OPTIMAL SOBOLEV REGULARITY OF ROOTS OF POLYNOMIALS

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Abstract. We study the regularity of the roots of complex univariate polynomials whose coefficients depend smoothly on parameters. We show that any continuous choice of a root of a $C^{n-1,1}$-curve of monic polynomials of degree n is locally absolutely continuous with locally p-integrable derivatives for every $1 \le p < n/(n-1)$, uniformly with respect to the coefficients. This result is optimal: in general, the derivatives of the roots of a smooth curve of monic polynomials of degree n are not locally $n/(n-1)$-integrable, and the roots may have locally unbounded variation if the coefficients are only of class $C^{n-1,\alpha}$ for $\alpha < 1$. We also prove a generalization of Ghisi and Gobbino's higher order Glaeser inequalities. We give three applications of the main results: local solvability of a system of pseudo-differential equations, a lifting theorem for mappings into orbit spaces of finite group representations, and a sufficient condition for multi-valued functions to be of Sobolev class $W^{1,p}$ in the sense of Almgren.

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1. Introduction

This paper is dedicated to the problem of determining the optimal regularity of the roots of univariate polynomials whose coefficients depend smoothly on parameters. There is a vast literature on this problem, but most contributions treat special cases:

- the polynomial is assumed to have only real roots ([9], [27], [45], [1], [21], [6], [7], [44], [8], [13], [31]),
- only radicals of functions are considered ([17], [11], [43], [12], [16]),
- it is assumed that the roots meet only of finite order, e.g., if the coefficients are real analytic or in some other quasianalytic class, ([10], [34], [35], [36], [39]),
- quadratic and cubic polynomials ([40]), etc.

In this paper we consider the general case: let $(\alpha, \beta) \subseteq \mathbb{R}$ be a bounded open interval and let

$$P_a(t)(Z) = P_{a(t)}(Z) = Z^n + \sum_{j=1}^{n} a_j(t) Z^{n-j}, \quad t \in (\alpha, \beta), \tag{1.1}$$

be a monic polynomial whose coefficients are complex valued smooth functions $a_j : (\alpha, \beta) \to \mathbb{C}$, $j = 1, \ldots, n$. It is not hard to see that P_a always admits a continuous system of roots (e.g. [20, Ch. II Theorem 5.2]), but in general the roots cannot satisfy a local Lipschitz condition. For a long time it was unclear whether the roots of P_a admit locally absolutely continuous parameterizations. This question was affirmatively solved in our recent paper [30]: there is a positive integer $k = k(n)$ and a rational number $p = p(n) > 1$ such that, if the coefficients are of class C^k, then each continuous root λ is locally absolutely continuous with derivative λ' being locally q-integrable for each $1 \le q < p$, uniformly with respect to the coefficients.

The problem of absolute continuity of the roots arose in the analysis of certain systems of pseudo-differential equations due to Spagnolo [41]; see Section 10.1. For the history of the problem we refer to the introduction of [30]. The main tool of [30] was the resolution of singularities. With this technique we could not determine the optimal parameters k and p.

1.1. **Main results.**

In the present paper we prove the optimal result by elementary methods. Our main result is the following theorem.

Theorem 1. *Let $(\alpha, \beta) \subseteq \mathbb{R}$ be a bounded open interval and let P_a be a monic polynomial (1.1) with coefficients $a_j \in C^{n-1,1}([\alpha, \beta])$, $j = 1, \ldots, n$. Let $\lambda \in C^0((\alpha, \beta))$ be a continuous root of P_a on (α, β). Then λ is absolutely continuous on (α, β) and belongs to the Sobolev space $W^{1,p}((\alpha, \beta))$ for every $1 \le p < n/(n-1)$. The derivative λ' satisfies*

$$\|\lambda'\|_{L^p((\alpha,\beta))} \le C(n, p) \max\{1, (\beta - \alpha)^{1/p}\} \max_{1 \le j \le n} \|a_j\|^{1/j}_{C^{n-1,1}([\alpha,\beta])}, \tag{1.2}$$

where the constant $C(n, p)$ depends only on n and p.

A well-known estimate for the Cauchy bound of a polynomial (cf. [28, p.56] or [33, (8.1.11)]) gives $|\lambda(t)| \le 2 \max_{1 \le j \le n} |a_j(t)|^{1/j}$ for all $t \in (\alpha, \beta)$, and hence

$$\|\lambda\|_{L^p((\alpha,\beta))} \le C(n)(\beta - \alpha)^{1/p} \max_{1 \le j \le n} \|a_j\|^{1/j}_{L^\infty((\alpha,\beta))}.$$

It follows that

$$\|\lambda\|_{W^{1,p}((\alpha,\beta))} \le C(n,p) \max\{1, (\beta-\alpha)^{1/p}\} \max_{1\le j\le n} \|a_j\|^{1/j}_{C^{n-1,1}([\alpha,\beta])}, \tag{1.3}$$

An application of Hölder's inequality yields the following corollary.

Corollary 1. *Every continuous root of P_a on (α,β) is Hölder continuous of exponent $\gamma = 1-1/p < 1/n$, and*

$$\|\lambda\|_{C^{0,\gamma}([\alpha,\beta])} \le C(n,p) \max\{1, (\beta-\alpha)^{1/p}\} \max_{1\le j\le n} \|a_j\|^{1/j}_{C^{n-1,1}([\alpha,\beta])}. \tag{1.4}$$

Proof. Indeed, $|\lambda(t)-\lambda(s)| \le |\int_s^t \lambda'\, d\tau| \le \|\lambda'\|_{L^p((\alpha,\beta))} |t-s|^{1-1/p}$. □

The result in Theorem 1 is best possible in the following sense:

- In general the roots of a polynomial of degree n cannot lie locally in $W^{1,n/(n-1)}$, even when the coefficients are real analytic. For instance, $Z^n = t$, $t \in \mathbb{R}$.
- If the coefficients are just in $C^{n-1,\delta}([\alpha,\beta])$ for every $\delta < 1$, then the roots need not have bounded variation in (α,β). See [16, Example 4.4].

A *curve* of complex monic polynomials (1.1) admits a continuous choice of its roots. This is no longer true if the dimension of the parameter space is at least two. In that case monodromy may prevent the existence of continuous roots. However, we obtain the following multiparameter result, where we impose the existence of a continuous root; see also Remark 8.

Theorem 2. *Let $U \subseteq \mathbb{R}^m$ be open and let*

$$P_a(x)(Z) = P_{a(x)}(Z) = Z^n + \sum_{j=1}^n a_j(x) Z^{n-j}, \quad x \in U, \tag{1.5}$$

be a monic polynomial with coefficients $a_j \in C^{n-1,1}(U)$, $j = 1,\dots,n$. Let $\lambda \in C^0(V)$ be a root of P_a on a relatively compact open subset $V \Subset U$. Then λ belongs to the Sobolev space $W^{1,p}(V)$ for every $1 \le p < n/(n-1)$. The distributional gradient $\nabla\lambda$ satisfies

$$\|\nabla\lambda\|_{L^p(V)} \le C(m,n,p,\mathcal{K}) \max_{1\le j\le n} \|a_j\|^{1/j}_{C^{n-1,1}(\overline{W})}, \tag{1.6}$$

where $\mathcal{K}$ is any finite cover of $\overline{V}$ by open boxes $\prod_{i=1}^m (\alpha_i,\beta_i)$ contained in U and $W = \bigcup \mathcal{K}$; the constant $C(m,n,p,\mathcal{K})$ depends only on m, n, p, and the cover $\mathcal{K}$.

Remark 1. For any two distinct points x and y in V such that the segment $[x,y]$ is contained in V, the root λ satisfies a Hölder condition

$$\frac{|\lambda(x)-\lambda(y)|}{|x-y|^\gamma} \le C(m,n,p,\operatorname{diam}(V)) \max_{1\le j\le n} \|a_j\|^{1/j}_{C^{n-1,1}([x,y])},$$

where $\gamma = 1-1/p < 1/n$. This follows easily from Theorem 2 and Remark 8.

The proof of Theorem 1 makes essential use of the recent result of Ghisi and Gobbino [16] who found the optimal regularity of radicals of functions (we will need a version for complex valued functions; see Section 3). But we independently prove and generalize Ghisi and Gobbino's higher order Glaeser inequalities (see Section 4.5) on which their result is based.

Theorem 3 (Ghisi and Gobbino [16])**.** *Let k be a positive integer, let $\alpha \in (0,1]$, let $I \subseteq \mathbb{R}$ be an open bounded interval, and let $f : I \to \mathbb{R}$ be a function. Assume that f is continuous and that there exists $g \in C^{k,\alpha}(\overline{I}, \mathbb{R})$ such that*

$$|f|^{k+\alpha} = |g|.$$

Let p be defined by $1/p + 1/(k+\alpha) = 1$. Then we have $f' \in L^p_w(I)$ and

$$\|f'\|_{p,w,I} \le C(k) \max\Big\{ \big(\text{Höld}_{\alpha,I}(g^{(k)})\big)^{1/(k+\alpha)} |I|^{1/p}, \|g'\|^{1/(k+\alpha)}_{L^\infty(I)} \Big\}, \tag{1.7}$$

where $C(k)$ is a constant that depends only on k.

Here $L^p_w(I)$ denotes the weak Lebesgue space equipped with the quasinorm $\| \cdot \|_{p,w,I}$ (see Section 2.2), and $\text{Höld}_{\alpha,I}(g^{(k)})$ is the α-Hölder constant of $g^{(k)}$ on I.

1.2. **Open problems.** We remark that our bound (1.2) is not invariant under rescaling, in contrast to (1.7). The reasons for this defect is linked to our method of proof.

Open Problem 1. Are there scale invariant estimates which could replace (1.2)?

We do not know whether, in the setting of Theorem 1, λ' is actually an element of $L^{n/(n-1)}_w((\alpha,\beta))$; as one could expect in view of Theorem 3. This has technical reasons and comes from the fact that $\| \cdot \|^p_{p,w,I}$ is not σ-additive.

Open Problem 2. Is λ' in the setting of Theorem 1 an element of $L^{n/(n-1)}_w((\alpha,\beta))$? If so is there an explicit bound for $\|\lambda'\|_{n/(n-1),w,(\alpha,\beta)}$ in terms of the coefficients a_j and the interval (α,β)?

The roots of (1.5) will in general not allow for continuous (and, a fortiori, $W^{1,1}_{\text{loc}}$) parameterizations if $m \ge 2$. It is thus natural to ask if the roots are representable locally by functions of bounded variation.

Open Problem 3. Are the roots of a polynomial $P_a(x)$, $x \in \mathbb{R}^m$, $m \ge 2$, with smooth complex valued coefficients representable by functions which locally have bounded variation? We can prove this for radicals of smooth functions.

1.3. **Strategy of the proof of Theorem 1.** Let us briefly describe the strategy of our proof of Theorem 1. It is by induction on the degree of the polynomial and its heart is Proposition 3 below.

First we reduce the polynomial P_a to Tschirnhausen form $P_{\tilde a}$ (indicated by adding *tilde*), where $\tilde a_1 \equiv 0$ (see Section 4.1). This has the benefit that near points t_0, where not all coefficients vanish, the polynomial $P_{\tilde a}$ splits,

$$P_{\tilde a}(t) = P_b(t) P_{b^*}(t), \quad t \in I, \quad (t_0 \in I),$$

thanks to the inverse function theorem. It is important for our proof that the splitting is *universal* (and independent of t_0). We achieve this by considering the polynomial

$$Q_{\underline{a}}(Z) := \tilde a_k^{-n/k} P_{\tilde a}(\tilde a_k^{1/k} Z) = Z^n + \sum_{j=2}^n \underline{a}_j Z^{n-j},$$

which splits locally near every $(\underline{a}_2, \ldots, \underline{a}_n) \in \mathbb{C}^{n-1} \cap \{\underline{a}_k = 1\}$.

We obtain a universal splitting by choosing a finite subcover of the compact set of points with $\underline{a}_k = 1$ and $|\underline{a}_j| \le 1$ for $j \ne k$. It induces a splitting of $P_{\tilde{a}}$ and gives formulas for the coefficients b_i (and b_i^*) in terms of $\tilde{a}_j$. See Sections 4.2 and 4.3. The differentiability class of the $\tilde{a}_j$ is preserved by the splitting.

After the Tschirnhausen transformation $P_b \rightsquigarrow P_{\tilde{b}}$, we split $P_{\tilde{b}}$ near points $t_1 \in I$, where not all $\tilde{b}_i$ vanish,

$$P_{\tilde{b}}(t) = P_c(t) P_{c^*}(t), \quad t \in J, \quad (t_1 \in J).$$

Again we use the universal splitting (now for polynomials of degree $n_b := \deg P_b$ in Tschirnhausen form). We get formulas for c_h (and c_h^*) in terms of $\tilde{b}_j$, and the differentiability class is preserved. Then we apply the Tschirnhausen transformation $P_c \rightsquigarrow P_{\tilde{c}}$ and so on.

The central idea underlying the induction is to show that, for $1 \le p < n/(n-1)$, we have an estimate of the form

$$\||J|^{-1}|\tilde{b}_\ell(t_1)|^{1/\ell}\|_{L^p(J)} + \sum_{h=2}^{n_c} \|(\tilde{c}_h^{1/h})'\|_{L^p(J)} \le C\Big(\||I|^{-1}|\tilde{a}_k(t_0)|^{1/k}\|_{L^p(J)} + \sum_{i=2}^{n_b} \|(\tilde{b}_i^{1/i})'\|_{L^p(J)}\Big), \tag{1.8}$$

for a universal constant $C = C(n,p)$ (where $n_c := \deg P_c$). Here k (resp. ℓ) is chosen such that $|\tilde{a}_k(t_0)|^{1/k} = \max_{2\le j\le n} |\tilde{a}_j(t_0)|^{1/j}$ (resp. $|\tilde{b}_\ell(t_1)|^{1/\ell} = \max_{2\le i\le n_b} |\tilde{b}_i(t_1)|^{1/i}$), that is the kth (resp. ℓth) correctly weighted coefficient is *dominant* at t_0 (resp. t_1).

In the derivation of (1.8) we make essential use of (1.7) and Lemma 4 below in order to bound the left-hand side by

$$|J|^{-1+1/p}|\tilde{b}_\ell(t_1)|^{1/\ell}.$$

Now the key to get (1.8) from this is that we can choose the interval J such that

$$D|J|^{-1+1/p}|\tilde{b}_\ell(t_1)|^{1/\ell} = |J|^{1/p}\Big(|I|^{-1}|\tilde{a}_k(t_0)|^{1/k} + \sum_{i=2}^{n_b} \|(\tilde{b}_i^{1/i})'\|_{L^1(J)}\Big) \tag{1.9}$$

where D is a universal constant.

We get the estimate (1.8) on neighborhoods J of all points $t_1 \in I$, where not all $\tilde{b}_i$ vanish. In order to glue these estimates we prove in Proposition 2 that there is a countable subcollection of intervals J such that every point in their union is covered at most by two intervals. In this gluing process we use the σ-additivity of $\|\cdot\|_{L^p}^p$. Since the L^p_w-quasinorm lacks this property, we are forced to switch from $L_w^{n/(n-1)}$- to L^p-bounds for $p < n/(n-1)$.

In the end we must estimate the right-hand side of (1.8) by a bound involving the $C^{n-1,1}$-norm of the $\tilde{a}_j$. At this stage we will not always have an identity corresponding to (1.9) (see Remark 7). We resolve this inconvenience by extending the coefficients $\tilde{a}_j$ to a larger interval and we force them to vanish at the boundary of this interval. This results in an identity of the type (1.9) for the $\tilde{a}_j$ instead of the $\tilde{b}_i$ (see Lemma 16). However, in this process we lose scale invariance of our bound (1.2).

1.4. **Structure of the paper.** The paper is structured as follows. We fix notation and recall facts on function spaces in Section 2. Ghisi and Gobbino's result on radicals (Theorem 3) is extended to complex valued functions in Section 3. We collect preliminaries on polynomials and define a universal splitting of such in Section 4. We derive bounds for the coefficients of a polynomial and generalize Ghisi and Gobbino's higher order Glaeser inequalities [16, Proposition 3.4] in Section 4.5, by applying these bounds to the Taylor polynomial. In Sections 5 and 6 we deduce estimates for the iterated derivatives of the coefficients before and after the splitting. Section 7 is dedicated to the proof of Proposition 2. The proof of Theorem 1 is finally carried out in Section 8; in Appendix A we illustrate the proof for polynomials of degree 3 and 4. We deduce Theorem 2 in Section 9. In Section 10 we provide three applications of our results: local solvability of a system of pseudo-differential equations, a lifting theorem for mappings into orbit spaces of finite group representations, and a sufficient condition for multi-valued functions to be of Sobolev class $W^{1,p}$ in the sense of Almgren [3].

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2. Function spaces

In this section we fix notation for function spaces and recall well-known facts.

2.1. **Hölder spaces.** Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. We denote by $C^0(\Omega)$ the space of continuous complex valued functions on Ω. For $k \in \mathbb{N} \cup \{\infty\}$ we set

$$C^k(\Omega) = \{f \in \mathbb{C}^\Omega : \partial^\alpha f \in C^0(\Omega), 0 \le |\alpha| \le k\},$$
$$C^k(\overline{\Omega}) = \{f \in C^k(\Omega) : \partial^\alpha f \text{ has a continuous extension to } \overline{\Omega}, 0 \le |\alpha| \le k\}.$$

For $\alpha \in (0,1]$ a function $f : \Omega \to \mathbb{C}$ belongs to $C^{0,\alpha}(\overline{\Omega})$ if it is α-Hölder continuous in Ω, i.e.,

$$\text{Höld}_{\alpha,\Omega}(f) := \sup_{x,y\in\Omega, x\ne y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty.$$

If f is Lipschitz, i.e., $f \in C^{0,1}(\overline{\Omega})$, we use

$$\text{Lip}_\Omega(f) = \text{Höld}_{1,\Omega}(f).$$

We define

$$C^{k,\alpha}(\overline{\Omega}) = \{f \in C^k(\overline{\Omega}) : \partial^\beta f \in C^{0,\alpha}(\overline{\Omega}), |\beta| = k\}.$$

Note that $C^{k,\alpha}(\overline{\Omega})$ is a Banach space when provided with the norm

$$\|f\|_{C^{k,\alpha}(\overline{\Omega})} := \sup_{\substack{|\beta|\le k\\ x\in\Omega}} |\partial^\beta f(x)| + \sup_{|\beta|=k} \text{Höld}_{\alpha,\Omega}(\partial^\beta f).$$

2.2. **Lebesgue spaces and weak Lebesgue spaces.** Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $1 \le p \le \infty$. We denote by $L^p(\Omega)$ the Lebesgue space with respect to the n-dimensional Lebesgue measure $\mathcal{L}^n$. For Lebesgue measurable sets $E \subseteq \mathbb{R}^n$ we denote by

$$|E| = \mathcal{L}^n(E)$$

its n-dimensional Lebesgue measure. Let p' denote the conjugate exponent of p defined by

$$\frac{1}{p} + \frac{1}{p'} = 1$$

with the convention $1' = \infty$ and $\infty' = 1$.

Let $1 \le p < \infty$ and let us assume that Ω is bounded. A measurable function $f : \Omega \to \mathbb{C}$ belongs to the weak L^p-space $L^p_w(\Omega)$ if

$$\|f\|_{p,w,\Omega} := \sup_{r \ge 0} r \, |\{x \in \Omega : |f(x)| > r\}|^{1/p} < \infty.$$

For $1 \le q < p < \infty$ we have (cf. [18, Ex. 1.1.11])

$$\|f\|_{q,w,\Omega} \le \|f\|_{L^q(\Omega)} \le \Big(\frac{p}{p-q}\Big)^{1/q} |\Omega|^{1/q-1/p} \|f\|_{p,w,\Omega} \tag{2.1}$$

and hence $L^p(\Omega) \subseteq L^p_w(\Omega) \subseteq L^q(\Omega) \subseteq L^q_w(\Omega)$ with strict inclusions. It will be convenient to *normalize* the L^p-norm and the L^p_w-quasinorm, i.e., we will consider

$$\|f\|^*_{L^p(\Omega)} := |\Omega|^{-1/p} \|f\|_{L^p(\Omega)},$$
$$\|f\|^*_{p,w,\Omega} := |\Omega|^{-1/p} \|f\|_{p,w,\Omega}.$$

Note that $\|1\|^*_{L^p(\Omega)} = \|1\|^*_{p,w,\Omega} = 1$. Then, for $1 \le q < p < \infty$,

$$\|f\|^*_{L^q(\Omega)} \le \|f\|^*_{L^p(\Omega)}, \tag{2.2}$$

$$\|f\|^*_{q,w,\Omega} \le \|f\|^*_{L^q(\Omega)} \le \Big(\frac{p}{p-q}\Big)^{1/q} \|f\|^*_{p,w,\Omega}. \tag{2.3}$$

We remark that $\|\cdot\|_{p,w,\Omega}$ is only a quasinorm: the triangle inequality fails, but for $f_j \in L^p_w(\Omega)$ we still have

$$\Big\| \sum_{j=1}^m f_j \Big\|_{p,w,\Omega} \le m \sum_{j=1}^m \|f_j\|_{p,w,\Omega}.$$

There exists a norm equivalent to $\|\cdot\|_{p,w,\Omega}$ which makes $L^p_w(\Omega)$ into a Banach space if $p > 1$.

The L^p_w-quasinorm is σ-subadditive: if $\{\Omega_j\}$ is a countable family of open sets with $\Omega = \bigcup \Omega_j$ then

$$\|f\|^p_{p,w,\Omega} \le \sum_j \|f\|^p_{p,w,\Omega_j} \qquad \text{for every } f \in L^p_w(\Omega). \tag{2.4}$$

But it is not σ-additive: for instance, for $h : (0,\infty) \to \mathbb{R}$, $h(t) := t^{-1/p}$, we have $\|h\|^p_{p,w,(0,\epsilon)} = 1$ for every $\epsilon > 0$, but $\|h\|^p_{p,w,(1,2)} = 1/2$.

2.3. **Sobolev spaces.** For $k \in \mathbb{N}$ and $1 \le p \le \infty$ we consider the Sobolev space

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) : \partial^\alpha f \in L^p(\Omega), 0 \le |\alpha| \le k\},$$

where $\partial^\alpha f$ denote distributional derivatives, with the norm

$$\|f\|_{W^{k,p}(\Omega)} := \sum_{|\alpha| \le k} \|\partial^\alpha f\|_{L^p(\Omega)}.$$

On bounded intervals $I \subseteq \mathbb{R}$ the Sobolev space $W^{1,1}(I)$ coincides with the space $AC(I)$ of absolutely continuous functions on I if we identify each $W^{1,1}$-functions with its unique continuous representative. Recall that a function $f : \Omega \to \mathbb{R}$ on an open subset $\Omega \subseteq \mathbb{R}$ is absolutely continuous if for every $\epsilon > 0$ there exists $\delta > 0$ so that $\sum_{i=1}^n |a_i - b_i| < \delta$ implies $\sum_{i=1}^n |f(a_i) - f(b_i)| < \epsilon$ whenever $[a_i, b_i]$, $i = 1, \ldots, n$, are non-overlapping intervals contained in Ω.

We shall also use $W^{k,p}_{\text{loc}}$, AC_{loc}, etc. with the obvious meaning.

2.4. **Extension lemma.** We will use the following extension lemma. The analogue for the L^p_w-quasinorm may be found in [30, Lemma 2.1] which is a slight generalization of [16, Lemma 3.2]. Here we need a version for the L^p-norm; the proof is the same.

Lemma 1. *Let $\Omega \subseteq \mathbb{R}$ be open and bounded, let $f : \Omega \to \mathbb{C}$ be continuous, and set $\Omega_0 := \{t \in \Omega : f(t) \ne 0\}$. Assume that $f|_{\Omega_0} \in AC_{\text{loc}}(\Omega_0)$ and that $f|_{\Omega_0}' \in L^p(\Omega_0)$ for some $p \ge 1$ (note that f is differentiable a.e. in Ω_0). Then the distributional derivative of f in Ω is a measurable function $f' \in L^p(\Omega)$ and*

$$\|f'\|_{L^p(\Omega)} = \|f|_{\Omega_0}'\|_{L^p(\Omega_0)}. \tag{2.5}$$

Proof. One shows that

$$\psi(t) := \begin{cases} f'(t) & \text{if } t \in \Omega_0, \\ 0 & \text{if } t \in \Omega \setminus \Omega_0, \end{cases}$$

represents the distributional derivative of f in Ω; for details see [30, Lemma 2.1]. □

3. Radicals of differentiable functions

We derive an analogue of Theorem 3 for complex valued functions.

Proposition 1. *Let $I \subseteq \mathbb{R}$ be a bounded interval, let $k \in \mathbb{N}_{>0}$, and $\alpha \in (0, 1]$. For each $g \in C^{k,\alpha}(\overline{I})$ we have*

$$|g'(t)| \le \Lambda_{k+\alpha}(t)|g(t)|^{1-1/(k+\alpha)}, \qquad \text{a.e. in } I, \tag{3.1}$$

for some $\Lambda_{k+\alpha} = \Lambda_{k+\alpha,g} \in L^p_w(I, \mathbb{R}_{\ge 0})$, where $p = (k + \alpha)'$, and such that

$$\|\Lambda_{k+\alpha}\|_{p,w,I} \le C(k) \max\Big\{ \big(\text{Höld}_{\alpha,I}(g^{(k)})\big)^{1/(k+\alpha)} |I|^{1/p}, \|g'\|_{L^\infty(I)}^{1/(k+\alpha)} \Big\}. \tag{3.2}$$

Proof. Analogous to the proof of [30, Proposition 3.1]. □

Corollary 2. *Let n be a positive integer and let $I \subseteq \mathbb{R}$ be an open bounded interval. Assume that $f : I \to \mathbb{C}$ is a continuous function such that $f^n = g \in C^{n-1,1}(\overline{I})$. Then we have $f' \in L^{n'}_w(I)$ and*

$$\|f'\|_{n',w,I} \le C(n) \max \Big\{ \big(\operatorname{Lip}_I(g^{(n-1)}) \big)^{1/n} |I|^{1/n'}, \|g'\|^{1/n}_{L^\infty(I)} \Big\}. \tag{3.3}$$

Proof. On the set $\Omega_0 = \{t \in I : f(t) \ne 0\}$, f is differentiable and satisfies

$$|f'(t)| = \frac{1}{n} \frac{|g'(t)|}{|g(t)|^{1-1/n}}.$$

So the assertion follows from Proposition 1 and the L^p_w-analogue of Lemma 1; see [30, Lemma 2.1]. □

Remark 2. Proposition 1 and hence also Corollary 2 are optimal in the following sense:

- $\Lambda_{k+\alpha}$ can in general not be chosen in L^p. Indeed, for $g : (-1,1) \to \mathbb{R}$, $g(t) = t$, we have $|g'||g|^{1/(k+\alpha)-1} = |t|^{-1/p}$ which is not p-integrable near 0; see [16, Example 4.3].
- If g is only in $C^{k,\beta}(\overline{I})$ for every $\beta < \alpha$, then (3.1) in general fails even for $\Lambda_{k+\alpha} \in L^1(I)$. We refer to [16, Example 4.4] for a non-negative function $g \in \bigcap_{\beta<\alpha} C^{k,\beta}(\overline{I}) \cap C^\infty(I)$ and $g \notin C^{k,\alpha}(\overline{I})$ whose non-negative $(k+\alpha)$-root has unbounded variation in I.

4. Preliminaries on polynomials

4.1. **Tschirnhausen transformation.** A monic polynomial

$$P_a(Z) = Z^n + \sum_{j=1}^n a_j Z^{n-j}, \quad a = (a_1, \ldots, a_n) \in \mathbb{C}^n,$$

is said to be in *Tschirnhausen form* if $a_1 = 0$. Every polynomial P_a can be transformed to a polynomial $P_{\tilde a}$ in Tschirnhausen form by the substitution $Z \mapsto Z - a_1/n$, which we refer to as the *Tschirnhausen transformation*,

$$P_{\tilde a}(Z) = P_a(Z - a_1/n) = Z^n + \sum_{j=2}^n \tilde a_j Z^{n-j}, \quad \tilde a = (\tilde a_2, \ldots, \tilde a_n) \in \mathbb{C}^{n-1}.$$

We have the formulas

$$\tilde a_j = \sum_{\ell=0}^j C_\ell \, a_\ell \, a_1{}^{j-\ell}, \quad j = 2, \ldots, n, \tag{4.1}$$

where C_ℓ are universal constants. The effect of the Tschirnhausen transformation will always be indicated by adding *tilde* to the coefficients, $P_a \rightsquigarrow P_{\tilde a}$.

We will identify the set of monic complex polynomials P_a of degree n with the set $\mathbb{C}^n$ (via $P_a \mapsto a$) and the set of monic complex polynomials $P_{\tilde a}$ of degree n in Tschirnhausen form with the set $\mathbb{C}^{n-1}$ (via $P_{\tilde a} \mapsto \tilde a$).

4.2. **Splitting.** The following well-known lemma (see e.g. [1] or [5]) is a consequence of the inverse function theorem.

Lemma 2. *Let $P_a = P_b P_c$, where P_b and P_c are monic complex polynomials without common root. Then for P near P_a we have $P = P_{b(P)} P_{c(P)}$ for analytic mappings of monic polynomials $P \mapsto b(P)$ and $P \mapsto c(P)$, defined for P near P_a, with the given initial values.*

Proof. The splitting $P_a = P_b P_c$ defines on the coefficients a polynomial mapping φ such that $a = \varphi(b, c)$, where $a = (a_i)$, $b = (b_i)$, and $c = (c_i)$. The Jacobian determinant $\det d\varphi(b, c)$ equals the resultant of P_b and P_c which is non-zero by assumption. Thus φ can be inverted locally. □

If $P_{\tilde a}$ is in Tschirnhausen form and if $\tilde a \neq 0$, then $P_{\tilde a}$ *splits*, i.e., $P_{\tilde a} = P_b P_c$ for monic polynomials P_b and P_c with positive degree and without common zero. For, if $\lambda_1, \ldots, \lambda_n$ denote the roots of $P_{\tilde a}$ and they all coincide, then since

$$\lambda_1 + \cdots + \lambda_n = \tilde a_1 = 0$$

they all must vanish, contradicting $\tilde a \neq 0$.

Let $\tilde a_2, \ldots, \tilde a_n$ denote the coordinates in $\mathbb{C}^{n-1}$ (= set of polynomials of degree n in Tschirnhausen form). Fix $k \in \{2, \ldots, n\}$ and let $\tilde p \in \mathbb{C}^{n-1} \cap \{\tilde a_k \neq 0\}$; $\tilde p$ corresponds to the polynomial $P_{\tilde a}$. We associate the polynomial

$$Q_{\underline{a}}(Z) := \tilde a_k^{-n/k} P_{\tilde a}(\tilde a_k^{1/k} Z) = Z^n + \sum_{j=2}^{n} \tilde a_k^{-j/k} \tilde a_j Z^{n-j},$$

$$\underline{a}_j := \tilde a_k^{-j/k} \tilde a_j, \quad j = 2, \ldots, n,$$

where some branch of the radical is fixed. Then $Q_{\underline{a}}$ is in Tschirnhausen form and $\underline{a}_k = 1$; it corresponds to a point $\underline{p} \in \mathbb{C}^{n-1} \cap \{\underline{a}_k = 1\}$. By Lemma 2 we have a splitting $Q_{\underline{a}} = Q_{\underline{b}} Q_{\underline{c}}$ on some open ball $B_\rho(\underline{p})$ centered at $\underline{p}$ with radius $\rho > 0$. In particular, there exist analytic functions ψ_i on $B_\rho(\underline{p})$ such that

$$\underline{b}_i = \psi_i\big(\tilde a_k^{-2/k} \tilde a_2, \tilde a_k^{-3/k} \tilde a_3, \ldots, \tilde a_k^{-n/k} \tilde a_n\big), \quad i = 1, \ldots, \deg Q_{\underline{b}}.$$

The splitting $Q_{\underline{a}} = Q_{\underline{b}} Q_{\underline{c}}$ induces a splitting $P_{\tilde a} = P_b P_c$, where

$$b_i = \tilde a_k^{i/k} \psi_i\big(\tilde a_k^{-2/k} \tilde a_2, \tilde a_k^{-3/k} \tilde a_3, \ldots, \tilde a_k^{-n/k} \tilde a_n\big), \quad i = 1, \ldots, n_b := \deg P_b; \tag{4.2}$$

likewise for c_j. Shrinking ρ slightly, we may assume that ψ_i and all its partial derivatives are bounded on $B_\rho(\underline{p})$. Let $\tilde b_j$ denote the coefficients of the polynomial $P_{\tilde b}$ resulting from P_b by the Tschirnhausen transformation. Then, by (4.1),

$$\tilde b_i = \tilde a_k^{i/k} \tilde \psi_i\big(\tilde a_k^{-2/k} \tilde a_2, \tilde a_k^{-3/k} \tilde a_3, \ldots, \tilde a_k^{-n/k} \tilde a_n\big), \quad i = 2, \ldots, n_b, \tag{4.3}$$

for analytic functions $\tilde\psi_i$ which, together with all their partial derivatives, are bounded on $B_\rho(\underline{p})$.

4.3. **Universal splitting of polynomials in Tschirnhausen form.** The set

$$K := \bigcup_{k=2}^{n} \{(\underline{a}_2, \ldots, \underline{a}_n) \in \mathbb{C}^{n-1} : \underline{a}_k = 1,\ |\underline{a}_j| \le 1 \text{ for } j \ne k\} \tag{4.4}$$

is compact. For each point $\underline{p} \in K$ there exists $\rho(\underline{p}) > 0$ such that we have a splitting $P_{\tilde{a}} = P_b P_c$ on the open ball $B_{\rho(\underline{p})}(\underline{p})$, and we fix this splitting; cf. Section 4.2. Choose a finite subcover of K by open balls $B_{\rho_\delta}(\underline{p}_\delta)$, $\delta \in \Delta$. Then there exists $\rho > 0$ such that for every $\underline{p} \in K$ there is a $\delta \in \Delta$ such that $B_\rho(\underline{p}) \subseteq B_{\rho_\delta}(\underline{p}_\delta)$.

To summarize, for each integer $n \ge 2$ we have fixed

- a finite cover $\mathcal{B}$ of K by open balls B,
- a splitting $P_{\tilde{a}} = P_b P_c$ on each $B \in \mathcal{B}$ together with analytic functions ψ_i and $\tilde{\psi}_i$ which are bounded on B along with all their partial derivatives,
- a positive number ρ such that for each $\underline{p} \in K$ there is a $B \in \mathcal{B}$ such that $B_\rho(\underline{p}) \subseteq B$ (note that ρ is a Lebesgue number of the cover $\mathcal{B}$).

We will refer to this data as a *universal splitting of polynomials of degree* n *in Tschirnhausen form* and to ρ as the *radius of the splitting.*

4.4. **Coefficient estimates.** The following estimates are crucial. (Here it is convenient to number the coefficients in reversed order.)

Lemma 3. *Let $m \ge 1$ be an integer and $\alpha \in (0,1]$. Let $P(x) = a_1 x + \cdots + a_m x^m \in \mathbb{C}[x]$ satisfy*

$$|P(x)| \le A(1 + M x^{m+\alpha}), \quad \text{for } x \in [0, B] \subseteq \mathbb{R}, \tag{4.5}$$

and constants $A, M \ge 0$ and $B > 0$. Then

$$|a_j| \le CA(1 + M^{j/(m+\alpha)} B^j) B^{-j}, \quad j = 1, \ldots, m, \tag{4.6}$$

for a constant C depending only on m and α.

Proof. The statement is well-known if $M = 0$; see [31, Lemma 3.4]. Assume that $M > 0$.

It suffices to consider the special case $A = B = 1$. The general case follows by applying the special case to $Q(x) = A^{-1} P(Bx) = b_1 x + \cdots + b_m x^m$, where $b_i = A^{-1} B^i a_i$.

Fix $k \in \{1, \ldots, m\}$ and write the inequality (4.5) in the form

$$|x^{-k} P(x)| \le x^{-k} + M x^{m+\alpha-k}. \tag{4.7}$$

The function on the right-hand side of (4.7) attains is minimum on $\{x > 0\}$ at the point

$$x_k = \big(\tfrac{k}{m+\alpha-k}\big)^{1/(m+\alpha)} M^{-1/(m+\alpha)}, \tag{4.8}$$

and this minimum is of the form $C_k M^{k/(m+\alpha)}$ for some C_k depending only on k, m, and α. Thus, provided that $x_k \le 1$, we get

$$|P(x_k)| \le C_k M^{k/(m+\alpha)} |x_k^k| = C_k \big(\tfrac{k}{m+\alpha-k}\big)^{k/(m+\alpha)}.$$

Suppose first that $x_k \le 1$ for all $k = 1, \ldots, m$ and consider

$$a_1 x_k + \cdots + a_m x_k^m = P(x_k), \quad k = 1, \ldots, m,$$

as a system of linear equations with the unknowns $a_j M^{-j/(m+\alpha)}$ and the (Vandermonde-like) matrix

$$L = \big(\big(\tfrac{k}{m+\alpha-k} \big)^{j/(m+\alpha)} \big)_{k,j=1}^{m}.$$

Then the vector of unknowns is given by

$$(a_1 M^{-1/(m+\alpha)}, \dots, a_m M^{-m/(m+\alpha)})^T = L^{-1}(P(x_1), P(x_2), \dots, P(x_m))^T.$$

By (4.4), we may conclude that

$$|a_j| \le C M^{j/(m+\alpha)}, \quad j = 1, \dots, m,$$

for a constant C depending only on m and α, that is (4.6).

If $x_k > 1$ then $M < k/(m+\alpha-k)$, by (4.8). Hence, using (4.5), for $x \in [0,1]$,

$$|P(x)| \le 1 + M x^{m+\alpha} \le 1 + \tfrac{k}{m+\alpha-k} \le \tfrac{m+\alpha}{\alpha}.$$

In this case we may apply the lemma with $M = 0$, $A = (m+\alpha)/\alpha$, and $B = 1$, and obtain

$$|a_j| \le C, \quad j = 1, \dots, m,$$

for a constant C depending only on m and α, which implies (4.6). □

As a consequence we get estimates for the intermediate derivatives of a finitely differentiable function in terms of the function and its highest derivative. For an interval $I \subseteq \mathbb{R}$ and a function $f : I \to \mathbb{C}$ we define

$$V_I(f) := \sup_{t,s \in I} |f(t) - f(s)|.$$

Lemma 4. *Let $I \subseteq \mathbb{R}$ be a bounded open interval, $m \in \mathbb{N}_{>0}$, and $\alpha \in (0,1]$. If $f \in C^{m,\alpha}(\overline{I})$, then for all $t \in I$ and $s = 1, \dots, m$,*

$$|f^{(s)}(t)| \le C|I|^{-s}\big(V_I(f) + V_I(f)^{(m+\alpha-s)/(m+\alpha)}(\text{Höld}_{\alpha,I}(f^{(m)}))^{s/(m+\alpha)}|I|^s\big), \tag{4.9}$$

for a universal constant C depending only on m and α.

Proof. We may suppose that $I = (-\delta, \delta)$. If $t \in I$ then at least one of the two intervals $[t, t \pm \delta)$, say $[t, t+\delta)$, is included in I. By Taylor's formula, for $t_1 \in [t, t+\delta)$,

$$\sum_{s=1}^{m} \frac{f^{(s)}(t)}{s!}(t_1 - t)^s = f(t_1) - f(t) - \int_0^1 \frac{(1-\tau)^{m-1}}{(m-1)!}\big(f^{(m)}(t + \tau(t_1 - t)) - f^{(m)}(t)\big)\, d\tau\, (t_1 - t)^m$$

and hence

$$\begin{aligned}\Big|\sum_{s=1}^{m} \frac{f^{(s)}(t)}{s!}(t_1 - t)^s\Big| &\le V_I(f) + \text{Höld}_{\alpha,I}(f^{(m)})(t_1 - t)^{m+\alpha} \\ &= V_I(f)\big(1 + V_I(f)^{-1}\,\text{Höld}_{\alpha,I}(f^{(m)})(t_1 - t)^{m+\alpha}\big).\end{aligned}$$

The assertion follows from Lemma 3. □

4.5. **Higher order Glaeser inequalities.** As a corollary of Lemma 4 we obtain a generalization of Ghisi and Gobbino's higher order Glaeser inequalities [16, Proposition 3.4].

Corollary 3. *Let $m \in \mathbb{N}_{>0}$ and $\alpha \in (0,1]$. Let $I = (t_0 - \delta, t_0 + \delta)$ with $t_0 \in \mathbb{R}$ and $\delta > 0$. If $f \in C^{m,\alpha}(\overline{I})$ is such that f and f' do not change their sign on I, then for all $s = 1, \ldots, m$,*

$$|f^{(s)}(t_0)| \le C|I|^{-s}\big(|f(t_0)| + |f(t_0)|^{(m+\alpha-s)/(m+\alpha)}(\text{Höld}_{\alpha,I}(f^{(m)}))^{s/(m+\alpha)}|I|^s\big), \tag{4.10}$$

for a universal constant C depending only on m and α.

Proof. For simplicity assume $t_0 = 0$. Changing f to $-f$ and t to $-t$ if necessary, we may assume that $f(t) \ge 0$ and $f'(t) \le 0$ for all $t \ge 0$. Then $V_{[0,\delta)}(f) \le f(0)$ and so (4.10) follows from (4.9). □

For $s = 1$ we recover [16, Proposition 3.4]. Indeed, for $s = 1$ we may write (4.10) as

$$|f'(t_0)| \le C|f(t_0)|^{(m+\alpha-1)/(m+\alpha)} \max\big\{|f(t_0)|^{1/(m+\alpha)}|I|^{-1}, (\text{Höld}_{\alpha,I}(f^{(m)}))^{1/(m+\alpha)}\big\}, \tag{4.11}$$

and the inequality in [16, Proposition 3.4] can be written as

$$|f'(t_0)| \le C|f(t_0)|^{(m+\alpha-1)/(m+\alpha)} \max\big\{|f'(t_0)|^{1/(m+\alpha)}|I|^{-1+1/(m+\alpha)}, (\text{Höld}_{\alpha,I}(f^{(m)}))^{1/(m+\alpha)}\big\}. \tag{4.12}$$

These two inequalities are equivalent in the following sense: if (4.11) holds with the constant $C > 0$ then (4.12) holds with the constant $\max\{C, C^{(m+\alpha-1)/(m+\alpha)}\}$, and, symmetrically, if (4.12) holds with the constant $C > 0$ then (4.11) holds with the constant $\max\{C, C^{(m+\alpha)/(m+\alpha-1)}\}$. For instance, suppose that (4.11) holds. If the second term in the maximum (in (4.11)) is dominant, then (4.12) holds with the same constant. If the first term is dominant in the maximum, that is $|f'(t_0)| \le C|f(t_0)||I|^{-1}$, then $|f'(t_0)|^{(m+\alpha-1)/(m+\alpha)} \le (C|f(t_0)||I|^{-1})^{(m+\alpha-1)/(m+\alpha)}$ and (4.12) holds with the constant $C^{(m+\alpha-1)/(m+\alpha)}$.

5. Estimates for the iterated derivatives of the coefficients

In the next three sections we collect the necessary tools for the proof of Theorem 1. In the current section we derive estimates for the derivatives of the coefficients of a $C^{n-1,1}$-curve of polynomials of degree n in Tschirnhausen form.

5.1. **Preparations for the splitting.** Let $I \subseteq \mathbb{R}$ be a bounded open interval and let

$$P_{\tilde{a}(t)}(Z) = Z^n + \sum_{j=2}^{n} \tilde{a}_j(t) Z^{n-j}, \quad t \in I, \tag{5.1}$$

be a monic complex polynomial in Tschirnhausen form with coefficients $\tilde{a}_j \in C^{n-1,1}(\overline{I})$, $j = 2, \ldots, n$. We make the following assumptions. Suppose that $t_0 \in I$ and $k \in \{2, \ldots, n\}$ are such that

$$|\tilde{a}_k(t_0)|^{1/k} = \max_{2 \le j \le n} |\tilde{a}_j(t_0)|^{1/j} \ne 0 \tag{5.2}$$

and that, for some positive constant $B < 1/3$,

$$\sum_{j=2}^{n} \|(\tilde a_j^{1/j})'\|_{L^1(I)} \le B|\tilde a_k(t_0)|^{1/k}. \tag{5.3}$$

By Corollary 2, every continuous selection f of the multi-valued function $\tilde a_j^{1/j}$ is absolutely continuous on I, and $\|f'\|_{L^1(I)}$ is independent of the choice of the selection (by (2.5)). (By a selection of a set-valued function $F : X \rightsquigarrow Y$ we mean a single-valued function $f : X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$.) So henceforth we shall fix one continuous selection of $\tilde a_j^{1/j}$ and, abusing notation, denote it by $\tilde a_j^{1/j}$ as well.

Lemma 5. *Assume that the polynomial* (5.1) *satisfies* (5.2)–(5.3). *Then for all $t \in I$ and $j = 2, \ldots, n$,*

$$|\tilde a_j^{1/j}(t) - \tilde a_j^{1/j}(t_0)| \le B|\tilde a_k(t_0)|^{1/k}, \tag{5.4}$$

$$\frac{2}{3} < 1 - B \le \Big|\frac{\tilde a_k(t)}{\tilde a_k(t_0)}\Big|^{1/k} \le 1 + B < \frac{4}{3}, \tag{5.5}$$

$$|\tilde a_j(t)|^{1/j} \le \frac{4}{3}|\tilde a_k(t_0)|^{1/k} \le 2|\tilde a_k(t)|^{1/k}. \tag{5.6}$$

Proof. First, (5.4) is a consequence of (5.3),

$$|\tilde a_j^{1/j}(t) - \tilde a_j^{1/j}(t_0)| = \Big|\int_{t_0}^{t} (\tilde a_j^{1/j})'\, ds\Big| \le \|(\tilde a_j^{1/j})'\|_{L^1(I)} \le B|\tilde a_k(t_0)|^{1/k}.$$

For $j = k$ it implies

$$\Big|\Big|\frac{\tilde a_k(t)}{\tilde a_k(t_0)}\Big|^{1/k} - 1\Big| \le B,$$

and thus (5.5). By (5.2), (5.4), and (5.5),

$$|\tilde a_j(t)|^{1/j} \le (1 + B)|\tilde a_k(t_0)|^{1/k} \le 2|\tilde a_k(t)|^{1/k},$$

that is (5.6). □

By (5.5), $\tilde a_k$ does not vanish on the interval I and so the curve

$$\begin{aligned} \underline{a} : I &\to \{(\underline{a}_2, \ldots, \underline{a}_n) \in \mathbb{C}^{n-1} : \underline{a}_k = 1\} \\ t &\mapsto \underline{a}(t) := (\tilde a_k^{-2/k}\tilde a_2, \ldots, \tilde a_k^{-n/k}\tilde a_n)(t) \end{aligned} \tag{5.7}$$

is well-defined.

Lemma 6. *Assume that the polynomial* (5.1) *satisfies* (5.2)–(5.3). *Then the length of the curve* (5.7) *is bounded by $3n^2\, 2^n B$.*

Proof. The estimates (5.4), (5.5), and (5.6) imply

$$|\tilde a_k^{-j/k}\tilde a_j'| \le 2^n|\tilde a_j^{-1+1/j}\tilde a_j'\tilde a_k^{-1/k}| \le 3n\, 2^{n-1}|(\tilde a_j^{1/j})'||\tilde a_k(t_0)|^{-1/k}$$
$$|(\tilde a_k^{-j/k})'\tilde a_j| \le n2^n|\tilde a_k^{-1/k}(\tilde a_k^{1/k})'| \le 3n\, 2^{n-1}|(\tilde a_k^{1/k})'||\tilde a_k(t_0)|^{-1/k},$$

and thus

$$|(\tilde{a}_k^{-j/k}\tilde{a}_j)'| \le 3n\, 2^{n-1}|\tilde{a}_k(t_0)|^{-1/k}\Big(|(\tilde{a}_j^{1/j})'| + |(\tilde{a}_k^{1/k})'|\Big).$$

Consequently, using (5.3),

$$\int_I |\underline{a}'|\, ds \le 3n^2\, 2^n B,$$

as required. □

5.2. **Estimates for the derivatives of the coefficients.** Let us replace (5.3) by the stronger assumption

$$M|I| + \sum_{j=2}^{n} \|(\tilde{a}_j^{1/j})'\|_{L^1(I)} \le B|\tilde{a}_k(t_0)|^{1/k}, \tag{5.8}$$

where

$$M = \max_{2\le j\le n} (\operatorname{Lip}_I(\tilde{a}_j^{(n-1)}))^{1/n}|\tilde{a}_k(t_0)|^{(n-j)/(kn)}. \tag{5.9}$$

Lemma 7. *Assume that the polynomial* (5.1) *satisfies* (5.2) *and* (5.8). *Then for all* $j = 2,\dots,n$ *and* $s = 1,\dots,n-1$*,*

$$\begin{aligned} \|\tilde{a}_j^{(s)}\|_{L^\infty(I)} &\le C(n)|I|^{-s}|\tilde{a}_k(t_0)|^{j/k}, \\ \operatorname{Lip}_I(\tilde{a}_j^{(n-1)}) &\le C(n)|I|^{-n}|\tilde{a}_k(t_0)|^{j/k}. \end{aligned} \tag{5.10}$$

Proof. The second estimate in (5.10) is immediate from (5.8). Let $t \in I$. By Lemma 4,

$$|\tilde{a}_j^{(s)}(t)| \le C|I|^{-s}\big(V_I(\tilde{a}_j) + V_I(\tilde{a}_j)^{(n-s)/n}\operatorname{Lip}_I(\tilde{a}_j^{(n-1)})^{s/n}|I|^s\big).$$

By (5.6),

$$V_I(\tilde{a}_j) \le 2\|\tilde{a}_j\|_{L^\infty(I)} \le 2\,(4/3)^n|\tilde{a}_k(t_0)|^{j/k}$$

and, by (5.8),

$$\max_{2\le j\le n}(\operatorname{Lip}_I(\tilde{a}_j^{(n-1)}))^{s/n}|\tilde{a}_k(t_0)|^{-js/(kn)}|I|^s = |\tilde{a}_k(t_0)|^{-s/k}M^s|I|^s \le 1.$$

Thus

$$\begin{aligned} &V_I(\tilde{a}_j) + V_I(\tilde{a}_j)^{(n-s)/n}\operatorname{Lip}_I(\tilde{a}_j^{(n-1)})^{s/n}|I|^s \\ &\quad\le |\tilde{a}_k(t_0)|^{j/k}\big(C_1 + C_2\operatorname{Lip}_I(\tilde{a}_j^{(n-1)})^{s/n}|\tilde{a}_k(t_0)|^{-js/(kn)}|I|^s\big) \\ &\quad\le C_3|\tilde{a}_k(t_0)|^{j/k}, \end{aligned}$$

for constants C_i that depend only on n. So also the first estimate in (5.10) is proved. □

6. The estimates after splitting

In this section we assume that our polynomial splits. We prove that the coefficients of each factor of the splitting satisfy estimates analogous to those in (5.10) on suitable subintervals.

6.1. **Estimates after splitting on I.** Assume that the polynomial (5.1) satisfies (5.2)–(5.3) and the estimates (5.10).

Additionally, we suppose that the curve $\underline{a}$ defined in (5.7) lies entirely in one of the balls $B_\rho(\underline{p})$ from Section 4.2 on which we have a splitting. Then $P_{\tilde a}$ splits on I,

$$P_{\tilde a}(t) = P_b(t) P_{b^*}(t), \quad t \in I. \tag{6.1}$$

By (4.2) and (4.3), the coefficients b_i are of the form

$$b_i = \tilde a_k^{i/k} \psi_i\big(\tilde a_k^{-2/k} \tilde a_2, \ldots, \tilde a_k^{-n/k} \tilde a_n\big), \quad i = 1, \ldots, n_b, \tag{6.2}$$

and after the Tschirnhausen transformation $P_b \rightsquigarrow P_{\tilde b}$, we get

$$\tilde b_i = \tilde a_k^{i/k} \tilde\psi_i\big(\tilde a_k^{-2/k} \tilde a_2, \ldots, \tilde a_k^{-n/k} \tilde a_n\big), \quad i = 2, \ldots, n_b, \tag{6.3}$$

where ψ_i and $\tilde\psi_i$ are the analytic functions specified in Section 4.2 and $n_b = \deg P_b$.

Lemma 8. *Assume that the polynomial* (5.1) *satisfies* (5.2)–(5.3)*,* (5.10)*, and* (6.1)–(6.3)*. Then for all $i = 2, \ldots, n_b$ and $s = 1, \ldots, n-1$,*

$$\begin{aligned} \|\tilde b_i^{(s)}\|_{L^\infty(I)} &\le C|I|^{-s} |\tilde a_k(t_0)|^{i/k}, \\ \operatorname{Lip}_I(\tilde b_i^{(n-1)}) &\le C|I|^{-n} |\tilde a_k(t_0)|^{i/k}, \end{aligned} \tag{6.4}$$

where C is a constant depending only on n and on the functions $\tilde\psi_i$.

Proof. Let us prove the first estimate in (6.4). Let F be any C^n-function defined on an open set U in $\mathbb{C}^{n-1}$ containing $\underline{a}(I)$ and satisfying $\|F\|_{C^n(\overline{U})} < \infty$. We claim that, for $s = 1, \ldots, n-1$,

$$\|\partial_t^s (F \circ \underline{a})\|_{L^\infty(I)} \le C|I|^{-s}, \tag{6.5}$$

where C is a constant depending only on n and $\|F\|_{C^n(\overline{U})}$. For any real exponent r, Faà di Bruno's formula implies

$$\partial_t^s\big(\tilde a_j^r\big) = \sum_{\ell \ge 1}^{s} \sum_{\gamma \in \Gamma(\ell,s)} c_{\gamma,\ell,r}\, \tilde a_j^{r-\ell} \tilde a_j^{(\gamma_1)} \cdots \tilde a_j^{(\gamma_\ell)} \tag{6.6}$$

where $\Gamma(\ell, s) = \{\gamma \in \mathbb{N}_{>0}^\ell : |\gamma| = s\}$ and

$$c_{\gamma,\ell,r} = \frac{s!}{\ell!\gamma!} r(r-1)\cdots(r-\ell+1).$$

By (5.10) and (5.5), this implies

$$\begin{aligned} \|\partial_t^s\big(\tilde a_j^r\big)\|_{L^\infty(I)} &\le \sum_{\ell \ge 1}^{s} \sum_{\gamma \in \Gamma(\ell,s)} c_{\gamma,\ell,r}\, \|\tilde a_j^{r-\ell}\|_{L^\infty(I)} \|\tilde a_j^{(\gamma_1)}\|_{L^\infty(I)} \cdots \|\tilde a_j^{(\gamma_\ell)}\|_{L^\infty(I)} \\ &\le C(n) \sum_{\ell \ge 1}^{s} \sum_{\gamma \in \Gamma(\ell,s)} c_{\gamma,\ell,r}\, |\tilde a_k(t_0)|^{(r-\ell)j/k} |I|^{-s} |\tilde a_k(t_0)|^{\ell j/k} \\ &\le C(n) |I|^{-s} |\tilde a_k(t_0)|^{rj/k}. \end{aligned} \tag{6.7}$$

Together with the Leibniz formula,

$$\partial_t^s\big(\tilde a_k^{-j/k}\tilde a_j\big) = \sum_{q=0}^{s}\binom{s}{q}\tilde a_j^{(q)}\partial_t^{s-q}\big(\tilde a_k^{-j/k}\big),$$

(6.7) and (5.10) lead to

$$\|\partial_t^s\big(\tilde a_k^{-j/k}\tilde a_j\big)\|_{L^\infty(I)} \le C(n)|I|^{-s}. \tag{6.8}$$

Again by the Leibniz formula,

$$\begin{aligned}
\partial_t(F\circ \underline{a}) &= \sum_{j=2}^{n}((\partial_{j-1}F)\circ \underline{a})\,\partial_t\big(\tilde a_k^{-j/k}\tilde a_j\big),\\
\partial_t^s(F\circ \underline{a}) &= \sum_{j=2}^{n}\partial_t^{s-1}\Big(((\partial_{j-1}F)\circ \underline{a})\,\partial_t\big(\tilde a_k^{-j/k}\tilde a_j\big)\Big)\\
&= \sum_{j=2}^{n}\sum_{p=0}^{s-1}\binom{s-1}{p}\partial_t^p((\partial_{j-1}F)\circ \underline{a})\,\partial_t^{s-p}\big(\tilde a_k^{-j/k}\tilde a_j\big).
\end{aligned}$$

For $s=1$ we immediately get (6.5). For $1<s\le n-1$, we may argue by induction on s. By induction hypothesis,

$$\|\partial_t^p((\partial_{j-1}F)\circ \underline{a})\|_{L^\infty(I)} \le C(n,\|\partial_{j-1}F\|_{C^s(\overline{U})})|I|^{-p},$$

for $p=1,\dots,s-1$. Together with (6.8) this entails (6.5).

Now the first part of (6.4) is a consequence of (6.3), (6.7) (for $j=k$ and $r=i/k$) and (6.5) (applied to $F=\tilde\psi_i$).

For the second part of (6.4) observe that for functions $f_1,\dots,f_m$ on I we have

$$\mathrm{Lip}_I(f_1f_2\cdots f_m) \le \sum_{i=1}^{m}\mathrm{Lip}_I(f_i)\|f_1\|_{L^\infty(I)}\cdots\widehat{\|f_i\|_{L^\infty(I)}}\cdots\|f_m\|_{L^\infty(I)}.$$

Applying it to (6.6) and using

$$\mathrm{Lip}_I(\tilde a_j^{r-\ell}) \le |r-\ell|\|\tilde a_j^{r-\ell-1}\|_{L^\infty(I)}\|\tilde a_j'\|_{L^\infty}$$

we find, as in the derivation of (6.7),

$$\mathrm{Lip}_I(\partial_t^{n-1}(\tilde a_j^r)) \le C(n,r)|I|^{-n}|\tilde a_k(t_0)|^{rj/k}.$$

As above this leads to

$$\mathrm{Lip}_I(\partial_t^{n-1}\big(\tilde a_k^{-j/k}\tilde a_j\big)) \le C(n)|I|^{-n}.$$

and

$$\mathrm{Lip}_I(\partial_t^{n-1}(F\circ \underline{a})) \le C(n,\|F\|_{C^n(\overline{U})})|I|^{-n},$$

and finally to the second part of (6.4). □

Remark 3. In the setup of Lemma 8 the same estimates hold for $\tilde b_i$ replaced by b_i. This follows by the same proof where ones uses (6.2) instead of (6.3). We shall only need the special case $i = s = 1$ which we state explicitly for later reference:

$$\|b_1'\|_{L^\infty(I)} \le C|I|^{-1}|\tilde a_k(t_0)|^{1/k}. \tag{6.9}$$

Lemma 9. *Assume that $\tilde b_i$, $i = 2,\ldots,m$, are $C^{n-1,1}$-functions, where $m \le n$, on an open bounded interval I which satisfy* (6.4) *for all $s = 1,\ldots,n-1$. Then, for all $1 \le p < m'$,*

$$\|(\tilde b_i^{1/i})'\|^*_{L^p(I)} \le C|I|^{-1}|\tilde a_k(t_0)|^{1/k}, \tag{6.10}$$

for a constant C which depends only on n, p, and the constant in (6.4).

Proof. By (3.3) and (6.4),

$$\begin{aligned}\|(\tilde b_i^{1/i})'\|_{i',w,I} &\le C(i)\max\Big\{\big(\mathrm{Lip}_I(\tilde b_i^{(i-1)})\big)^{1/i}|I|^{1/i'}, \|\tilde b_i'\|^{1/i}_{L^\infty(I)}\Big\}\\ &\le C|I|^{-1+1/i'}|\tilde a_k(t_0)|^{1/k},\end{aligned}$$

or equivalently,

$$\|(\tilde b_i^{1/i})'\|^*_{i',w,I} \le C|I|^{-1}|\tilde a_k(t_0)|^{1/k}.$$

In view of (2.3), this entails (6.10). □

6.2. **Special subintervals of I and estimates on them.** Assume that the polynomial (5.1) satisfies (5.2)–(5.3), (5.10), and (6.1)–(6.3). Suppose that $t_1 \in I$ and $\ell \in \{2,\ldots,n_b\}$ are such that

$$|\tilde b_\ell(t_1)|^{1/\ell} = \max_{2\le i\le n_b} |\tilde b_i(t_1)|^{1/i} \ne 0. \tag{6.11}$$

By (5.6) and (6.3), for all $t \in I$ and $i = 2,\ldots,n_b$,

$$|\tilde b_i(t)| \le C_1|\tilde a_k(t_0)|^{i/k}, \tag{6.12}$$

where the constant C_1 depends only on the functions $\tilde\psi_i$. Thanks to (6.12) we can choose a constant $D < 1/3$ and an open interval J with $t_1 \in J \subseteq I$ such that

$$|J||I|^{-1}|\tilde a_k(t_0)|^{1/k} + \sum_{i=2}^{n_b} \|(\tilde b_i^{1/i})'\|_{L^1(J)} = D|\tilde b_\ell(t_1)|^{1/\ell}. \tag{6.13}$$

It suffices to take $D < C_1^{-1}$ where C_1 is the constant in (6.12); note that $\tilde b_i^{1/i}$ is absolutely continuous by Corollary 2.

Remark 4. The identity (6.13) will be crucial for the proof of Theorem 1.

We will now see that on the interval J the estimates of Section 5 hold for $\tilde b_i$ instead of $\tilde a_j$.

Lemma 10. *Assume that the polynomial* (5.1) *satisfies* (5.2)–(5.3)*,* (5.10)*,* (6.1)–(6.3)*, and* (6.11)*. Let D and J be as in* (6.13)*. Then the functions $\tilde b_i$ on J satisfy the conclusions of*

Lemmas 5, 6, and 7. More precisely, for all $t \in J$ and $i = 2, \ldots, n_b$,

$$|\tilde b_i^{1/i}(t) - \tilde b_i^{1/i}(t_1)| \le D|\tilde b_\ell(t_1)|^{1/\ell}, \tag{6.14}$$

$$\frac{2}{3} < 1 - D \le \Big|\frac{\tilde b_\ell(t)}{\tilde b_\ell(t_1)}\Big|^{1/\ell} \le 1 + D < \frac{4}{3}, \tag{6.15}$$

$$|\tilde b_i(t)|^{1/i} \le \frac{4}{3}|\tilde b_\ell(t_1)|^{1/\ell} \le 2|\tilde b_\ell(t)|^{1/\ell}. \tag{6.16}$$

The length of the curve

$$J \ni t \mapsto \underline{b}(t) := (\tilde b_\ell^{-2/\ell}\tilde b_2, \ldots, \tilde b_\ell^{-n_b/\ell}\tilde b_{n_b})(t) \tag{6.17}$$

is bounded by $3n_b^2 2^{n_b} D$. For all $i = 2, \ldots, n_b$ and $s = 1, \ldots, n-1$,

$$\begin{aligned} \|\tilde b_i^{(s)}\|_{L^\infty(J)} &\le C|J|^{-s}|\tilde b_\ell(t_1)|^{i/\ell}, \\ \operatorname{Lip}_J(\tilde b_i^{(n-1)}) &\le C|J|^{-n}|\tilde b_\ell(t_1)|^{i/\ell}, \end{aligned} \tag{6.18}$$

for a universal constant C depending only on n and $\tilde\psi_i$.

Proof. The proof of (6.14)–(6.16) is analogous to the proof of Lemma 5; use (6.11) and (6.13) instead of (5.2) and (5.3). The bound for the length of the curve $J \ni t \mapsto \underline{b}(t)$ (which is well-defined by (6.15)) follows from (6.13) and (6.14)–(6.16); see the proof of Lemma 6.

Let us prove (6.18). By (6.4), for $t \in I$ and $i = 2, \ldots, n_b$ (note that $n_b < n$),

$$|\tilde b_i^{(i)}(t)| \le C|I|^{-i}|\tilde a_k(t_0)|^{i/k}, \tag{6.19}$$

where $C = C(n, \tilde\psi_i)$. Thus, for $t \in J$ and $s = 1, \ldots, i$,

$$\begin{aligned} |\tilde b_i^{(s)}(t)| &\le C|J|^{-s}\big(V_J(\tilde b_i) + V_J(\tilde b_i)^{(i-s)/i}\|\tilde b_i^{(i)}\|_{L^\infty(J)}^{s/i}|J|^s\big) && \text{by Lemma 4} \\ &\le C_1|J|^{-s}\Big(|\tilde b_\ell(t_1)|^{i/\ell} + |\tilde b_\ell(t_1)|^{(i-s)/\ell}|J|^s|I|^{-s}|\tilde a_k(t_0)|^{s/k}\Big) && \text{by (6.16) and (6.19)} \\ &\le C_2|J|^{-s}|\tilde b_\ell(t_1)|^{i/\ell} && \text{by (6.13),} \end{aligned}$$

for constants $C = C(i)$ and $C_h = C_h(n, \tilde\psi_i)$. For $s > i$ (including $s = n$), we have $(|J||I|^{-1})^s \le (|J||I|^{-1})^i$ and thus

$$|I|^{-s}|\tilde a_k(t_0)|^{i/k} \le |J|^{-s}\big(|J||I|^{-1}|\tilde a_k(t_0)|^{1/k}\big)^i \le |J|^{-s}|\tilde b_\ell(t_1)|^{i/\ell},$$

where the second inequality follows from (6.13). Hence (6.4) implies (6.18). □

7. A special cover by intervals

In this section we prove a technical result which will allow us to glue local L^p-estimates to global ones in the proof of Theorem 1.

7.1. **Intervals of first and second kind.** Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $\tilde{b}_i \in C^{n_b-1,1}(\overline{I})$, $i = 2, \ldots, n_b$. For each point t_1 in

$$I' := I \setminus \{t \in I : \tilde{b}_2(t) = \cdots = \tilde{b}_{n_b}(t) = 0\}$$

there exists $\ell \in \{2, \ldots, n_b\}$ such that (6.11). Assume that there are positive constants $D < 1/3$ and L such that for all $t_1 \in I'$ there is an open interval $J = J(t_1)$ with $t_1 \in J \subseteq I$ such that

$$L|J| + \sum_{i=2}^{n_b} \|(\tilde{b}_i^{1/i})'\|_{L^1(J)} = D|\tilde{b}_\ell(t_1)|^{1/\ell}. \tag{7.1}$$

Note that (6.11) and (7.1) imply (6.15) (cf. the proof of Lemma 10); in particular, we have $J \subseteq I'$.

Let us consider the functions

$$\varphi_{t_1,+}(s) := L(s - t_1) + \sum_{i=2}^{n_b} \|(\tilde{b}_i^{1/i})'\|_{L^1([t_1,s))}, \quad s \geq t_1,$$

$$\varphi_{t_1,-}(s) := L(t_1 - s) + \sum_{i=2}^{n_b} \|(\tilde{b}_i^{1/i})'\|_{L^1((s,t_1])}, \quad s \leq t_1.$$

Then $\varphi_{t_1,\pm} \geq 0$ are monotonic continuous functions defined for small $\pm(s - t_1) \geq 0$ and satisfying $\varphi_{t_1,\pm}(t_1) = 0$. We let $\varphi_{t_1,\pm}$ grow until $\varphi_{t_1,-}(s_-) + \varphi_{t_1,+}(s_+) = D|\tilde{b}_\ell(t_1)|^{1/\ell}$, that is (7.1) with $J = (s_-, s_+)$. And we do this *symmetrically* whenever possible:

(i) We say that the interval $J = (s_-, s_+)$ is of *first kind* if

$$\varphi_{t_1,-}(s_-) = \varphi_{t_1,+}(s_+) = \frac{D}{2}|\tilde{b}_\ell(t_1)|^{1/\ell}. \tag{7.2}$$

(ii) If (7.2) is not possible, i.e., we reach the boundary of the interval I before either $\varphi_{t_1,-}$ or $\varphi_{t_1,+}$ has grown to the value $(D/2)|\tilde{b}_\ell(t_1)|^{1/\ell}$, then we say that $J = (s_-, s_+)$ is of *second kind*.

Remark 5. We may always assume that the interval $J(t_1)$ is of first kind, if such a choice for t_1 exists.

7.2. **A special subcover.** The goal of this section is to prove the following proposition.

Proposition 2. *Let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $\tilde{b}_i \in C^{n_b-1,1}(\overline{I})$, $i = 2, \ldots, n_b$. For each point t_1 in I' fix $\ell \in \{2, \ldots, n_b\}$ such that (6.11). Let $\{J(t_1)\}_{t_1 \in I'}$ be a collection of open intervals $J = J(t_1)$ with $t_1 \in J \subseteq I$ such that:*

(1) *There are positive constants $D < 1/3$ and L such that for all $t_1 \in I'$ we have (7.1) for $J = J(t_1)$.*
(2) *The interval $J(t_1)$ is of first kind, i.e., (7.2) holds, if such a choice for t_1 exists.*

Then the collection $\{J(t_1)\}_{t_1 \in I'}$ has a countable subcollection $\mathcal{J}$ that still covers I' and such that every point in I' belongs to at most two intervals in $\mathcal{J}$. In particular,

$$\sum_{J \in \mathcal{J}} |J| \leq 2|I'|.$$

Remark 6. It is essential for us that $\mathscr{J}$ is a subcollection and not a refinement; by shrinking the intervals we would lose equality in (7.1). We will need this proposition for glueing local L^p-estimates to global ones.

We can treat the connected components of I' separately. So let (α, β) be any connected component of I' and let $\mathcal{I} := \{J(t_1)\}_{t_1 \in (\alpha,\beta)}$. The function $\tilde{b} := (\tilde{b}_2, \ldots, \tilde{b}_{n_b})$ may or may not vanish at the endpoints of (α, β). We distinguish three cases:

(i) $\tilde{b}$ vanishes at both endpoints,

$$\tilde{b}(\alpha) = \tilde{b}(\beta) = 0. \tag{7.3}$$

(ii) $\tilde{b}$ vanishes at one endpoint, say α, but not at the other,

$$\tilde{b}(\alpha) = 0,\ \tilde{b}(\beta) \neq 0. \tag{7.4}$$

(iii) $\tilde{b}$ does not vanishes at either endpoint,

$$\tilde{b}(\alpha) \neq 0,\ \tilde{b}(\beta) \neq 0. \tag{7.5}$$

We shall need the following two lemmas.

Lemma 11. *We have:*

(1) *If $\tilde{b}(\alpha) = 0$, then no interval $J \in \mathcal{I}$ has left endpoint α and $|J(t_1)| \to 0$ as $t_1 \to \alpha$. If $\tilde{b}(\beta) = 0$, then no interval $J \in \mathcal{I}$ has right endpoint β and $|J(t_1)| \to 0$ as $t_1 \to \beta$.*

(2) *If $\tilde{b}(\alpha) \neq 0$, then there exists an interval $J \in \mathcal{I}$ of second kind (with endpoint α). If $\tilde{b}(\beta) \neq 0$, then there exists an interval $J \in \mathcal{I}$ of second kind (with endpoint β).*

Proof. (1) By (6.15), $\tilde{b}$ is non-zero at both endpoints of J. That $|J(t_1)| \to 0$ as t_1 tends to an endpoint, where $\tilde{b}$ vanishes, is immediate from (7.1).

(2) Suppose that $\tilde{b}(\beta) \neq 0$. If all intervals $J(t_1)$ in $\mathcal{I}$ were of first kind then, by (7.1) and (7.2),

$$\varphi_{t_1,+}(\beta) \geq \frac{D}{2}|\tilde{b}_\ell(t_1)|^{1/\ell} = \frac{D}{2} \max_{2 \leq i \leq n_b} |\tilde{b}_i(t_1)|^{1/i}, \quad t_1 \in (\alpha, \beta). \tag{7.6}$$

But $\varphi_{t_1,+}(\beta) \to 0$ as $t_1 \to \beta$, while the right-hand side of (7.6) tends to a positive constant, a contradiction. □

Lemma 12. *Let $J \in \mathcal{I}$ and let $t_1 \notin J$ be such that $J(t_1)$ is of first kind. Then $J \not\subseteq J(t_1)$.*

Proof. Let $J = J(s_1) = (\alpha_{s_1}, \beta_{s_1})$ and assume without loss of generality that $\beta_{s_1} \leq t_1$. Suppose that $J(s_1) \subseteq J(t_1)$. Since $J(t_1) = (\alpha_{t_1}, \beta_{t_1})$ is of first kind (cf. (7.2)), we have

$$L(t_1 - \alpha_{t_1}) + \sum_{i=2}^{n_b} \|(\tilde{b}_i^{1/i})'\|_{L^1((\alpha_{t_1},t_1])} = \varphi_{t_1,-}(\alpha_{t_1}) = \frac{D}{2}|\tilde{b}_{\ell_{t_1}}(t_1)|^{1/\ell_{t_1}} < D|\tilde{b}_{\ell_{s_1}}(s_1)|^{1/\ell_{s_1}},$$

because by (6.15) and (6.16) (which follow from (6.11) and (7.1)),

$$|\tilde{b}_{\ell_{t_1}}(t_1)|^{1/\ell_{t_1}} < \frac{3}{2}|\tilde{b}_{\ell_{t_1}}(s_1)|^{1/\ell_{t_1}} \leq 2|\tilde{b}_{\ell_{s_1}}(s_1)|^{1/\ell_{s_1}}.$$

But this leads to a contradiction in view of (7.1). □

Let us now prove Proposition 2.

Case (i). By (7.3) and Lemma 11, each $J \in \mathcal{I}$ is an interval of first kind.

Choose any interval $J(t_1)$, $t_1 \in (\alpha, \beta)$, and denote it by $J_0 = (\alpha_0, \beta_0)$. Define recursively (for $\gamma \in \mathbb{Z}$)

$$J_\gamma = (\alpha_\gamma, \beta_\gamma) := \begin{cases} J(\beta_{\gamma-1}) & \text{if } \gamma \ge 1, \\ J(\alpha_{\gamma+1}) & \text{if } \gamma \le -1. \end{cases}$$

By Lemma 12, we have $\alpha < \alpha_\gamma < \alpha_{\gamma+1}$ and $\beta_\gamma < \beta_{\gamma+1} < \beta$ for all γ. Let us show that the collection $\mathcal{J} = \{J_\gamma\}_{\gamma \in \mathbb{Z}}$ covers (α, β). Suppose that, say, $\tau := \sup_\gamma \beta_\gamma < \beta$. By (7.1) and since all intervals are of first kind (cf. (7.2)),

$$L(\tau - \beta_\gamma) + \sum_{i=2}^{n_b} \|(\tilde{b}_i^{1/i})'\|_{L^1((\beta_\gamma, \tau])} \ge \frac{D}{2} \max_{2 \le i \le n_b} |\tilde{b}_i(\beta_\gamma)|^{1/i}.$$

But the left-hand side tends to 0 as $\gamma \to +\infty$, whereas the right-hand side converges to $(D/2) \max_{2 \le i \le n_b} |\tilde{b}_i(\tau)|^{1/i} > 0$, a contradiction.

Now Proposition 2 follows from Lemma 11 and the following lemma.

Lemma 13. *Let $\mathcal{J} = \{J_\gamma\}_{\gamma \in \mathbb{Z}}$ be a countable collection of bounded open intervals $J_\gamma = (\alpha_\gamma, \beta_\gamma) \subseteq \mathbb{R}$ such that*

(1) *$\bigcup \mathcal{J} = (\alpha, \beta)$ is a bounded open interval,*
(2) *$\alpha < \alpha_\gamma < \alpha_{\gamma+1}$ and $\beta_\gamma < \beta_{\gamma+1} < \beta$ for all $\gamma \in \mathbb{Z}$,*
(3) *$|J_\gamma| \to 0$ as $\gamma \to \pm\infty$.*

Then there is a subcollection $\mathcal{J}_0 \subseteq \mathcal{J}$ with $\bigcup \mathcal{J}_0 = (\alpha, \beta)$ and such that every point in (α, β) belongs to at most two intervals in $\mathcal{J}_0$.

Proof. The assumptions imply that the sequence of left endpoints (α_γ) converges to β as $\gamma \to \infty$, and the sequence of right endpoints (β_γ) converges to α as $\gamma \to -\infty$. Thus, there exists $\gamma_1 > 0$ such that $\alpha_{\gamma_1} < \beta_0 \le \alpha_{\gamma_1+1}$, there exists $\gamma_2 > \gamma_1$ such that $\alpha_{\gamma_2} < \beta_{\gamma_1} \le \alpha_{\gamma_2+1}$, and iteratively, there exists $\gamma_j > \gamma_{j-1}$ such that $\alpha_{\gamma_j} < \beta_{\gamma_{j-1}} \le \alpha_{\gamma_j+1}$. Symmetrically, there exist integers $\gamma_{j-1} < \gamma_j < 0$ ($j \in \mathbb{Z}_{<0}$) such that $\beta_{\gamma_{j-1}-1} \le \alpha_{\gamma_j} < \beta_{\gamma_{j-1}}$. Set $\gamma_0 := 0$ and define

$$\mathcal{J}_0 := \{J_{\gamma_j}\}_{j \in \mathbb{Z}}.$$

By construction $\mathcal{J}_0$ still covers (α, β) and the left and right endpoints of the intervals J_{γ_j} are interlacing,

$$\cdots < \beta_{\gamma_{j-2}} < \alpha_{\gamma_j} < \beta_{\gamma_{j-1}} < \alpha_{\gamma_{j+1}} < \beta_{\gamma_j} < \alpha_{\gamma_{j+2}} < \cdots$$

Thus $\mathcal{J}_0$ has the required properties. □

Proposition 2 is proved in Case (i).

Case (ii). By (7.4) and Lemma 11, the collection $\mathcal{I}$ contains an interval of second kind. Since $\tilde{b}(\alpha) = 0$, all intervals of second kind in $\mathcal{I}$ must have endpoint β. Thus,

$$\tau := \inf\{t_1 : J(t_1) \in \mathcal{I} \text{ is of second kind}\} > \alpha,$$

because $|J(t_1)| \to 0$ as $t \to \alpha$ by Lemma 11. The interval $J(\tau)$ is of first kind (being of second kind is an open condition). There is an interval $J_0 = (\alpha_0, \beta_0 = \beta)$ of second kind in $\mathcal{I}$ with $J(\tau) \cap J_0 \neq \emptyset$. Let us denote $J(\tau)$ by $J_{-1} = (\alpha_{-1}, \beta_{-1})$ and define recursively

$$J_\gamma = (\alpha_\gamma, \beta_\gamma) := J(\alpha_{\gamma+1}), \quad \gamma \le -1.$$

The arguments in Case (i) imply that the collection $\mathcal{J} := \{J_\gamma\}_{\gamma \le 0}$ is a countable cover of (α, β) satisfying $\alpha < \alpha_\gamma < \alpha_{\gamma+1}$ and $|J_\gamma| \to 0$.

Proposition 2 follows from (an obvious modification of) Lemma 13. This ends Case (ii).

Case (iii). In this case $\mathcal{I}$ has a finite subcollection $\mathcal{J}$ that still covers (α, β). Indeed, by (7.5) and Lemma 11, the collection $\mathcal{I}$ contains intervals of second kind with endpoints α and β, say, (α, δ) and (ϵ, β). If their intersection is non-empty we are done. Otherwise there are finitely many intervals in $\mathcal{I}$ that cover the compact interval $[\delta, \epsilon]$.

Proposition 2 follows from the following lemma.

Lemma 14. *Every finite collection $\mathcal{J}$ of open intervals with $\bigcup \mathcal{J} = (\alpha, \beta)$ has a subcollection that still covers (α, β) and every point in (α, β) belongs to at most two intervals in the subcollection.*

Proof. The collection $\mathcal{J}$ contains an interval with endpoint α; let $J_0 = (\alpha = \alpha_0, \beta_0)$ be the biggest among them. If $\beta_0 < \beta$, let $J_1 = (\alpha_1, \beta_1)$ denote the interval among all intervals in $\mathcal{J}$ containing β_0 whose right endpoint is maximal. If $\beta_1 < \beta$, let $J_2 = (\alpha_2, \beta_2)$ denote the interval among all intervals in $\mathcal{J}$ containing β_1 whose right endpoint is maximal, etc. This yields a finite cover of (α, β) by intervals $J_i = (\alpha_i, \beta_i)$, $i = 0, 1, \ldots, N$, such that $\alpha_0 < \alpha_1 < \cdots < \alpha_N$. Define

$$i_1 := \max_{\alpha_i < \beta_0} i, \qquad i_j := \max_{\alpha_i < \beta_{i_{j-1}}} i, \quad j \ge 2.$$

Then $\{J_0, J_{i_1}, J_{i_2}, \ldots, J_N\}$ has the required properties. □

The proof of Proposition 2 is complete.

8. Proof of Theorem 1

We suppose henceforth that for each integer n a universal splitting of polynomials of degree n in Tschirnhausen form in the sense of Section 4.3 has been fixed. Whenever we speak of a splitting we mean the fixed universal splitting. Accordingly, we will apply the following convention:

> *All dependencies of constants on data of the universal splitting, like ρ, $\tilde{\psi}_i$, etc., (see Section 4.3) will no longer be explicitly stated. For simplicity it will henceforth be subsumed by saying that the constants depend on the degree of the polynomials. The constants which are universal in this sense will be denoted by C and may vary from line to line.*

The heart of the proof of Theorem 1 is the following proposition. It comprises the inductive argument on the degree.

Proposition 3. *Let $I \subseteq \mathbb{R}$ be a bounded open interval and let $P_{\tilde a}$ be a monic polynomial of degree $n_{\tilde a}$ in Tschirnhausen form with coefficients of class $C^{n_{\tilde a}-1,1}(\overline{I})$. Let $t_0 \in I$ and $k \in \{2,\ldots,n_{\tilde a}\}$ be such that*

(1) $|\tilde a_k(t_0)|^{1/k} = \max_{2\le j\le n_{\tilde a}} |\tilde a_j(t_0)|^{1/j} \ne 0$,
(2) $\sum_{j=2}^{n_{\tilde a}} \|(\tilde a_j^{1/j})'\|_{L^1(I)} \le B|\tilde a_k(t_0)|^{1/k}$ *for some constant* $B < 1/3$,
(3) *for all* $j = 2,\ldots,n_{\tilde a}$ *and* $s = 1,\ldots,n_{\tilde a}-1$,

$$\|\tilde a_j^{(s)}\|_{L^\infty(I)} \le C|I|^{-s}|\tilde a_k(t_0)|^{j/k},$$
$$\mathrm{Lip}_I(\tilde a_j^{(n_{\tilde a}-1)}) \le C|I|^{-n_{\tilde a}}|\tilde a_k(t_0)|^{j/k},$$

where $C = C(n_{\tilde a})$.
(4) *Assume that* $P_{\tilde a}$ *splits on* I, *i.e.,* $P_{\tilde a}(t) = P_b(t)P_{b^*}(t)$ *for* $t \in I$, *where* b_i *and* b_i^* *are given by* (4.2).

Then every continuous root $\mu \in C^0(I)$ *of* $P_{\tilde b}$ *is absolutely continuous and satisfies*

$$\|\mu'\|_{L^p(I)} \le C\Big(\||I|^{-1}|\tilde a_k(t_0)|^{1/k}\|_{L^p(I)} + \sum_{i=2}^{n_b}\|(\tilde b_i^{1/i})'\|_{L^p(I)}\Big), \tag{8.1}$$

for all $1 \le p < (n_{\tilde a})'$ *and a constant* C *depending only on* $n_{\tilde a}$ *and* p.

The proof of Theorem 1 is divided into three steps.

Step 1: We check that a monic polynomial in Tschirnhausen form satisfying the assumptions of Theorem 1 also satisfies those of Proposition 3.
Step 2: We prove Proposition 3.
Step 3: We finish the proof of Theorem 1. The goal is to estimate the right-hand side of (8.1) in terms of the $\tilde a_j$.

Step 1: The assumptions of Theorem 1 imply those of Proposition 3. Let $(\alpha,\beta) \subseteq \mathbb{R}$ be a bounded open interval and let

$$P_{\tilde a(t)}(Z) = Z^n + \sum_{j=2}^{n} \tilde a_j(t)Z^{n-j}, \quad t \in (\alpha,\beta), \tag{8.2}$$

be a monic polynomial in Tschirnhausen form with coefficients $\tilde a_j \in C^{n-1,1}([\alpha,\beta])$, $j = 2,\ldots,n$.

Let ρ be the radius of the fixed universal splitting of polynomials of degree n in Tschirnhausen form (cf. Section 4.3). We fix a universal positive constant B satisfying

$$B < \min\Big\{\frac{1}{3}, \frac{\rho}{3n^2 2^n}\Big\}. \tag{8.3}$$

Fix $t_0 \in (\alpha,\beta)$ and $k \in \{2,\ldots,n\}$ such that (5.2) holds, i.e.,

$$|\tilde a_k(t_0)|^{1/k} = \max_{2\le j\le n} |\tilde a_j(t_0)|^{1/j} \ne 0 \tag{8.4}$$

This is possible unless $\tilde{a} \equiv 0$ in which case nothing is to prove. Choose a maximal open interval $I \subseteq (\alpha, \beta)$ containing t_0 such that we have (5.8), i.e.,

$$M|I| + \sum_{j=2}^{n} \|(\tilde{a}_j^{1/j})'\|_{L^1(I)} \le B|\tilde{a}_k(t_0)|^{1/k}, \tag{8.5}$$

with M given by (5.9). In particular, all conclusions of Section 5 hold true.

Consider the point $\underline{p} = \underline{a}(t_0)$, where $\underline{a}$ is the curve defined in (5.7). By (8.4), $\underline{p}$ is an element of the set K defined in (4.4). By the properties of the universal splitting specified in Section 4.3, the ball $B_\rho(\underline{p})$ is contained in some ball of the finite cover $\mathcal{B}$ of K. By Lemma 6 and (8.3), the length of the curve $\underline{a}|_I$ is bounded by ρ. Thus we have a splitting on I,

$$P_{\tilde{a}}(t) = P_b(t) P_{b^*}(t), \quad t \in I.$$

The coefficients b_i of P_b are given by (6.2), and, after the Tschirnhausen transformation $P_b \rightsquigarrow P_{\tilde{b}}$, the coefficients $\tilde{b}_i$ of $P_{\tilde{b}}$ are given by (6.3). (Similar formulas hold for b_i^* and $\tilde{b}_i^*$.)

In summary, the restriction of the curve of polynomials $P_{\tilde{a}}$ to the interval I satisfies all assumptions and thus all conclusions of Sections 5 and 6. In particular, the assumptions of Proposition 3 are satisfied. Thus we have proved the following lemma.

Lemma 15. *Let $(\alpha, \beta) \subseteq \mathbb{R}$ be a bounded open interval and let $P_{\tilde{a}}$ be a polynomial* (8.2) *in Tschirnhausen form with coefficients $\tilde{a}_j \in C^{n-1,1}([\alpha, \beta])$, $j = 2, \ldots, n$. Let B be a positive constant satisfying* (8.3). *Let $t_0 \in (\alpha, \beta)$ and $k \in \{2, \ldots, n\}$ be such that* (8.4) *holds. Let I be an open interval with $t_0 \in I \subseteq (\alpha, \beta)$ and satisfying* (8.5). *Then the assumptions* (1)–(4) *of Proposition 3 are fulfilled.*

Step 2: Induction on the degree. Let us prove Proposition 3.

We proceed by induction on the degree $n = n_{\tilde{a}}$. The assumptions of the proposition amount exactly to the assumptions (5.1)–(5.3), (5.10), and (6.1)–(6.3). Thus we may rely on all conclusions of Sections 5 and 6.

Induction basis. Proposition 3 trivially holds for polynomials of degree 1. Using the result of Ghisi and Gobbino, i.e., Corollary 2, one can also check that Proposition 3 is valid for polynomials of the form $P_{\tilde{a}}(Z) = Z^n - \tilde{a}_n$, $n \ge 2$, because they can be split into the product of linear factors $P_{\tilde{a}}(Z) = \prod_{\xi^n = 1}(Z - \xi \tilde{a}_n^{1/n})$. But we do not need to consider this case separately, since it will appear implicitly in the inductive step.

Inductive step. By (5.5), $\tilde{a}_k$ does not vanish on I, and thus b_i and $\tilde{b}_i$ belong to $C^{n-1,1}(\overline{I})$. Let us set

$$I' := I \setminus \{t \in I : \tilde{b}_2(t) = \cdots = \tilde{b}_{n_b}(t) = 0\}.$$

For each $t_1 \in I'$ choose $\ell \in \{2, \ldots, n_b\}$ such that (6.11) holds. By Section 6.2, there is an open interval $J = J(t_1)$, $t_1 \in J \subseteq I'$, such that (6.13). The constant D in (6.13) can be chosen sufficiently small such that on J we have a splitting

$$P_{\tilde{b}}(t) = P_c(t) P_{c^*}(t), \quad t \in J;$$

in fact, it suffices to choose

$$D < \min\Big\{\frac{1}{3}, \frac{\sigma}{3 n_b^2 2^{n_b}}, C_1^{-1}\Big\}, \tag{8.6}$$

where C_1 is the constant in (6.12) and where σ is the radius of the universal splitting of polynomials of degree n_b in Tschirnhausen form. Indeed, the length of the curve $\underline{b}|_J$ is bounded by σ, which follows from Lemma 10, and the arguments in Section 4.3 and in Step 1 applied to $P_{\tilde b}$.

By Proposition 2 (where (6.13) plays the role of (7.1)), we may conclude that there is a countable family $\{(J_\gamma, t_\gamma, \ell_\gamma)\}$ of open intervals $J_\gamma \subseteq I'$, of points $t_\gamma \in J_\gamma$, and of integers $\ell_\gamma \in \{2, \ldots, n_b\}$ satisfying

$$|\tilde b_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma} = \max_{2 \le i \le n_b} |\tilde b_i(t_\gamma)|^{1/i} \ne 0, \tag{8.7}$$

$$|J_\gamma||I|^{-1}|\tilde a_k(t_0)|^{1/k} + \sum_{i=2}^{n_b} \|(\tilde b_i^{1/i})'\|_{L^1(J_\gamma)} = D|\tilde b_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma}, \tag{8.8}$$

$$P_{\tilde b}(t) = P_{c_\gamma}(t) P_{c_\gamma^*}(t), \quad t \in J_\gamma, \tag{8.9}$$

$$\bigcup_\gamma J_\gamma = I', \quad \sum_\gamma |J_\gamma| \le 2|I'|. \tag{8.10}$$

In particular, for every γ, the polynomial $P_{\tilde b}(t) = P_{c_\gamma}(t)P_{c_\gamma^*}(t)$, $t \in J_\gamma$, satisfies the assumptions of Proposition 3; note that (3) in Proposition 3 corresponds to (6.18).

Let $\mu \in C^0(I)$ be a continuous root of $P_{\tilde b}$. We may assume without loss of generality that in J_γ,

$$\tilde\mu(t) := \mu(t) + \frac{c_{\gamma 1}(t)}{n_{c_\gamma}}, \quad t \in J_\gamma, \tag{8.11}$$

is a root of $P_{\tilde c_\gamma}$, where $n_{c_\gamma} := \deg P_{c_\gamma}$. Since $n_{c_\gamma} < n_b < n_{\tilde a}$, the induction hypothesis implies that $\tilde \mu$ is absolutely continuous and satisfies

$$\|\tilde\mu'\|_{L^p(J_\gamma)} \le C\Big(\||J_\gamma|^{-1}|\tilde b_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma}\|_{L^p(J_\gamma)} + \sum_{h=2}^{n_{c_\gamma}} \|(\tilde c_{\gamma h}^{1/h})'\|_{L^p(J_\gamma)}\Big), \tag{8.12}$$

for all $1 \le p < (n_b)'$, where C is a constant depending only on n_b and p.

L^p-estimates on I. To finish the proof of Proposition 3 we have to show that the estimates (8.12) on the subintervals J_γ imply the bound (8.1) on I. To this end we claim that, for all p with $1 \le p < (n_{c_\gamma})'$,

$$\sum_{h=2}^{n_{c_\gamma}} \|(\tilde c_{\gamma h}^{1/h})'\|^*_{L^p(J_\gamma)} \le C|J_\gamma|^{-1}|\tilde b_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma}, \tag{8.13}$$

for a constant C that depends only on $n_{\tilde a}$ and p.

By the properties of the universal splitting (cf. Sections 4.2 and 4.3), the coefficients $c_{\gamma h}$ of P_{c_γ} are of the form

$$c_{\gamma h} = \tilde b_{\ell_\gamma}^{h/\ell_\gamma} \theta_h\big(\tilde b_{\ell_\gamma}^{-2/\ell_\gamma}\tilde b_2, \ldots, \tilde b_{\ell_\gamma}^{-n_b/\ell_\gamma}\tilde b_{n_b}\big), \quad h = 1, \ldots, n_{c_\gamma},$$

and after the Tschirnhausen transformation $P_{c_\gamma} \rightsquigarrow P_{\tilde c_\gamma}$, see (4.3),

$$\tilde c_{\gamma h} = \tilde b_{\ell_\gamma}^{h/\ell_\gamma} \tilde\theta_h\big(\tilde b_{\ell_\gamma}^{-2/\ell_\gamma}\tilde b_2, \ldots, \tilde b_{\ell_\gamma}^{-n_b/\ell_\gamma}\tilde b_{n_b}\big), \quad h = 2, \ldots, n_{c_\gamma},$$

where θ_h, respectively, $\tilde\theta_h$, are analytic functions with bounded partial derivatives of all orders. By (6.15), $\tilde b_{\ell_\gamma}$ does not vanish on J_γ and thus $c_{\gamma h}$ and $\tilde c_{\gamma h}$ belong to $C^{n_{\tilde a}-1,1}(\overline{J}_\gamma)$. By Lemma 8 (applied to $\tilde c_{\gamma h}$, J_γ, $|\tilde b_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma}$ instead of $\tilde b_i$, I, $|\tilde a_k(t_0|^{1/k}$), we find that, for $h = 2, \ldots, n_{c_\gamma}$ and $s = 1, \ldots, n_{\tilde a} - 1$,

$$\begin{aligned} \|\tilde c_{\gamma h}^{(s)}\|_{L^\infty(J_\gamma)} &\le C|J_\gamma|^{-s}|\tilde b_{\ell_\gamma}(t_\gamma)|^{h/\ell_\gamma}, \\ \operatorname{Lip}_{J_\gamma}(\tilde c_{\gamma h}^{(n_{\tilde a}-1)}) &\le C|J_\gamma|^{-n_{\tilde a}}|\tilde b_{\ell_\gamma}(t_\gamma)|^{h/\ell_\gamma}, \end{aligned}$$

where $C = C(n_{\tilde a})$. Then Lemma 9 yields (8.13).

Now (8.13), (8.8), and (2.2) allow us to estimate the right-hand side of (8.12):

$$\begin{aligned} \||J_\gamma|^{-1}&|\tilde b_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma}\|^*_{L^p(J_\gamma)} + \sum_{h=2}^{n_{c_\gamma}} \|(\tilde c_{\gamma h}^{1/h})'\|^*_{L^p(J_\gamma)} \\ &\le (1+C)|J_\gamma|^{-1}|\tilde b_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma} \\ &= (1+C)D^{-1}\Big(\||I|^{-1}|\tilde a_k(t_0)|^{1/k}\|^*_{L^1(J_\gamma)} + \sum_{i=2}^{n_b} \|(\tilde b_i^{1/i})'\|^*_{L^1(J_\gamma)}\Big) \\ &\le (1+C)D^{-1}\Big(\||I|^{-1}|\tilde a_k(t_0)|^{1/k}\|^*_{L^p(J_\gamma)} + \sum_{i=2}^{n_b} \|(\tilde b_i^{1/i})'\|^*_{L^p(J_\gamma)}\Big) \end{aligned}$$

and therefore

$$\begin{aligned} \||J_\gamma|^{-1}|\tilde b_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma}\|^p_{L^p(J_\gamma)} &+ \sum_{h=2}^{n_{c_\gamma}} \|(\tilde c_{\gamma h}^{1/h})'\|^p_{L^p(J_\gamma)} \\ &\le CD^{-p}\Big(\||I|^{-1}|\tilde a_k(t_0)|^{1/k}\|^p_{L^p(J_\gamma)} + \sum_{i=2}^{n_b} \|(\tilde b_i^{1/i})'\|^p_{L^p(J_\gamma)}\Big), \end{aligned} \tag{8.14}$$

for a constant C that depends only on $n_{\tilde a}$ and p.

By Remark 3 (applied to $\tilde c_{\gamma h}$, J_γ, $|\tilde b_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma}$ instead of $\tilde b_i$, I, $|\tilde a_k(t_0|^{1/k}$), we have

$$\|c'_{\gamma 1}\|_{L^\infty(J_\gamma)} \le C|J_\gamma|^{-1}|\tilde b_{\ell_\gamma}(t_\gamma)|^{1/\ell_\gamma},$$

where $C = C(n_{\tilde a})$. Thus, using (8.8) and (2.2), we find (as in the derivation of (8.14))

$$\|c'_{\gamma 1}\|^p_{L^p(J_\gamma)} \le CD^{-p}\Big(\||I|^{-1}|\tilde a_k(t_0)|^{1/k}\|^p_{L^p(J_\gamma)} + \sum_{i=2}^{n_b} \|(\tilde b_i^{1/i})'\|^p_{L^p(J_\gamma)}\Big), \tag{8.15}$$

for a constant C that depends only on $n_{\tilde a}$ and p.

Let us now glue the bounds on J_γ to a bound on I. By (8.10), (8.12), (8.14), and (8.15),

$$\sum_\gamma \|\tilde\mu'\|^p_{L^p(J_\gamma)} \le CD^{-p}\Big(\||I|^{-1}|\tilde a_k(t_0)|^{1/k}\|^p_{L^p(I)} + \sum_{i=2}^{n_b} \|(\tilde b_i^{1/i})'\|^p_{L^p(I)}\Big), \tag{8.16}$$

and

$$\sum_\gamma \|c_{\gamma 1}'\|^p_{L^p(J_\gamma)} \le C D^{-p}\Big(\||I|^{-1}|\tilde a_k(t_0)|^{1/k}\|^p_{L^p(I)} + \sum_{i=2}^{n_b} \|(\tilde b_i^{1/i})'\|^p_{L^p(I)}\Big), \tag{8.17}$$

for a constant C that depends only on $n_{\tilde a}$ and p. By (8.10), (8.11), (8.16), and (8.17), we may conclude that μ is absolutely continuous on I' and

$$\|\mu'\|_{L^p(I')} \le C D^{-1}\Big(\||I|^{-1}|\tilde a_k(t_0)|^{1/k}\|_{L^p(I)} + \sum_{i=2}^{n_b} \|(\tilde b_i^{1/i})'\|_{L^p(I)}\Big),$$

for a constant C that depends only on $n_{\tilde a}$ and p. Since μ vanishes on $I \setminus I'$, Lemma 1 implies that μ is absolutely continuous on I and satisfies (8.1), since $D = D(n_{\tilde a})$ by (8.6). This completes the proof of Proposition 3.

Step 3: End of the proof of Theorem 1. We have seen in Lemma 15 that for a polynomial $P_{\tilde a}$ in Tschirnhausen form (8.2) satisfying (8.4) and (8.5) the assumptions of Proposition 3 hold with the constant B fulfilling (8.3). Our next goal is to estimate the right-hand side of (8.1) in terms of the $\tilde a_j$.

By Lemma 8, we have (6.4), and thus, by Lemma 9, we get for all p with $1 \le p < (n_b)'$,

$$\||I|^{-1}|\tilde a_k(t_0)|^{1/k}\|^*_{L^p(I)} + \sum_{i=2}^{n_b} \|(\tilde b_i^{1/i})'\|^*_{L^p(I)} \le C|I|^{-1}|\tilde a_k(t_0)|^{1/k} \tag{8.18}$$

where the constant C depends only on n and p.

At this stage two cases may occur:

(i) Either we have equality in (8.5), i.e.,

$$M|I| + \sum_{j=2}^{n} \|(\tilde a_j^{1/j})'\|_{L^1(I)} = B|\tilde a_k(t_0)|^{1/k}. \tag{8.19}$$

(ii) Or $I = (\alpha,\beta)$ and

$$M|I| + \sum_{j=2}^{n} \|(\tilde a_j^{1/j})'\|_{L^1(I)} < B|\tilde a_k(t_0)|^{1/k}. \tag{8.20}$$

Case (ii) entails an unpleasant blow-up of the bounds if $\beta - \alpha \to 0$ as explained in the following remark. We will explain below how to avoid this phenomenon.

Remark 7. In Case (ii) we have a splitting $P_{\tilde a} = P_b P_{b^*}$ on the whole interval $I = (\alpha, \beta)$; cf. Step 1. Thus, (8.18) becomes

$$\||(\beta-\alpha)^{-1}|\tilde a_k(t_0)|^{1/k}\|_{L^p((\alpha,\beta))} + \sum_{i=2}^{n_b} \|(\tilde b_i^{1/i})'\|_{L^p((\alpha,\beta))} \le C(\beta-\alpha)^{-1+1/p}|\tilde a_k(t_0)|^{1/k}$$

which can be bounded by

$$C(\beta-\alpha)^{-1+1/p} \max_{2\le j\le n} \|\tilde a_j\|^{1/j}_{L^\infty((\alpha,\beta))}. \tag{8.21}$$

Similarly, (6.9) implies that $\|b_1'\|_{L^p((\alpha,\beta))}$ is bounded by (8.21). If $\lambda \in C^0((\alpha,\beta))$ is a continuous root of $P_{\tilde a}$ then we may assume that it is a root of P_b, and hence $\lambda = \mu - b_1/n_b$, for a continuous root $\mu \in C^0((\alpha,\beta))$ of $P_{\tilde b}$. By (8.1), we may conclude that λ is absolutely continuous on (α,β) and

$$\|\lambda'\|_{L^p((\alpha,\beta))} \le C(\beta-\alpha)^{-1+1/p} \max_{2\le j\le n} \|\tilde a_j\|^{1/j}_{L^\infty((\alpha,\beta))}, \tag{8.22}$$

where $C = C(n,p)$. But the bound for $\|\lambda'\|_{L^p((\alpha,\beta))}$ in (8.22) tends to infinity if $\beta-\alpha \to 0$ unless $p=1$.

The next lemma provides a way to enforce Case (i).

Lemma 16. *Let $-\infty < \alpha < \beta < \infty$. Let $\tilde a_j \in C^{n-1,1}([\alpha,\beta])$, for $2,\ldots,n$. Let $\hat\alpha := \alpha - 1$ and $\hat\beta := \beta + 1$. The functions $\tilde a_j$ can be extended to functions, again denoted by $\tilde a_j$, defined on $(\hat\alpha,\hat\beta)$ such that the following holds. We have*

$$\|\tilde a_j\|_{C^{n-1,1}([\hat\alpha,\hat\beta])} \le C\, \|\tilde a_j\|_{C^{n-1,1}([\alpha,\beta])}, \tag{8.23}$$

for some universal constant C independent of (α,β). For each $t_0 \in (\hat\alpha,\hat\beta)$ and $k \in \{2,\ldots,n\}$ satisfying (8.4) *there is an open interval $I \subseteq (\hat\alpha,\hat\beta)$ containing t_0 such that* (8.19) *holds true with B specified in* (8.3) *and M defined in* (5.9)*.*

Proof. Using a simple version of Whitney's extension theorem (cf. [42, Theorem 4, p.177]), we may extend the functions $\tilde a_j \in C^{n-1,1}([\alpha,\beta])$ to functions, again denoted by $\tilde a_j$, defined on $\mathbb{R}$ such that $\tilde a_j, \tilde a_j', \ldots, \tilde a_j^{(n-1)}$ are continuous and bounded on $\mathbb{R}$ and $\operatorname{Lip}_{\mathbb{R}}(\tilde a_j^{(n-1)}) < \infty$. More precisely,

$$\max_{0\le i\le n-1} \|\tilde a_j^{(i)}\|_{L^\infty(\mathbb{R})} + \operatorname{Lip}_{\mathbb{R}}(\tilde a_j^{(n-1)}) \le C\, \|\tilde a_j\|_{C^{n-1,1}([\alpha,\beta])},$$

for some universal constant C independent of (α,β). Choose a smooth function $\varphi : \mathbb{R} \to [0,1]$ such that $\varphi(t) = 1$ for $t \le 0$ and $\varphi(t) = 0$ for $t \ge 1$. Then

$$\psi(t) := \varphi(\alpha - t)\varphi(t-\beta), \quad t \in \mathbb{R},$$

is a smooth function which is 1 on the interval $[\alpha,\beta]$ and 0 outside the interval $[\hat\alpha,\hat\beta]$. By multiplying all functions $\tilde a_j$ with the cut-off function ψ, we may assume that each $\tilde a_j$ vanishes somewhere in $[\hat\alpha,\hat\beta]$ and the Leibniz formula implies (8.23) for a constant C depending only on φ.

If there is a point $s = s(j) \in [\hat\alpha,\hat\beta]$ such that $\tilde a_j(s) = 0$, then, for $t \in [\hat\alpha,\hat\beta]$,

$$|\tilde a_j^{1/j}(t)| = \Big|\int_s^t (\tilde a_j^{1/j})'\, d\tau\Big| \le \|(\tilde a_j^{1/j})'\|_{L^1((\hat\alpha,\hat\beta))}$$

and hence

$$\max_{2\le j\le n} \|\tilde a_j\|^{1/j}_{L^\infty((\hat\alpha,\hat\beta))} \le \sum_{j=2}^n \|(\tilde a_j^{1/j})'\|_{L^1((\hat\alpha,\hat\beta))}. \tag{8.24}$$

Since $B < 1$ (by (8.3)), (8.24) enforces Case (i): for each $t_0 \in (\hat\alpha,\hat\beta)$ and $k \in \{2,\ldots,n\}$ satisfying (8.4) there is an open interval $I \subseteq (\hat\alpha,\hat\beta)$ containing t_0 such that (8.19) holds true. $\square$

Lemma 17. *Let $P_{\tilde a}$ be a monic polynomial of degree n in Tschirnhausen form (8.2) with coefficients of class $C^{n-1,1}([\hat\alpha,\hat\beta])$. Let $t_0 \in (\hat\alpha,\hat\beta)$, $k \in \{2,\cdots,n\}$, and let $I \subseteq (\hat\alpha,\hat\beta)$ be an open interval containing t_0 such that (8.4) and (8.19) hold with the constant B fulfilling (8.3) and M defined by (5.9). Then any continuous root $\lambda \in C^0(I)$ of $P_{\tilde a}$ on I is absolutely continuous on I and satisfies*

$$\|\lambda'\|_{L^p(I)} \le C\Big(\hat A\|1\|_{L^p(I)} + \sum_{j=2}^n \|(\tilde a_j^{1/j})'\|_{L^p(I)}\Big), \tag{8.25}$$

where

$$\hat A := \max_{2\le j\le n} \|\tilde a_j\|^{1/j}_{C^{n-1,1}([\hat\alpha,\hat\beta])}. \tag{8.26}$$

Proof. By Lemma 15 (for $(\hat\alpha,\hat\beta)$ instead of (α,β)), the assumptions of Proposition 3 are satisfied. In particular, we have a splitting $P_{\tilde a}(t) = P_b(t)P_{b^*}(t)$ for $t \in I$. We may assume without loss of generality that λ is a root of P_b. Then it has the form

$$\lambda(t) = -\frac{b_1(t)}{n_b} + \mu(t), \quad t \in I, \tag{8.27}$$

where μ is a continuous root of $P_{\tilde b}$. By Proposition 3, μ is absolutely continuous on I and satisfies (8.1).

Using (8.19) and (2.2) to estimate (8.18) (as in the derivation of (8.14)), we arrive at

$$\||I|^{-1}|a_k(t_0)|^{1/k}\|_{L^p(I)} + \sum_{i=2}^{n_b} \|(\tilde b_i^{1/i})'\|_{L^p(I)} \le C\Big(M\|1\|_{L^p(I)} + \sum_{j=2}^n \|(\tilde a_j^{1/j})'\|_{L^p(I)}\Big), \tag{8.28}$$

for a constant C that depends only on n and p; note that $B = B(n)$ by (8.3). Thus, by (8.1) and (8.28),

$$\|\mu'\|_{L^p(I)} \le C\Big(M\|1\|_{L^p(I)} + \sum_{j=2}^n \|(\tilde a_j^{1/j})'\|_{L^p(I)}\Big).$$

By (2.2), (6.9), (8.19), and (8.28), we have the same bound for $\|b_1'\|_{L^p(I)}$, and, in view of (8.27), we conclude that λ is absolutely continuous on I and satisfies

$$\|\lambda'\|_{L^p(I)} \le C\Big(M\|1\|_{L^p(I)} + \sum_{j=2}^n \|(\tilde a_j^{1/j})'\|_{L^p(I)}\Big).$$

The constant M, defined in (5.9), which depends on t_0 and I can be bounded by $\hat A$ defined in (8.26); in fact,

$$M = \max_{2\le j\le n} (\mathrm{Lip}_I(\tilde a_j^{(n-1)}))^{1/n} |\tilde a_k(t_0)|^{(n-j)/(kn)} \le \max_{2\le j\le n} \hat A^{j/n}\hat A^{(n-j)/n} = \hat A.$$

This entails (8.25). $\square$

Proof of Theorem 1. Let $(\alpha,\beta) \subseteq \mathbb{R}$ be a bounded open interval and let

$$P_{a(t)}(Z) = Z^n + \sum_{j=1}^n a_j(t) Z^{n-j}, \quad t \in (\alpha,\beta), \tag{8.29}$$

be a monic polynomial with coefficients $a_j \in C^{n-1,1}([\alpha,\beta])$, $j = 1,\dots,n$.

Without loss of generality we may assume that $n \ge 2$ and that $P_a = P_{\tilde a}$ is in Tschirnhausen form, i.e., $\tilde a_1 = 0$. We shall see at the end of the proof how to get the bound (1.2) from a corresponding bound involving the $\tilde a_j$. If $\{\lambda_j(t)\}_{j=1}^n$, $t \in (\alpha,\beta)$, is any system of the roots of $P_{\tilde a}$ (not necessarily continuous), then, since $\tilde a_1 = 0$, for fixed $t \in (\alpha,\beta)$,

$$\forall_{i,j}\ \lambda_i(t) = \lambda_j(t) \quad\Longleftrightarrow\quad \forall_i\ \lambda_i(t) = 0 \quad\Longleftrightarrow\quad \forall_i\ \tilde a_i(t) = 0. \tag{8.30}$$

Let $\lambda \in C^0((\alpha,\beta))$ be a continuous root of $P_{\tilde a}$. We use Lemma 16 to extend $P_{\tilde a}$ to the interval $[\hat\alpha,\hat\beta]$. We extend λ continuously to the interval $(\hat\alpha,\hat\beta)$ such that $P_{\tilde a(t)}(\lambda(t)) = 0$ for all $t \in (\hat\alpha,\hat\beta)$. Then, by Lemma 17 and Proposition 2 (applied to $\tilde a_j$ instead of $\tilde b_i$ and (8.19) instead of (7.1)), we can cover the complement in $(\hat\alpha,\hat\beta)$ of the points t satisfying (8.30) by a countable family $\mathcal I$ of open intervals I on which (8.25) holds and such that $\sum_{I\in\mathcal I}|I| \le 2(\hat\beta-\hat\alpha)$. Since λ vanishes on the points t satisfying (8.30), Lemma 1 yields that λ is absolutely continuous on $(\hat\alpha,\hat\beta)$ and satisfies

$$\|\lambda'\|_{L^p((\hat\alpha,\hat\beta))} \le C\Big(\hat A\|1\|_{L^p((\hat\alpha,\hat\beta))} + \sum_{j=2}^n \|(\tilde a_j^{1/j})'\|_{L^p((\hat\alpha,\hat\beta))}\Big),$$

and, using (3.3), we obtain

$$\|\lambda'\|_{L^p((\hat\alpha,\hat\beta))} \le C\Big(\hat A(\hat\beta-\hat\alpha)^{1/p} + \sum_{j=2}^n \max\Big\{(\mathrm{Lip}_{(\hat\alpha,\hat\beta)}(\tilde a_j^{(j-1)}))^{1/j}(\hat\beta-\hat\alpha)^{1-1/j}, \|\tilde a_j'\|^{1/j}_{L^\infty((\hat\alpha,\hat\beta))}\Big\}\Big),$$

where $C = C(n,p)$.

Now let us restrict to the interval (α,β) again, and set

$$\tilde A := \max_{2\le j\le n} \|\tilde a_j\|^{1/j}_{C^{n-1,1}([\alpha,\beta])}.$$

By (8.23) and (8.26), we have $\hat A \le C\,\tilde A$ for a universal constant C. Moreover, $\hat\beta-\hat\alpha = \beta-\alpha+2$ and $1-1/j < 1/p$ for all $j \le n$. Consequently,

$$\|\lambda'\|_{L^p((\alpha,\beta))} \le C(n,p)\max\{1,(\beta-\alpha)^{1/p}\}\tilde A. \tag{8.31}$$

Finally we determine the bound in terms of the a_j (i.e., *before* the Tschirnhausen transformation). Let $\check\lambda := \lambda - a_1/n$, i.e., $\check\lambda$ is a continuous root of P_a, and set

$$A := \max_{1\le j\le n} \|a_j\|^{1/j}_{C^{n-1,1}([\alpha,\beta])}.$$

Thanks to the weighted homogeneity of the formulas (4.1), $\tilde A \le C(n)A$. Thus, by (8.31),

$$\begin{aligned}
\|\check\lambda'\|_{L^p((\alpha,\beta))} &\le \|\lambda'\|_{L^p((\alpha,\beta))} + \|a_1'\|_{L^p((\alpha,\beta))}\\
&\le C(n,p)\max\{1,(\beta-\alpha)^{1/p}\}\tilde A + (\beta-\alpha)^{1/p}\|a_1'\|_{L^\infty((\alpha,\beta))}\\
&\le C(n,p)\max\{1,(\beta-\alpha)^{1/p}\}A,
\end{aligned}$$

that is (1.2). The proof of Theorem 1 is complete. □

9. Proof of Theorem 2

Theorem 2 follows from Theorem 1 by the arguments given in the proof of [30, Theorem 4.1]. We provide full details in order to see that the constant in the bound (1.6) depends only on the cover $\mathcal{K}$ of $\overline{V}$ (apart from m, n, and p); this will be important in forthcoming work.

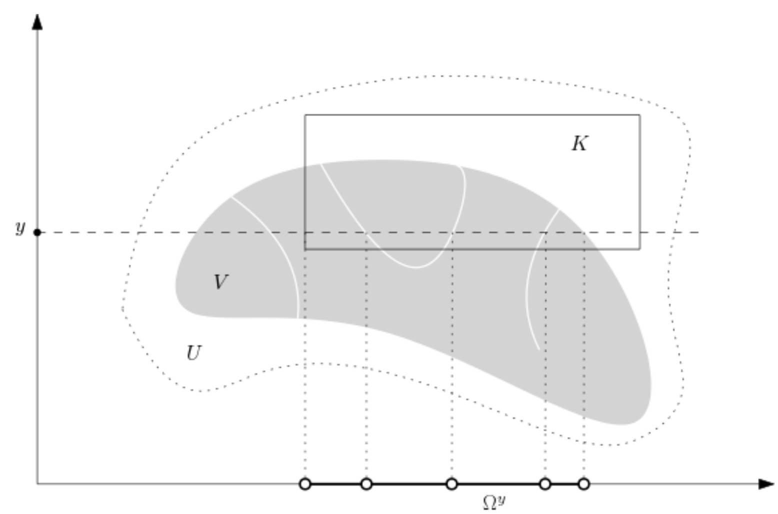
Proof of Theorem 2. By Theorem 1, λ is absolutely continuous along affine lines parallel to the coordinate axes (restricted to V). So λ possesses the partial derivatives $\partial_i \lambda$, $i = 1, \ldots, m$, which are defined almost everywhere and are measurable.

Set $x = (t, y)$, where $t = x_1$, $y = (x_2, \ldots, x_m)$, and let V_1 be the orthogonal projection of V on the hyperplane $\{x_1 = 0\}$. For each $y \in V_1$ we denote by $V^y := \{t \in \mathbb{R} : (t, y) \in V\}$ the corresponding section of V; note that V^y is open in $\mathbb{R}$.

We may cover $\overline{V}$ by finitely many open boxes $K = I_1 \times \cdots \times I_m$ contained in U. Let K be fixed and set $L = I_2 \times \cdots \times I_m$. Fix $y \in V_1 \cap L$ and let λ_j^y, $j = 1, \ldots, n$, be a continuous system of the roots of $P_a(\ ,y)$ on $\Omega^y := V^y \cap I_1$ such that $\lambda(\ ,y) = \lambda_1^y$; it exists since $\lambda(\ ,y)$ can be completed to a continuous system of the roots of $P_a(\ ,y)$ on each connected component of Ω^y by [38, Lemma 6.17]. Our goal is to bound

$$\|\partial_t \lambda(\ ,y)\|_{L^p(\Omega^y)} = \|(\lambda_1^y)'\|_{L^p(\Omega^y)}$$

uniformly with respect to $y \in V_1 \cap L$.



To this end let $\mathcal{C}^y$ denote the set of connected components J of the open subset $\Omega^y \subseteq \mathbb{R}$. For each $J \in \mathcal{C}^y$ we extend the system of roots $\lambda_j^y|_J$, $j = 1, \ldots, n$, continuously to I_1, i.e., we choose continuous functions $\lambda_j^{y,J}$, $j = 1, \ldots, n$, on I_1 such that $\lambda_j^{y,J}|_J = \lambda_j^y|_J$ for all j and

$$P_a(t, y)(Z) = \prod_{j=1}^n (Z - \lambda_j^{y,J}(t)), \ t \in I_1.$$

This is possible since $\lambda_j^y|_J$ has a continuous extension to the endpoints of the (bounded) interval J, by [22, Lemma 4.3], and can then be extended on the left and on the right of J by a continuous system of the roots of $P_a(\ ,y)$ after suitable permutations.

By Theorem 1, for each $y \in V_1 \cap L$, $J \in \mathcal{C}^y$, and $j = 1, \ldots, n$, the function $\lambda_j^{y,J}$ is absolutely continuous on I_1 and $(\lambda_j^{y,J})' \in L^p(I_1)$, for $1 \le p < n/(n-1)$, with

$$\|(\lambda_j^{y,J})'\|_{L^p(I_1)} \le C(n, p, |I_1|) \max_{1 \le i \le n} \|a_i\|^{1/i}_{C^{n-1,1}(\overline{K})}. \tag{9.1}$$

Let $J, J_0 \in \mathcal{C}^y$ be arbitrary. By [30, Lemma 3.6], $(\lambda_j^y)'$ as well as $(\lambda_j^{y,J_0})'$ belong to $L^p(J)$ and we have

$$\sum_{j=1}^n \|(\lambda_j^y)'\|^p_{L^p(J)} = \sum_{j=1}^n \|(\lambda_j^{y,J})'\|^p_{L^p(J)} = \sum_{j=1}^n \|(\lambda_j^{y,J_0})'\|^p_{L^p(J)}.$$

Thus,

$$\begin{aligned}\sum_{j=1}^n \|(\lambda_j^y)'\|^p_{L^p(\Omega^y)} &= \sum_{J \in \mathcal{C}^y} \sum_{j=1}^n \|(\lambda_j^y)'\|^p_{L^p(J)} = \sum_{J \in \mathcal{C}^y} \sum_{j=1}^n \|(\lambda_j^{y,J_0})'\|^p_{L^p(J)} \\ &= \sum_{j=1}^n \|(\lambda_j^{y,J_0})'\|^p_{L^p(\Omega^y)} \le \sum_{j=1}^n \|(\lambda_j^{y,J_0})'\|^p_{L^p(I_1)}.\end{aligned}$$

In particular, by (9.1),

$$\|\partial_t \lambda(\ ,y)\|_{L^p(\Omega^y)} = \|(\lambda_1^y)'\|_{L^p(\Omega^y)} \le C(n,p,K) \max_{1 \le i \le n} \|a_i\|^{1/i}_{C^{n-1,1}(\overline{K})},$$

and so, by Fubini's theorem,

$$\begin{aligned}\int_{V \cap K} |\partial_1 \lambda(x)|^p \, dx &= \int_{V_1 \cap L} \int_{\Omega^y} |\partial_1 \lambda(t,y)|^p \, dt \, dy \\ &\le \Big(C(n,p,K) \max_{1 \le i \le n} \|a_i\|^{1/i}_{C^{n-1,1}(\overline{K})} \Big)^p \int_{V_1 \cap L} dy,\end{aligned}$$

and thus

$$\|\partial_1 \lambda\|_{L^p(V \cap K)} \le C(n,p,K) \max_{1 \le i \le n} \|a_i\|^{1/i}_{C^{n-1,1}(\overline{K})}.$$

The other partial derivatives $\partial_i \lambda$, $i \ge 2$, are treated analogously. This implies (1.6), where W is the (finite) union of the boxes K. □

Remark 8. This can be improved slightly if $\overline{V}$ has just finitely many recesses: in this case the constant in (1.6) depends only on m, n, p, and $\operatorname{diam}(\overline{V})$. For simplicity let us assume that $\overline{V}$ is convex. Then in the previous proof we need not restrict to the open boxes K. Instead of I_1 we may work with the interval $\overline{V^y}$ and (9.1) can be replaced by

$$\|(\lambda_j^{y,J})'\|_{L^p(\overline{V^y})} \le C(n,p) \max\{1, \operatorname{diam}(\overline{V})^{1/p}\} \max_{1 \le i \le n} \|a_i\|^{1/i}_{C^{n-1,1}(\overline{V})}. \tag{9.2}$$

10. Applications

In this section we present three applications of our main results Theorems 1 and 2. First we improve upon a result due to Spagnolo [41] on local solvability of certain systems of pseudo-differential equations. Secondly, we obtain a lifting theorem for differentiable mappings into orbit spaces of finite group representations. As a third application we give a sufficient condition for a multi-valued function to be of Sobolev class $W^{1,p}$ in the sense of Almgren. We also want to point out that our results were used in [4].

10.1. **Local solvability of pseudo-differential equations.** In [41] Spagnolo proved that the pseudo-differential $n \times n$ system

$$u_t + iA(t,D_x)u + B(t,D_x)u = f(t,x), \qquad (t,x) \in I \times U \subseteq \mathbb{R} \times \mathbb{R}^m, \tag{10.1}$$

where $A \in C^\infty(I, S^1(\mathbb{R}^m))^{n\times n}$, $B \in C^0(I, S^0(\mathbb{R}^m))^{n\times n}$ are matrix symbols of order 1 and 0, respectively, and $A(t,\xi)$ is homogeneous of degree 1 in ξ for $|\xi| \ge 1$, is locally solvable in the Gevrey class G^s for $1 \le s \le n/(n-1)$ and semi-globally solvable in G^s for $1 < s < n/(n-1)$ under the following assumptions: the eigenvalues of $A(t,\xi)$ admit a parameterization $\tau_1(t,\xi),\ldots,\tau_n(t,\xi)$ such that each $\tau_j(t,\xi)$ is absolutely continuous in t, uniformly with respect to ξ, i.e.,

$$|\partial_t \tau_j(t,\xi)| \le \mu(t,\xi)(1+|\xi|^2)^{1/2}, \qquad \text{with } \mu(\ ,\xi) \text{ equi-integrable on } I, \tag{$\mathcal{A}_1$}$$

and for each ξ the imaginary parts of the $\tau_j(t,\xi)$ do not change sign for varying t and j, i.e.,

$$\forall \xi \quad \text{either } \operatorname{Im}\tau_j(t,\xi) \ge 0, \quad \forall t,j, \quad \text{or } \operatorname{Im}\tau_j(t,\xi) \le 0, \quad \forall t,j. \tag{$\mathcal{A}_2$}$$

Theorem 1 implies that the assumption ($\mathcal{A}_1$) is always satisfied. Indeed, this follows by applying Theorem 1 to the characteristic polynomial of the matrix $(1+|\xi|^2)^{-1/2}A(t,\xi)$ and noting that the entries of $(1+|\xi|^2)^{-1/2}A(t,\xi)$ and its iterated partial derivatives with respect to t are globally bounded in ξ, since $A(t,\xi)$ is a symbol of order 1.

In particular, the scalar equation

$$\partial_t^n u + \sum_{j=1}^n a_j(t,D_x)\partial_t^{n-j} u = f(t,x), \tag{10.2}$$

where u, f are scalar functions and $a_j(t,D_x)$ is a pseudo-differential operator of order j with principal symbol $a_j^0(t,\xi)$ smooth in t, is locally solvable in G^s for $1 \le s \le n/(n-1)$ and semi-globally solvable in G^s for $1 < s < n/(n-1)$ provided that the roots $\tau_1(t,\xi),\ldots,\tau_n(t,\xi)$ of

$$(iZ)^n + \sum_{j=1}^n a_j^0(t,\xi)(iZ)^{n-j} = 0$$

satisfy assumption ($\mathcal{A}_2$); cf. [41, Corollary 2].

A crucial tool in the proof is the technique of quasi-diagonalization for a Sylvester matrix, introduced by [19] for weakly hyperbolic problems and then refined by [14].

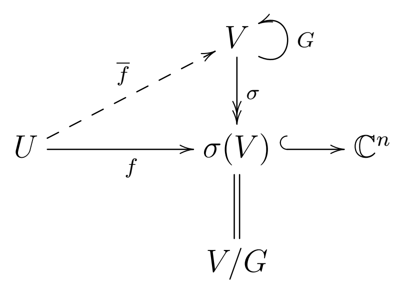
Actually, by Theorem 1, the above conclusions hold provided that the matrix symbol $A(t,\xi)$ is just of class $C^{n-1,1}$ in time t.

Theorem 4. *The pseudo-differential $n \times n$ system* (10.1)*, where $A \in C^{n-1,1}(I, S^1(\mathbb{R}^m))^{n\times n}$, $B \in C^0(I, S^0(\mathbb{R}^m))^{n\times n}$, and $A(t,\xi)$ is homogeneous of degree* 1 *in ξ for $|\xi| \ge 1$, is locally solvable in the Gevrey class G^s for $1 \le s \le n/(n-1)$ and semi-globally solvable in G^s for $1 < s < n/(n-1)$ provided that the eigenvalues $\tau_1(t,\xi),\ldots,\tau_n(t,\xi)$ of $A(t,\xi)$ satisfy* ($\mathcal{A}_2$)*.*

Proof. Theorem 1 implies ($\mathcal{A}_1$) provided that $A(t,\xi)$ is $C^{n-1,1}$ in t. Then the proof in [41] yields the result. □

10.2. **Lifting mappings from orbit spaces.** Let G be a finite group and let $\rho : G \to \mathrm{GL}(V)$ be a representation of G in a finite dimensional complex vector space V. By Hilbert's theorem, the algebra $\mathbb{C}[V]^G$ of G-invariant polynomials on V is finitely generated. We consider the categorical quotient $V/\!\!/G$, i.e., the affine algebraic variety with coordinate ring $\mathbb{C}[V]^G$, and the morphism $\pi : V \to V/\!\!/G$ defined by the embedding $\mathbb{C}[V]^G \to \mathbb{C}[V]$. Since G is finite, $V/\!\!/G$ coincides with the orbit space V/G. Let $\sigma_1, \ldots, \sigma_n$ be a system of homogeneous generators of $\mathbb{C}[V]^G$ with positive degrees $d_1, \ldots, d_n$. Then we can identify π with the mapping of invariants $\sigma = (\sigma_1, \ldots \sigma_n) : V \to \sigma(V) \subseteq \mathbb{C}^n$ and the orbit space V/G with the image $\sigma(V)$.

Let $U \subseteq \mathbb{R}^m$ be open, and $k \in \mathbb{N}$. Consider a mapping $f \in C^{k-1,1}(U, \sigma(V))$, i.e., f is of Hölder class $C^{k-1,1}$ as mapping $U \to \mathbb{C}^n$ with the image $f(U)$ contained in $\sigma(V) \subseteq \mathbb{C}^n$. We say that a mapping $\overline{f} : U \to V$ is a *lift of f over σ* if $f = \sigma \circ \overline{f}$. It is natural to ask how *regular* a lift of f can be chosen. This question is independent of the choice of generators of $\mathbb{C}[V]^G$, since any two choices differ by a polynomial diffeomorphism. This and similar problems were studied in [2], [22], [23], [24], [25], [26], [37], [32].



The subject of this paper, i.e., optimal regularity of roots of polynomials, is just a special case of this problem: let the symmetric group S_n act on $\mathbb{C}^n$ by permuting the coordinates. Then $\mathbb{C}[\mathbb{C}^n]^{\mathrm{S}_n}$ is generated by the elementary symmetric polynomials $\sigma_j(z) = \sum_{i_1<\cdots<i_j} z_{i_1} \cdots z_{i_j}$, $\mathbb{C}^n/\mathrm{S}_n = \sigma(\mathbb{C}^n) = \mathbb{C}^n$, and $f : U \to \sigma(\mathbb{C}^n)$ amounts to a family of complex monic polynomials P_f with coefficients $(-1)^j f_j$, $j = 1, \ldots, n$, in view of Vieta's formulas. Lifting f over σ precisely means choosing the roots of P_f.

As an application of our main Theorems 1 and 2 we obtain the following lifting result for finite groups. Following Noether's proof of Hilbert's theorem we associate with ρ a suitable polynomial and use the regularity result for its roots.

In the following $Gv := \{gv : g \in G\}$ denotes the orbit through v.

Theorem 5. *Let $\rho : G \to \mathrm{GL}(V)$ be a complex finite dimensional representation of a finite group G. Let $\sigma_1, \ldots, \sigma_n$ be a system of homogeneous generators of $\mathbb{C}[V]^G$. Decompose $V = \bigoplus_{i=1}^{\ell} V_i$ into irreducible subrepresentations of G, and let*

$$k := \max_{i=1,\ldots,\ell} \min_{v \in V_i \setminus \{0\}} |Gv|.$$

Then:

(1) *If $c \in C^{k-1,1}(I, \sigma(V))$, where $I \subseteq \mathbb{R}$ is a compact interval, then any continuous lift $\overline{c} \in C^0(I, V)$ of c is absolutely continuous and belongs to the Sobolev space $W^{1,p}(I, V)$*

for every $1 \le p < k/(k-1)$. *If* $\mathcal{C}$ *is a bounded subset of* $C^{k-1,1}(I,\sigma(V))$, *then* $\overline{\mathcal{C}} := \{\overline{c} \in C^0(I,V) : \sigma \circ \overline{c} \in \mathcal{C}\}$ *is bounded in* $W^{1,p}(I,V)$ *for every* $1 \le p < k/(k-1)$.

(2) *If* $f \in C^{k-1,1}(U,\sigma(V))$, *where* $U \subseteq \mathbb{R}^m$ *is open, and* $\overline{f} \in C^0(\Omega,V)$ *is a continuous lift of* f *on a relatively compact open subset* $\Omega \Subset U$, *then* $\overline{f}$ *belongs to the Sobolev space* $W^{1,p}(\Omega,V)$ *for every* $1 \le p < k/(k-1)$. *If* $\mathcal{F}$ *is a bounded subset of* $C^{k-1,1}(U,\sigma(V))$, *then* $\overline{\mathcal{F}} := \{\overline{f} \in C^0(\Omega,V) : \sigma \circ \overline{f} \in \mathcal{F}\}$ *is bounded in* $W^{1,p}(\Omega,V)$ *for every* $1 \le p < k/(k-1)$.

Note that there always exists a continuous lift $\overline{c}$ of $c \in C^0(I,\sigma(V))$; see [26, Theorem 5.1].

Proof. By treating the irreducible subrepresentations separately, we may assume without loss of generality that ρ is irreducible. Fix a non-zero vector $v \in V$ such that $|Gv|$ is minimal. Choose a G-invariant Hermitian inner product $\langle\ ,\ \rangle$ on V, and associate with $g \in G$ the linear form $\ell_g : V \to \mathbb{C}$ defined by $\ell_g(x) := \langle x, gv\rangle$. Choose a numbering of the left coset $G/G_v = \{g_1,\ldots,g_k\}$, where $G_v = \{g \in G : gv = v\}$ and $k = |Gv|$, and set $\ell_i := \ell_{g_i}$ for $i = 1,\ldots,k$. Then the action of G on G/G_v by left multiplication induces a permutation of the set $\{g_1,\ldots,g_k\}$, and thus

$$a_j := (-1)^j \sum_{1 \le i_1 < \cdots < i_j \le k} \ell_{i_1} \cdots \ell_{i_j}, \quad j = 1,\ldots,k,$$

are G-invariant polynomials on V. So $a_j = p_j \circ \sigma$ for polynomials $p_j \in \mathbb{C}[\mathbb{C}^n]$, and the polynomial $P_a \in \mathbb{C}[V]^G[Z]$ given by

$$P_{a(x)}(Z) = Z^k + \sum_{j=1}^k a_j(x) Z^{k-j} = \prod_{j=1}^k (Z - \ell_j(x)), \quad x \in V,$$

factors through the polynomial $P_p \in \mathbb{C}[\mathbb{C}^n][Z]$, i.e., $P_a = P_{p\circ\sigma}$. Applying Theorem 1 to $P_{p(c(t))}$, $t \in I$, we find that $t \mapsto \ell_i(\overline{c}(t)) = \langle \overline{c}(t), g_i v\rangle$, $i = 1,\ldots,k$, belongs to $W^{1,p}(I)$ for each $1 \le p < k/(k-1)$. Since ρ is irreducible, the orbit Gv spans V and (1) follows. Analogously, (2) follows from Theorem 2. $\square$

As a consequence one obtains a similar result for polar representations of reductive algebraic groups, since the lifting problem can be reduced to the action of the corresponding generalized Weyl group which is finite; cf. [26] or [37].

10.3. **Multi-valued Sobolev functions.** In [3] Almgren developed a theory of n-valued Sobolev functions and proved the existence of n-valued minimizers of the Dirichlet energy functional. See also [15] for simpler proofs.

An n-valued function is a mapping with values in the set $\mathcal{A}_n(\mathbb{R}^\ell)$ of unordered n-tuples of points in $\mathbb{R}^\ell$. Let us denote by $[x] = [x_1,\ldots,x_n]$ the unordered n-tuple consisting of $x_1,\ldots,x_n \in \mathbb{R}^\ell$; then $[x_1,\ldots,x_n] = [x_{\sigma(1)},\ldots,x_{\sigma(n)}]$ for each permutation $\sigma \in \mathrm{S}_n$. The set $\mathcal{A}_n(\mathbb{R}^\ell) = \{[x] = [x_1,\ldots,x_n] : x_i \in \mathbb{R}^\ell\}$ forms a complete metric space when endowed with the metric

$$d([x],[y]) := \min_{\sigma \in \mathrm{S}_n} \Big(\sum_{i=1}^n |x_i - y_{\sigma(i)}|^2 \Big)^{1/2}.$$

Almgren proved that there is an integer $N = N(n,\ell)$, a positive constant $C = C(n,\ell)$, and an injective mapping $\Delta : \mathcal{A}_n(\mathbb{R}^\ell) \to \mathbb{R}^N$ such that $\operatorname{Lip}(\Delta) \le 1$ and $\operatorname{Lip}(\Delta|^{-1}_{\Delta(\mathcal{A}_n(\mathbb{R}^\ell))}) \le C$; moreover, there is a Lipschitz retraction of $\mathbb{R}^N$ onto $\Delta(\mathcal{A}_n(\mathbb{R}^\ell))$.

One can use this bi-Lipschitz embedding to define Sobolev spaces of n-valued functions: for open $U \subseteq \mathbb{R}^m$ and $1 \le p \le \infty$ define

$$W^{1,p}(U, \mathcal{A}_n(\mathbb{R}^\ell)) := \{f : U \to \mathcal{A}_n(\mathbb{R}^\ell) : \Delta \circ f \in W^{1,p}(U, \mathbb{R}^N)\}.$$

For an intrinsic definition see [15, Definition 0.5 and Theorem 2.4].

Let us identify $\mathbb{R}^2 \cong \mathbb{C}$. Theorem 1 implies a sufficient condition for an n-valued function $U \to \mathcal{A}_n(\mathbb{C})$ to belong to the Sobolev spaces $W^{1,p}(U, \mathcal{A}_n(\mathbb{C}))$ for every $1 \le p < n/(n-1)$; see Theorem 6 below.

We shall use the following terminology. By a *parameterization* of an n-valued function $f : U \to \mathcal{A}_n(\mathbb{C})$ we mean a function $\varphi : U \to \mathbb{C}^n$ such that $f(x) = [\varphi(x)] = [\varphi_1(x), \ldots, \varphi_n(x)]$ for all $x \in U$. Let $\pi : \mathbb{C}^n \to \mathcal{A}_n(\mathbb{C})$ be defined by $\pi(z) := [z]$; it is a Lipschitz mapping with $\operatorname{Lip}(\pi) = 1$. Then a parameterization of f amounts to a lift φ of f over π, i.e., $f = \pi \circ \varphi$. The elementary symmetric polynomials induce a bijective mapping $a : \mathcal{A}_n(\mathbb{C}) \to \mathbb{C}^n$,

$$a_j([z_1, \ldots, z_n]) := (-1)^j \sum_{i_1 < \cdots < i_j} z_{i_1} \cdots z_{i_j}, \quad 1 \le j \le n.$$

In other words, monic complex polynomials of degree n are in one-to-one correspondence with their unordered n-tuples of roots.

Theorem 6. *Let $U \subseteq \mathbb{R}^m$ be open and let $f : U \to \mathcal{A}_n(\mathbb{C})$ be continuous. If $a \circ f \in C^{n-1,1}(U, \mathbb{C}^n)$, then $f \in W^{1,p}(V, \mathcal{A}_n(\mathbb{C}))$ for each relatively compact open $V \Subset U$ and each $1 \le p < n/(n-1)$. Moreover,*

$$\|\nabla(\Delta \circ f)\|_{L^p(V)} \le C(m,n,p,\mathcal{K},\Delta)\big(1 + \max_{1\le j\le n} \|a_j \circ f\|^{1/j}_{C^{n-1,1}(\overline{W})}\big),$$

where $\mathcal{K}$ is any finite cover of $\overline{V}$ by open boxes $\prod_{i=1}^m (\alpha_i, \beta_i)$ contained in U and $W = \bigcup \mathcal{K}$.

Proof. Fix $V \Subset U$. We must show that $\Delta \circ f$ is an element of $W^{1,p}(V, \mathbb{R}^N)$. Clearly, $\Delta \circ f : U \to \mathbb{R}^N$ is continuous. The set V can be covered by finitely many open boxes $K = \prod_{i=1}^m I_i$ contained in U. Let e_i be the ith standard unit vector in $\mathbb{R}^m$. Denote by K_i the orthogonal projection of K onto the hyperplane $e_i^\perp$. For each $y \in K_i$ we have $I_i = \{t \in \mathbb{R} : y + te_i \in K\}$.

By Theorem 1, $I_i \ni t \mapsto f(y + te_i)$ admits an absolutely continuous parameterization $\varphi_{i,y}$ such that, for $1 \le p < n/(n-1)$,

$$\|\varphi'_{i,y}\|_{L^p(I_i)} \le C(n,p) \max\{1, |I_i|^{1/p}\} \max_{1\le j\le n} \|a_j \circ f\|^{1/j}_{C^{n-1,1}(\overline{K})}.$$

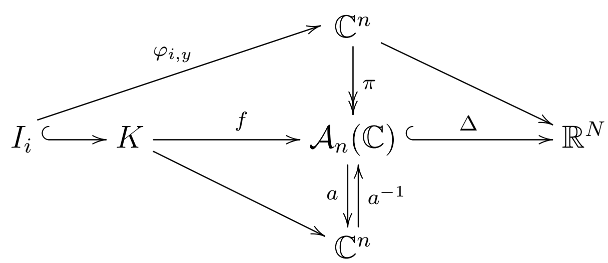
Thus, $I_i \ni t \mapsto \Delta(f(y + te_i)) = \Delta(\pi(\varphi_{i,y}(t)))$ is absolutely continuous and

$$\|(\Delta \circ \pi \circ \varphi_{i,y})'\|_{L^p(I_i)} \le C(m,n,p,|I_i|,\Delta)\big(1 + \max_{1\le j\le n} \|a_j \circ f\|^{1/j}_{C^{n-1,1}(\overline{K})}\big),$$

since composition with the Lipschitz mapping $\Delta \circ \pi$ maps $W^{1,p}(I_i, \mathbb{C}^n)$ to $W^{1,p}(I_i, \mathbb{R}^N)$ in a bounded way; see [29, Theorem 1]. By Fubini's theorem,

$$\int_K |\partial_i(\Delta \circ f)|^p \, dx = \int_{K_i} \int_{I_i} |(\Delta \circ \pi \circ \varphi_{i,y})'|^p \, dt \, dy,$$

and the statement follows. □



In particular, the roots of a polynomial P_a of degree n with coefficients $a_j \in C^{n-1,1}(U)$, $j = 1, \ldots, n$, form an n-valued function $\lambda : U \to \mathcal{A}_n(\mathbb{C})$ which belongs to $W^{1,p}_{\text{loc}}(U, \mathcal{A}_n(\mathbb{C}))$ for each $1 \le p < n/(n-1)$; in fact, it is well-known that $\lambda : U \to \mathcal{A}_n(\mathbb{C})$ is continuous (cf. [20] or [33, Theorem 1.3.1]). Theorem 6 implies that the push-forward

$$(a^{-1})_* : C^{n-1,1}(U, \mathbb{C}^n) \to \bigcap_{1 \le p < n/(n-1)} W^{1,p}_{\text{loc}}(U, \mathcal{A}_n(\mathbb{C})).$$

is a bounded mapping.

We remark that much more is true in the case of *real* n-valued functions. In this situation the elementary symmetric polynomials induce a bijective mapping $a : \mathcal{A}_n(\mathbb{R}) \to H_n$, where H_n is a closed semialgebraic subset of $\mathbb{R}^n$, namely, the space of *hyperbolic* polynomials of degree n (i.e., polynomials with all roots real). Then the mapping

$$(a^{-1})_* : C^{n-1,1}(U, H_n) \to C^{0,1}(U, \mathcal{A}_n(\mathbb{R})),$$

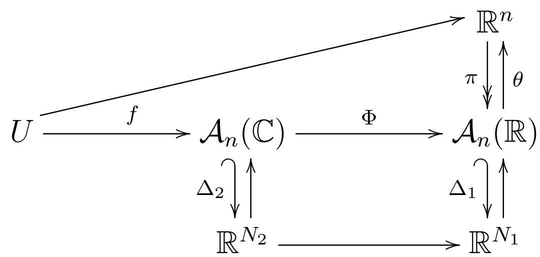
is bounded. It is easy to see that the projection $\pi : \mathbb{R}^n \to \mathcal{A}_n(\mathbb{R})$ admits a continuous section θ, for instance, by ordering the components increasingly. Then we have a bounded mapping

$$(\theta \circ a^{-1})_* : C^{n-1,1}(U, H_n) \to C^{0,1}(U, \mathbb{R}^n).$$

All this essentially follows from Bronshtein's theorem [9]; see [31].

Remark 9. Let $\Phi : \mathcal{A}_n(\mathbb{C}) \to \mathcal{A}_n(\mathbb{R})$ be a Lipschitz function. If $f \in W^{1,p}(U, \mathcal{A}_n(\mathbb{C}))$, then $\Phi \circ f \in W^{1,p}(U, \mathcal{A}_n(\mathbb{R}))$ and it admits a parameterization $\theta \circ \Phi \circ f \in W^{1,p}(U, \mathbb{R}^n)$. This follows (again by [29, Theorem 1]) from the following diagram in which all vertical arrows are Lipschitz: the arrows in the lower row by Almgren's results, and θ is Lipschitz, since

$d([x],[y]) = |\theta([x]) - \theta([y])|$ for $[x],[y] \in \mathcal{A}_n(\mathbb{R})$.



Every Lipschitz function $\phi : \mathbb{C} \to \mathbb{R}$ induces a Lipschitz functions $\Phi : \mathcal{A}_n(\mathbb{C}) \to \mathcal{A}_n(\mathbb{R})$ by setting $\Phi([z]) := [\phi(z_1), \ldots, \phi(z_n)]$. In particular, we can take $\varphi(z) = |z|$, $\varphi(z) = \mathrm{Re}(z)$, or $\varphi(z) = \mathrm{Im}(z)$. In view of Theorem 6, we may conclude that the real and imaginary parts of the roots of a monic polynomial P_a of degree n with coefficients in $C^{n-1,1}(U)$ admit continuous parameterization that are of class $W^{1,p}_{\mathrm{loc}}(U, \mathbb{R}^n)$ for each $1 \le p < n/(n-1)$. The same holds for the absolute values. But note that real and imaginary parts of the roots do not allow continuous parameterizations simultaneously!

Appendix A. Illustration of the proof in simple cases

Let us illustrate the proof of Theorem 1 for polynomials P_a of degree 3 and 4. For simplicity we assume that P_a is in Tschirnhausen form.

Degree 3. In degree 3 Proposition 3 is trivial: the factors of a splitting are at most of degree 2; so (8.1) reduces to $\|\mu'\|_{L^p(I)} = \|(\tilde{b}_2^{1/2})'\|_{L^p(I)}$ if $n_b = 2$ and $\mu \equiv 0$ if $n_b = 1$.

Let $(\alpha, \beta) \subseteq \mathbb{R}$ be a bounded open interval. Let

$$P_{\tilde{a}(t)}(Z) = Z^3 + \tilde{a}_2(t) Z + \tilde{a}_3(t), \quad t \in (\alpha, \beta),$$

be a monic polynomial of degree 3 in Tschirnhausen form with coefficients $\tilde{a}_2, \tilde{a}_3 \in C^{2,1}([\alpha, \beta])$. We may use Lemma 16 to extend $\tilde{a}_2, \tilde{a}_3$ to functions in $C^{2,1}([\hat{\alpha}, \hat{\beta}])$, where $\hat{\alpha} = \alpha - 1$ and $\hat{\beta} = \beta + 1$, such that

- (8.23) holds for $n = 3$, and
- for $t_0 \in (\alpha, \beta)$ and $k \in \{2, 3\}$ satisfying

$$|\tilde{a}_k(t_0)|^{1/k} = \max_{j=2,3} |\tilde{a}_j(t_0)|^{1/j} \neq 0, \tag{A.1}$$

  and a constant B satisfying (8.3) for $n = 3$, there is an open interval $I \subseteq (\hat{\alpha}, \hat{\beta})$ containing t_0 such that

$$M|I| + \|(\tilde{a}_2^{1/2})'\|_{L^1(I)} + \|(\tilde{a}_3^{1/3})'\|_{L^1(I)} = B|\tilde{a}_k(t_0)|^{1/k}, \tag{A.2}$$

  where

$$M = \max_{j=2,3} (\mathrm{Lip}_I(\tilde{a}_j^{(2)}))^{1/3} |\tilde{a}_k(t_0)|^{(3-j)/(3k)}. \tag{A.3}$$

We have a splitting $P_{\tilde{a}}(t) = P_b(t) P_{b^*}(t)$, $t \in I$ (see Lemma 15).

Case $n_b = 2$. In this case

$$P_{b(t)}(Z) = Z^2 + b_1(t)Z + b_2(t), \quad t \in I,$$

and after Tschirnhausen transformation

$$P_{\tilde b(t)}(Z) = Z^2 + \tilde b_2(t), \quad t \in I.$$

The coefficients b_1, b_2, and $\tilde b_2$ are given by (6.2) and (6.3) for $n_b = 2$. They are of class $C^{2,1}(\overline{I})$ since $\tilde a_k$ does not vanish on I (by (5.5)). If $\mu \in C^0(I)$ is a continuous root of $P_{\tilde b}$, then Lemma 8 and Lemma 9 imply

$$\|\mu'\|^*_{L^p(I)} = \|(\tilde b_2^{1/2})'\|^*_{L^p(I)} \le C(p)|I|^{-1}|\tilde a_k(t_0)|^{1/k}, \quad 1 \le p < 2. \tag{A.4}$$

Moreover, by (6.9),

$$\|b_1'\|^*_{L^p(I)} \le C|I|^{-1}|\tilde a_k(t_0)|^{1/k}. \tag{A.5}$$

Case $n_b = 1$. In this case $P_{b(t)}(Z) = Z + b_1(t)$, $P_{\tilde b(t)}(Z) = Z$, and $\mu \equiv 0$. In particular, (A.4) and (A.5) are still valid.

Let $\lambda \in C^0((\alpha,\beta))$ be a continuous root of $P_{\tilde a}$. We extend λ continuously to $(\hat\alpha,\hat\beta)$ such that λ is a root of $P_{\tilde a}$ on $(\hat\alpha,\hat\beta)$. Assume that, on I, λ is a root of P_b; then

$$\lambda(t) = -\frac{b_1(t)}{n_b} + \mu(t), \quad t \in I.$$

By (A.4), (A.5), and (2.2),

$$\begin{aligned}\|\lambda'\|^*_{L^p(I)} &\le C(p)|I|^{-1}|\tilde a_k(t_0)|^{1/k}\\ &= C(p)B^{-1}\Big(M + \|(\tilde a_2^{1/2})'\|^*_{L^1(I)} + \|(\tilde a_3^{1/3})'\|^*_{L^1(I)}\Big)\\ &\le C(p)B^{-1}\Big(\hat A + \|(\tilde a_2^{1/2})'\|^*_{L^p(I)} + \|(\tilde a_3^{1/3})'\|^*_{L^p(I)}\Big),\end{aligned}$$

where $\hat A := \max_{j=2,3} \|\tilde a_j\|^{1/j}_{C^{2,1}([\hat\alpha,\hat\beta])}$ which dominates M as defined in (A.3) (see the proof of Lemma 17). By Proposition 2 (applied to $\tilde a_j$ instead of $\tilde b_i$ and (A.2) instead of (7.1)) and Lemma 1, we may conclude that λ is absolutely continuous on $(\hat\alpha,\hat\beta)$ and satisfies

$$\|\lambda'\|_{L^p((\hat\alpha,\hat\beta))} \le C(p)\Big(\hat A(\hat\beta-\hat\alpha)^{1/p} + \|(\tilde a_2^{1/2})'\|_{L^p((\hat\alpha,\hat\beta))} + \|(\tilde a_3^{1/3})'\|_{L^p((\hat\alpha,\hat\beta))}\Big);$$

the constant B is universal. Using (3.3) and (8.23), we find

$$\|\lambda'\|_{L^p((\alpha,\beta))} \le C(p)\max\{1,(\beta-\alpha)^{1/p}\}\max_{j=2,3}\|\tilde a_j\|^{1/j}_{C^{2,1}([\alpha,\beta])}, \quad 1 \le p < 3/2.$$

Degree 4. In degree 4 the interesting case is when after splitting one of the factors has degree 3. Then the conclusion of Proposition 3 is obtained by a second splitting which further reduces the degree.

Let $(\alpha,\beta) \subseteq \mathbb{R}$ be a bounded open interval. Let

$$P_{\tilde a(t)}(Z) = Z^4 + \tilde a_2(t)Z^2 + \tilde a_3(t)Z + \tilde a_4(t), \quad t \in (\alpha,\beta),$$

be a monic polynomial of degree 4 in Tschirnhausen form with coefficients $\tilde{a}_2, \tilde{a}_3, \tilde{a}_4 \in C^{3,1}([\alpha, \beta])$. As in degree 3 we may assume that $\tilde{a}_2, \tilde{a}_3, \tilde{a}_4$ are functions in $C^{3,1}([\hat{\alpha}, \hat{\beta}])$ (where $\hat{\alpha} = \alpha - 1$ and $\hat{\beta} = \beta + 1$) such that

- (8.23) holds for $n = 4$, and
- for $t_0 \in (\alpha, \beta)$ and $k \in \{2, 3, 4\}$ satisfying
$$|\tilde{a}_k(t_0)|^{1/k} = \max_{j=2,3,4} |\tilde{a}_j(t_0)|^{1/j} \neq 0, \tag{A.6}$$
and a constant B satisfying (8.3) for $n = 4$, there is an open interval $I \subseteq (\hat{\alpha}, \hat{\beta})$ containing t_0 such that
$$M|I| + \sum_{j=2}^{4} \|(\tilde{a}_j^{1/j})'\|_{L^1(I)} = B|\tilde{a}_k(t_0)|^{1/k}, \tag{A.7}$$
where
$$M = \max_{j=2,3,4} (\mathrm{Lip}_I(\tilde{a}_j^{(3)}))^{1/4} |\tilde{a}_k(t_0)|^{(4-j)/(4k)}. \tag{A.8}$$

We have a splitting $P_{\tilde{a}}(t) = P_b(t) P_{b^*}(t)$, $t \in I$.

Case $n_b = 3$. In this case
$$P_{b(t)}(Z) = Z^3 + b_1(t)Z^2 + b_2(t)Z + b_3(t), \quad t \in I,$$
and after Tschirnhausen transformation
$$P_{\tilde{b}(t)}(Z) = Z^3 + \tilde{b}_2(t)Z + \tilde{b}_3(t), \quad t \in I.$$
The coefficients b_1, b_2, b_3 and $\tilde{b}_2$, $\tilde{b}_3$ are given by (6.2) and (6.3) for $n_b = 3$. They are of class $C^{3,1}(\overline{I})$ since $\tilde{a}_k$ does not vanish on I.

In this situation we have to work harder to obtain the conclusion of Proposition 3: we must split again. Let $I' := I \setminus \{t \in I : \tilde{b}_2(t) = \tilde{b}_3(t) = 0\}$. For each $t_1 \in I'$ choose $\ell \in \{2, 3\}$ such that
$$|\tilde{b}_\ell(t_0)|^{1/\ell} = \max_{j=2,3} |\tilde{b}_i(t_0)|^{1/i} \neq 0.$$
There is an open interval $J = J(t_1)$, $t_1 \in J \subseteq I'$, such that
$$|J||I|^{-1}|\tilde{a}_k(t_0)|^{1/k} + \|(\tilde{b}_2^{1/2})'\|_{L^1(J)} + \|(\tilde{b}_3^{1/3})'\|_{L^1(J)} = D|\tilde{b}_\ell(t_1)|^{1/\ell}, \tag{A.9}$$
for a constant D satisfying (8.6) for $n_b = 3$. Then we have a splitting $P_{\tilde{b}}(t) = P_c(t)P_{c^*}(t)$, $t \in J$; see Section 6.2 and p. 25.

Let $\mu \in C^0(I)$ be a continuous root of $P_{\tilde{b}}$. We may assume that
$$\tilde{\mu}(t) := \mu(t) + \frac{c_1(t)}{n_c}, \quad t \in J,$$
is a root of $P_{\tilde{c}}$ in J. We have $n_c \leq 2$. If $n_c = 2$, then, in analogy to (A.4) and (A.5),
$$\|\tilde{\mu}'\|^*_{L^p(J)} = \|(\tilde{c}_2^{1/2})'\|^*_{L^p(J)} \leq C(p)|J|^{-1}|\tilde{b}_\ell(t_1)|^{1/\ell}, \quad 1 \leq p < 2. \tag{A.10}$$
and
$$\|c_1'\|^*_{L^p(J)} \leq C|J|^{-1}|\tilde{b}_\ell(t_1)|^{1/\ell}. \tag{A.11}$$

In the case that $n_c = 1$ we have $P_{c(t)}(Z) = Z + c_1(t)$, $P_{\tilde c(t)}(Z) = Z$, and $\tilde\mu \equiv 0$. In particular, (A.10) and (A.11) are still valid.

Thus, (A.9), (A.10), (A.11), and (2.2) imply

$$\begin{aligned}\|\mu'\|^*_{L^p(J)} &\le C(p)|J|^{-1}|\tilde b_\ell(t_1)|^{1/\ell}\\ &= C(p)D^{-1}\Big(|I|^{-1}|\tilde a_k(t_0)|^{1/k} + \|(\tilde b_2^{1/2})'\|^*_{L^1(J)} + \|(\tilde b_3^{1/3})'\|^*_{L^1(J)}\Big)\\ &\le C(p)D^{-1}\Big(|I|^{-1}|\tilde a_k(t_0)|^{1/k} + \|(\tilde b_2^{1/2})'\|^*_{L^p(J)} + \|(\tilde b_3^{1/3})'\|^*_{L^p(J)}\Big).\end{aligned}$$

Using Proposition 2 to extract a countable subcollection of $\{J(t_1)\}_{t_1\in I'}$, σ-additivity of $\|\cdot\|^p_{L^p}$ to glue the L^p-estimates, and Lemma 1 to extend the estimate to I, we obtain

$$\|\mu'\|_{L^p(I)} \le C(p)\Big(\||I|^{-1}|\tilde a_k(t_0)|^{1/k}\|_{L^p(I)} + \|(\tilde b_2^{1/2})'\|_{L^p(I)} + \|(\tilde b_3^{1/3})'\|_{L^p(I)}\Big),$$

that is the conclusion of Proposition 3 (the constant D is universal). With Lemma 8 and Lemma 9 we may conclude

$$\|\mu'\|^*_{L^p(I)} \le C(p)|I|^{-1}|\tilde a_k(t_0)|^{1/k}, \quad 1 \le p < 3/2. \tag{A.12}$$

Case $n_b \le 2$. In this case (A.12) follows from (A.4) and (A.5).

Let $\lambda \in C^0((\alpha,\beta))$ be a continuous root of $P_{\tilde a}$. We extend λ continuously to $(\hat\alpha,\hat\beta)$ such that λ is a root of $P_{\tilde a}$ on $(\hat\alpha,\hat\beta)$. Assume that, on I, λ is a root of P_b; then

$$\lambda(t) = -\frac{b_1(t)}{n_b} + \mu(t), \quad t \in I.$$

By (A.12), (A.5), (A.7), and (2.2),

$$\begin{aligned}\|\lambda'\|^*_{L^p(I)} &\le C(p)|I|^{-1}|\tilde a_k(t_0)|^{1/k}\\ &= C(p)B^{-1}\Big(M + \sum_{j=2}^4 \|(\tilde a_j^{1/j})'\|^*_{L^1(I)}\Big)\\ &\le C(p)B^{-1}\Big(\hat A + \sum_{j=2}^4 \|(\tilde a_j^{1/j})'\|^*_{L^p(I)}\Big),\end{aligned}$$

where $\hat A := \max_{j=2,3,4} \|\tilde a_j\|^{1/j}_{C^{3,1}([\hat\alpha,\hat\beta])}$ dominates M as defined in (A.8). As in the end of the proof for degree 3, we may use Proposition 2 and Lemma 1 to glue the L^p-estimates, and (3.3) and (8.23) to conclude

$$\|\lambda'\|_{L^p((\alpha,\beta))} \le C(p)\max\{1,(\beta-\alpha)^{1/p}\} \max_{j=2,3,4} \|\tilde a_j\|^{1/j}_{C^{3,1}([\alpha,\beta])}, \quad 1 \le p < 4/3.$$

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