g^{(l+1)}(F(\tau,y))\big(B^{\gamma,0}\big) \\ &\quad = g^{(l+1)}(F(\tau,x))(B^{\gamma,l+1}) - g^{(l+1)}(F(\tau,y))(B^{\gamma,l+1}) \\ &\qquad + \sum_{h=1}^{l+1} g^{(l+1)}(F(\tau,y))(B^{\gamma,h}) - g^{(l+1)}(F(\tau,y))(B^{\gamma,h-1}). \end{aligned}$$

For the first summand

$$\begin{aligned} &\big\|g^{(l+1)}(F(\tau,x))(B^{\gamma,l+1}) - g^{(l+1)}(F(\tau,y))(B^{\gamma,l+1})\big\|_{L_m} \\ &\quad \le \big\|g^{(l+1)}(F(\tau,x)) - g^{(l+1)}(F(\tau,y))\big\|_{L_{l+1}} (1 + \|f(\tau,\cdot)\|_m)^m \|\partial_t f(\tau,\cdot)\|_{m-j} \\ &\quad \le \begin{cases} \|g\|_{m+1} [F(\tau,\cdot)]_1 \|x-y\| (1 + \|f(\tau,\cdot)\|_m)^m \|\partial_t f(\tau,\cdot)\|_m & \text{if } l < m, \\ \|g\|_{m+1,\alpha} [F(\tau,\cdot)]_1^\alpha \|x-y\|^\alpha (1 + \|f(\tau,\cdot)\|_m)^m \|\partial_t f(\tau,\cdot)\|_m & \text{if } l = m. \end{cases} \end{aligned}$$

For the other summands we observe that, by multilinearity,

$$g^{(l+1)}(F(\tau,y))(B^{\gamma,h}) - g^{(l+1)}(F(\tau,y))(B^{\gamma,h-1}) = g^{(l+1)}(F(\tau,y))\big(\sharp\big),$$

where

$$\sharp = \big(\dots, d_x^{\gamma_{h-1}} F(\tau,x), d_x^{\gamma_h} F(\tau,x) - d_x^{\gamma_h} F(\tau,y), d_x^{\gamma_{h+1}} F(\tau,y), \dots\big).$$

Hence, if $h \le l$,

$$\begin{aligned} &\big\|g^{(l+1)}(F(\tau,y))(B^{\gamma,h}) - g^{(l+1)}(F(\tau,y))(B^{\gamma,h-1})\big\|_{L_{m+1}} \\ &\quad \le \|g\|_{m+1} (1 + \|f(\tau,\cdot)\|_m)^{m-1} \|f(\tau,\cdot)\|_{m,\alpha} \|x-y\|^\alpha \|\partial_t f(\tau,\cdot)\|_m, \end{aligned}$$

and, if $h = l+1$,

$$\begin{aligned} &\big\|g^{(l+1)}(F(\tau,y))(B^{\gamma,h}) - g^{(l+1)}(F(\tau,y))(B^{\gamma,h-1})\big\|_{L_{m+1}} \\ &\quad \le \|g\|_{m+1} (1 + \|f(\tau,\cdot)\|_m)^m \|\partial_t f(\tau,\cdot)\|_{m,\alpha} \|x-y\|^\alpha. \end{aligned}$$

The theorem follows. □

3. Groups of Hölder diffeomorphisms

3.1. **The (non-topological) group $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$.** Let $n \in \mathbb{N}_{\geq 1}$ and $\beta \in (0,1]$. Let us define the set of orientation preserving diffeomorphisms of $\mathbb{R}^d$ which differ from the identity by a $C_0^{n,\beta}$-mapping:

$$\mathcal{D}^{n,\beta}(\mathbb{R}^d) := \big\{\Phi \in \mathrm{Id} + C_0^{n,\beta}(\mathbb{R}^d, \mathbb{R}^d) : \det d\Phi(x) > 0 \ \forall x \in \mathbb{R}^d\big\}. \tag{3.1}$$

We will show that $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ is a group (with respect to composition).

We endow $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ with the topology given by the metric

$$d(\Phi_1, \Phi_2) := \|\Phi_1 - \Phi_2\|_{n,\beta}$$

and denote by $B^{n,\beta}(\Phi, r)$ the open ball of radius r and center Φ in $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$. We use the same notation for balls in $C_0^{n,\beta}(\mathbb{R}^d, \mathbb{R}^d)$ which causes no problems since $\mathrm{Id} \notin C_0^{n,\beta}(\mathbb{R}^d, \mathbb{R}^d)$.

Since the determinant is multiplicative, it is an easy consequence of Theorem 2.7 that $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ is a monoid with respect to composition.

Lemma 3.1. *$\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ consists of C^n-diffeomorphisms of $\mathbb{R}^d$. The first n derivatives of the inverse of an element of $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ are again globally bounded.*

Proof. Let $\Phi = \mathrm{Id} + \phi \in \mathcal{D}^{n,\beta}(\mathbb{R}^d)$. First we have to make sure that Φ is bijective. This is an immediate consequence of [24, Cor. 4.3], which states that a C^1 mapping converging to infinity at infinity with non-vanishing jacobian determinant is already a C^1 diffeomorphism. The inverse mapping theorem shows that Φ^{-1} is actually C^n. Boundedness of the first n derivatives of $\Phi^{-1} - \mathrm{Id}$ follows as in [22, p. 535-536]. ☐

Lemma 3.2 ([22, p. 535])**.** *The operator norm of an invertible linear operator $A : \mathbb{R}^d \to \mathbb{R}^d$ satisfies $\|A^{-1}\| \leq |\det A|^{-1} \|A\|^{d-1}$.*

Lemma 3.3. *Let $\Phi_0 = \mathrm{Id} + \phi_0 \in \mathcal{D}^{n,\beta}(\mathbb{R}^d)$. Then:*

1. $\varepsilon := \inf_{x \in \mathbb{R}^d} \det d\Phi_0(x) > 0$.
2. *There is $\delta > 0$ such that $\inf_{x \in \mathbb{R}^d} \det d\Phi(x) \geq \varepsilon/2$ for all $\Phi \in \mathrm{Id} + B^{n,\beta}(\phi_0, \delta)$.*
3. *There are $\delta, C > 0$ such that $\sup_{x \in \mathbb{R}^d} \|d\Phi^{-1}(x)\| \leq C$ for all $\Phi \in B^{n,\beta}(\Phi_0, \delta)$.*

Proof. (1) Observe that $d\Phi_0(x) \to \mathbb{1}$ as $\|x\| \to \infty$. Thus $\det d\Phi_0(x) \to 1$ as $\|x\| \to \infty$, which implies $\varepsilon := \inf_{x \in \mathbb{R}^d} \det d\Phi_0(x) > 0$.

(2) This follows from the fact that the determinant is uniformly continuous on each ball in the space of $d \times d$ matrices.

(3) Let $\delta > 0$ be as in (2). Then, for all $\Phi \in B^{n,\beta}(\Phi_0, \delta)$,

$$\|d\Phi^{-1}(\Phi(x))\| = \|(d\Phi(x))^{-1}\| \leq \frac{\|d\Phi(x)\|^{d-1}}{|\det d\Phi(x)|} \leq \frac{2}{\epsilon}(\|\Phi_0\|_{n,\beta} + \delta)^{d-1},$$

by Lemma 3.2. Since Φ is bijective, the proof is complete. ☐

Lemma 3.3 shows that $\mathcal{D}^{n,\beta}(\mathbb{R}^d) - \mathrm{Id}$ is an open subset of $C_0^{n,\beta}(\mathbb{R}^d, \mathbb{R}^d)$. Thus, for $\Phi_0 = \mathrm{Id} + \phi_0 \in \mathcal{D}^{n,\beta}(\mathbb{R}^d)$ and for sufficiently small $r > 0$,

$$B^{n,\beta}(\Phi_0, r) = \mathrm{Id} + B^{n,\beta}(\phi_0, r).$$

We interpret $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ as a Banach manifold modelled on $C_0^{n,\beta}(\mathbb{R}^d, \mathbb{R}^d)$ with global chart $\Phi \mapsto \Phi - \mathrm{Id}$.

Theorem 3.4. *Let $n \in \mathbb{N}_{\geq 1}$ and $\beta \in (0,1]$. Then $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ is a group. In general, left translations are discontinuous.*

The theorem will follow from Lemma 3.5 and Proposition 3.6.

Lemma 3.5. *In general, left translations in $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ are discontinuous.*

Proof. The construction is taken from [3, 6.4]. We prove the claim in the case $d = 1$. Let $\chi \in C_c^\infty(\mathbb{R})$ be 1 on $[-1,1]$, and set $\psi(x) := x^n|x|^\beta\chi(x)$. Then $\psi \in C_0^{n,\beta}(\mathbb{R},\mathbb{R})$. In addition, let $\Phi_k(x) := x + \chi(x)/k$. Since $\mathcal{D}^{n,\beta}(\mathbb{R}) - \mathrm{Id}$ is open, we have $\Phi_k \in \mathcal{D}^{n,\beta}(\mathbb{R})$, for sufficiently large k, and $\Phi_k \to \mathrm{Id}$ in $\mathcal{D}^{n,\beta}(\mathbb{R})$ as $k \to \infty$. It is easy to see that, for $|x| < 1$,

$$\psi^{(n)}(x) = (n+\beta)\cdots(1+\beta)|x|^\beta =: C_{n,\beta}|x|^\beta. \tag{3.2}$$

Thus, for large k,

$$(\psi \circ \Phi_k)^{(n)}\Big(-\frac{1}{k}\Big) = C_{n,\beta}\Big|-\frac{1}{k}+\frac{1}{k}\Big|^\beta = 0,$$

and

$$(\psi \circ \Phi_k)^{(n)}(0) = \frac{C_{n,\beta}}{k^\beta}.$$

Hence

$$\big((\psi\circ\Phi_k)^{(n)} - (\psi\circ\mathrm{Id})^{(n)}\big)\Big(-\frac{1}{k}\Big) - \big((\psi\circ\Phi_k)^{(n)} - (\psi\circ\mathrm{Id})^{(n)}\big)(0) = -\frac{2C_{n,\beta}}{k^\beta},$$

which immediately gives $\|\psi\circ\Phi_k - \psi\circ\mathrm{Id}\,\|_{n,\beta} \geq 2C_{n,\beta}$. Since $\mathcal{D}^{n,\beta}(\mathbb{R}) - \mathrm{Id}$ is open, there is some small $r > 0$ such that $\mathrm{Id} + r\psi \in \mathcal{D}^{n,\beta}(\mathbb{R})$. □

The next proposition completes the proof of Theorem 3.4.

Proposition 3.6. *$\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ is closed under inversion. The chart representation*

$$\mathrm{inv}_c : (\mathcal{D}^{n,\beta}(\mathbb{R}^d) - \mathrm{Id}) \to (\mathcal{D}^{n,\beta}(\mathbb{R}^d) - \mathrm{Id}), \quad \phi \mapsto (\mathrm{Id}+\phi)^{-1} - \mathrm{Id}$$

is locally bounded.

Proof. For $\Phi = \mathrm{Id}+\phi \in \mathcal{D}^{n,\beta}(\mathbb{R}^d)$ and $\Phi^{-1} =: \mathrm{Id}+\tau$ we have $(\mathrm{Id}+\tau)\circ(\mathrm{Id}+\phi) = \mathrm{Id}$, i.e.,

$$\tau(x+\phi(x)) = -\phi(x), \quad x \in \mathbb{R}^d. \tag{3.3}$$

It follows that $\det d\Phi^{-1}(x) > 0$ for all x and that $\tau \circ (\mathrm{Id}+\phi) \in C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d)$. By Lemma 3.1, Φ^{-1} is n-times differentiable with globally bounded derivatives.

Let $\Phi_0 = \mathrm{Id}+\phi_0 \in \mathcal{D}^{n,\beta}(\mathbb{R}^d)$ and $\Phi_0^{-1} =: \mathrm{Id}+\tau_0$. Choose $\delta > 0$ such that $B^{n,\beta}(\phi_0,\delta) \subseteq (\mathcal{D}^{n,\beta}(\mathbb{R}^d) - \mathrm{Id})$ (recall that $\mathcal{D}^{n,\beta}(\mathbb{R}^d) - \mathrm{Id}$ is open) and such that the conclusion of Lemma 3.3 holds.

Claim 1. *$\mathrm{inv}_c(B^{n,\beta}(\phi_0,\delta))$ is bounded in $C_0^n(\mathbb{R}^d,\mathbb{R}^d)$.*

By Lemma 3.1, we know that inv_c maps into $C_b^n(\mathbb{R}^d,\mathbb{R}^d)$. An inspection of Faà di Bruno's formula (2.7) shows that it actually maps into $C_0^n(\mathbb{R}^d,\mathbb{R}^d)$. Let $\phi \in B^{n,\beta}(\phi_0,\delta)$ and $\tau = \mathrm{inv}_c(\phi) \in C_0^n(\mathbb{R}^d,\mathbb{R}^d)$ so that (3.3) implies

$$\|\tau(x+\phi(x))\| \leq \|\phi(x)\| \leq \|\phi-\phi_0\|_{n,\beta} + \|\phi_0\|_{n,\beta} \leq \delta + \|\phi_0\|_{n,\beta}$$

for all x. Since $\mathrm{Id}+\phi$ is bijective, this gives

$$\|\,\mathrm{inv}_c(\phi)\|_0 = \|\tau\|_0 \leq \delta + \|\phi_0\|_{n,\beta}, \quad \phi \in B^{n,\beta}(\phi_0,\delta).$$

We prove by induction on k that for all $k \le n$ there are constants $D_k = D_k(\phi_0, \delta)$ such that

$$\|\tau\|_k = \|\operatorname{inv}_c(\phi)\|_k \le D_k, \quad \phi \in B^{n,\beta}(\phi_0, \delta). \tag{3.4}$$

By Faà di Bruno's formula (2.7),

$$\begin{aligned} d^k(\tau \circ \Phi)(x) &= \tau^{(k)}(\Phi(x))(d\Phi(x), \dots, d\Phi(x)) \\ &\quad + \operatorname{sym} \sum_{l=1}^{k-1} \sum_{\gamma \in \Gamma(l,k)} c_\gamma \tau^{(l)}(\Phi(x))\big(\Phi^{(\gamma_1)}(x), \dots, \Phi^{(\gamma_l)}(x)\big). \end{aligned} \tag{3.5}$$

By the induction hypothesis, for $\phi \in B^{n,\beta}(\phi_0, \delta)$ and $l \le k-1$,

$$\begin{aligned} &\big\|\tau^{(l)}(\Phi(x))\big(\Phi^{(\gamma_1)}(x), \dots, \Phi^{(\gamma_l)}(x)\big)\big\|_{L_k} \\ &\le \|\tau^{(l)}(\Phi(x))\|_{L_l} \|\Phi^{(\gamma_1)}(x)\|_{L_{\gamma_1}} \cdots \|\Phi^{(\gamma_l)}(x)\|_{L_{\gamma_l}} \\ &\le D_{k-1}(1 + \|\phi\|_{n,\beta})^k \\ &\le D_{k-1}(1 + \delta + \|\phi_0\|_{n,\beta})^k. \end{aligned}$$

In addition, $\|d^k(\tau \circ \Phi)(x)\|_{L_k} \le \|\phi\|_{n,\beta} \le \delta + \|\phi_0\|_{n,\beta}$, by (3.3). It follows that there is some constant $D_k = D_k(\phi_0, \delta)$ such that

$$\|\tau^{(k)}(\Phi(x))(d\Phi(x), \dots, d\Phi(x))\|_{L_k} \le D_k, \quad \phi \in B^{n,\beta}(\phi_0, \delta),\ x \in \mathbb{R}^d. \tag{3.6}$$

Since

$$\|\tau^{(k)}(\Phi(x))\|_{L_k} \le \|\tau^{(k)}(\Phi(x))(d\Phi(x), \dots, d\Phi(x))\|_{L_k} \|(d\Phi(x))^{-1}\|_{L_1}^k,$$

(3.6) and Lemma 3.3 imply (3.4) and hence Claim 1.

Claim 2. *$\operatorname{inv}_c(B^{n,\beta}(\phi_0, \delta))$ is bounded in $C_0^{n,\beta}(\mathbb{R}^d, \mathbb{R}^d)$.*

Observe that, since Φ is a bijection of $\mathbb{R}^d$,

$$\begin{aligned} [\tau]_{n,\beta} &= \sup_{x \ne y} \frac{\|d^n\tau(\Phi(x)) - d^n\tau(\Phi(y))\|_{L_n}}{\|x-y\|^\beta} \frac{\|x-y\|^\beta}{\|\Phi(x) - \Phi(y)\|^\beta} \\ &\le \sup_{x \ne y} \frac{\|d^n\tau(\Phi(x)) - d^n\tau(\Phi(y))\|_{L_n}}{\|x-y\|^\beta} \Big(\sup_{x \ne y} \frac{\|x-y\|}{\|\Phi(x) - \Phi(y)\|} \Big)^\beta \\ &= \sup_{x \ne y} \frac{\|d^n\tau(\Phi(x)) - d^n\tau(\Phi(y))\|_{L_n}}{\|x-y\|^\beta} \operatorname{Lip}(\Phi^{-1})^\beta \\ &\le \sup_{x \ne y} \frac{\|d^n\tau(\Phi(x)) - d^n\tau(\Phi(y))\|_{L_n}}{\|x-y\|^\beta} (1 + D_1)^\beta, \end{aligned}$$

for all $\phi \in B^{n,\beta}(\phi_0, \delta)$, by (3.4). For $k \le n$, let

$$A^k = A^k(x,y) := (\underbrace{d\Phi(x), \dots, d\Phi(x)}_{k\text{-times}}, \underbrace{d\Phi(y), \dots, d\Phi(y)}_{(n-k)\text{-times}}).$$

Then

$$\begin{aligned} &\|d^n\tau(\Phi(y))(A^k) - d^n\tau(\Phi(y))(A^{k-1})\|_{L_n} \\ &\le \|d^n\tau(\Phi(y))\|_{L_n} \|d\Phi(x)\|_{L_1}^{k-1} \|d\Phi(y)\|_{L_1}^{n-k} \|d\phi(x) - d\phi(y)\|_{L_1} \\ &\le \|\tau\|_n (1 + \|\phi\|_1)^{n-1} 2 \|\phi\|_{n,\beta} \|x-y\|^\beta \\ &\le 2D_n(1 + \delta + \|\phi_0\|_{n,\beta})^n \|x-y\|^\beta, \end{aligned} \tag{3.7}$$

where we use $\|d\phi(x) - d\phi(y)\|_{L_1} \le \|\phi\|_{1,\beta}\|x-y\|^\beta$ for the case $n = 1$ (which holds by definition). For the case $n \ge 2$ we use the mean value inequality. (If $\|x-y\| \le 1$ then $\|x-y\| \le \|x-y\|^\beta$, otherwise $\|d\phi(x)-d\phi(y)\|_{L_1} \le 2\|\phi\|_{n,\beta} \le 2\|\phi\|_{n,\beta}\|x-y\|$.) By Lemma 3.3,

$$\begin{aligned}\|d^n\tau(\Phi(x)) - d^n\tau(\Phi(y))\|_{L_n} &\le \|d^n\tau(\Phi(x))(A^n) - d^n\tau(\Phi(y))(A^n)\|_{L_n}\|d\Phi(x)^{-1}\|_{L_1}^n \\ &\le C\|d^n\tau(\Phi(x))(A^n) - d^n\tau(\Phi(y))(A^0)\|_{L_n} \\ &\quad + C\sum_{k=1}^n \|d^n\tau(\Phi(y))(A^k) - d^n\tau(\Phi(y))(A^{k-1})\|_{L_n}.\end{aligned}$$

We may use (3.7) to estimate the second term on the right-hand side. Thus, to end the proof of Claim 2, and hence of the proposition, it remains to show the following.

Claim 3. *There exists a constant C such that*

$$\|d^n\tau(\Phi(x))(A^n) - d^n\tau(\Phi(y))(A^0)\|_{L_n} \le C\|x-y\|^\beta,$$

for all $\phi \in B^{n,\beta}(\phi_0,\delta)$ and all $x, y \in \mathbb{R}^d$.

For any $\gamma \in \mathbb{N}^l_{>0}$ and $0 \le j \le l$ let

$$A^{\gamma,j} = A^{\gamma,j}(x,y) := \big(\Phi^{(\gamma_1)}(x), \dots, \Phi^{(\gamma_j)}(x), \Phi^{(\gamma_{j+1})}(y), \dots, \Phi^{(\gamma_l)}(y)\big).$$

Then, by Faà di Bruno's formula (2.7),

$$\begin{aligned} d^n(\tau\circ\Phi)(x) - d^n(\tau\circ\Phi)(y) &= \tau^{(n)}(\Phi(x))(A^n) - \tau^{(n)}(\Phi(y))(A^0) \\ &\quad + \operatorname{sym}\sum_{l=1}^{n-1}\sum_{\gamma\in\Gamma(l,n)} c_\gamma\big(\tau^{(l)}(\Phi(x))(A^{\gamma,l}) - \tau^{(l)}(\Phi(y))(A^{\gamma,0})\big). \end{aligned} \tag{3.8}$$

By (3.3), there is a constant C such that

$$\|d^n(\tau\circ\Phi)(x) - d^n(\tau\circ\Phi)(y)\|_{L_n} \le C\|x-y\|^\beta, \quad \phi \in B^{n,\beta}(\phi_0,\delta). \tag{3.9}$$

Moreover,

$$\begin{aligned} &\big\|\tau^{(l)}(\Phi(x))(A^{\gamma,l}) - \tau^{(l)}(\Phi(y))(A^{\gamma,0})\big\|_{L_n} \\ &\quad\le \big\|\tau^{(l)}(\Phi(x))(A^{\gamma,l}) - \tau^{(l)}(\Phi(y))(A^{\gamma,l})\big\|_{L_n} \\ &\qquad + \sum_{k=1}^l \big\|\tau^{(l)}(\Phi(y))(A^{\gamma,k}) - \tau^{(l)}(\Phi(y))(A^{\gamma,k-1})\big\|_{L_n}. \end{aligned}$$

For the first summand, since $l < n$,

$$\begin{aligned} &\big\|\tau^{(l)}(\Phi(x))(A^{\gamma,l}) - \tau^{(l)}(\Phi(y))(A^{\gamma,l})\big\|_{L_n} \\ &\quad\le \big\|\tau^{(l)}(\Phi(x)) - \tau^{(l)}(\Phi(y))\big\|_{L_l}(1+\|\phi\|_{n,\beta})^n \\ &\quad\le \|\tau\|_n(1+\|\phi\|_1)\|x-y\|(1+\|\phi\|_{n,\beta})^n, \end{aligned}$$

and $\|\tau\|_n \le D_n$ for $\phi \in B^{n,\beta}(\phi_0,\delta)$, by Claim 1. For the other summands observe that

$$\begin{aligned} &\tau^{(l)}(\Phi(y))(A^{\gamma,k}) - \tau^{(l)}(\Phi(y))(A^{\gamma,k-1}) \\ &\quad= \tau^{(l)}(\Phi(y))\big(\dots, \Phi^{(\gamma_{k-1})}(x), (\Phi^{(\gamma_k)}(x) - \Phi^{(\gamma_k)}(y)), \Phi^{(\gamma_{k+1})}(y), \dots\big), \end{aligned}$$

whence

$$\begin{aligned}&\left\|\tau^{(l)}(\Phi(y))(A^{\gamma,k})-\tau^{(l)}(\Phi(y))(A^{\gamma,k-1})\right\|_{L_n}\\&\quad\leq\begin{cases}\|\tau\|_n(1+\|\phi\|_n)^{n-1}\|\phi\|_n\|x-y\| & \text{if } l>1,\\ \|\tau\|_1\|\phi\|_{n,\beta}\|x-y\|^\beta & \text{if } l=1.\end{cases}\end{aligned}$$

Altogether this means that we find a constant K such that for all $\phi\in B^{n,\beta}(\phi_0,\delta)$ and all $x,y\in\mathbb{R}^d$

$$\left\|\tau^{(l)}(\Phi(x))(A^{\gamma,l})-\tau^{(l)}(\Phi(y))(A^{\gamma,0})\right\|_{L_n}\leq K\|x-y\|^\beta;$$

indeed, if $\|x-y\|\leq 1$ then $\|x-y\|\leq\|x-y\|^\beta$, otherwise the estimate follows from the triangle inequality and Claim 1. Together with (3.8) and (3.9) this implies Claim 3. □

For later use we prove the following.

Proposition 3.7. *Let $n\in\mathbb{N}_{\geq 1}$ and $0<\alpha<\beta\leq 1$. Then, for all $\phi_0\in(\mathcal{D}^{n,\beta}(\mathbb{R}^d)-\mathrm{Id})$ there exists $\delta>0$ such that for all $\phi_1,\phi_2\in B^{n,\beta}(\phi_0,\delta)\subseteq(\mathcal{D}^{n,\beta}(\mathbb{R}^d)-\mathrm{Id})$,*

$$\|\operatorname{inv}_c(\phi_1)-\operatorname{inv}_c(\phi_2)\|_{n,\alpha}\leq M\|\phi_1-\phi_2\|_{n,\alpha}^{\beta-\alpha}, \tag{3.10}$$

where $M=M(n,\phi_0,\delta)$. In particular,

$$\operatorname{inv}_c:(\mathcal{D}^{n,\beta}(\mathbb{R}^d)-\mathrm{Id})\to(\mathcal{D}^{n,\alpha}(\mathbb{R}^d)-\mathrm{Id}),\quad \phi\mapsto(\mathrm{Id}+\phi)^{-1}-\mathrm{Id}$$

is continuous.

Proof. Choose $\delta>0$ such that $B^{n,\beta}(\phi_0,\delta)\subseteq(\mathcal{D}^{n,\beta}(\mathbb{R}^d)-\mathrm{Id})$. Let $\phi_1,\phi_2\in B^{n,\beta}(\phi_0,\delta)$. We write $\tau_i=\operatorname{inv}_c(\phi_i)$ and $\Phi_i=\mathrm{Id}+\phi_i$, for $i=0,1,2$. Then

$$\begin{aligned}\tau_1-\tau_2&=\Phi_1^{-1}\circ\Phi_2\circ\Phi_2^{-1}-\Phi_1^{-1}\circ\Phi_1\circ\Phi_2^{-1}\\&=\phi_2\circ(\mathrm{Id}+\tau_2)-\phi_1\circ(\mathrm{Id}+\tau_2)\\&\quad+\tau_1\circ(\mathrm{Id}+\phi_2)\circ(\mathrm{Id}+\tau_2)-\tau_1\circ(\mathrm{Id}+\phi_1)\circ(\mathrm{Id}+\tau_2).\end{aligned}$$

By Theorem 2.7,

$$\|\phi_2\circ(\mathrm{Id}+\tau_2)-\phi_1\circ(\mathrm{Id}+\tau_2)\|_{n,\alpha}\leq M(n)\|\phi_1-\phi_2\|_{n,\alpha}(1+\|\tau_2\|_{n,\alpha})^{n+1},$$

and by Theorem 2.7, Theorem 2.8 (and Lemma 2.2),

$$\begin{aligned}&\|\tau_1\circ(\mathrm{Id}+\phi_2)\circ(\mathrm{Id}+\tau_2)-\tau_1\circ(\mathrm{Id}+\phi_1)\circ(\mathrm{Id}+\tau_2)\|_{n,\alpha}\\&\quad\leq M(n)\|\tau_1\circ(\mathrm{Id}+\phi_2)-\tau_1\circ(\mathrm{Id}+\phi_1)\|_{n,\alpha}(1+\|\tau_2\|_{n,\alpha})^{n+1}\\&\quad\leq M(n,\|\phi_0\|_{n,\alpha},\delta)\|\tau_1\|_{n,\beta}\|\phi_1-\phi_2\|_{n,\alpha}^{\beta-\alpha}(1+\|\tau_2\|_{n,\alpha})^{n+1}.\end{aligned}$$

Since $\|\tau_i\|_{n,\beta}$ is uniformly bounded for $\phi_i\in B^{n,\beta}(\phi_0,\delta)$ if $\delta>0$ is chosen sufficiently small, by Proposition 3.6, this implies the assertion. □

3.2. **Intermediate Hölder spaces.** As we have already seen in Lemma 3.5, the group $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ is not topological (with respect to the topology given as a Banach manifold modelled on $C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d)$). Nevertheless we know that the left translations become continuous if the outer mapping is only slightly more regular than the space it acts on. This observation motivates the following definitions.

Let E, F be Banach spaces, $U\subseteq E$ open, and $n\in\mathbb{N}$. For $\beta\in(0,1]$ define

$$C_b^{n,\beta-}(U,F):=\bigcap_{\alpha\in(0,\beta)}C_b^{n,\alpha}(U,F),$$

and for $\beta \in [0,1)$,

$$C_b^{n,\beta+}(U,F) := \bigcup_{\alpha\in(\beta,1)} C_b^{n,\alpha}(U,F).$$

If $\beta \in (0,1)$ we have the strict inclusions

$$C_b^{n,\beta+}(U,F) \subsetneq C_b^{n,\beta}(U,F) \subsetneq C_b^{n,\beta-}(U,F).$$

We endow $C_b^{n,\beta-}(U,F)$ and $C_b^{n,\beta+}(U,F)$ with their natural projective and inductive locally convex limit topologies, respectively.

Then $C_b^{n,\beta-}(U,F)$ is a Fréchet space with a generating system of seminorms $\mathcal{P} = \{\|\cdot\|_{n,\alpha} : \alpha \in (0,\beta)\}$, or a countable subfamily thereof, like $\{\|\cdot\|_{n,\beta-1/k} : k \ge k_0\}$. The balls $B_\alpha^{n,\beta-}(f_0,\varepsilon) := \{f \in C_b^{n,\beta-}(U,F) : \|f-f_0\|_{n,\alpha} < \varepsilon\}$ satisfy

$$B_{\alpha_2}^{n,\beta-}(f_0,\varepsilon) \subseteq B_{\alpha_1}^{n,\beta-}(f_0,2\varepsilon) \quad \text{ if } \alpha_1 < \alpha_2,$$

by Lemma 2.2. Thus $\{B_\alpha^{n,\beta-}(f_0,\varepsilon) : \alpha < \beta,\, \varepsilon > 0\}$ forms a neighborhood base of $f_0 \in C_b^{n,\beta-}(U,F)$.

In analogy we define $C_0^{n,\beta\pm}$ and $C^{n,\beta\pm}$.

Lemma 3.8. *$C_b^{n,\beta+}(U,F)$ and $C_0^{n,\beta+}(E,F)$ are compactly regular (LB)-spaces.*

Proof. It suffices, by [23, Satz 1], to verify condition (M) of [25]: There exists a sequence of increasing 0-neighborhoods $B_p \subseteq C_b^{n,\beta+1/p}(U,F)$ such that for each p there exists an $m \ge p$ for which the topologies of $C_b^{n,\beta+1/k}(U,F)$ and of $C_b^{n,\beta+1/m}(U,F)$ coincide on B_p for all $k \ge m$.

For $\alpha \le \alpha'$ we have $\|f\|_{n,\alpha} \le 2\|f\|_{n,\alpha'}$, by Lemma 2.2. It suffices to show that for $\beta < \alpha_2 < \alpha_1 < \alpha$, $\epsilon > 0$, and $f \in B^{n,\alpha}(0,1)$ there exists $\delta > 0$ such that $B^{n,\alpha_2}(f,\delta) \cap B^{n,\alpha}(0,1) \subseteq B^{n,\alpha_1}(f,\epsilon)$.

Let $g \in B^{n,\alpha_2}(f,\delta) \cap B^{n,\alpha}(0,1)$. Then $\|g-f\|_{n,\alpha_2} < \delta$ and $\|g\|_{n,\alpha} < 1$. By Lemma 2.1,

$$\|g-f\|_{n,\alpha_1} \le \|g-f\|_{n,\alpha_2}^{\frac{\alpha-\alpha_1}{\alpha-\alpha_2}} \|g-f\|_{n,\alpha}^{\frac{\alpha_1-\alpha_2}{\alpha-\alpha_2}} < \delta^{\frac{\alpha-\alpha_1}{\alpha-\alpha_2}} 2^{\frac{\alpha_1-\alpha_2}{\alpha-\alpha_2}}.$$

So it is clear that we may find δ as required. □

Consequently, $C_b^{n,\beta+}(U,F)$ and $C_0^{n,\beta+}(U,F)$ are complete (thus convenient), webbed, and ultra-bornological.

3.3. **$C^{0,\omega}$-mappings between convenient vector spaces.** Let $\omega : [0,\infty) \to [0,\infty)$ be a subadditive increasing modulus of continuity ($\lim_{t\to 0}\omega(t) = \omega(0) = 0$). By a $C^{0,\omega}$-curve c we mean a function defined on the real line with values in a convenient vector space F such that for each bounded interval $I \subseteq \mathbb{R}$,

$$\Big\{\frac{c(t)-c(s)}{\omega(|t-s|)} : t,s \in I,\, t \ne s\Big\}$$

is bounded in F. We say that a mapping between convenient vector spaces is $C^{0,\omega}$, if it maps C^∞-curves to $C^{0,\omega}$-curves. The c^∞-topology coincides with the final topology of all $C^{0,\omega}$-curves (which follows from the proof of [15, 2.13]), and so a $C^{0,\omega}$-mapping is continuous with respect to the c^∞-topology. The following lemma shows that between Banach spaces the notion of $C^{0,\omega}$-mapping coincides with the usual definition.

Lemma 3.9. *Let E, F be Banach spaces, $U \subseteq E$ open. A mapping $f : U \to F$ is $C^{0,\omega}$ if and only if $f(x) - f(y)/\omega(\|x-y\|)$ is locally bounded.*

Proof. Suppose that there is $z \in U$ and $x_n \ne y_n \in U$ such that $\|x_n - z\| \le 4^{-n}$, $\|y_n - z\| \le 4^{-n}$, and $\|f(x_n) - f(y_n)\| \ge n 2^n \omega(\|x_n - y_n\|)$. By [15, 12.2], there is a C^∞-curve c and a convergent sequence of real numbers t_n such that $c(t+t_n) = x_n + t\frac{(y_n - x_n)}{2^n\|x_n - y_n\|}$ for all $0 \le t \le s_n := 2^n\|x_n - y_n\|$. Then, by subadditivity of ω,

$$\frac{\|(f\circ c)(t_n+s_n) - (f\circ c)(t_n)\|}{\omega(s_n)} = \frac{\|f(x_n) - f(y_n)\|}{\omega(2^n\|x_n - y_n\|)} \ge n.$$

The converse implication follows from subadditivity and monotonicity of ω, since C^∞-curves are locally Lipschitz. □

This lemma can be found in [5], [15, 12.7], or [18, Lemma] in the Hölder (or Lipschitz) case $\omega(t) = t^\gamma$.

Definition 3.10. We say that ω is a *slowly vanishing* modulus of continuity if ω is increasing, subadditive, and satisfies

$$\liminf_{t\downarrow 0} \frac{\omega(t)}{t^\gamma} > 0 \quad \text{for all } \gamma > 0.$$

For instance, ω defined by $\omega(t) := -(\log t)^{-1}$, if $0 < t < e^{-2}$, $\omega(t) := 1/2$, if $t \ge e^{-2}$, and $\omega(0) := 0$, is a slowly vanishing modulus of continuity.

3.4. **The $C^{0,\omega}$ Lie groups $\mathcal{D}^{n,\beta-}(\mathbb{R}^d)$ and $\mathcal{D}^{n,\beta+}(\mathbb{R}^d)$.** Let $n \in \mathbb{N}_{\ge 1}$. We define

$$\mathcal{D}^{n,\beta\pm}(\mathbb{R}^d) := \big\{\Phi \in \mathrm{Id} + C_0^{n,\beta\pm}(\mathbb{R}^d,\mathbb{R}^d) : \det d\Phi(x) > 0 \ \forall x \in \mathbb{R}^d\big\},$$

where $\beta \in (0,1]$ if $\pm = -$ and $\beta \in [0,1)$ if $\pm = +$. Then $\mathcal{D}^{n,\beta\pm}(\mathbb{R}^d) - \mathrm{Id}$ is an open subset of $C_0^{n,\beta\pm}(\mathbb{R}^d,\mathbb{R}^d)$. We take this interpretation as defining property for the topology, i.e., $V \subseteq \mathcal{D}^{n,\beta\pm}(\mathbb{R}^d)$ is open if and only if $(V - \mathrm{Id})$ is open in $(\mathcal{D}^{n,\beta\pm}(\mathbb{R}^d) - \mathrm{Id}) \subseteq C_0^{n,\beta\pm}(\mathbb{R}^d,\mathbb{R}^d)$.

Clearly, $\mathcal{D}^{n,\beta\pm}(\mathbb{R}^d)$ forms a group, by Theorem 3.4. We will now prove that $\mathcal{D}^{n,\beta\pm}(\mathbb{R}^d)$ are $C^{0,\omega}$ Lie groups for any slowly vanishing modulus of continuity ω.

Theorem 3.11. *Let $n \in \mathbb{N}_{\ge 1}$. Let ω be a slowly vanishing modulus of continuity. Then $\mathcal{D}^{n,\beta-}(\mathbb{R}^d)$, for $\beta \in (0,1]$, and $\mathcal{D}^{n,\beta+}(\mathbb{R}^d)$, for $\beta \in [0,1)$, are $C^{0,\omega}$ Lie groups. In particular, $\mathcal{D}^{n,\beta-}(\mathbb{R}^d)$, for $\beta \in (0,1]$, is a topological group (with respect to its natural Fréchet topology).*

Proof. Let us first consider $\mathcal{D}^{n,\beta-}(\mathbb{R}^d)$, for $\beta \in (0,1]$. Let $g, f \in C^\infty(\mathbb{R}, C_0^{n,\beta-}(\mathbb{R}^d,\mathbb{R}^d))$ and let $I \subseteq \mathbb{R}$ be a compact interval. Then the sets $g(I)$, $f(I)$ are bounded in $C_0^{n,\beta-}(\mathbb{R}^d,\mathbb{R}^d)$ and thus in every $C_0^{n,\alpha}(\mathbb{R}^d,\mathbb{R}^d)$ for $\alpha < \beta$. If $\alpha < \tilde\alpha < \beta$, then, by (2.6),

$$\begin{aligned}&\|g(t)\circ(\mathrm{Id}+f(t)) - g(s)\circ(\mathrm{Id}+f(s))\|_{n,\alpha}\\ &\qquad \le M\big(\|g(t)-g(s)\|_{n,\alpha} + \|f(t)-f(s)\|_{n,\alpha}\big)^{\tilde\alpha-\alpha} \le \tilde M |t-s|^{\tilde\alpha-\alpha},\end{aligned}$$

for $t, s \in I$. There is $\epsilon > 0$ and $C = C(\alpha,\tilde\alpha)$ such that $|t-s|^{\tilde\alpha-\alpha} \le C\omega(|t-s|)$ if $|t-s| \le \epsilon$. Since ω is increasing, we may conclude that, for $t \mapsto h(t) := g(t)\circ(\mathrm{Id}+f(t))$,

$$\Big\{\frac{h(t)-h(s)}{\omega(|t-s|)} : s \ne t \in I\Big\} \tag{3.11}$$

is bounded in $C_0^{n,\alpha}(\mathbb{R}^d,\mathbb{R}^d)$. So the composition is $C^{0,\omega}$ on $\mathcal{D}^{n,\beta-}(\mathbb{R}^d)$.

Let us turn to the inversion in $\mathcal{D}^{n,\beta-}(\mathbb{R}^d)$. Let $f \in C^\infty(\mathbb{R}, C_0^{n,\beta-}(\mathbb{R}^d,\mathbb{R}^d))$. Fix $\alpha < \tilde\alpha < \beta$ and $t_0 \in \mathbb{R}$. Let $\delta > 0$ be such that $B^{n,\tilde\alpha}(f(t_0),\delta) \subseteq (\mathcal{D}^{n,\tilde\alpha}(\mathbb{R}^d) - \mathrm{Id})$. There is a neighborhood I of t_0 such that $f(I) \subseteq B^{n,\tilde\alpha}(f(t_0),\delta)$. By Proposition 3.7 (after possibly shrinking δ), for all $t, s \in I$,

$$\|\operatorname{inv}_c(f(t)) - \operatorname{inv}_c(f(s))\|_{n,\alpha} \le M \|f(t) - f(s)\|_{n,\alpha}^{\tilde\alpha-\alpha},$$

where $M = M(n, f(t_0), \delta)$. Finishing the arguments in the same way as for the composition, we conclude that the inversion is $C^{0,\omega}$ on $\mathcal{D}^{n,\beta-}(\mathbb{R}^d)$.

This implies that $\mathcal{D}^{n,\beta-}(\mathbb{R}^d)$ is a topological group, since the underlying Fréchet topology and the c^∞-topology coincide. Of course, it also follows directly from Corollary 2.9 and Proposition 3.7.

Now let us consider $\mathcal{D}^{n,\beta+}(\mathbb{R}^d)$, for $\beta \in [0,1)$. Let $g, f \in C^\infty(\mathbb{R}, C_0^{n,\beta+}(\mathbb{R}^d,\mathbb{R}^d))$. For any compact interval $I \subseteq \mathbb{R}$, the images $g(I)$, $f(I)$ are bounded in $C_0^{n,\beta+}(\mathbb{R}^d,\mathbb{R}^d)$. Since $C_0^{n,\beta+}(\mathbb{R}^d,\mathbb{R}^d)$ is a compactly regular (LB)-space, there is some $\alpha_0 > \beta$ such that $g(I)$, $f(I)$ are bounded in $C_0^{n,\alpha_0}(\mathbb{R}^d,\mathbb{R}^d)$, and thus also in every $C_0^{n,\alpha}(\mathbb{R}^d,\mathbb{R}^d)$, for $\alpha \in (\beta,\alpha_0]$. Let $\alpha,\tilde\alpha \in (\beta,\alpha_0]$ with $\alpha < \tilde\alpha$. Then the arguments above show that the set (3.11) is bounded in $C_0^{n,\alpha}(\mathbb{R}^d,\mathbb{R}^d)$, and thus in $C_0^{n,\beta+}(\mathbb{R}^d,\mathbb{R}^d)$. So the composition is $C^{0,\omega}$ on $\mathcal{D}^{n,\beta+}(\mathbb{R}^d)$. Similarly for the inversion. □

Remark 3.12. We do not know whether $\mathcal{D}^{n,\beta+}(\mathbb{R}^d)$ is a topological group with respect to its natural inductive locally convex topology, since the c^∞-topology is finer in this case.

Groups with continuous left translations and smooth right translations were dubbed half-Lie groups in [20]. The chart representations of the right translations in $\mathcal{D}^{n,\beta\pm}(\mathbb{R}^d)$ are affine and bounded, by Theorem 2.7, and thus smooth. Hence, $\mathcal{D}^{n,\beta-}(\mathbb{R}^d)$ is a half-Lie group.

The next result shows that the $C^{0,\omega}$-regularity of the group operations in $D^{n,\beta\pm}(\mathbb{R}^d)$ is optimal.

Proposition 3.13. *Let $n \in \mathbb{N}_{\ge 1}$.*

1. *For all $\beta \in (0,1]$, $\mathcal{D}^{n,\beta-}(\mathbb{R}^d)$ is a half-Lie group. There are left translations in $\mathcal{D}^{n,\beta-}(\mathbb{R}^d)$ which are not locally Hölder continuous of any order $\gamma > 0$.*
2. *Let $\beta \in [0,1)$. For any $\gamma > 0$, there are left translations in $\mathcal{D}^{n,\beta+}(\mathbb{R}^d)$ which are not locally Hölder continuous of order γ.*

Proof. (1) Let $\chi \in C_c^\infty(\mathbb{R})$ be 1 on $[-1,1]$ and satisfy $\chi'(x) > -1$ for all $x \in \mathbb{R}$, and set $\psi(x) := x^n|x|^\beta\chi(x) \in C_0^{n,\beta}(\mathbb{R},\mathbb{R}) \subseteq C_0^{n,\beta-}(\mathbb{R},\mathbb{R})$. We will show that $\theta(t) := \psi \circ (\mathrm{Id} + t\chi)$, for small $t \in \mathbb{R}$, is not locally Hölder continuous of order γ into $C_0^{n,\alpha}(\mathbb{R},\mathbb{R})$ for any $\alpha > \beta - \gamma$. This implies the assertion, since $\mathrm{Id} + r\psi \in \mathcal{D}^{n,\beta-}(\mathbb{R})$ if $r > 0$ is small enough. We must show that, for any small interval $I \ni 0$, the set

$$\Big\{\frac{\theta(t)^{(n)}(x) - \theta(t)^{(n)}(y) - \theta(s)^{(n)}(x) + \theta(s)^{(n)}(y)}{|x-y|^\alpha |s-t|^\gamma} : x \ne y \in \mathbb{R}, s \ne t \in I\Big\}$$

is unbounded. If $|x| < 1$, then for small t (cf. (3.2)),

$$\theta(t)^{(n)}(x) = \psi^{(n)}(x+t) = C_{n,\beta}\,|x+t|^\beta.$$

For $t = x = 0$ and $|y| \le 1$ the expression reads (up to a constant factor)

$$\frac{-|y|^\beta - |s|^\beta + |y+s|^\beta}{|y|^\alpha |s|^\gamma}$$

and upon setting $y = -s$, we get $-2|s|^{\beta-\alpha-\gamma}$ which is unbounded near $s = 0$.

(2) Let $\gamma > 0$ be given. For $\alpha > \beta$ let $\psi_\alpha(x) := x^n |x|^\alpha \chi(x)$. Then, as seen above, $\theta_\alpha(t) := \psi_\alpha \circ (\mathrm{Id} + t\chi)$, for small $t \in \mathbb{R}$, is not locally Hölder continuous of order γ into $C_0^{n,\alpha_1}(\mathbb{R},\mathbb{R})$ for any $\alpha_1 \in (\beta,\alpha)$ with $\gamma > \alpha - \alpha_1$. It follows that $(\psi_\alpha)_\star$ is not locally Hölder continuous of order γ, provided that $\alpha - \beta < \gamma$. Indeed, if

$$\Big\{ \frac{\theta_\alpha(t) - \theta_\alpha(s)}{|s-t|^\gamma} : s \ne t \in I \Big\}$$

were bounded in $C_0^{n,\beta+}(\mathbb{R},\mathbb{R})$, then it would be so in some step $C_0^{n,\alpha_1}(\mathbb{R},\mathbb{R})$, by Lemma 3.8. □

The results of this section are summarized in Table 1.

	group	$C^{0,\omega}$-Lie group	topological group	half-Lie group	Lie group
$\mathcal{D}^{n,\beta}(\mathbb{R}^d)$	yes	no	no	no	no
$\mathcal{D}^{n,\beta-}(\mathbb{R}^d)$	yes	yes	yes	yes	no
$\mathcal{D}^{n,\beta+}(\mathbb{R}^d)$	yes	yes	?	?	no

TABLE 1. Here $n \in \mathbb{N}_{\ge 1}$ and ω is any slowly vanishing modulus of continuity. In the first two rows $\beta \in (0,1]$, in the third row $\beta \in [0,1)$.

4. Hölder spaces are ODE closed

4.1. **Flows of time-dependent Hölder vector fields.** Let $n \in \mathbb{N}_{\ge 1}$ and $\beta \in (0,1]$. By a *strong time-dependent $C_0^{n,\beta}$-vector field* we mean a Bochner integrable function $u : [0,1] \to C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d)$. We will write $I := [0,1]$ and

$$\|u\|_{L^1(I,C^{n,\beta})} := \int_0^1 \|u(t)\|_{n,\beta}\, dt.$$

The space $L^1(I, C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d))$ of (equivalence classes with respect to a.e. coincidence of) Bochner integrable function $u : I \to C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d)$ equipped with this norm is a Banach space.

Let $\alpha \le \beta$. We say that a continuous mapping $\Phi : I \to \mathcal{D}^{n,\alpha}(\mathbb{R}^d)$ is a *strong $\mathcal{D}^{n,\alpha}$-flow* of u if for all $t \in I$ we have

$$\Phi(t) = \mathrm{Id} + \int_0^t u(s) \circ \Phi(s)\, ds \tag{4.1}$$

in $\mathcal{D}^{n,\alpha}(\mathbb{R}^d)$, where the integral is the Bochner integral.

Since evaluation ev_x at $x \in \mathbb{R}^d$ is continuous and linear on $C_0^{n,\alpha}(\mathbb{R}^d,\mathbb{R}^d)$ and it thus commutes with the Bochner integral, (4.1) entails

$$\Phi^\wedge(t,x) = x + \int_0^t u^\wedge(s, \Phi^\wedge(s,x))\, ds, \quad x \in \mathbb{R}^d. \tag{4.2}$$

We say that $\Phi^\wedge : I \times \mathbb{R}^d \to \mathbb{R}^d$ is the *pointwise flow* of $u^\wedge$ if it satisfies (4.2). So, if u has a strong $\mathcal{D}^{n,\alpha}$-flow Φ, then $u^\wedge$ has a pointwise flow which is continuous in

t and differs from the identity by a $C_0^{n,\alpha}$-mapping in x. Conversely, the existence of a pointwise flow with this properties will entail the existence of a strong $\mathcal{D}^{n,\alpha}$-flow only if the Bochner integral in (4.1) exists. Since the Hölder spaces are non-separable, strong measurability of $t \mapsto u(t) \circ \Phi(t)$ may fail and the integral in (4.1) may not exist, if the left translation $u(t)_\star$ is not continuous. This is exactly what happens if $\beta = \alpha$.

Luckily we can work with pointwise estimates which enable us to prove that time-dependent $C_0^{n,\beta}$-vector fields have unique pointwise flows Φ such that $\Phi^\vee \in C(I, \mathcal{D}^{n,\beta}(\mathbb{R}^d))$ (no loss of regularity!). The proof actually works for a wider class of vector fields, so-called *pointwise time-dependent* $C_0^{n,\beta}$*-vector fields*, which shall be introduced in the next subsection.

We shall see in Section 5.1 that the unique pointwise flow $\Phi^\vee \in C(I, \mathcal{D}^{n,\beta}(\mathbb{R}^d))$ of a strong time-dependent $C_0^{n,\beta}$-vector field u lifts to a strong $\mathcal{D}^{n,\alpha}$-flow, for each $\alpha < \beta$.

4.2. **Trouvé group and ODE closedness.** Let $I = [0, 1]$ and let E be a Banach space of mappings $\mathbb{R}^d \to \mathbb{R}^d$ which is continuously embedded in $C_0^1(\mathbb{R}^d, \mathbb{R}^d)$.

Definition 4.1. We say that a mapping $u : I \times \mathbb{R}^d \to \mathbb{R}^d$ is a *pointwise time-dependent* E*-vector field* if the following conditions are satisfied.

- $u(t, \cdot) \in E$ for every $t \in I$.
- $u(\cdot, x)$ is measurable for every $x \in \mathbb{R}^d$.
- $I \ni t \to \|u(t, \cdot)\|_E$ is (Lebesgue) integrable.

Let us denote the set of all pointwise time-dependent E-vector fields by $\mathfrak{X}_E(I, \mathbb{R}^d)$. We remark that instead of the third condition we could also require that $\|u^\vee\|_E$ is dominated a.e. by some non-negative function $m \in L^1(I)$.

Clearly, $u \in L^1(I, E)$ implies $u^\wedge \in \mathfrak{X}_E(I, \mathbb{R}^d)$; the converse is in general not true, in particular, if E is non-separable and strong measurability and measurability are not the same; see Example 4.2 below. We will continue to write

$$\|u^\vee\|_{L^1(I,E)} = \int_0^1 \|u^\vee(t)\|_E \, dt, \qquad \text{for } u \in \mathfrak{X}_E(I, \mathbb{R}^d),$$

even though $\|u^\vee\|_{L^1(I,E)}$ might be finite while $u^\vee : I \to E$ is not Bochner integrable; this will lead to no confusion.

Example 4.2. Let $\chi \in C_c^\infty(\mathbb{R})$ be 1 on $[-1, 1]$, and let $\psi(x) := x^n |x|^\beta \chi(x)$, then ψ lies in $C_0^{n,\beta}(\mathbb{R}, \mathbb{R})$ (cf. Lemma 3.5). Let $u : I \times \mathbb{R} \to \mathbb{R}$ be defined by $u(t, x) = \psi(x - t)$; u is clearly a pointwise time-dependent $C_0^{n,\beta}$-vector field. But $u^\vee \notin L^1(I, C_0^{n,\beta}(\mathbb{R}))$: indeed, for fixed $t, s \in I$, $t \neq s$, (cf. (3.2))

$$\begin{aligned}
[u^\vee(t) - u^\vee(s)]_{n,\beta} &= \sup_{x \neq y} \frac{|\psi^{(n)}(x - t) - \psi^{(n)}(y - t) - \psi^{(n)}(x - s) + \psi^{(n)}(y - s)|}{|x - y|^\beta} \\
&\geq C_{n,\beta} \sup_{x,y \in I,\, x \neq y} \frac{\big||x - t|^\beta - |y - t|^\beta - |x - s|^\beta + |y - s|^\beta\big|}{|x - y|^\beta} \\
&\geq 2C_{n,\beta} > 0 \qquad (\text{choose } x = t,\ y = s).
\end{aligned}$$

It follows that the image $u^\vee(I)$ is not essentially separable in $C_0^{n,\beta}(\mathbb{R})$, and so $u^\vee$ is not strongly measurable, by the Pettis measurability theorem (cf. [4, p. 42]).

It is well-known that pointwise time-dependent C_0^n-vector fields u have unique pointwise flows $\Phi = \Phi_u : I \times \mathbb{R}^d \to \mathbb{R}^d$ such that $\Phi^\vee : I \to \operatorname{Id} + C_0^n(\mathbb{R}^d, \mathbb{R}^d)$ is continuous, and $\Phi^\vee(t)$ is a C^n-diffeomorphism at any time t; see e.g. [28, 8.7, 8.8, 8.9] and the arguments in the proof of Theorem 4.6 below.

Definition 4.3. Let E be a Banach space of mappings $\mathbb{R}^d \to \mathbb{R}^d$ which is continuously embedded in $C_0^1(\mathbb{R}^d, \mathbb{R}^d)$. Then

$$\mathcal{G}_E := \left\{ \Phi_u^\vee(1) : u \in \mathfrak{X}_E(I, \mathbb{R}^d) \right\}$$

is a group with respect to composition; cf. [28, 8.14]. We call $\mathcal{G}_E$ the *Trouvé group* of E.

Definition 4.4. We say that E is *ODE closed* if $\mathcal{G}_E \subseteq \operatorname{Id} + E$.

Remark 4.5. It is clear that, more generally, we could take (mutatis mutandis) any locally convex space E of mappings $\mathbb{R}^d \to \mathbb{R}^d$ which is continuously embedded in $C_0^1(\mathbb{R}^d, \mathbb{R}^d)$ in the above definitions.

Furthermore, this leads to the notion of *ODE hull* of E, i.e., the intersection of all locally convex spaces F of mappings $\mathbb{R}^d \to \mathbb{R}^d$ which are continuously embedded in $C_0^1(\mathbb{R}^d, \mathbb{R}^d)$ and continuously contain E, endowed with the natural projective topology. The ODE hull is well-defined, because C_0^1 is ODE-closed, and it is evidently ODE closed.

4.3. **The Trouvé group of $C_0^{n,\beta}(\mathbb{R}^d, \mathbb{R}^d)$.** Let $n \in \mathbb{N}_{\geq 1}$ and $\beta \in (0, 1]$. In this section we show that the Trouvé group of $C_0^{n,\beta}(\mathbb{R}^d, \mathbb{R}^d)$,

$$\mathcal{G}_{n,\beta}(\mathbb{R}^d) := \mathcal{G}_{C_0^{n,\beta}(\mathbb{R}^d, \mathbb{R}^d)},$$

coincides with the connected component of the identity in $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$. In particular, $C_0^{n,\beta}$ is ODE closed. Let us use the short notation

$$\mathfrak{X}_{n,\beta}(I, \mathbb{R}^d) := \mathfrak{X}_{C_0^{n,\beta}}(I, \mathbb{R}^d).$$

We want to stress that this is an example of an ODE closed space on which left translations $g_\star$ are *not* continuous.

Theorem 4.6. *Let $n \in \mathbb{N}_{\geq 1}$ and $\beta \in (0, 1]$. Let $u \in \mathfrak{X}_{n,\beta}(I, \mathbb{R}^d)$. Then u has a unique pointwise flow $\Phi : I \times \mathbb{R}^d \to \mathbb{R}^d$ such that $\Phi^\vee : I \to \mathcal{D}^{n,\beta}(\mathbb{R}^d)$ is continuous. In particular, $C_0^{n,\beta}$ is ODE closed.*

Proof. In fact, cf. [28, 8.7, 8.8, 8.9], the pointwise flow exists, $x \mapsto \Phi(t, x)$ is C_b^n, and for all $t \in I$,

$$\|\Phi(t, \cdot) - \operatorname{Id}\|_n \leq C_1 e^{C_2 \|u\|_{L^1(I, C_b^n)}}. \tag{4.3}$$

Set $W_x^n(t) := d_x^n \Phi(t, x)$ and $V_{x,y}^n(t) := (W_x^n(t) - W_y^n(t))/\|x - y\|^\beta$. Then $W_x^n(t)$ satisfies

$$\partial_t W_x^n(t) = d_x^n\big(u^\vee(t)(\Phi(t, x))\big), \quad W_x^n(0) = \begin{cases} \mathbb{1}, & \text{for } n = 1 \\ 0, & \text{for } n \geq 2 \end{cases}.$$

Upon setting

$$A^{\gamma,j}(t) = A^{\gamma,j}(x, y)(t) := \big(W_x^{\gamma_1}(t), \ldots, W_x^{\gamma_j}(t), W_y^{\gamma_{j+1}}(t), \ldots, W_y^{\gamma_l}(t)\big).$$

and using Faà di Bruno's formula (2.7), this ODE takes the form

$$\partial_t W^n_x(t) = \operatorname{sym} \sum_{l=1}^{n} \sum_{\gamma\in\Gamma(l,n)} c_\gamma u^\vee(t)^{(l)}(\Phi(t,x))(A^{\gamma,l}(t)),$$

and analogously,

$$\partial_t W^n_y(t) = \operatorname{sym} \sum_{l=1}^{n} \sum_{\gamma\in\Gamma(l,n)} c_\gamma u^\vee(t)^{(l)}(\Phi(t,y))(A^{\gamma,0}(t)).$$

It follows that $V^n_{x,y}(t)$ satisfies $V^n_{x,y}(0) = 0$ and

$$\partial_t V^n_{x,y}(t) = A_x(t)\cdot V^n_{x,y}(t) + b^n_{x,y}(t) + \operatorname{sym} \sum_{l=2}^{n} \sum_{\gamma\in\Gamma(l,n)} c_\gamma \frac{u^\vee(t)^{(l)}(\Phi(t,x))\cdot A^{\gamma,l}(t) - u^\vee(t)^{(l)}(\Phi(t,y))\cdot A^{\gamma,0}(t)}{\|x-y\|^\beta}, \tag{4.4}$$

where $A_x(t) = du^\vee(t)(\Phi(t,x))$ and $b^n_{x,y}(t) := \frac{A_x(t)-A_y(t)}{\|x-y\|^\beta}\cdot W^n_y(t)$. It can be easily seen, using (4.3), that

$$\|A_x(t)\|_{L_1} \le \|u^\vee(t)\|_1, \quad \|b^n_{x,y}(t)\|_{L_1} \le \|u^\vee(t)\|_{1,\beta}[\Phi^\vee(t)]^\beta_1[\Phi^\vee(t)]_n \le C_3\|u^\vee(t)\|_{1,\beta}. \tag{4.5}$$

Similar arguments (see, e.g., the proof of Proposition 3.6) show that all remaining terms in the sum can be estimated by $\|u(t)\|_{n,\beta}$ times a constant uniformly in x,y. An application of Gronwall's inequality implies that $[\Phi^\vee(t)]_{n,\beta}$ is bounded in t, showing that $\Phi^\vee(t) \in \mathrm{Id} + C^{n,\beta}_b(\mathbb{R}^d,\mathbb{R}^d)$ for all t. Finally, we may conclude, integrating (4.4) and using similar estimates, that

$$[\Phi^\vee(t)-\Phi^\vee(t_0)]_{n,\beta} = \sup_{x\ne y}\|V^n_{x,y}(t) - V^n_{x,y}(t_0)\|_{L_n} \le C_4 \int_{t_0}^t \|u^\vee(s)\|_{n,\beta}\,ds$$

which tends to 0 as $t\to t_0$. In an analogous way one sees that $\|\Phi^\vee(t)-\Phi^\vee(t_0)\|_n \to 0$ as $t\to t_0$. This shows continuity in time.

It remains to prove that $\Phi^\vee(I) \subseteq \mathcal{D}^{n,\beta}(\mathbb{R}^d)$. For fixed $x\in\mathbb{R}^d$, the mapping $I \ni t \mapsto \det d\Phi(t,x)$ is continuous with image in $\mathbb{R}\setminus\{0\}$ (since $\Phi^\vee(t)$ is a C^1-diffeomorphism of $\mathbb{R}^d$ for each t). Since $\Phi^\vee(0) = \mathrm{Id}$, we may conclude that $\det d\Phi(t,x) > 0$ for all $x\in\mathbb{R}^d$ and all $t\in I$.

Finally, let us check that $\phi^\vee(t) = \Phi^\vee(t) - \mathrm{Id} \in C^{n,\beta}_0(\mathbb{R}^d,\mathbb{R}^d)$ for all $t\in I$. Suppose that $\|\phi(t,x)\| \not\to 0$ as $\|x\|\to\infty$. Then there is $\epsilon>0$ and a sequence $x_k\in\mathbb{R}^d$ such that $\|x_k\|\to\infty$ and $\|\phi(t,x_k)\|\ge\epsilon$. Since $\Phi^\vee(s)$ is a diffeomorphism of $\mathbb{R}^d$, for all $s\in I$,

$$\sup_{x\in\mathbb{R}^d}\|u(s,x+\phi(s,x))\| = \sup_{y\in\mathbb{R}^d}\|u(s,y)\| = \|u^\vee(s)\|_0$$

and so the dominated convergence theorem implies that

$$\|\phi(t,x_k)\| \le \int_0^t \|u(s,x_k+\phi(s,x_k))\|\,ds \to 0,$$

because $\|x_k+\phi(s,x_k)\|\to\infty$ as $k\to\infty$ and $\|u(s,x)\|\to 0$ as $\|x\|\to\infty$, for each $s\in I$; a contradiction. To see that $\|d^k_x\phi(t,x)\|\to 0$ as $\|x\|\to\infty$, for $1\le k\le n$, we argue similarly: Since $s\mapsto\phi^\vee(s)$ is continuous into $C^{n,\beta}_b(\mathbb{R}^d,\mathbb{R}^d)$, there is a constant C such that $\sup_{s\in I}\|\phi^\vee(s)\|_{n,\beta} < C$. Thus Faà di Bruno's formula (2.7) implies that $\sup_{x\in\mathbb{R}^d}\|d^k_x(u(s,x+\phi(s,x)))\|$ is bounded above by $\|u(s)\|_k$ times a constant (independent of s and x) for all $s \in I$. Then the dominated convergence theorem implies the assertion as before. $\square$

Theorem 4.7. *Let $n \in \mathbb{N}_{\ge 1}$ and $\beta \in (0,1]$. Then*

$$\mathcal{G}_{n,\beta}(\mathbb{R}^d) = \mathcal{D}^{n,\beta}(\mathbb{R}^d)_0, \tag{4.6}$$

where $\mathcal{D}^{n,\beta}(\mathbb{R}^d)_0$ denotes the connected component of the identity in $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$.

Proof. The inclusion $\mathcal{G}_{n,\beta}(\mathbb{R}^d) \subseteq \mathcal{D}^{n,\beta}(\mathbb{R}^d)_0$ follows from Theorem 4.6.

Let us prove $\mathcal{D}^{n,\beta}(\mathbb{R}^d)_0 \subseteq \mathcal{G}_{n,\beta}(\mathbb{R}^d)$. Since $\mathcal{D}^{n,\beta}(\mathbb{R}^d)_0$ is connected and locally path-connected, it is path-connected, and each $\Phi \in \mathcal{D}^{n,\beta}(\mathbb{R}^d)_0$ can be connected by a polygon with the identity.

Let $\Phi = \mathrm{Id} + \phi \in \mathcal{D}^{n,\beta}(\mathbb{R}^d)_0$ be such that $\gamma(t) := (1-t)\,\mathrm{Id} + t\Phi \in \mathcal{D}^{n,\beta}(\mathbb{R}^d)$ for all $t \in I$. Then $\gamma(t)(x) = x + t\phi(x)$, and

$$u(t,x) := (\gamma'(t) \circ \gamma(t)^{-1})(x) = \phi(\gamma(t)^{-1}(x))$$

is a time-dependent vector field such that:

- $u(t,\cdot) \in C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d)$ for all $t \in I$, since $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ is a group, by Theorem 3.4.
- $u(\cdot,x)$ is a Borel function for every $x \in \mathbb{R}^d$; indeed, if $\gamma(t)^{-1}(x) =: x + \tau(t,x)$ then τ satisfies the implicit equation
$$\tau(t,x) + t\phi(x + \tau(t,x)) = 0,$$
and is C^n by the implicit function theorem.
- We have $\int_0^1 \|u(t,\cdot)\|_{n,\beta}\, dt < \infty$, since inversion is locally bounded on $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ and left translation maps bounded sets to bounded sets, see Theorem 2.7 and Proposition 3.6.

That means that $u \in \mathfrak{X}_{n,\beta}(I,\mathbb{R}^d)$ and hence $\Phi \in \mathcal{G}_{n,\beta}(\mathbb{R}^d)$.

Suppose we are given a polygon in $\mathcal{D}^{n,\beta}(\mathbb{R}^d)$ with vertices $\mathrm{Id}, \Phi_1, \ldots, \Phi_n$. Then $\Phi_1 \in \mathcal{G}_{n,\beta}(\mathbb{R}^d)$, by the previous paragraph. Consider the line segment γ connecting Φ_1 and Φ_2. Then $t \mapsto \gamma(t) \circ \Phi_1^{-1}$ connects Id with $\Phi_2 \circ \Phi_1^{-1}$. So, by the above, $\Phi_2 \circ \Phi_1^{-1} \in \mathcal{G}_{n,\beta}(\mathbb{R}^d)$ and hence $\Phi_2 \in \mathcal{G}_{n,\beta}(\mathbb{R}^d)$, since $\mathcal{G}_{n,\beta}(\mathbb{R}^d)$ is a group. By iteration all vertices Φ_j belong to $\mathcal{G}_{n,\beta}(\mathbb{R}^d)$. $\square$

Remark 4.8. Analyzing the proof one finds that the identity (4.6) still holds if in the definition of the Trouvé group we restrict to $u \in \mathfrak{X}_{n,\beta}(I,\mathbb{R}^d)$ which are piecewise C^n in time t. (Discontinuities in t guarantee that $\mathcal{G}_{n,\beta}(\mathbb{R}^d)$ is a group.)

4.4. **The Trouvé group of $C_0^{n,\beta\pm}(\mathbb{R}^d,\mathbb{R}^d)$.** We define *pointwise time-dependent $C_0^{n,\beta\pm}$-vector fields* to be the elements of

$$\mathfrak{X}_{n,\beta-}(I,\mathbb{R}^d) := \bigcap_{\alpha\in(0,\beta)} \mathfrak{X}_{n,\alpha}(I,\mathbb{R}^d), \quad \mathfrak{X}_{n,\beta+}(I,\mathbb{R}^d) := \bigcup_{\alpha\in(\beta,1)} \mathfrak{X}_{n,\alpha}(I,\mathbb{R}^d),$$

respectively, and the corresponding Trouvé groups by

$$\mathcal{G}_{n,\beta\pm}(\mathbb{R}^d) := \big\{\Phi_u^\vee(1) : u \in \mathfrak{X}_{n,\beta\pm}(I,\mathbb{R}^d)\big\}.$$

Theorem 4.9. *Let $n \in \mathbb{N}_{\ge 1}$. For $\beta \in (0,1]$, $C_0^{n,\beta-}$ is ODE closed and*

$$\mathcal{G}_{n,\beta-}(\mathbb{R}^d) = \mathcal{D}^{n,\beta-}(\mathbb{R}^d)_0, \tag{4.7}$$

and, for $\beta \in [0,1)$, $C_0^{n,\beta+}$ is ODE closed and

$$\mathcal{G}_{n,\beta+}(\mathbb{R}^d) = \mathcal{D}^{n,\beta+}(\mathbb{R}^d)_0, \tag{4.8}$$

In particular, $\mathcal{G}_{n,\beta\pm}(\mathbb{R}^d)$ has a $C^{0,\omega}$ Lie group structure, for every slowly vanishing modulus of continuity ω. Moreover, $\mathcal{G}_{n,\beta-}(\mathbb{R}^d)$ has a topological group structure and a half-Lie group structure.

Proof. This follows from Theorem 4.6, Theorem 4.7, and Theorem 3.11. To see, e.g., (4.8), note that

$$\mathcal{G}_{n,\beta+}(\mathbb{R}^d) = \bigcup_{\alpha>\beta} \mathcal{G}_{n,\alpha}(\mathbb{R}^d) = \bigcup_{\alpha>\beta} \mathcal{D}^{n,\alpha}(\mathbb{R}^d)_0$$

is path-connected in $\mathcal{D}^{n,\beta+}(\mathbb{R}^d) = \bigcup_{\alpha>\beta} \mathcal{D}^{n,\alpha}(\mathbb{R}^d)$ and thus contained in $\mathcal{D}^{n,\beta+}(\mathbb{R}^d)_0$. The inclusion $\mathcal{D}^{n,\beta+}(\mathbb{R}^d)_0 \subseteq \mathcal{G}_{n,\beta+}(\mathbb{R}^d)$ follows from the proof of Theorem 4.7: the line segment γ factors to some step of the inductive limit defining $\mathcal{D}^{n,\beta+}(\mathbb{R}^d)$, by Lemma 3.8. □

Clearly, Remark 4.8 also applies in this situation.

Let us summarize the results of this section in Table 2.

	ODE closed	Trouvé group
$C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d)$	yes	$\mathcal{D}^{n,\beta}(\mathbb{R}^d)_0$
$C_0^{n,\beta-}(\mathbb{R}^d,\mathbb{R}^d)$	yes	$\mathcal{D}^{n,\beta-}(\mathbb{R}^d)_0$
$C_0^{n,\beta+}(\mathbb{R}^d,\mathbb{R}^d)$	yes	$\mathcal{D}^{n,\beta+}(\mathbb{R}^d)_0$

TABLE 2. Here $n \in \mathbb{N}_{\geq 1}$. In the first two rows $\beta \in (0,1]$, in the third row $\beta \in [0,1)$.

5. Continuity of the flow map

Let $n \in \mathbb{N}_{\geq 1}$ and $\beta \in (0,1]$. By the results of the last section every $u \in \mathfrak{X}_{n,\beta}(I,\mathbb{R}^d)$ (thus every $u^\vee \in L^1(I, C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d))$) has a unique pointwise flow Φ with $\Phi^\vee \in C(I,\mathcal{D}^{n,\beta}(\mathbb{R}^d))$. The goal of this section is to show the following:

(1) If $u^\vee \in L^1(I, C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d))$ and $\alpha < \beta$, then $\Phi^\vee$ is the unique strong $\mathcal{D}^{n,\alpha}$-flow of u, i.e.,

$$\Phi^\vee(t) = \mathrm{Id} + \int_0^t u^\vee(s) \circ \Phi^\vee(s)\, ds, \quad t \in I, \tag{5.1}$$

in $\mathcal{D}^{n,\alpha}(\mathbb{R}^d)$.

(2) The flow map $L^1(I, C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d)) \to C(I,\mathcal{D}^{n,\alpha}(\mathbb{R}^d))$, $u^\vee \mapsto \Phi^\vee$, is bounded for all $n \geq 1$ and continuous, even $C^{0,\beta-\alpha}$, if $n \geq 2$.

We recall that the Bochner integral in (5.1) might not exist if $\alpha = \beta$, because strong measurability of $s \mapsto u^\vee(s) \circ \Phi^\vee(s)$ may fail if $u^\vee(s)_\star$ is not continuous; see Remark 5.2 below.

5.1. **Existence of the strong $\mathcal{D}^{n,\alpha}$-flow.** First we show that the Bochner integral in (5.1) exists if $\beta > \alpha$.

Lemma 5.1. *Let $n \in \mathbb{N}_{\geq 1}$ and $0 < \alpha < \beta \leq 1$. Let $u \in L^1(I, C_0^{n,\beta}(\mathbb{R}^d, \mathbb{R}^d))$ and let $\Phi : I \to \mathcal{D}^{n,\alpha}(\mathbb{R}^d)$ be continuous. Then:*

- (i) *The mapping $I \times C_0^{n,\alpha}(\mathbb{R}^d, \mathbb{R}^d) \to C_0^{n,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$, $(t, \phi) \mapsto u(t) \circ (\mathrm{Id} + \phi)$, has the Carathéodory property. The function $t \mapsto u(t) \circ \Phi(t)$ belongs to $L^1(I, C_0^{n,\alpha}(\mathbb{R}^d, \mathbb{R}^d))$.*
- (ii) *If $\mathrm{ev}_x \Phi(t) =: \Phi^\wedge(t, x)$ is the pointwise flow of u for the initial condition $\Phi^\wedge(0, x) = x$ for all x, then Φ is the strong $\mathcal{D}^{n,\alpha}$-flow of u.*

Proof. (i) That $(t, \phi) \mapsto u(t) \circ (\mathrm{Id} + \phi)$ has the Carathéodory property follows easily from Theorem 2.7 and Theorem 2.8. Together with continuity of Φ, an application of [1, Lemma 2.2] yields strong measurability of $t \mapsto u(t) \circ \Phi(t)$. Integrability follows from Theorem 2.7.

(ii) Recall that ev_x is continuous and linear on $C_0^{n,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$ and thus commutes with the Bochner integral. Observe also that $s \mapsto u(s) \circ \Phi(s)$ is Bochner integrable in $C_0^{n,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$, by (i). Since $\Phi^\wedge(t, x)$ is the pointwise flow, we have

$$\mathrm{ev}_x \Phi(t) = \Phi^\wedge(t, x) = x + \int_0^t u(s)(\Phi^\wedge(s, x))\, ds = \mathrm{ev}_x \Big(\mathrm{Id} + \int_0^t u(s) \circ \Phi(s)\, ds \Big).$$

Since the family of evaluation maps is point separating on $C_0^{n,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$, we are done. □

Remark 5.2. In general, the function $t \mapsto u(t) \circ \Phi(t)$ cannot belong to $L^1(I, C_0^{n,\beta}(\mathbb{R}^d, \mathbb{R}^d))$, even if u is a constant and Φ is continuous into $\mathcal{D}^{N,1}(\mathbb{R}^d)$ for all $N \geq n$. Indeed: Let $\chi \in C_c^\infty(\mathbb{R})$ be 1 on $[-1, 1]$, and let $\psi(x) := x^n |x|^\beta \chi(x)$ be the $C_0^{n,\beta}$-function from the proof of Lemma 3.5 and Example 4.2. Taking $u(t) := \psi$ and $\Phi(t)(x) := x + t\chi(x)$ we get $(u(t) \circ \Phi(t))(x) = \psi(x + t)$ if $x \in [-1, 1]$. So Example 4.2 shows that $t \mapsto u(t) \circ \Phi(t)$ is not strongly measurable.

Theorem 5.3. *Let $n \in \mathbb{N}_{\geq 1}$ and $0 < \alpha < \beta \leq 1$. Let $u \in L^1(I, C_0^{n,\beta}(\mathbb{R}^d, \mathbb{R}^d))$. Then u has a unique strong $\mathcal{D}^{n,\alpha}$-flow Φ.*

Proof. By Theorem 4.6, u has a unique pointwise flow $\Phi : I \times \mathbb{R}^d \to \mathbb{R}^d$ such that $\Phi^\vee \in C(I, \mathcal{D}^{n,\beta}(\mathbb{R}^d))$. For $\alpha < \beta$, we have $\Phi^\vee \in C(I, \mathcal{D}^{n,\alpha}(\mathbb{R}^d))$. By Lemma 5.1, $\Phi^\vee$ is the unique strong $\mathcal{D}^{n,\alpha}$-flow of u, i.e., it satisfies (5.1). □

Remark 5.4. One can use Carathéodory's solution theory for ODEs on Banach spaces which are Bochner integrable in time (cf. Section 2.3) to give an alternative proof which, however, does not work for $n = 1$! Indeed, using Lemma 5.1, Corollary 2.11, and Theorem 2.4 one can show that u has a unique strong $\mathcal{D}^{n-1,\alpha}$-flow Φ. That the flow $\Phi = \mathrm{Id} + \phi$ is actually strongly $\mathcal{D}^{n,\alpha}$-valued follows from the observation that $d\phi$ satisfies the linear ODE

$$d\phi(t) = d \int_0^t u(s) \circ (\mathrm{Id} + \phi(s))\, ds = \int_0^t du(s) \circ (\mathrm{Id} + \phi(s)) \cdot (\mathbb{1} + d\phi)(s)\, ds,$$

and from Theorem 2.5.

5.2. **Continuity of the flow map.** First we prove that the flow map is bounded.

Proposition 5.5. *Let $n \in \mathbb{N}_{\ge 1}$ and $\beta \in (0,1]$. The flow map $L^1(I, C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d)) \to C(I, \mathcal{D}^{n,\beta}(\mathbb{R}^d))$, $u \mapsto \Phi$, is bounded.*

In the proof we use only pointwise estimates; thus the result still holds if $u^\wedge$ varies in $\mathfrak{X}_{n,\beta}(I,\mathbb{R}^d)$ endowed with the norm $\|u\|_{L^1(I,C_b^{n,\beta})}$.

Proof. We first claim that $u \mapsto \phi := \Phi - \mathrm{Id}$ is bounded into $C(I, C_b^n(\mathbb{R}^d,\mathbb{R}^d))$. We proceed by induction on n. For simplicity of notation we simply write $\phi(t,x)$ instead of $\phi^\wedge(t,x)$, etc. Clearly

$$\|\phi(t,x)\| \le \int_0^t \|u(s,\Phi(s,x))\|\,ds \le \|u\|_{L^1(I,C_b^{n,\beta})}$$

and hence

$$\|\phi\|_{C(I,C_b^0)} \le \|u\|_{L^1(I,C_b^{n,\beta})}.$$

Assume that $u \mapsto \phi$ is bounded into $C(I, C_b^{n-1}(\mathbb{R}^d,\mathbb{R}^d))$. By Faà di Bruno's formula (2.7),

$$\begin{aligned} d_x^n(u(s)\circ\Phi(s))(x) &= d_x u(s)(\Phi(s,x))(d_x^n\Phi(s,x)) \\ &\quad + \operatorname{sym}\sum_{l=2}^n \sum_{\gamma\in\Gamma(l,n)} c_\gamma u(s)^{(l)}(\Phi(s,x))\big(d_x^{\gamma_1}\Phi(s,x),\dots,d_x^{\gamma_l}\Phi(s,x)\big). \end{aligned}$$

Hence

$$\begin{aligned} &\|d_x^n(u(s)\circ\Phi(s))\|_0 \\ &\quad \le \|u(s)\|_1[\Phi(s)]_n + \sum_{l=2}^n \sum_{\gamma\in\Gamma(l,n)} c_\gamma \|u(s)\|_l [\Phi(s)]_{\gamma_1}\cdots[\Phi(s)]_{\gamma_l} \end{aligned}$$

and so, by induction hypothesis and since $[\Phi(s)]_n \le 1 + [\phi(s)]_n$,

$$\begin{aligned} [\phi(t)]_n &\le \int_0^t \|d_x^n(u(s)\circ\Phi(s))\|_0\,ds \\ &\le \int_0^t \|u(s)\|_1[\phi(s)]_n\,ds + C\|u\|_{L^1(I,C_b^{n,\beta})}. \end{aligned}$$

Gronwall's lemma implies that

$$\|\phi\|_{C(I,C_b^n)} \le C\|u\|_{L^1(I,C_b^{n,\beta})}\exp(\|u\|_{L^1(I,C_b^{n,\beta})}),$$

and the claim is proved.

It remains to show that $u \mapsto \phi$ is bounded into $C(I, C_b^{n,\alpha}(\mathbb{R}^d,\mathbb{R}^d))$. To this end consider

$$\begin{aligned} &d_x^n(u(s)\circ\Phi(s))(x) - d_x^n(u(s)\circ\Phi(s))(y) \\ &\quad = \operatorname{sym}\sum_{l=1}^n \sum_{\gamma\in\Gamma(l,n)} c_\gamma\big(u(s)^{(l)}(\Phi(s,x))(A^{\gamma,l}(x,y)) - u(s)^{(l)}(\Phi(s,y))(A^{\gamma,0}(x,y))\big), \end{aligned}$$

where

$$A^{\gamma,j} = A^{\gamma,j}(x,y) := \big(d_x^{\gamma_1}\Phi(s,x),\dots,d_x^{\gamma_j}\Phi(s,x), d_x^{\gamma_{j+1}}\Phi(s,y),\dots,d_x^{\gamma_l}\Phi(s,y)\big).$$

Then

$$\begin{aligned}
&\big\|u(s)^{(l)}(\Phi(s,x))(A^{\gamma,l}) - u(s)^{(l)}(\Phi(s,y))(A^{\gamma,0})\big\|_{L_n} \\
&\quad\le \big\|u(s)^{(l)}(\Phi(s,x))(A^{\gamma,l}) - u(s)^{(l)}(\Phi(s,y))(A^{\gamma,l})\big\|_{L_n} \\
&\qquad + \sum_{k=1}^{l} \big\|u(s)^{(l)}(\Phi(s,y))(A^{\gamma,k}) - u(s)^{(l)}(\Phi(s,y))(A^{\gamma,k-1})\big\|_{L_n}.
\end{aligned}$$

For the first summand

$$\begin{aligned}
&\big\|u(s)^{(l)}(\Phi(s,x))(A^{\gamma,l}) - u(s)^{(l)}(\Phi(s,y))(A^{\gamma,l})\big\|_{L_n} \\
&\quad\le \big\|u(s)^{(l)}(\Phi(s,x)) - u(s)^{(l)}(\Phi(s,y))\big\|_{L_n}(1+\|\phi(s)\|_n)^n \\
&\quad\le \begin{cases} \|u(s)\|_n[\Phi(s)]_1\|x-y\|(1+\|\phi(s)\|_n)^n & \text{if } l<n, \\ \|u(s)\|_{n,\beta}[\Phi(s)]_1^\beta\|x-y\|^\beta(1+\|\phi(s)\|_n)^n & \text{if } l=n. \end{cases}
\end{aligned}$$

For the other summands observe that

$$\begin{aligned}
&u(s)^{(l)}(\Phi(s,y))(A^{\gamma,k}) - u(s)^{(l)}(\Phi(s,y))(A^{\gamma,k-1}) \\
&= u(s)^{(l)}(\Phi(s,y))\big(\dots, d_x^{\gamma_{k-1}}\Phi(s,x), d_x^{\gamma_k}\Phi(s,x) - d_x^{\gamma_k}\Phi(s,y), d_x^{\gamma_{k+1}}\Phi(s,y), \dots\big),
\end{aligned}$$

whence, if $l \ge 2$ and hence $\gamma_k \le n-1$,

$$\begin{aligned}
&\big\|u(s)^{(l)}(\Phi(s,y))(A^{\gamma,k}) - u(s)^{(l)}(\Phi(s,y))(A^{\gamma,k-1})\big\|_{L_n} \\
&\quad\le \|u(s)\|_n(1+\|\phi(s)\|_n)^{n-1}\|\phi(s)\|_n\|x-y\|.
\end{aligned}$$

For $l=1$, we have

$$\begin{aligned}
&\big\|du(s)(\Phi(s,y))(d_x^n\Phi(s,x)) - du(s)(\Phi(s,y))(d_x^n\Phi(s,y))\big\|_{L_n} \\
&\quad\le \|u(s)\|_n\|\phi(s)\|_{n,\beta}\|x-y\|^\beta.
\end{aligned}$$

These estimates, together with the fact that $u \mapsto \phi$ is bounded into $C(I, C_b^n(\mathbb{R}^d,\mathbb{R}^d))$, imply

$$\begin{aligned}
\|\phi(t)\|_{n,\beta} &\le \int_0^t \|u(s)\circ\Phi(s)\|_{n,\beta}\,ds \\
&\le C\int_0^t \|u(s)\|_{n,\beta}\|\phi(s)\|_{n,\beta}\,ds + C\|u\|_{L^1(I,C_b^{n,\beta})},
\end{aligned}$$

and Gronwall's inequality yields the assertion. □

Theorem 5.6. *Let $n \in \mathbb{N}_{\ge 2}$ and $0<\alpha<\beta\le 1$. Then the flow map $L^1(I, C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d)) \to C(I,\mathcal{D}^{n,\alpha}(\mathbb{R}^d))$, $u\mapsto\Phi$, is continuous, even $C^{0,\beta-\alpha}$.*

We do not know if the theorem also holds for $n=1$ or for $\alpha=\beta$.

Proof. Fix $u_0 \in L^1(I, C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d))$ and let $u,v \in L^1(I, C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d))$ be in the ball with radius $\delta>0$ and center u_0 in $L^1(I, C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d))$. Consider the corresponding flows $\Phi = \mathrm{Id}+\phi, \Psi = \mathrm{Id}+\psi \in C(I,\mathcal{D}^{n,\alpha}(\mathbb{R}^d))$. By Proposition 5.5, there is a constant $C = C(u_0,\delta)>0$ such that

$$\|\phi\|_{C(I,C_b^{n,\alpha})} \le C, \quad \|\psi\|_{C(I,C_b^{n,\alpha})} \le C.$$

By Corollary 2.11, Theorem 2.7, and Theorem 5.3,

$$\begin{aligned}\|\phi(t)-\psi(t)\|_{n-1,\alpha} &\le \int_0^t \|u(s)\circ\Phi(s)-v(s)\circ\Psi(s)\|_{n-1,\alpha}\,ds\\ &\le \int_0^t \|u(s)\circ\Phi(s)-u(s)\circ\Psi(s)\|_{n-1,\alpha}+\|(u(s)-v(s))\circ\Psi(s)\|_{n-1,\alpha}\,ds\\ &\le C_1\int_0^t \|u(s)\|_{n,\beta}\|\phi(s)-\psi(s)\|_{n-1,\alpha}+\|u(s)-v(s)\|_{n-1,\alpha}\,ds\end{aligned}$$

and so, by Gronwall's lemma (and Lemma 2.2),

$$\begin{aligned}\|\phi(t)-\psi(t)\|_{n-1,\alpha} &\le C_1\|u-v\|_{L^1(I,C_b^{n,\beta})}\exp(C_1\|u\|_{L^1(I,C_b^{n,\beta})})\\ &=: C_2\|u-v\|_{L^1(I,C_b^{n,\beta})}.\end{aligned}\tag{5.2}$$

This proves that $u\mapsto\Phi$ is continuous into $C(I,\mathcal{D}^{n-1,\alpha}(\mathbb{R}^d))$. Applying $d=d_x$,

$$\begin{aligned}\|d\phi(t)-d\psi(t)\|_{n-1,\alpha} &\le \int_0^t \|(du(s)\circ\Phi(s))d\Phi(s)-(dv(s)\circ\Psi(s))d\Psi(s)\|_{n-1,\alpha}\,ds\\ &\le \int_0^t \|(du(s)\circ\Phi(s))(d\Phi(s)-d\Psi(s))\|_{n-1,\alpha}\,ds\\ &\quad+\int_0^t \|(du(s)\circ\Phi(s)-dv(s)\circ\Psi(s))d\Psi(s)\|_{n-1,\alpha}\,ds.\end{aligned}$$

By Proposition 2.6 and Theorem 2.7,

$$\begin{aligned}&\|(du(s)\circ\Phi(s))(d\Phi(s)-d\Psi(s))\|_{n-1,\alpha}\\ &\quad\le 2^n\|du(s)\circ\Phi(s)\|_{n-1,\alpha}\|d\Phi(s)-d\Psi(s)\|_{n-1,\alpha}\\ &\quad\le 2^nM\|du(s)\|_{n-1,\alpha}(1+\|\phi(s)\|_{n-1,\alpha})^n\|d\Phi(s)-d\Psi(s)\|_{n-1,\alpha}\\ &\quad\le C_3\|u(s)\|_{n,\beta}\|d\Phi(s)-d\Psi(s)\|_{n-1,\alpha}\end{aligned}$$

and

$$\begin{aligned}&\|(du(s)\circ\Phi(s)-dv(s)\circ\Psi(s))d\Psi(s)\|_{n-1,\alpha}\\ &\quad\le 2^n\|du(s)\circ\Phi(s)-dv(s)\circ\Psi(s)\|_{n-1,\alpha}\|d\Psi(s)\|_{n-1,\alpha}\\ &\quad\le C_4\|du(s)\circ\Phi(s)-dv(s)\circ\Psi(s)\|_{n-1,\alpha}.\end{aligned}$$

By Theorem 2.7, Theorem 2.8, and (5.2),

$$\begin{aligned}&\|du(s)\circ\Phi(s)-dv(s)\circ\Psi(s)\|_{n-1,\alpha}\\ &\le \|(du(s)-dv(s))\circ\Phi(s)\|_{n-1,\alpha}+\|dv(s)\circ\Phi(s)-dv(s)\circ\Psi(s)\|_{n-1,\alpha}\\ &\le M\big(\|du(s)-dv(s)\|_{n-1,\alpha}(1+\|\phi(s)\|_{n-1,\alpha})^n+\|v(s)\|_{n,\beta}\|\phi(s)-\psi(s)\|_{n-1,\alpha}^{\beta-\alpha}\big)\\ &\le C_5\big(\|u(s)-v(s)\|_{n,\beta}+\|v(s)\|_{n,\beta}\|\phi(s)-\psi(s)\|_{n-1,\alpha}^{\beta-\alpha}\big).\end{aligned}$$

Together with (5.2) this gives

$$\begin{aligned}&\int_0^t \|du(s)\circ\Phi(s)-dv(s)\circ\Psi(s)\|_{n-1,\alpha}\,ds\\ &\qquad\le C_6\big(\|u-v\|_{L^1(I,C_b^{n,\beta})}+\|u-v\|_{L^1(I,C_b^{n,\beta})}^{\beta-\alpha}\big)\\ &\qquad\le C_7\|u-v\|_{L^1(I,C_b^{n,\beta})}^{\beta-\alpha},\end{aligned}$$

provided that $\|u-v\|_{L^1(I,C_b^{n,\beta})} \le 1$. Consequently,

$$\|d\phi(t)-d\psi(t)\|_{n-1,\alpha} \le C_3 \int_0^t \|u(s)\|_{n,\beta}\|d\phi(s)-d\psi(s)\|_{n-1,\alpha}\,ds + C_7\|u-v\|^{\beta-\alpha}_{L^1(I,C_b^{n,\beta})}.$$

Then Gronwall's inequality implies

$$\begin{aligned}\|d\phi(t)-d\psi(t)\|_{n-1,\alpha} &\le C_7\|u-v\|^{\beta-\alpha}_{L^1(I,C_b^{n,\beta})}\exp(C_3\|u\|_{L^1(I,C_b^{n,\beta})})\\ &\le C_8\|u-v\|^{\beta-\alpha}_{L^1(I,C_b^{n,\beta})},\end{aligned}$$

for all $t\in I$, and the assertion follows. □

5.3. **Flows of strong time-dependent $C_0^{n,\beta-}$-vector fields.** Let $n\in\mathbb{N}_{\ge1}$.

By a *strong time-dependent $C_0^{n,\beta-}$-vector field*, for $\beta\in(0,1]$, we mean a function $u: I\to C_0^{n,\beta-}(\mathbb{R}^d,\mathbb{R}^d)$ such that $u\in L^1(I,C_0^{n,\alpha}(\mathbb{R}^d,\mathbb{R}^d))$ for all $\alpha<\beta$. We denote the space of all strong time-dependent $C_0^{n,\beta-}$-vector fields by $L^1(I,C_0^{n,\beta-}(\mathbb{R}^d,\mathbb{R}^d))$ and equip it with the fundamental system of seminorms $\{\|\cdot\|_{L^1(I,C_b^{n,\alpha})} : \alpha<\beta\}$.

Clearly, for every strong time-dependent $C_0^{n,\beta-}$-vector field u, $u^\wedge$ is a pointwise time-dependent $C_0^{n,\beta-}$-vector field (as defined in Section 4.4); the converse is not true in general.

By Proposition 5.5, the flow map $L^1(I,C_0^{n,\beta-}(\mathbb{R}^d,\mathbb{R}^d))\to C(I,\mathcal{D}^{n,\beta-}(\mathbb{R}^d))$ is bounded, for all $n\in\mathbb{N}_{\ge1}$, $\beta\in(0,1]$.

Theorem 5.7. *Let $n\in\mathbb{N}_{\ge2}$. For $\beta\in(0,1]$, the flow map $L^1(I,C_0^{n,\beta-}(\mathbb{R}^d,\mathbb{R}^d))\to C(I,\mathcal{D}^{n,\beta-}(\mathbb{R}^d))$, $u\mapsto\Phi$, is continuous and $C^{0,\omega}$, for any slowly vanishing modulus of continuity ω.*

Proof. This is immediate from Theorem 5.6 and from the estimates in its proof. □

Remark 5.8. One could define *strong time-dependent $C_0^{n,\beta+}$-vector fields*, for $\beta\in[0,1)$, to be the elements of the (LB)-space $\bigcup_{\alpha\in(\beta,1)} L^1(I,C_0^{n,\alpha}(\mathbb{R}^d,\mathbb{R}^d))$. Then, by Theorem 4.6, we have a flow map

$$\bigcup_{\alpha\in(\beta,1)} L^1(I,C_0^{n,\alpha}(\mathbb{R}^d,\mathbb{R}^d))\to\bigcup_{\alpha\in(\beta,1)} C(I,C_0^{n,\alpha}(\mathbb{R}^d,\mathbb{R}^d))\subseteq C(I,C_0^{n,\beta+}(\mathbb{R}^d,\mathbb{R}^d)).$$

Note that this is not clear for $u\in L^1(I,C_0^{n,\beta+}(\mathbb{R}^d,\mathbb{R}^d))$, since such u may not factor to some step in the inductive limit defining $C_0^{n,\beta+}(\mathbb{R}^d,\mathbb{R}^d)$. Is this flow map $C^{0,\omega}$, for slowly vanishing moduli of continuity ω? This would follow from Theorem 5.6 if the (LB)-space $\bigcup_{\alpha\in(\beta,1)} L^1(I,C_0^{n,\alpha}(\mathbb{R}^d,\mathbb{R}^d))$ were regular.

Appendix A. Proofs for Section 2.4

Proposition 2.6 is precisely [3, 4.2].

Proof of Theorem 2.7. We prove the assertion by induction on m. First observe that $d(g\circ(\mathrm{Id}+f)) = dg\circ(\mathrm{Id}+f)\cdot(\mathbb{1}+df) = dg\circ(\mathrm{Id}+f)+dg\circ(\mathrm{Id}+f)\cdot df$. We have

$$\begin{aligned}\|dg(x+f(x))-dg(y+f(y))\|_{L_1} &\le \|dg\|_{0,\alpha}\|x-y+f(x)-f(y)\|^\alpha\\ &\le \|g\|_{1,\alpha}(1+\|f\|_1)^\alpha\|x-y\|^\alpha,\end{aligned} \tag{A.1}$$

and

$$\begin{aligned}
&\|dg(x+f(x))\cdot df(x)-dg(y+f(y))\cdot df(y)\|_{L_1}\\
&\qquad\le \|dg(x+f(x))\cdot df(x)-dg(x+f(x))\cdot df(y)\|_{L_1}\\
&\qquad\qquad+\|dg(x+f(x))\cdot df(y)-dg(y+f(y))\cdot df(y)\|_{L_1}\\
&\qquad\le \|dg(x+f(x))\|_{L_1}\|df(x)-df(y)\|_{L_1}\\
&\qquad\qquad+\|dg(x+f(x))-dg(y+f(y))\|_{L_1}\|df(y)\|_{L_1}\\
&\qquad\le \|g\|_{1,\alpha}\|f\|_{1,\alpha}\|x-y\|^\alpha+\|g\|_{1,\alpha}(1+\|f\|_1)^\alpha\|x-y\|^\alpha\|f\|_{1,\alpha}.
\end{aligned}$$

Thus,

$$\|dg\circ(\mathrm{Id}+f)\cdot df\|_{0,\alpha}\le 2\|g\|_{1,\alpha}(1+\|f\|_{1,\alpha})^{1+\alpha},$$

and since the same bound is trivially also valid for $\|g\circ(\mathrm{Id}+f)\|_0$, the case $m=1$ is proved.

Now assume the statement holds for $m-1$. Then

$$\|d(g\circ(\mathrm{Id}+f))\|_{m-1,\alpha}\le\|dg\circ(\mathrm{Id}+f)\|_{m-1,\alpha}+\|dg\circ(\mathrm{Id}+f)\cdot df\|_{m-1,\alpha}.$$

The inductive assumption implies

$$\|dg\circ(\mathrm{Id}+f)\|_{m-1,\alpha}\le M\|dg\|_{m-1,\alpha}(1+\|f\|_{m-1,\alpha})^{m-1+\alpha},$$

and using Proposition 2.6, we get

$$\|dg\circ(\mathrm{Id}+f)\cdot df\|_{m-1,\alpha}\le 2^m\|dg\circ(\mathrm{Id}+f)\|_{m-1,\alpha}\cdot\|df\|_{m-1,\alpha}$$

which now adds up to (2.4). □

Proof of Theorem 2.8. We proceed by induction on m. First observe that we have

$$\|g_\star(f_1)-g_\star(f_2)\|_0\le\|g\|_1\|f_1-f_2\|_0.$$

Moreover, by Proposition 2.6,

$$\begin{aligned}
&\|d(g_\star(f_1))-d(g_\star(f_2))\|_{0,\alpha}\\
&\quad=\|dg\circ(\mathrm{Id}+f_1)\cdot(\mathbb{1}+df_1)-dg\circ(\mathrm{Id}+f_2)\cdot(\mathbb{1}+df_2)\|_{0,\alpha}\\
&\quad=\|dg\circ(\mathrm{Id}+f_1)\cdot(df_1-df_2)-(dg\circ(\mathrm{Id}+f_2)-dg\circ(\mathrm{Id}+f_1))\cdot(\mathbb{1}+df_2)\|_{0,\alpha}\\
&\quad\le 2\|dg\circ(\mathrm{Id}+f_1)\|_{0,\alpha}\|df_1-df_2\|_{0,\alpha}\\
&\qquad+\|dg\circ(\mathrm{Id}+f_2)-dg\circ(\mathrm{Id}+f_1)\|_{0,\alpha}(1+2\|df_2\|_{0,\alpha}).
\end{aligned}$$

As an intermediate step we use Lemma 2.1 and (A.1) to estimate

$$\begin{aligned}
&\|dg\circ(\mathrm{Id}+f_2)-dg\circ(\mathrm{Id}+f_1)\|_{0,\alpha}\\
&\quad\le\|dg\circ(\mathrm{Id}+f_2)-dg\circ(\mathrm{Id}+f_1)\|_0^{\frac{\beta-\alpha}{\beta}}\|dg\circ(\mathrm{Id}+f_2)-dg\circ(\mathrm{Id}+f_1)\|_{0,\beta}^{\frac{\alpha}{\beta}}\\
&\quad\le(\|g\|_{1,\beta}\|f_1-f_2\|_0^\beta)^{\frac{\beta-\alpha}{\beta}}(\|dg\circ(\mathrm{Id}+f_1)\|_{0,\beta}+\|dg\circ(\mathrm{Id}+f_2)\|_{0,\beta})^{\frac{\alpha}{\beta}}\\
&\quad\le(\|g\|_{1,\beta}\|f_1-f_2\|_0^\beta)^{\frac{\beta-\alpha}{\beta}}(\|g\|_{1,\beta}((1+\|f_1\|_1)^\beta+(1+\|f_2\|_1)^\beta))^{\frac{\alpha}{\beta}}\\
&\quad\le\|g\|_{1,\beta}(2+\|f_1\|_1+\|f_2\|_1)\|f_1-f_2\|_0^{\beta-\alpha}.
\end{aligned}$$

Consequently, if $R>0$, $f_1,f_2\in B^{1,\alpha}(f_0,R)$, and hence $\|f_1-f_2\|_{1,\alpha}\le(1+2R)\|f_1-f_2\|_{1,\alpha}^{\beta-\alpha}$, then

$$\|d(g_\star(f_1))-d(g_\star(f_2))\|_{0,\alpha}\le M\|g\|_{1,\beta}\|f_1-f_2\|_{1,\alpha}^{\beta-\alpha},$$

where $M = M(\|f_0\|_{1,\alpha}, R)$, and hence

$$\|g_\star(f_1) - g_\star(f_2)\|_{1,\alpha} \leq M\|g\|_{1,\beta}\|f_1 - f_2\|_{1,\alpha}^{\beta-\alpha}$$

which proves the case $m = 1$.

Now assume we have already proven the desired result for $m - 1$. Then, as in the case $m = 1$, we have

$$\begin{aligned}&\|d(g_\star(f_1)) - d(g_\star(f_2))\|_{m-1,\alpha}\\ &\qquad \leq 2^m\|dg \circ (\mathrm{Id} + f_1)\|_{m-1,\alpha}\|df_1 - df_2\|_{m-1,\alpha}\\ &\qquad\quad + \|dg \circ (\mathrm{Id} + f_2) - dg \circ (\mathrm{Id} + f_1)\|_{m-1,\alpha}(1 + 2^m\|df_2\|_{m-1,\alpha}).\end{aligned}$$

By the inductive assumption,

$$\|dg \circ (\mathrm{Id} + f_2) - dg \circ (\mathrm{Id} + f_1)\|_{m-1,\alpha} \leq M\|g\|_{m,\beta}\|f_1 - f_2\|_{m-1,\alpha}^{\beta-\alpha}.$$

Together with Theorem 2.7, which makes it possible to extract $\|g\|_{m,\beta}$ from the term $\|dg \circ (\mathrm{Id} + f_1)\|_{m-1,\alpha}$, and using that $\|f_1 - f_2\|_{m,\alpha} \leq (1 + 2R)\|f_1 - f_2\|_{m,\alpha}^{\beta-\alpha}$ for $f_1, f_2 \in B^{m,\alpha}(f_0, R)$, we may conclude (2.5). □

Proof of Corollary 2.9. This follows easily from Theorem 2.7, Theorem 2.8, and

$$g_1 \circ (\mathrm{Id} + f_1) - g_2 \circ (\mathrm{Id} + f_2) = f_1^\star(g_1 - g_2) + (g_2)_\star(f_1) - (g_2)_\star(f_2). \qquad \square$$

Proof of Theorem 2.10. By Theorem 2.8, the mapping $(dg)_\star : C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d) \to C_b^{m,\alpha}(\mathbb{R}^d, L(\mathbb{R}^d, \mathbb{R}^d))$, $\phi \mapsto dg \circ (\mathrm{Id} + \phi)$ is continuous. Consider the mapping

$$\begin{aligned} l : C_b^{m,\alpha}(\mathbb{R}^d, L(\mathbb{R}^d, \mathbb{R}^d)) &\to L(C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d), C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d))\\ u &\mapsto l(u)(\eta) := (x \mapsto u(x)(\eta(x)))\end{aligned}$$

which is continuous and linear. We claim that $d(g_\star)$ exists and satisfies $d(g_\star) = l \circ (dg)_\star$. This implies the proposition.

First note that for $\psi_0 \in C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$ and $\phi \in C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$,

$$(l \circ (dg)_\star)(\psi_0)(\phi)(x) = dg \circ (\mathrm{Id} + \psi_0)(x) \cdot \phi(x),$$

where $\cdot$ denotes the action of the linear map $dg \circ (\mathrm{Id} + \psi_0)(x) \in L(\mathbb{R}^d, \mathbb{R}^d)$ to the vector $\phi(x) \in \mathbb{R}^d$.

Take $\phi \in C_b^{m,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$ with $\|\phi\|_{m,\alpha} \leq 1$. By Theorem 2.8 (applied to dg), for $\psi_1 \in B^{m,\alpha}(\psi_0, 1)$,

$$\|(dg)_\star(\psi_0) - (dg)_\star(\psi_1)\|_{m,\alpha} \leq M\|g\|_{m+1,\beta}\|\psi_0 - \psi_1\|_{m,\alpha}^{\beta-\alpha}, \tag{A.2}$$

For $\varepsilon < 1$ we have $\psi_0 + \varepsilon\phi \in B^{m,\alpha}(\psi_0, 1)$ for all $\phi \in B^{m,\alpha}(0,1)$. Now, by Proposition 2.6 and (A.2),

$$\begin{aligned}
&\frac{1}{\varepsilon}\|g_\star(\psi_0+\varepsilon\phi) - g_\star(\psi_0) - (l \circ (dg)_\star)(\psi_0)(\varepsilon\phi)\|_{m,\alpha} \\
&\quad = \frac{1}{\varepsilon}\|g \circ (\mathrm{Id} + \psi_0 + \varepsilon\phi) - g \circ (\mathrm{Id} + \psi_0) - \varepsilon(dg \circ (\mathrm{Id} + \psi_0)) \cdot \phi\|_{m,\alpha} \\
&\quad = \Big\| \int_0^1 (dg \circ (\mathrm{Id} + \psi_0 + s\varepsilon\phi) - dg \circ (\mathrm{Id} + \psi_0)) \cdot \phi \, ds \Big\|_{m,\alpha} \\
&\quad \le \int_0^1 2\|dg \circ (\mathrm{Id} + \psi_0 + s\varepsilon\phi) - dg \circ (\mathrm{Id} + \psi_0)\|_{m,\alpha} \|\phi\|_{m,\alpha} \, ds \\
&\quad \le \int_0^1 2M\|g\|_{m+1,\beta} \|\varepsilon s\phi\|_{m,\alpha}^{\beta-\alpha} \, ds \\
&\quad \le 2M\|g\|_{m+1,\beta}\varepsilon^{\beta-\alpha}
\end{aligned}$$

which tends to 0 uniformly in $\phi \in B^{m,\alpha}(0,1)$ as $\varepsilon \to 0$. The claim is proved. □

Proof of Corollary 2.11. Let $\gamma(s) := (1-s)f_1 + sf_2$ for $s \in [0,1]$. Using Lemma 2.3 and Theorem 2.10, we get

$$\begin{aligned}
g_\star(f_1) - g_\star(f_2) &= \int_0^1 \frac{d}{ds}(g_\star \circ \gamma)(s)\, ds = \int_0^1 d(g_\star)(\gamma(s)) \cdot \gamma'(s)\, ds \\
&= \int_0^1 dg \circ (\mathrm{Id} + \gamma(s)) \cdot (f_2 - f_1)\, ds.
\end{aligned}$$

Thus, by Proposition 2.6 and Theorem 2.7,

$$\begin{aligned}
&\|g_\star(f_1) - g_\star(f_2)\|_{m,\alpha} \\
&\quad \le \int_0^1 \|dg \circ (\mathrm{Id} + \gamma(s)) \cdot (f_2 - f_1)\|_{m,\alpha}\, ds \\
&\quad \le \int_0^1 M\|dg\|_{m,\beta}(1 + \|\gamma(s)\|_{m,\alpha})^{m+1}\|f_2 - f_1\|_{m,\alpha}\, ds \\
&\quad \le M\|g\|_{m+1,\beta}(1 + \max_{i=1,2}\|f_i\|_{m,\alpha})^{m+1}\|f_2 - f_1\|_{m,\alpha}.
\end{aligned}$$
□

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