Recognizing (ultra)differentiable functions on closed sets

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Recognizing (ultra)differentiable functions on open sets. Let $f : U \to \mathbb{R}$ be a function defined in an open set $U \subseteq \mathbb{R}^d$. Then f induces a map $f_* : U^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$, $f_*(c) = f \circ c$, whose invariance properties encode the regularity of f:

(i) f is smooth ($\mathcal{C}^\infty$) if and only if $f_*\mathcal{C}^\infty(\mathbb{R}, U) \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$; due to [1].
(ii) f is $\mathcal{C}^{k,\alpha}$ if and only if $f_*\mathcal{C}^\infty(\mathbb{R}, U) \subseteq \mathcal{C}^{k,\alpha}(\mathbb{R}, \mathbb{R})$; see [5], [4], [8].
(iii) f is $\mathcal{C}^M$ if and only if $f_*\mathcal{C}^M(\mathbb{R}, U) \subseteq \mathcal{C}^M(\mathbb{R}, \mathbb{R})$, where M is a non-quasianalytic weight sequence; see [9].

By $\mathcal{C}^{k,\alpha}$ ($k \in \mathbb{N}$, $\alpha \in (0, 1]$) we denote the class of $\mathcal{C}^k$-functions whose partial derivatives of order k satisfy a local α-Hölder condition. Let us now define $\mathcal{C}^M$.

Ultradifferentiable functions of class $\mathcal{C}^M$. Let $M = (M_k)$ be a positive sequence. The *Denjoy–Carleman class* $\mathcal{C}^M(U, \mathbb{R}^m)$ is the set of all $f \in \mathcal{C}^\infty(U, \mathbb{R}^m)$ such that for all compact $K \subseteq U$,

$$\exists C, \rho > 0\, \forall n \in \mathbb{N}\, \forall x \in K : \|f^{(n)}(x)\|_{L_n(\mathbb{R}^d,\mathbb{R}^m)} \le C\rho^n n!\, M_n. \tag{1}$$

For the constant sequence $M_k = 1$, we recover the real analytic class $\mathcal{C}^\omega(U, \mathbb{R}^m)$.

We will impose some regularity properties on M: An increasing log-convex sequence $M = (M_k)$ with $M_0 = 1$ is called a *weight sequence*. A weight sequence M is called *non-quasianalytic* if

$$\sum_k \frac{M_k}{(k+1)M_{k+1}} < \infty; \tag{2}$$

otherwise it is said to be *quasianalytic*. We say that M has *moderate growth* if there is a constant $C > 0$ such that $M_{j+k} \le C^{j+k} M_j M_k$ for all j, k.

If M is a weight sequence, then $\mathcal{C}^M$ contains $\mathcal{C}^\omega$ and is stable under composition. By the Denjoy–Carleman theorem, M is non-quasianalytic if and only if there are $\mathcal{C}^M$-functions with compact support. Clearly, (iii) fails for quasianalytic weight sequences M. The moderate growth condition will be important below.

On closed fat sets with Hölder boundary. What about (i), (ii), and (iii) for functions defined in *non-open* subsets $X \subseteq \mathbb{R}^d$? For arbitrary $X \subseteq \mathbb{R}^d$ we define

$$\mathcal{A}^\infty(X) := \big\{f : X \to \mathbb{R} : f_*\{c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^d) : c(\mathbb{R}) \subseteq X\} \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})\big\},$$
$$\mathcal{A}^M(X) := \big\{f : X \to \mathbb{R} : f_*\{c \in \mathcal{C}^M(\mathbb{R}, \mathbb{R}^d) : c(\mathbb{R}) \subseteq X\} \subseteq \mathcal{C}^M(\mathbb{R}, \mathbb{R})\big\},$$
$$\mathcal{A}^\infty_M(X) := \big\{f : X \to \mathbb{R} : f_*\{c \in \mathcal{C}^M(\mathbb{R}, \mathbb{R}^d) : c(\mathbb{R}) \subseteq X\} \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})\big\}.$$

If $X \subseteq \mathbb{R}^d$ is a non-empty open set, then (i) and (iii) amount to

$$\mathcal{A}^\infty(X) = \mathcal{C}^\infty(X), \quad \mathcal{A}^M(X) = \mathcal{C}^M(X). \tag{3}$$

Clearly, some restrictions on X are necessary if one hopes for identities as in (3) on non-open sets X, not to mention definitions of $\mathcal{C}^\infty$ and $\mathcal{C}^M$. We will say that a non-empty closed subset $X \subseteq \mathbb{R}^d$ is *fat* if it has dense interior, i.e., $X = \overline{\operatorname{int}(X)}$. For such X we define (see also Remark 2 below)

$$(4)\quad \mathcal{C}^\infty(X) := \left\{ f : X \to \mathbb{R} \;\middle|\; \begin{array}{c} f|_{\operatorname{int}(X)} \in \mathcal{C}^\infty, \\ \forall n \in \mathbb{N} : (f|_{\operatorname{int}(X)})^{(n)} \text{ extends continuously to } X \end{array} \right\}.$$

For a weight sequence $M = (M_k)$, let

$$(5)\qquad \mathcal{C}^M(X) := \big\{ f \in \mathcal{C}^\infty(X) : (1) \text{ holds for all compact } K \subseteq X \big\}.$$

Question 1. *When do we have* $\mathcal{A}^\infty(X) = \mathcal{C}^\infty(X)$ *and* $\mathcal{A}^M(X) = \mathcal{C}^M(X)$*?*

Interestingly, the analogue for finite differentiability (ii) fails even on the closed half-space, which is a consequence of Glaeser's inequality. That the identities in Question 1 are not always true is shown by the following example.

Example 1. Let $p : [0,\infty) \to [0,\infty)$ be a strictly increasing $\mathcal{C}^\infty$-function which is infinitely flat at 0. Consider the ∞-*flat cusp* $X = \{(x,y) \in \mathbb{R}^2 : x \ge 0,\, 0 \le y \le p(x)\}$ and the function $f : X \to \mathbb{R}$ defined by $f(x,y) = \sqrt{x^2 + y}$. Then $f \notin \mathcal{C}^\infty(X)$, but $f \in \mathcal{A}^\infty(X)$. The latter follows from a division theorem of [6].

On the positive side, [7] proved that $\mathcal{A}^\infty(X) = \mathcal{C}^\infty(X)$ holds for convex sets X with non-empty interior. We will extend this result to a larger family of sets.

Let $\mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R}$ with Euclidean coordinates $x = (x', x_d)$. Let $\alpha \in (0,1]$, and $r, h > 0$. Consider the *truncated open cusp*

$$\Gamma_\alpha(r,h) := \big\{ (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : |x'| < r,\, h(|x'|/r)^\alpha < x_d < h \big\}.$$

An open set $U \subseteq \mathbb{R}^d$ is said to have the *uniform cusp property of index* α (we write (UCP_α) for short), if for each $x \in \partial U$ there exist $\epsilon, r, h > 0$ and $A \in \mathrm{O}(d)$ such that for all $y \in \overline{U} \cap B(x,\epsilon)$ we have $y + A\Gamma_\alpha(r,h) \subseteq U$.

Remark 1. A bounded open set $U \subseteq \mathbb{R}^d$ has (UCP_α) if and only if U has equi-α-Hölder boundary; cf. [3]. In particular, U has (UCP_1) if and only if it is a Lipschitz domain. If $\alpha < 1$ then the Hausdorff dimension of ∂U can be larger than $d - 1$.

Theorem 1. *Let* $M = (M_k)$ *be a non-quasianalytic weight sequence. Let* $X \subseteq \mathbb{R}^d$ *be a closed fat set. If* $\operatorname{int}(X)$ *has* (UCP_α) *for some* α*, then*

$$(6)\qquad \mathcal{A}^\infty(X) = \mathcal{A}^\infty_M(X) = \mathcal{C}^\infty(X).$$

If $\operatorname{int}(X)$ *has* (UCP_1)*, then*

$$(7)\qquad \mathcal{A}^M(X) = \mathcal{C}^M(X).$$

On closed fat subanalytic sets. Using rectilinearization of subanalytic sets we obtain the following consequences of Theorem 1.

Theorem 2. *Let* $M = (M_k)$ *be a non-quasianalytic weight sequence. Let* $X \subseteq \mathbb{R}^d$ *be a closed fat subanalytic set. There is a locally finite collection of real analytic* *mappings $\varphi_j : U_j \to \mathbb{R}^d$, where each φ_j is the composite of a finite sequence of local blow-ups with smooth centers and U_j is open in $\mathbb{R}^d$, such that, for all j,*

$$\varphi_j^* \mathcal{A}^\infty(X) \subseteq \mathcal{C}^\infty(\varphi_j^{-1}(X)), \tag{8}$$

$$\varphi_j^* \mathcal{A}^M(X) \subseteq \mathcal{C}^M(\varphi_j^{-1}(X)). \tag{9}$$

If f is $\mathcal{C}^\infty$, φ real analytic, and the composite $f \circ \varphi$ is $\mathcal{C}^M$, then in general f need not be $\mathcal{C}^M$. Under suitable conditions one can however expect that f is $\mathcal{C}^{M^a}$ for some positive integer a independent of M (where $(M^a)_k := (M_k)^a$). Combining a result of [2] (which makes this precise) with Theorem 2 we deduce the following.

Let $M = (M_k)$ be a weight sequence. Let $X \subseteq \mathbb{R}^d$ be a closed fat set. We define

$$\mathcal{A}^{\widehat{M}}(X) := \bigcap_{a>0} \mathcal{A}^{M^a}(X) \quad \text{and} \quad \mathcal{C}^{\widehat{M}}(X) := \bigcap_{a>0} \mathcal{C}^{M^a}(X).$$

Theorem 3. *Let $M = (M_k)$ be a weight sequence of moderate growth such that M^a is non-quasianalytic for all $a > 0$. Let $X \subseteq \mathbb{R}^d$ be a closed fat subanalytic set. Then*

$$\mathcal{C}^\infty(X) \cap \mathcal{A}^{\widehat{M}}(X) = \mathcal{C}^{\widehat{M}}(X). \tag{10}$$

Example 2. The sequence $M_k = k!$ satisfies the assumptions of Theorem 3. In that case $\mathcal{C}^{\widehat{M}}$ is the intersection of all Gevrey classes.

Remark 2. Often a function on a closed set $X \subseteq \mathbb{R}^d$ is declared to be $\mathcal{C}^\infty$ if it is the restriction of a $\mathcal{C}^\infty$-function on $\mathbb{R}^d$. For general closed fat sets, this differs from the notion of smoothness defined in (4). But in the cases considered here (i.e., $\operatorname{int}(X)$ has (UCP_α) for some α, or X is subanalytic) the two notions coincide.

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