ADDENDUM TO: “LIFTING SMOOTH CURVES OVER INVARIANTS FOR REPRESENTATIONS OF COMPACT LIE GROUPS, III” [J. LIE THEORY 16 (2006), NO. 3, 579–600.]

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Abstract. We improve the main results in [10] using a recent refinement of Bronshtein’s theorem [5] due to Colombini, Orrú, and Pernazza [6]. They are then in general best possible both in the hypothesis and in the outcome. As a consequence we obtain a result on lifting smooth mappings in several variables.

A recent refinement of Bronshtein’s theorem [5] and of some of its consequences due to Colombini, Orrú, and Pernazza [6] (namely theorem 1 below) allows to essentially improve our main results in [10]; see theorem 2 and corollary 3 below. The improvement consists in weakening the hypothesis considerably: In [10] we needed a curve $c$ to be of class (i) $C^k$ in order to admit a differentiable lift with locally bounded derivative, (ii) $C^{k+d}$ in order to admit a $C^1$-lift, and (iii) $C^{k+2d}$ in order to admit a twice differentiable lift.

It turns out that theorem 2 and corollary 3 are in general best possible both in the hypothesis and in the outcome. In theorem 4 and corollary 5 we deduce some results on lifting smooth mappings in several variables.

Refinement of Bronshtein’s theorem. Bronshtein’s theorem [5] (see also Wakahayashi’s version [15]) states that, for a curve of monic hyperbolic polynomials

$$P(t)(x) = x^n + \sum_{j=1}^{n} (-1)^j a_j(t)x^{n-j}.$$  

with coefficients $a_j \in C^n(\mathbb{R})$ (1 ≤ $j$ ≤ n), there exist differentiable functions $\lambda_j$ (1 ≤ $j$ ≤ n) with locally bounded derivatives which parameterize the roots of $P$. A polynomial is called hyperbolic if all its roots are real.

The following theorem refines Bronshtein’s theorem [5] and also a result of Mandai [14] and a result of Kriegl, Losik, and Michor [8]. In [14] the coefficients are required to be of class $C^{2n}$ for $C^1$-roots, and in [8] they are assumed to be $C^{3n}$ for twice differentiable roots.

1. Theorem ([6, 2.1]). Consider a curve $P$ of monic hyperbolic polynomials (1). Then:

(i) If $a_j \in C^n(\mathbb{R})$ (1 ≤ $j$ ≤ n), then there exist functions $\lambda_j \in C^1(\mathbb{R})$ (1 ≤ $j$ ≤ n) which parameterize the roots of $P$.

(ii) If $a_j \in C^{2n}(\mathbb{R})$ (1 ≤ $j$ ≤ n), then the roots of $P$ may be chosen twice differentiable.

Counterexamples (e.g. in [6, section 4]) show that in this result the assumptions on $P$ cannot be weakened.

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**Improvement of the results in [10].** Let \( \rho : G \to O(V) \) be an orthogonal representation of a compact Lie group \( G \) in a real finite dimensional Euclidean vector space \( V \). Choose a minimal system of homogeneous generators \( \sigma_1, \ldots, \sigma_n \) of the algebra \( \mathbb{R}[V]^G \) of \( G \)-invariant polynomials on \( V \). Define

\[
d = d(\rho) := \max\{\deg \sigma_i : 1 \leq i \leq n\},
\]

which is independent of the choice of the \( \sigma_i \) (see [10, 2.4]).

If \( G \) is a finite group, we write \( V = V_1 \oplus \cdots \oplus V_l \) as orthogonal direct sum of irreducible subspaces \( V_i \). We choose \( v_i \in V_i \setminus \{0\} \) such that the cardinality of the corresponding isotropy group \( G_{v_i} \) is maximal, and put

\[
k = k(\rho) := \max\{d(\rho), |G|/|G_{v_i}| : 1 \leq i \leq l\}.
\]

The mapping \( \sigma = (\sigma_1, \ldots, \sigma_n) : V \to \mathbb{R}^n \) induces a homeomorphism between the orbit space \( V/G \) and the image \( \sigma(V) \). Let \( c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n \) be a smooth curve in the orbit space (smooth as a curve in \( \mathbb{R}^n \)). A curve \( \tilde{c} : \mathbb{R} \to V \) is called lift of \( c \) if \( \sigma \circ \tilde{c} = c \). The problem of lifting curves smoothly over invariants is independent of the choice of the \( \sigma_i \) (see [10, 2.2]).

2. **Theorem.** Let \( \rho : G \to O(V) \) be a representation of a finite group \( G \). Let \( d = d(\rho) \) and \( k = k(\rho) \). Consider a curve \( c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n \) in the orbit space of \( \rho \). Then:

(i) If \( c \) is of class \( C^k \), then any differentiable lift \( \tilde{c} : \mathbb{R} \to V \) of \( c \) (which always exists) is actually \( C^1 \).

(ii) If \( c \) is of class \( C^{k+d} \), then there exists a global twice differentiable lift \( \tilde{c} : \mathbb{R} \to V \) of \( c \).

**Proof.** (i) Let \( \tilde{c} \) be any differentiable lift of \( c \). Note that the existence of \( \tilde{c} \) is guaranteed for any \( C^d \)-curve \( c \), by [1]. In the proof of [10, 8.1] we construct curves of monic hyperbolic polynomials \( t \mapsto P(t) \) which have the regularity of \( \tilde{c} \) and whose roots are parameterized by \( t \mapsto \langle v_i \mid g\tilde{c}(t) \rangle \ (g \in G_{v_i} \setminus G) \).

If \( c \) is of class \( C^k \), then theorem [1(i)] provides \( C^1 \)-roots of \( t \mapsto P(t) \). By the proof of [10, 4.2] we obtain that the parameterization \( t \mapsto \langle v_i \mid g\tilde{c}(t) \rangle \) is \( C^1 \) as well. Hence \( \tilde{c} \) is a \( C^1 \)-lift of \( c \). Alternatively, the proof of [1(i)] in [5] actually shows that any differentiable choice of roots is \( C^1 \).

(ii) Let \( c \) be of class \( C^{k+d} \). The existence of a global twice differentiable lift \( \tilde{c} \) follows from the proof of [10, 5.1 and 5.2], where we use (i) instead of [10, 4.2].

3. **Corollary.** Let \( \rho : G \to O(V) \) be a polar representation of a compact Lie group \( G \). Let \( \Sigma \subseteq V \) be a section, \( W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma) \) its generalized Weyl group, and \( \rho_\Sigma : W(\Sigma) \to O(\Sigma) \) the induced representation. Let \( d = d(\rho_\Sigma) \) and \( k = k(\rho_\Sigma) \). Consider a curve \( c : \mathbb{R} \to V/G = \sigma(\Sigma) \subseteq \mathbb{R}^n \) in the orbit space of \( \rho \). Then:

(i) If \( c \) is of class \( C^k \), then there exists a global orthogonal \( C^1 \)-lift \( \tilde{c} : \mathbb{R} \to V \) of \( c \).

(ii) If \( c \) is of class \( C^{k+d} \), then there exists a global orthogonal twice differentiable lift \( \tilde{c} : \mathbb{R} \to V \) of \( c \).

The examples which show that the hypothesis in [11] are best possible also imply that in general the hypothesis in [2] and [3] cannot be improved.

On the other hand the outcome of [2] and [3] cannot be refined either: A \( C^\infty \)-curve \( c \) does in general not allow a \( C^{1,\alpha} \)-lift for any \( \alpha > 0 \). See [7], [1], [4]. But see also [3] and [10, remark 4.2].

Note that the improvement affects also [13] part 6.
Lifting smooth mappings in several variables. From theorem 2 we can deduce a lifting result for mappings in several variables.

4. Theorem. Let \( \rho : G \to O(V) \) be a representation of a finite group \( G \), \( d = d(\rho) \), and \( k = k(\rho) \). Let \( U \subseteq \mathbb{R}^q \) be open. Consider a mapping \( f : U \to V/G = \sigma(V) \subseteq \mathbb{R}^n \) of class \( C^k \). Then any continuous lift \( \bar{f} : U \to V \) of \( f \) is actually locally Lipschitz.

Proof. Let \( c : \mathbb{R} \to U \) be a \( C^\infty \)-curve. By theorem 2(i) the curve \( f \circ c \) admits a \( C^1 \)-lift \( \bar{f} \circ c \). A further continuous lift of \( f \circ c \) is formed by \( \bar{f} \circ c \). By [12, 5.3] we can conclude that \( f \circ c \) is locally Lipschitz. So we have shown that \( \bar{f} \) is locally Lipschitz along \( C^\infty \)-curves. By Boman [2] (see also [11, 12.7]) that implies that \( \bar{f} \) is locally Lipschitz. □

In general there will not always exist a continuous lift of \( f \) (for instance, if \( G \) is a finite rotation group and \( f \) is defined near 0). However, if \( G \) is a finite reflection group, then any continuous \( f \) allows a continuous lift (since the orbit space can be embedded homeomorphically in \( V \)).

5. Corollary. Let \( \rho : G \to O(V) \) be a polar representation of a compact connected Lie group \( G \). Let \( \Sigma \subseteq V \) be a section, \( W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma) \) its generalized Weyl group, \( \rho_\Sigma : W(\Sigma) \to O(\Sigma) \) the induced representation, \( d = d(\rho_\Sigma) \), and \( k = k(\rho_\Sigma) \). Let \( U \subseteq \mathbb{R}^q \) be open. Consider a mapping \( f : U \to V/G = \sigma(V) \subseteq \mathbb{R}^n \) of class \( C^k \). Then there exists an orthogonal lift \( \tilde{f} : U \to V \) of \( f \) which is actually locally Lipschitz.

Proof. The Weyl group \( W(\Sigma) \) is a finite reflection group, since \( G \) is connected. □

References

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