# A NEW PROOF OF BRONSHTEIN'S THEOREM

ADAM PARUSIŃSKI AND ARMIN RAINER

Abstract. We give a new self-contained proof of Bronshtein's theorem, that any continuous root of a $C^{n-1,1}$-family of monic hyperbolic polynomials of degree $n$ is locally Lipschitz, and obtain explicit bounds for the Lipschitz constant of the root in terms of the coefficients. As a by-product we reprove the recent result of Colombini, Orrú, and Pernazza, that a $C^n$-curve of hyperbolic polynomials of degree $n$ admits a $C^1$-system of its roots.

## 1. Introduction

Choosing regular roots of polynomials whose coefficients depend on parameters is a classical much studied problem with important connections to various fields such as algebraic geometry, partial differential equations, and perturbation theory.

This problem is of special interest for *hyperbolic* polynomials whose roots are all real. Probably the first result in this direction was obtained by Glaeser [9] who studied the square root of a nonnegative smooth function. The most important and most difficult result in this field is Bronshtein's theorem [6]: any continuous root of a $C^{p-1,1}$-curve of monic hyperbolic polynomials, where $p$ is the maximal multiplicity of the roots, is locally Lipschitz with uniform Lipschitz constants; cf. Theorem 2.1. A multiparameter version follows immediately; see Theorem 2.2. A different proof was later given by Wakabayashi [23] who actually proved a more general Hölder version; for a refinement of Bronshtein's method in order to show this generalization see Tarama [22]. Kurdyka and Paunescu [11] used resolution of singularities to show that the roots of a hyperbolic polynomial whose coefficients are real analytic functions in several variables admit a parameterization which is locally Lipschitz; in one variable we have Rellich's classical theorem [20] that the roots may be parameterized by real analytic functions.

A $C^p$-curve of monic hyperbolic polynomials with at most $p$-fold roots admits a differentiable system of its roots. Using Bronshtein's theorem, Mandai [12] showed that the roots can be chosen $C^1$ if the coefficients are $C^{2p}$, and Kriegl, Losik, and Michor [10] found twice differentiable roots provided that the coefficients are $C^{3p}$. Recently, Colombini, Orrú and Pernazza [7] proved that $C^p$ (resp. $C^{2p}$) coefficients suffice for $C^1$ (resp. twice differentiable) roots and that this statement is best possible.

In this paper we present a new proof of Bronshtein's theorem. Our proof is simple and elementary. The main tool is the splitting principle, a criterion that allows to factorize

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polynomials under elementary assumptions. The coefficients of the factors can be expressed in a simple way in terms of the coefficients of the original polynomials, so that the bounds on the coefficients and their derivatives can be also carried over. Thanks to this we obtain explicit bounds on the Lipschitz constant of the roots. As a by-product we give a new proof of the aforementioned result of Colombini, Orrú, and Pernazza on the existence of $C^1$-roots; see Theorem 2.4.

Note that the statements of Theorem 2.1, Theorem 2.2, and Theorem 2.4 are best possible in the following sense. If the coefficients are just $C^{p-1,1}$ then the roots need not admit a differentiable parameterization. Moreover, the roots can in general not be parameterized by $C^{1,\alpha}$-functions for any $\alpha > 0$ even if the coefficients are $C^\infty$. Some better conclusions can be obtained if additional assumptions are made; see [1], [3], [4], [5], [15], [17].

**Convention.** We will denote by $C(n,\ldots)$ *any* constant depending only on $n,\ldots$; it may change from line to line. Specific constants will bear a subscript like $C_1(n)$ or $C_2(n)$.

## 2. Bronshtein's theorem

Let $I \subseteq \mathbb{R}$ be an open interval and consider a monic polynomial

$$P_a(t)(Z) = P_{a(t)}(Z) = Z^n + \sum_{j=1}^n a_j(t) Z^{n-j}, \quad t \in I.$$

We say that $P_a(t)$, $t \in I$, is a *$C^{p-1,1}$-curve of hyperbolic polynomials* if $(a_j)_{j=1}^n \in C^{p-1,1}(I,\mathbb{R}^n)$ and all roots of $P_a(t)$ are real for each $t \in I$.

Note that ordering the roots of a hyperbolic polynomial $P_a(Z) = Z^n + \sum_{j=1}^n a_j Z^{n-j}$ increasingly, $\lambda_1^\uparrow(a) \le \lambda_2^\uparrow(a) \le \cdots \le \lambda_n^\uparrow(a)$, provides a continuous mapping $\lambda^\uparrow = (\lambda_j^\uparrow)_{j=1}^n : H_n \to \mathbb{R}^n$ on the space of hyperbolic polynomials of degree $n$, see e.g. [1, Lemma 4.1], which can be identified with a closed semialgebraic subset $H_n \subseteq \mathbb{R}^n$, see e.g. [13].

By a *system of the roots* of $P_a(t)$, $t \in I$, we mean any $n$-tuple $\lambda = (\lambda_j)_{j=1}^n : I \to \mathbb{R}^n$ satisfying

$$P_a(t)(Z) = \prod_{j=1}^n (Z - \lambda_j(t)), \quad t \in I.$$

Note that any *continuous root* $\mu_1$ of $P_a(t)$, $t \in I$, i.e., $\mu_1 \in C^0(I,\mathbb{R})$ and $P_a(t)(\mu_1(t)) = 0$ for all $t \in I$, can be completed to a continuous system of the roots $\mu = (\mu_j)_{j=1}^n$, cf. [16, Lemma 6.17].

**Theorem 2.1** (Bronshtein's theorem)**.** *Let $P_a(t)$, $t \in I$, be a $C^{p-1,1}$-curve of hyperbolic polynomials of degree $n$, where $p$ is the maximal multiplicity of the roots of $P_a$. Then any continuous root of $P_a$ is locally Lipschitz.*

*Moreover if $p = n$ then for any pair of intervals $I_0 \Subset I_1 \Subset I$ and for any continuous root $\lambda(t)$ its Lipschitz constant can be bounded as follows*

$$\begin{aligned} \text{(2.1)} \qquad \operatorname{Lip}_{I_0}(\lambda) &\le C(n, I_0, I_1) \big( \max_i \|a_i\|^{\frac{1}{i}}_{C^{n-1,1}(\overline{I}_1)} \big) \\ &\le \tilde{C}(n, I_0, I_1) \big( 1 + \max_i \|a_i\|_{C^{n-1,1}(\overline{I}_1)} \big), \end{aligned}$$

*where the constants $C(n, I_0, I_1)$, $\tilde{C}(n, I_0, I_1)$ depend only on $n$ and the intervals $I_0, I_1$. (More precise bounds are stated in Subsection 4.6.)*

*If $p < n$ then there exist uniform bounds on the Lipschitz constant provided the multiplicities of roots are at most $p$ "in a uniform way". These bounds are stated in Subsection 4.7.*

For an open subset $U \subseteq \mathbb{R}^m$ and $p \in \mathbb{N}_{\geq 1}$, we denote by $C^{p-1,1}(U)$ the space of all functions $f \in C^{p-1}(U)$ so that each partial derivative $\partial^\alpha f$ of order $|\alpha| = p - 1$ is locally Lipschitz. It is a Fréchet space with the following system of seminorms,

$$\|f\|_{C^{p-1,1}(K)} = \|f\|_{C^{p-1}(K)} + \sup_{|\alpha|=p-1} \mathrm{Lip}_K(\partial^\alpha f), \quad \mathrm{Lip}_K(f) = \sup_{\substack{x,y\in K\\ x\neq y}} \frac{|f(x) - f(y)|}{\|x - y\|},$$

where $K$ ranges over (a countable exhaustion of) the compact subsets of $U$; on $\mathbb{R}^m$ we consider the 2-norm $\| \; \| = \| \; \|_2$.

By Rademacher's theorem, the partial derivatives of order $p$ of a function $f \in C^{p-1,1}(U)$ exist almost everywhere and coincide almost everywhere with the corresponding weak partial derivatives.

Theorem 2.1 readily implies the following multiparameter version.

**Theorem 2.2.** *Let $U \subseteq \mathbb{R}^m$ be open and let $P_a(x)$, $x \in U$, be a $C^{p-1,1}$-family of hyperbolic polynomials of degree $n$, where $p$ is the maximal multiplicity of the roots of $P_a$. Then any continuous root of $P_a$ is locally Lipschitz.*

*Moreover, if $p = n$ for any pair of relatively compact subsets $U_0 \Subset U_1 \Subset U$ and for any continuous root $\lambda(x)$ its Lipschitz constant can be bounded as follows*

$$\begin{aligned} \mathrm{Lip}_{U_0}(\lambda) &\leq C(m, n, U_0, U_1) \big( \max_i \|a_i\|^{\frac{1}{i}}_{C^{n-1,1}(\overline{U}_1)} \big) \\ &\leq \tilde{C}(m, n, U_0, U_1) \big(1 + \max_i \|a_i\|_{C^{n-1,1}(\overline{U}_1)}\big), \end{aligned} \tag{2.2}$$

*where the constants $C(m, n, U_0, U_1)$, $\tilde{C}(m, n, U_0, U_1)$ depend only on $m$, $n$, and the sets $U_0, U_1$.*

*Proof.* Let $\lambda$ be a continuous root of $P_a$. Without loss of generality we may assume that $U_0$ and $U_1$ are open boxes parallel to the coordinate axes, $U_i = \prod_{j=1}^m I_{i,j}$, $i = 0, 1$, with $I_{0,j} \Subset I_{1,j}$ for all $j$. Let $x, y \in U_0$ and set $h := y - x$. Let $\{e_j\}_{j=1}^m$ denote the standard unit vectors in $\mathbb{R}^m$. For any $z$ in the orthogonal projection of $U_0$ on the hyperplane $x_j = 0$ consider the function $\lambda_{z,j} : I_{0,j} \to \mathbb{R}$ defined by $\lambda_{z,j}(t) := \lambda(z + te_j)$. By Theorem 2.1, each $\lambda_{z,j}$ is Lipschitz and $C := \sup_{z,j} \mathrm{Lip}_{I_{0,j}}(\lambda_{z,j}) < \infty$. Thus

$$|\lambda(x) - \lambda(y)| \leq \sum_{j=0}^{m-1} \Big|\lambda\big(x + \sum_{k=1}^{j} h_k e_k\big) - \lambda\big(x + \sum_{k=1}^{j+1} h_k e_k\big)\Big| \leq C\|h\|_1 \leq C\sqrt{m}\|h\|_2.$$

The bounds (2.2) follow from (2.1). □

**Corollary 2.3.** *Let $U \subseteq \mathbb{R}^m$ be open. The push forward $(\lambda^\uparrow)_* : C^{n-1,1}(U, \mathbb{R}^n) \supseteq C^{n-1,1}(U, H_n) \to C^{0,1}(U, \mathbb{R}^n)$ is bounded.*

Next we suppose that $P_a(t)$, $t \in I$, is a $C^p$-curve of hyperbolic polynomials of degree $n$, where $p$ is the maximal multiplicity of the roots of $P_a$. Then the roots can be chosen $C^1$. We will give a new proof of this recent result of [7], see Theorem 2.4.

For a function $f(t)$ we denote by $f'^-(t_0)$ (resp. $f'^+(t_0)$) the left (resp. right) derivative of $f$ at the point $t_0$.

**Theorem 2.4.** *Let $P_a(t)$, $t \in I$, be a $C^p$-curve of hyperbolic polynomials of degree $n$, where $p$ is the maximal multiplicity of the roots of $P_a$. Then:*

1. *Any continuous root $\lambda(t)$ of $P_a$ has both one-sided derivatives at every $t \in I$.*
2. *These derivatives are continuous: for every $t_0 \in I$ we have*
$$\lim_{t \to t_0^-} \lambda'^{\pm}(t) = \lambda'^-(t_0) \qquad \lim_{t \to t_0^+} \lambda'^{\pm}(t) = \lambda'^+(t_0).$$
3. *There exists a differentiable system of the roots.*
4. *Any differentiable root is $C^1$.*

## 3. Preliminaries

3.1. **Tschirnhausen transformation.** A monic polynomial

$$P_a(Z) = Z^n + \sum_{j=1}^{n} a_j Z^{n-j}, \quad a = (a_1, ..., a_n) \in \mathbb{R}^n_a$$

is said to be in *Tschirnhausen form* if $a_1 = 0$. Every $P_a$ can be transformed to such a form by the substitution $Z \mapsto Z - \frac{a_1}{n}$, which we refer to as *Tschirnhausen transformation*,

$$P_{\tilde{a}}(Z) = P_a(Z - \frac{a_1}{n}) = Z^n + \sum_{j=2}^{n} \tilde{a}_j Z^{n-j}, \quad \tilde{a} = (\tilde{a}_2, ..., \tilde{a}_n) \in \mathbb{R}^{n-1}_{\tilde{a}}. \tag{3.1}$$

We identify the set of monic real polynomials $P_a$ of degree $n$ with $\mathbb{R}^n_a$, where $a = (a_1, a_2, \ldots, a_n)$, and those in Tschirnhausen form with $\mathbb{R}^{n-1}_{\tilde{a}}$. In what follows we write the effect of the Tschirnhausen transformation on a polynomial $P_a$ simply by adding tilde, $P_{\tilde{a}}$.

Thus let $P_{\tilde{a}}$ be a monic polynomial in Tschirnhausen form. Then

$$s_2 = -2\tilde{a}_2 = \lambda_1^2 + \cdots + \lambda_n^2,$$

where the $s_i$ denote the Newton polynomials in the roots $\lambda_j$ of $P_a$. Thus, for a hyperbolic polynomial $P_{\tilde{a}}$ in Tschirnhausen form,

$$s_2 = -2\tilde{a}_2 \geq 0.$$

**Lemma 3.1.** *The coefficients of a hyperbolic polynomial $P_{\tilde{a}}$ in Tschirnhausen form satisfy*

$$|\tilde{a}_i|^{\frac{1}{i}} \leq |s_2|^{\frac{1}{2}} = \sqrt{2}\, |\tilde{a}_2|^{\frac{1}{2}}, \quad i = 2, \ldots, n.$$

*Proof.* Newton's identities give $|\tilde{a}_i| \leq \frac{1}{i} \sum_{j=2}^{i} |s_j||\tilde{a}_{i-j}|$, where $\tilde{a}_0 = 1$, which together with

$$|s_i|^{\frac{1}{i}} \leq |s_2|^{\frac{1}{2}}, \quad i = 2, \ldots, n, \tag{3.2}$$

will imply the result by induction on $i$. To show (3.2) we note that it is equivalent to

$$(\lambda_1^i + \cdots + \lambda_n^i)^2 \leq (\lambda_1^2 + \cdots + \lambda_n^2)^i. \tag{3.3}$$

Each mixed term $\lambda_\ell^i\lambda_m^i$ on the left-hand side of (3.3) may be estimated by the sum of all $\lambda_\ell^a\lambda_m^b$ terms with $a,b>0$ on the right-hand side of (3.3), in fact

$$2\lambda_\ell^i\lambda_m^i = 2\lambda_\ell^2\lambda_m^2\lambda_\ell^{i-2}\lambda_m^{i-2} \le \lambda_\ell^2\lambda_m^2(\lambda_\ell^{2(i-2)}+\lambda_m^{2(i-2)}) \le \sum_{j=1}^{i-1}\binom{i}{j}\lambda_\ell^{2j}\lambda_m^{2(i-j)}.$$

This implies the statement. □

3.2. **Splitting.** The following well-known lemma (see e.g. [1] or [2]) is an easy consequence of the inverse function theorem.

**Lemma 3.2.** *Let $P_a = P_bP_c$, where $P_b$ and $P_c$ are monic complex polynomials without common root. Then for $P$ near $P_a$ we have $P = P_{b(P)}P_{c(P)}$ for analytic mappings $\mathbb{R}^n_a \ni P \mapsto b(P) \in \mathbb{R}^{\deg P_b}_b$ and $\mathbb{R}^n_a \ni P \mapsto c(P) \in \mathbb{R}^{\deg P_c}_c$, defined for $P$ near $P_a$, with the given initial values.*

*Proof.* The product $P_a = P_bP_c$ defines on the coefficients a polynomial mapping $\varphi$ such that $a = \varphi(b,c)$, where $a=(a_i)$, $b=(b_i)$, and $c=(c_i)$. The Jacobian determinant $\det d\varphi(b,c)$ equals the resultant of $P_b$ and $P_c$ which is nonzero by assumption. Thus $\varphi$ can be inverted locally. □

If $P_{\tilde a}$ is in Tschirnhausen form and $\tilde a \neq 0$ then, the sum of its roots being equal to zero, it always splits. The space of hyperbolic polynomials of degree $n$ in Tschirnhausen form can be identified with a closed semialgebraic subset $H_n$ of $\mathbb{R}^{n-1}_{\tilde a}$. By Lemma 3.1, the set $H_n^0 := H_n \cap \{\tilde a_2 = -1\}$ is compact.

Let $p \in H_n \cap \{\tilde a_2 \neq 0\}$. Then the polynomial

$$Q_{\underline{a}}(Z) := |\tilde a_2|^{-\frac{n}{2}} P_{\tilde a}(|\tilde a_2|^{\frac12} Z) = Z^n - Z^{n-2} + |\tilde a_2|^{-\frac32}\tilde a_3 Z^{n-3} + \cdots + |\tilde a_2|^{-\frac{n}{2}}\tilde a_n$$

is hyperbolic and, by Lemma 3.2, it *splits*, i.e., $Q_{\underline{a}} = Q_{\underline{b}}Q_{\underline{c}}$ and $\deg Q_{\underline{b}}, \deg Q_{\underline{c}} < n$, on some open ball $B_p(r)$ centered at $p$. Thus, there exist real analytic functions $\psi_i$ so that, on $B_p(r)$,

$$\underline{b}_i = \psi_i\big(|\tilde a_2|^{-\frac32}\tilde a_3, \ldots, |\tilde a_2|^{-\frac{n}{2}}\tilde a_n\big), \quad i = 1,\ldots,\deg P_b;$$

likewise for $\underline{c}_j$. The splitting $Q_{\underline{a}} = Q_{\underline{b}}Q_{\underline{c}}$ induces a splitting $P_{\tilde a} = P_bP_c$, where

$$b_i = |\tilde a_2|^{\frac{i}{2}}\psi_i\big(|\tilde a_2|^{-\frac32}\tilde a_3, \ldots, |\tilde a_2|^{-\frac{n}{2}}\tilde a_n\big), \quad i = 1,\ldots,\deg P_b; \tag{3.4}$$

likewise for $c_j$. Shrinking $r$ slightly, we may assume that all partial derivatives of $\psi_i$ are separately bounded on $B_p(r)$. We denote by $\tilde b_j$ the coefficients of the polynomial $P_{\tilde b}$ resulting from $P_b$ by the Tschirnhausen transformation.

**Lemma 3.3.** *In this situation we have $|\tilde b_2| \le 2n|\tilde a_2|$.*

*Proof.* Let $(\lambda_j)_{j=1}^k$ denote the roots of $P_b$ and $(\lambda_j)_{j=1}^n$ those of $P_a$. Then, as $|b_1| \le \sum_{j=1}^k |\lambda_j| \le (k \sum_{j=1}^k \lambda_j^2)^{1/2}$ and thus $|\lambda_j||b_1| \le k \sum_{j=1}^k \lambda_j^2$,

$$\begin{aligned}2|\tilde b_2| &= \sum_{j=1}^k \Big(\lambda_j + \frac{b_1}{k}\Big)^2 \le \frac{1}{k^2} \sum_{j=1}^k (k^2\lambda_j^2 + b_1^2 + 2k|\lambda_j||b_1|) \\ &\le \frac{1}{k^2} \sum_{j=1}^k \big(k^2\lambda_j^2 + k\sum_{\ell=1}^k \lambda_\ell^2 + 2k^2 \sum_{\ell=1}^k \lambda_\ell^2\big) = 2(k+1)\sum_{j=1}^k \lambda_j^2 \le 2n \sum_{j=1}^n \lambda_j^2 = 4n|\tilde a_2|,\end{aligned}$$

as required. □

3.3. **Coefficient estimates.** We shall need the following estimates. (Here it is convenient to number the coefficients in reversed order.)

**Lemma 3.4.** *Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{C}[x]$ satisfy $|P(x)| \le A$ for $x \in [0,B] \subseteq \mathbb{R}$. Then*

$$|a_j| \le (2n)^{n+1} A B^{-j}, \quad j = 0, \ldots, n.$$

*Proof.* We show the lemma for $A = B = 1$. The general statement follows by applying this special case to the polynomial $A^{-1}P(By)$, $y = B^{-1}x$. Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be equidistant points. By Lagrange's interpolation formula (e.g. [14, (1.2.5)]),

$$P(x) = \sum_{k=0}^n P(x_k) \prod_{\substack{j=0\\ j\ne k}}^n \frac{x - x_j}{x_k - x_j},$$

and therefore

$$a_j = \sum_{k=0}^n P(x_k) \prod_{\substack{j=0\\ j\ne k}}^n (x_k - x_j)^{-1} (-1)^{n-j} \sigma_{n-j}^k,$$

where $\sigma_j^k$ is the $j$th elementary symmetric polynomial in $(x_\ell)_{\ell\ne k}$. The statement follows. □

A better constant can be obtained using Chebyshev polynomials; cf. [14, Thm. 16.3.1-2].

3.4. **Consequences of Taylor's theorem.** The following two lemmas are classical. We include them for the reader's convenience.

**Lemma 3.5.** *Let $I \subseteq \mathbb{R}$ be an open interval and let $f \in C^{1,1}(\overline{I})$ be nonnegative or nonpositive. For any $t_0 \in I$ and $M > 0$ such that $I_{t_0}(M^{-1}) := \{t : |t - t_0| < M^{-1}|f(t_0)|^{\frac12}\} \subseteq I$ and $M \ge (\mathrm{Lip}_{I_{t_0}(M^{-1})}(f'))^{\frac12}$ we have*

$$|f'(t_0)| \le \big(M + M^{-1}\,\mathrm{Lip}_{I_{t_0}(M^{-1})}(f')\big)|f(t_0)|^{\frac12} \le 2M|f(t_0)|^{\frac12}.$$

*Proof.* Suppose that $f$ is nonnegative; otherwise consider $-f$. It follows that the inequality holds true at the zeros of $f$. Let us assume that $f(t_0) > 0$. The statement follows from

$$0 \le f(t_0 + h) = f(t_0) + f'(t_0)h + \int_0^1 (1-s) f''(t_0 + hs)\, ds\, h^2$$

with $h = \pm M^{-1}|f(t_0)|^{\frac12}$. □

**Lemma 3.6.** *Let $f \in C^{m-1,1}(\overline{I})$. There is a universal constant $C(m)$ such that for all $t \in I$ and $k = 1, \ldots, m$,*

$$|f^{(k)}(t)| \le C(m)|I|^{-k}\big(\|f\|_{L^\infty(I)} + \mathrm{Lip}_I(f^{(m-1)})|I|^m\big). \tag{3.5}$$

*Proof.* We may suppose that $I = (-\delta, \delta)$. If $t \in I$ then at least one of the two intervals $[t, t \pm \delta)$, say $[t, t + \delta)$, is included in $I$. By Taylor's formula, for $t_1 \in [t, t + \delta)$,

$$\begin{aligned}\Big|\sum_{k=0}^{m-1} \frac{f^{(k)}(t)}{k!}(t_1 - t)^k\Big| &\le |f(t_1)| + \Big|\int_0^1 \frac{(1-s)^{m-1}}{(m-1)!} f^{(m)}(t + s(t_1 - t))\, ds\, (t_1 - t)^m\Big| \\ &\le \|f\|_{L^\infty(I)} + \mathrm{Lip}_I(f^{(m-1)})\delta^m,\end{aligned}$$

and for $k \le m - 1$ we may conclude (3.5) by Lemma 3.4. For $k = m$, (3.5) is trivially satisfied. $\square$

## 4. Proof of Theorem 2.1

4.1. **First reductions.** We assume that the maximal multiplicity $p$ of the roots equals the degree $n$ of $P_a$. If $p < n$ then we may use Lemma 3.2 to split $P_a$ locally in factors that have this property. We discuss it in more detail at the end of the proof.

So let $P_a(t)$, $t \in I$, be a $C^{n-1,1}$-curve of hyperbolic polynomials of degree $n$. Without loss of generality we may assume that $n \ge 2$ and that $P_a = P_{\tilde{a}}$ is in Tschirnhausen form. Let $(\lambda_j(t))_{j=1}^n$, $t \in I$, be any continuous system of the roots of $P_{\tilde{a}}$. Then

$$\tilde{a}_2(t) = 0 \quad \Longleftrightarrow \quad \lambda_1(t) = \cdots = \lambda_n(t) = 0.$$

We shall show that, for any relatively compact open subinterval $I_0 \Subset I$ and any $t_0 \in I_0 \setminus \tilde{a}_2^{-1}(0)$, there exists a neighborhood $I_{t_0}$ of $t_0$ in $I_0 \setminus \tilde{a}_2^{-1}(0)$ so that each $\lambda_j$ is Lipschitz on $I_{t_0}$ and the Lipschitz constant $\mathrm{Lip}_{I_{t_0}}(\lambda_j)$ satisfies

$$\mathrm{Lip}_{I_{t_0}}(\lambda_j) \le C(n, I_0, I_1)\,\big(\max_i \|\tilde{a}_i\|^{\frac{1}{i}}_{C^{n-1,1}(\overline{I}_1)}\big),$$

where $I_1$ is any open interval satisfying $I_0 \Subset I_1 \Subset I$. Here, recall, $C(n, I_0, I_1)$ stands for a universal constant depending only on $n$, $I_0$, and $I_1$.

This will imply Theorem 2.1 by the following lemma.

**Lemma 4.1.** *Let $I \subseteq \mathbb{R}$ be an open interval. If $f \in C^0(I)$ and each $t_0 \in I \setminus f^{-1}(0)$ has a neighborhood $I_{t_0} \subseteq I \setminus f^{-1}(0)$ so that $L := \sup_{t_0 \in I \setminus f^{-1}(0)} \mathrm{Lip}_{I_{t_0}}(f) < \infty$, then $f$ is Lipschitz on $I$ and $\mathrm{Lip}_I(f) = L$.*

*Proof.* Let $t, s \in I$. It is easy to see that $|f(t) - f(s)| \le L|t - s|$ if $t$ and $s$ belong to the same connected component $J$ of $I \setminus f^{-1}(0)$. By continuity, this estimate also holds on the closed interval $\overline{J}$. If $t \in \overline{J}_1$ and $s \in \overline{J}_2$, $t < s$, and $\overline{J}_1 \cap \overline{J}_2 = \emptyset$, let $r_i$ be the endpoint of $\overline{J}_i$ so that $s \le r_1 < r_2 \le t$. Then

$$|f(t) - f(s))| \le |f(t) - f(r_2)| + |f(r_1) - f(s)| \le L|t - s|.$$

Clearly, $\mathrm{Lip}_I(f) = L$. $\square$

4.2. **Convenient assumption.** The proof of the statement in Subsection 4.1 will be carried out by induction on the degree of $P_a$. We replace the assumption of Theorem 2.1 by a new assumption that will be more convenient for the inductive step. Before we state it we need a bit of notation.

For open intervals $I_0$ and $I_1$ so that $I_0 \Subset I_1 \Subset I$, we set

$$I_i' := I_i \setminus \tilde a_2^{-1}(0), \quad i = 0,1.$$

For $t_0 \in I_0'$ and $r > 0$ consider the interval

$$I_{t_0}(r) := \big(t_0 - r|\tilde a_2(t_0)|^{\frac12}, t_0 + r|\tilde a_2(t_0)|^{\frac12}\big).$$

**Assumption.** Let $I_0 \Subset I_1$ be open intervals. Suppose that $(\tilde a_i)_{i=2}^n \in C^{n-1,1}(\overline I_1, \mathbb{R}^{n-1})$ are the coefficients of a hyperbolic polynomial $P_{\tilde a}$ of degree $n$ in Tschirnhausen form. Assume that there is a constant $A > 0$, so that for all $t_0 \in I_0'$, $t \in I_{t_0}(A^{-1})$, $i = 2, \ldots, n$, $k = 0, \ldots, n$,

$$I_{t_0}(A^{-1}) \subseteq I_1, \tag{A.1}$$

$$2^{-1} \le \frac{\tilde a_2(t)}{\tilde a_2(t_0)} \le 2, \tag{A.2}$$

$$|\tilde a_i^{(k)}(t)| \le C(n)\, A^k\, |\tilde a_2(t)|^{\frac{i-k}{2}}, \tag{A.3}$$

where $C(n)$ is a universal constant. For $k = n$, (A.3) is understood to hold almost everywhere, by Rademacher's theorem.

Condition (A.3) implies that

$$\big|\partial_t^k\big(|\tilde a_2|^{-\frac{i}{2}} \tilde a_i\big)(t)\big| \le C(n)\, A^k\, |\tilde a_2(t)|^{-\frac{k}{2}}. \tag{A.4}$$

More generally, if we assign $\tilde a_i$ the weight $i$ and $|\tilde a_2|^{\frac12}$ the weight 1 and let $L(x_2, \ldots, x_n, y) \in \mathbb{R}[x_2, \ldots, x_n, y, y^{-1}]$ be weighted homogeneous of degree $d$, then

$$\big|\partial_t^k L\big(\tilde a_2, \ldots, \tilde a_n, |\tilde a_2|^{\frac12}\big)(t)\big| \le C(n, L)\, A^k\, |\tilde a_2(t)|^{\frac{d-k}{2}}.$$

4.3. **Inductive step.** Let $P_{\tilde a}$, $I_0$, $I_1$, $A$, $t_0$ be as in Assumption. We will show by induction on $\deg P_{\tilde a}$ that any continuous system of the roots of $P_{\tilde a}$ is Lipschitz on $I_0$ with Lipschitz constant bounded from above by $C(n)\, A$. First we establish the following.

- For some constant $C_1(n) > 1$, the polynomial $P_{\tilde a}(t)$ splits on the interval $I_{t_0}(C_1(n)^{-1}A^{-1})$, that is we have $P_{\tilde a}(t) = P_b(t) P_c(t)$, where $P_b$ and $P_c$ are $C^{n-1,1}$-curves of hyperbolic polynomials of degree strictly smaller than $n$.
- After applying the Tschirnhausen transformation $P_b \rightsquigarrow P_{\tilde b}$, the coefficients $(\tilde b_i)_{i=2}^{\deg P_b}$ satisfy (A.1)–(A.3) for suitable neighborhoods $J_0$, $J_1$ of $t_0$, and a constant $B = C(n)\, A$ in place of $A$.

We restrict our curve of hyperbolic polynomials $P_{\tilde a}$ to $I_{t_0}(A^{-1})$ and consider

$$\underline a := \big(-1, |\tilde a_2|^{-\frac32} \tilde a_3, \ldots, |\tilde a_2|^{-\frac{n}{2}} \tilde a_n\big) : I_{t_0}(A^{-1}) \to \mathbb{R}^{n-1}_{\underline a}.$$

Then $\underline a$ is continuous, by (A.2), and bounded, by Lemma 3.1. Moreover, by (A.4) and (A.2), there is a universal constant $C_1(n)$ so that, for $t \in I_{t_0}(A^{-1})$,

$$\|\underline a'(t)\| \le C_1(n)\, A\, |\tilde a_2(t_0)|^{-\frac12}. \tag{4.1}$$

According to Subsection 3.2, choose a finite cover of $H_n^0$ by open balls $B_{p_\alpha}(r)$, $\alpha \in \Delta$, on which we have a splitting $P_{\tilde{a}} = P_b P_c$ with coefficients of $P_b$ given by (3.4). There exists $r_1 > 0$ such that for any $p \in H_n^0$ there is $\alpha \in \Delta$ so that $B_p(r_1) \subseteq B_{p_\alpha}(r)$; $2r_1$ is a Lebesgue number of the cover $\{B_{p_\alpha}(r)\}_{\alpha\in\Delta}$. Then, if $C_1(n)$ is the constant from (4.1),

$$J_1 := I_{t_0}(r_1 C_1(n)^{-1} A^{-1}) \subseteq \underline{a}^{-1}(B_{\underline{a}(t_0)}(r_1)), \tag{4.2}$$

and on $J_1$ we have a splitting $P_{\tilde{a}}(t) = P_b(t)P_c(t)$ with $b_i$ given by (3.4). Fix $r_0 < r_1$ and let

$$J_0 := I_{t_0}(r_0 C_1(n)^{-1} A^{-1}). \tag{4.3}$$

(Here we assume without loss of generality that $r_1 \le C_1(n)$.)

Let us show that the coefficients $(\tilde{b}_i)_{i=2}^{\deg P_b}$ of $P_{\tilde{b}}$ satisfy (A.1)–(A.3) for the intervals $J_1$ and $J_0$ from (4.2) and (4.3). To this end we set

$$J_i' := J_i \setminus \tilde{b}_2^{-1}(0), \quad i = 0, 1,$$

consider, for $t_1 \in J_0'$ and $r > 0$,

$$J_{t_1}(r) := \big(t_1 - r|\tilde{b}_2(t_1)|^{\frac{1}{2}}, t_1 + r|\tilde{b}_2(t_1)|^{\frac{1}{2}}\big),$$

and prove the following lemma.

**Lemma 4.2.** *There exists a constant $\tilde{C} = \tilde{C}(n, r_0, r_1) > 1$ such that for $B = \tilde{C}A$ and for all $t_1 \in J_0'$, $t \in J_{t_1}(B^{-1})$, $i = 2, \dots, \deg P_b$, $k = 0, \dots, n$,*

$$J_{t_1}(B^{-1}) \subseteq J_1, \tag{B.1}$$

$$2^{-1} \le \frac{\tilde{b}_2(t)}{\tilde{b}_2(t_1)} \le 2, \tag{B.2}$$

$$|\tilde{b}_i^{(k)}(t)| \le C(n)\, B^k\, |\tilde{b}_2(t)|^{\frac{i-k}{2}}. \tag{B.3}$$

*for some universal constant $C(n)$.*

*Proof.* If

$$B \ge (r_1 - r_0)^{-1}\, 2\sqrt{n}\, C_1(n)\, A,$$

then by Lemma 3.3 and (A.2),

$$B^{-1}|\tilde{b}_2(t_1)|^{\frac{1}{2}} \le (r_1 - r_0)\, C_1(n)^{-1}\, A^{-1}\, |\tilde{a}_2(t_0)|^{\frac{1}{2}},$$

and hence (B.1) follows from (4.2) and (4.3), since $t_1 \in J_0$.

Next we claim that, on $J_1$,

$$\big|\partial_t^k \psi_i\big(|\tilde{a}_2|^{-\frac{3}{2}}\tilde{a}_3, \dots, |\tilde{a}_2|^{-\frac{n}{2}}\tilde{a}_n\big)\big| \le C(n)\, A^k\, |\tilde{a}_2|^{-\frac{k}{2}}. \tag{4.4}$$

To see this we differentiate the following equation $(k-1)$ times, apply induction on $k$, and use (A.4),

$$\partial_t \psi_i\big(|\tilde{a}_2|^{-\frac{3}{2}}\tilde{a}_3, \dots, |\tilde{a}_2|^{-\frac{n}{2}}\tilde{a}_n\big) = \sum_{j=3}^{n} (\partial_{j-2}\psi_i)(\underline{a})\partial_t\big(|\tilde{a}_2|^{-\frac{j}{2}}\tilde{a}_j\big); \tag{4.5}$$

recall that all partial derivatives of the $\psi_i$'s are separately bounded on $\underline{a}(J_1)$ and these bounds are universal. From (3.4) and (4.4) we obtain, on $J_1$ and for all $i = 1, \dots, \deg P_b$, $k = 0, \dots, n$,

$$|b_i^{(k)}| \le C(n)\, A^k\, |\tilde{a}_2|^{\frac{i-k}{2}}, \tag{4.6}$$

thus, as the Tschirnhausen transformation preserves the weights of the coefficients, cf. (A.4),

$$|\tilde{b}_i^{(k)}| \le C(n)\, A^k\, |\tilde{a}_2|^{\frac{i-k}{2}},$$

and so, by Lemma 3.3,

$$|\tilde{b}_i^{(k)}| \le C(n)\, A^k\, |\tilde{b}_2|^{\frac{i-k}{2}} \qquad \text{if } i-k \le 0.$$

This shows (B.3) for $i \le k$. (B.3) for $k = 0$ follows from Lemma 3.1. (B.2) and the remaining inequalities of (B.3), i.e., for $0 < k < i$, follow now from Lemma 4.3 below. □

**Lemma 4.3.** *There exists a constant $C(n) \ge 1$ such that the following holds. If* (A.1) *and* (A.3) *for $k = 0$ and $k = i$, $i = 2, ..., n$, are satisfied, then so are* (A.2) *and* (A.3) *for $k < i$, $i = 2, ..., n$, after replacing $A$ by $C(n)A$.*

*Proof.* By assumption, $\mathrm{Lip}_{I_{t_0}(A^{-1})}(\tilde{a}_2')) \le C(n)A^2$. Thus, by Lemma 3.5 for $f = \tilde{a}_2$ and $M = C(n)^{\frac{1}{2}}A$, we get

$$|\tilde{a}_2'(t_0)| \le 2M|\tilde{a}_2(t_0)|^{\frac{1}{2}}.$$

It follows that, for $t \in I_{t_0}((6M)^{-1})$,

$$\frac{|\tilde{a}_2(t) - \tilde{a}_2(t_0)|}{|\tilde{a}_2(t_0)|} \le \frac{|\tilde{a}_2'(t_0)|}{|\tilde{a}_2(t_0)|}|t-t_0| + \int_0^1 (1-s)|\tilde{a}_2''(t_0 + s(t-t_0))|ds\, \frac{|t-t_0|^2}{|\tilde{a}_2(t_0)|} \le \frac{1}{2} \tag{4.7}$$

That implies (A.2). The other inequalities follow from Lemma 3.6. □

4.4. **End of inductive step.** In $J_1$, any continuous root $\lambda_j$ of $P_{\tilde{a}}$, where $P_{\tilde{a}}$ is in Tschirnhausen form, is a root of either $P_b$ or $P_c$. Say it is a root of $P_b$. Then it has the form

$$\lambda_j(t) = -\frac{b_1(t)}{\deg P_b} + \mu_j(t), \tag{4.8}$$

where $\mu_j$ is a continuous root of $P_{\tilde{b}}$ defined on a neighborhood of $t_0$. By the inductive assumption we may assume that $\mu_j$ is Lipschitz with Lipschitz constant bounded from above by $C(n)B$. Hence $\lambda_j$ is Lipschitz with Lipschitz constant bounded from above by $C(n)A$ (the constant $C(n)$ changes), as $B = \tilde{C}\,A$ and by (4.6) for $i = k = 1$. This ends the inductive step.

4.5. $P_{\tilde{a}}$ **satisfies Assumption.** Now we show that $P_{\tilde{a}}$ always satisfies Assumption. The choice of $A$ will provide the upper bound on the Lipschitz constant of the roots.

**Proposition 4.4.** *Let $P_{\tilde{a}}(t)$, $t \in I$, be a $C^{n-1,1}$-curve of hyperbolic polynomials of degree $n$ in Tschirnhausen form, and let $I_0$ and $I_1$ be open intervals satisfying $I_0 \Subset I_1 \Subset I$. Then its coefficients $(\tilde{a}_i)_{i=2}^n$ satisfy* (A.1)–(A.3).

*Proof.* Let $\delta$ denote the distance between the endpoints of $I_0$ and those of $I_1$. Set

$$A_1 := \max\big\{\delta^{-1}\|\tilde a_2\|^{\frac12}_{L^\infty(I_1)}, (\mathrm{Lip}_{I_1}(\tilde a_2'))^{\frac12}\big\}, \quad A_2 := \max_i \big\{M_i\|\tilde a_2\|^{\frac{n-i}{2}}_{L^\infty(I_1)}\big\}^{\frac1n}, \tag{4.9}$$

where $M_i = \mathrm{Lip}_{I_1}(\tilde a_i^{(n-1)})$. Then we may choose

$$A \ge A_0 = 6\max\{A_1, A_2\}. \tag{4.10}$$

For (A.1)–(A.2) to be satisfied we need only $A \ge 6A_1$. Indeed, clearly, for $t_0 \in I_0'$,

$$I_{t_0}(A_1^{-1}) \subseteq I_1. \tag{4.11}$$

Then Lemma 3.5 implies that

$$|\tilde a_2'(t_0)| \le 2A_1\,|\tilde a_2(t_0)|^{\frac12}.$$

It follows that, for $t_0 \in I_0'$ and $t \in I_{t_0}((6A_1)^{-1})$, we have (4.7) and hence (A.2). If $t \in I_{t_0}(A^{-1})$ then Lemma 3.6, Lemma 3.1, and (A.2) imply (A.3). This ends the proof of Proposition 4.4. □

4.6. **Bounds for** $p = n$**.** Let $\lambda(t) \in C^0(I)$ be a root of $P_{\tilde a}$ that is in Tschirnhausen form and let $I_0 \Subset I_1 \Subset I$. By the inductive step 4.3, Proposition 4.4, and Lemma 4.1 we have the following bounds

$$\begin{aligned}\mathrm{Lip}_{I_0}(\lambda) &\le C(n)\max\Big\{\delta^{-1}\|\tilde a_2\|^{\frac12}_{L^\infty(I_1)}, (\mathrm{Lip}_{I_1}(\tilde a_2'))^{\frac12}, \max_i\big\{M_i\|\tilde a_2\|^{\frac{n-i}{2}}_{L^\infty(I_1)}\big\}^{\frac1n}\Big\} \\ &\le C(n, I_0, I_1)\,\big(\max_i\|\tilde a_i\|^{\frac1i}_{C^{n-1,1}(\bar I_1)}\big) \\ &\le C(n, I_0, I_1)\,\big(1 + \max_i\|\tilde a_i\|_{C^{n-1,1}(\bar I_1)}\big),\end{aligned} \tag{4.12}$$

where $\delta$ is the distance between the endpoints of $I_0$ and those of $I_1$, and $M_i = \mathrm{Lip}_{I_1}(\tilde a_i^{(n-1)})$. Then the bounds stated in Theorem 2.1 follow from

$$\max_i\|\tilde a_i\|^{\frac1i}_{C^{n-1,1}(\bar I_1)} \le C(n)\,\big(\max_i\|a_i\|^{\frac1i}_{C^{n-1,1}(\bar I_1)}\big) \le C(n)\,\big(1 + \max_i\|a_i\|_{C^{n-1,1}(\bar I_1)}\big).$$

The first inequality follows from the (weighted) homogeneity of the formulas for $\tilde a_i$ in terms of $(a_1, \dots, a_n)$. (The opposite inequality does not hold in general. Adding a constant to all the roots of $P_a$ does not change the associated Tschirnhausen form $P_{\tilde a}$ but changes the norm of the coefficients of $P_a$.)

4.7. **The case** $2 \le p < n$**.** To show that the roots are Lipschitz it suffices, using Lemma 3.2, to split $P_{\tilde a}$ locally in factors of degree smaller than or equal to $p$ and apply the case $n = p$.

In order to have a uniform bound we need to know that the multiplicities of roots are at most $p$ "uniformly". For this we order the roots of $P_{\tilde a}$ increasingly, $\lambda_1(t) \le \lambda_2(t) \le \cdots \le \lambda_n(t)$, and consider

$$\alpha(t) := \frac{|\lambda_n(t) - \lambda_1(t)|}{\min_{i=1,\dots,n-p}|\lambda_{i+p}(t) - \lambda_i(t)|}, \quad \alpha_I := \sup_{t\in I}\alpha(t).$$

We note that the numerator $|\lambda_n(t)-\lambda_1(t)|$ is of the same size as $|\tilde a_2(t)|^{\frac12}$, for $P_{\tilde a}$ in Tschirnhausen form, since then $\lambda_1(t)$ and $\lambda_n(t)$ have opposite signs and

$$n|\lambda_n(t)-\lambda_1(t)| \ge s_2(t)^{\frac12} = \sqrt{2}|\tilde a_2(t)|^{\frac12} \ge \frac12 |\lambda_n(t)-\lambda_1(t)|.$$

There are the following changes in the way we proceed. First in the proof of Proposition 4.4 we have to modify the formula for $A_2$ as follows

$$A_2 := \max\Big\{\max_{i\le p}\big\{M_i\|\tilde a_2\|_{L^\infty(I_1)}^{\frac{p-i}{2}}\big\}^{\frac1p}, \max_{i>p}\big\{M_i\, m_2^{\frac{p-i}{2}}\big\}^{\frac1p}\Big\}, \tag{4.13}$$

where $M_i = \operatorname{Lip}_{I_1}(\tilde a_i^{(p-1)})$ and $m_2 = \min_{t\in \overline{I}_0} |\tilde a_2(t)|$.

In (A.3) of Assumption we may consider only the derivatives of order $k\le p$. Therefore the argument of the inductive step (proof of Lemma 4.2) changes as follows. The first part of the proof of Lemma 4.2 does not change. Then we need (B.3) for $i=k$ in order to apply Lemma 4.3. This is not available if $i>p$, which happens if $\deg P_b > p$, and then we have only

$$|\tilde b_i^{(p)}| \le C(n)\, A^p\, |\tilde a_2|^{\frac{i-p}{2}} \le C(n)\, A_b^p\, |\tilde b_2|^{\frac{i-p}{2}},$$

where we may take $A_b = C(n)|\tilde a_2(t_1)/\tilde b_2(t_1)|^{\frac{n-p}{2p}} A$. Then, by Lemma 3.6, we conclude Lemma 4.2 with $A$ replaced by $A_b$. This modification is no longer necessary when $\deg P_b \le p$. Thus during the induction process, say, $P_{\tilde a}\to P_{\tilde b}\to\cdots\to P_{\tilde d}\to P_{\tilde e}$ with $\deg P_e\le p$, for the intervals $I_{t_0}(A^{-1})\supset I_{t_1}(A_b^{-1})\supset\cdots\supset I_{t_s}(A_d^{-1})$, the constant $A$ is replaced by

$$\tilde A = C(n)\alpha(t_s)^{\frac{n-p}{p}} A \ge C(n)\Big(\Big|\frac{\tilde a_2(t_s)}{\tilde b_2(t_s)}\Big|\cdot\Big|\frac{\tilde b_2(t_s)}{\tilde c_2(t_s)}\Big|\cdots\Big|\frac{\tilde c_2(t_s)}{\tilde d_2(t_s)}\Big|\Big)^{\frac{n-p}{2p}} A.$$

Finally this gives the following bounds on the Lipschitz constant of each of the roots

$$\begin{aligned} &\operatorname{Lip}_{I_0}(\lambda) \\ &\le C(n)\,\alpha_{I_1}^{\frac{n-p}{p}} \max\Big\{\delta^{-1}\|\tilde a_2\|_{L^\infty(I_1)}^{\frac12}, (\operatorname{Lip}_{I_1}(\tilde a_2'))^{\frac12}, \max_{i\le p}\big\{M_i\|\tilde a_2\|_{L^\infty(I_1)}^{\frac{p-i}{2}}\big\}^{\frac1p}, \max_{i>p}\big\{M_i\, m_2^{\frac{p-i}{2}}\big\}^{\frac1p}\Big\} \\ &\le C(n,I_0,I_1)\,\alpha_{I_1}^{\frac{n-p}{p}}\,\big(1+m_2^{\frac{p-n}{2p}}\big)\big(1+\max_i \|\tilde a_i\|_{C^{p-1,1}(\overline{I}_1)}\big). \end{aligned} \tag{4.14}$$

This completes the proof of Theorem 2.1. ☐

## 5. Proof of Theorem 2.4

5.1. **${}^pC^m$-functions.** In the proof of Theorem 2.4 we shall need a result for functions defined near $0\in\mathbb{R}$ that become $C^m$ when multiplied with the monomial $t^p$.

**Definition 5.1.** Let $p,m\in\mathbb{N}$ with $p\le m$. A continuous complex valued function $f$ defined near $0\in\mathbb{R}$ is called a *${}^pC^m$-function* if $t\mapsto t^p f(t)$ belongs to $C^m$.

Let $I\subseteq\mathbb{R}$ be an open interval containing 0. Then $f: I\to\mathbb{C}$ is ${}^pC^m$ if and only if it has the following properties, cf. [21, 4.1], [18, Satz 3], or [19, Thm 4]:

- $f\in C^{m-p}(I)$,
- $f|_{I\setminus\{0\}}\in C^m(I\setminus\{0\})$,

- $\lim_{t\to 0} t^k f^{(m-p+k)}(t)$ exists as a finite number for all $0 \le k \le p$.

**Proposition 5.2.** *If $g = (g_1, \dots, g_n)$ is ${}^pC^m$ and $F$ is $C^m$ near $g(0) \in \mathbb{C}^n$, then $F \circ g$ is ${}^pC^m$.*

*Proof.* Cf. [19, Thm 9] or [17, Prop 3.2]. Clearly $g$ and $F \circ g$ are $C^{m-p}$ near 0 and $C^m$ off 0. By Faà di Bruno's formula [8], for $1 \le k \le p$ and $t \ne 0$,

$$\frac{t^k (F\circ g)^{(m-p+k)}(t)}{(m-p+k)!} = \sum_{\ell\ge 1}\sum_{\alpha\in A} \frac{t^{k-|\beta|}}{\ell!} d^\ell F(g(t))\Big(\frac{t^{\beta_1} g^{(\alpha_1)}(t)}{\alpha_1!}, \dots, \frac{t^{\beta_\ell} g^{(\alpha_\ell)}(t)}{\alpha_\ell!}\Big)$$

$$A := \{\alpha \in \mathbb{N}^\ell_{>0} : \alpha_1 + \cdots + \alpha_\ell = m-p+k\}$$

$$\beta_i := \max\{\alpha_i - m + p, 0\}, \quad |\beta| = \beta_1 + \cdots + \beta_\ell \le k,$$

whose limit as $t \to 0$ exists as a finite number by assumption. □

Let us prove Theorem 2.4. We suppose that $P_a$ is in Tschirnhausen form $P_a = P_{\tilde a}$. It suffices to consider the case $n = p$. We show that every $t_0 \in I$ has a neighborhood in $I$ on which (1) and (2) (of Theorem 2.4) hold. If $\tilde a_2(t_0) \ne 0$ then $P_{\tilde a}$ splits on a neighborhood of $t_0$ and we may proceed by induction on $\deg P_a$. If $\tilde a_2(t_0) = 0$ then $\tilde a_2'(t_0) = 0$ and we distinguish two cases

- *Case (i):* $\tilde a_2(t_0) = \tilde a_2'(t_0) = \tilde a_2''(t_0) = 0$.
- *Case (ii):* $\tilde a_2(t_0) = \tilde a_2'(t_0) = 0$ and $\tilde a_2''(t_0) \ne 0$.

To simplify the notation we suppose $t_0 = 0$. Fix a continuous root $\lambda(t)$ defined in a neighborhood of 0.

5.2. **Proof of (1).** In Case (i), $\lambda(t) = o(t)$ and hence $\lambda$ is differentiable at 0 and $\lambda'(0) = 0$. In Case (ii), $\tilde a_2(t) \sim t^2$ and hence $\tilde a_i(t) = O(t^i)$. Therefore,

$$\underline{a}(t) := \big(t^{-2}\tilde a_2(t), t^{-3}\tilde a_3(t), \dots, t^{-n}\tilde a_n(t)\big) : I_1 \to \mathbb{R}^{n-1}_{\underline{a}}$$

defined on a neighborhood $I_1$ of 0 is continuous. By Lemma 3.2, $P_{\underline{a}}$ splits. The splitting $P_{\underline{a}} = P_{\underline{b}} P_{\underline{c}}$ induces a splitting $P_{\tilde a} = P_b P_c$, where the $b_i$ are given by

$$b_i = t^i \psi_i\big(t^{-2}\tilde a_2, \dots, t^{-n}\tilde a_n\big), \quad i = 1, \dots, \deg P_b; \tag{5.1}$$

and similar formulas hold for $\tilde b_i$. Then $b_i$ and $\tilde b_i$ are of class $C^i$ at 0, by Proposition 5.2, and of class $C^n$ in the complement of 0. Moreover we may choose the splitting such that $\lambda(t)$ for $t \ge 0$ is a root of $P_b$, and all the roots of $P_{\underline{b}(0)}$ are equal. The latter gives

$$\tilde b_2(0) = \tilde b_2'(0) = \tilde b_2''(0) = 0.$$

Thus, $\lambda(t)$ can be expressed as in (4.8) with $b_1$ of class $C^1$ and $\mu_j$ differentiable at 0 ($\mu_j'(0) = 0$). This finishes the proof of (1).

5.3. **Proof of (2).** This is the heart of the proof. In Case (i) the continuity of the one-sided derivatives at 0 follows from (4.12) and the following lemma.

**Lemma 5.3.** *Suppose Case (i) holds. Then for any $\varepsilon > 0$ there is $\delta > 0$ such that for $I_0 = (-\delta, \delta)$ and $I_1 = (-2\delta, 2\delta)$ and $A_0$ defined by (4.10) we have $A_0 \le \varepsilon$.*

*Proof.* This follows immediately from the formula (4.10). □

To show the continuity in Case (ii) we need a similar result for $P_{\tilde b}$.

**Lemma 5.4.** *Suppose Case (ii) holds. Then, under the assumptions of Subsection 5.2, for any $\varepsilon > 0$ there is a neighborhood $I_\varepsilon$ of $0$ in $I$ such that for every $t_0 \in I_\varepsilon \setminus \{0\}$ the conditions* (A.1)–(A.3) *are satisfied for $P_{\tilde b}$ with $A \le \varepsilon$.*

*Proof.* Since $P_{\tilde b}$ is not necessarily of class $C^{\deg P_b}$ we cannot use directly Lemma 5.3 and the induction on $\deg P_a$. But the proof is similar and we sketch it below.

Let $I_1 = I_\delta = (-\delta, \delta)$ and $I_0 = (-\frac{\delta}{2}, \frac{\delta}{2})$. Since $\tilde b_2''(0) = 0$ and $\tilde b_2(t)$ is of class $C^2$, the constant $A_1$ of (4.9) for $\tilde b$ can be made arbitrarily small, provided $\delta$ is chosen sufficiently small. This is what we need to get (A.1)–(A.2) with arbitrarily small $A$.

By Lemma 3.1, $\tilde b_i^{(k)}(0) = 0$ for $i = 2, ..., \deg P_b$, $k = 0, ..., i$. Fix $A > 0$. Since every $\tilde b_i$ is of class $C^i$, there is a neighborhood $I_\delta$ in which (A.3) holds for $i = 2, ..., n$, $k = i$, and then, by Lemma 4.3, in a smaller neighborhood, also for $i = 2, ..., n$, $k \le i$.

Finally, given $A > 0$ we show (A.3) for $i < k \le n$ and $\delta$ sufficiently small. Let $\hat A$ denote the constant $A$ for which (A.1)–(A.3) holds for $P_a$. By (4.6),

$$|\tilde b_i^{(k)}(t)| \le C(n)\hat A^k\, |\tilde a_2(t)|^{\frac{i-k}{2}} \le C(n)\hat A^k\, \varphi(t)|\tilde b_2(t)|^{\frac{i-k}{2}},$$

which gives the required result since, for $k > i$, $\varphi(t) = |\tilde b_2(t)/\tilde a_2(t)|^{\frac{k-i}{2}} = o(1)$. □

5.4. **Proof of (3).** We proceed by induction on $n$. The case $n = 1$ is obvious. So assume $n > 1$. Set $F = \{t \in I : \tilde a_2(t) = \tilde a_2''(t) = 0\}$. Its complement is a countable union of disjoint open intervals, $I \setminus F = \bigcup_k I_k$. At each $t_0 \in I \setminus F$ the polynomial $P_{\tilde a}$ splits, and, by the induction hypothesis, there exists a local differentiable system of the roots of $P_{\tilde a}$ near $t_0$. We may infer that there exists a differentiable system on each interval $I_k$. For, if the (say) right endpoint $t_1$ of the domain $I_\lambda$ of $\lambda = (\lambda_j)_{j=1}^n$ belongs to $I_k$, there exists a local system $\mu = (\mu_j)_{j=1}^n$ with $t_1 \in I_\mu$. We may choose $t_2 \in I_\lambda \cap I_\mu$ and extend $(\lambda_j)_j$ by $(\mu_{\sigma(j)})_j$ on the right of $t_2$ beyond $t_1$, where $\sigma$ is a suitable permutation. Extending by 0 on $F$ yields a differentiable system $(\lambda_j)_j$ of the roots on $I$ (the derivatives vanish on $F$).

5.5. **Proof of (4).** It follows immediately from (2).

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Adam Parusiński: Univ. Nice Sophia Antipolis, CNRS, LJAD, UMR 7351, 06108 Nice, France
*E-mail address*: adam.parusinski@unice.fr

Armin Rainer: Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Wien, Austria
*E-mail address*: armin.rainer@univie.ac.at