Selberg integrals and symmetric functions

Seamus Albion
The University of Queensland

(with Ole Warnaar)

AEC Sommerschool 2019

What is the Selberg integral?

Our story begins with the beta integral

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

valid for $\alpha, \beta \in \mathbb{C}$ such that $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$, which was first proved by Euler in 1730.

What does a k-dimensional analogue look like? Of course there is the trivial

$$\int_{[0,1]^k} \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1}\,\mathrm{d}t_1\cdots\mathrm{d}t_k = \prod_{i=1}^{k}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

but this is not very satisfactory.

A (more) satifactory answer comes in the form of the Selberg integral, first proved by Selberg in 1944. We have

$$S_k(\alpha,\beta;\gamma) := \int_{[0,1]^k} \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\leqslant i<j\leqslant k} |t_i - t_j|^{2\gamma}\, \mathrm{d}t_1\cdots\mathrm{d}t_k$$

$$= \prod_{i=1}^{k} \frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(\beta+(i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+(k+i-2)\gamma)\Gamma(1+\gamma)},$$

for $\alpha,\beta,\gamma\in\mathbb{C}$ such that $\mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0$ and

$$\mathrm{Re}(\gamma) > -\min\left\{\frac{1}{k}, \frac{\mathrm{Re}(\alpha)}{k-1}, \frac{\mathrm{Re}(\beta)}{k-1}\right\}.$$

The Selberg integral has appeared in random matrix theory, analytic number theory, enumerative combinatorics, conformal field theory,...

Three generalisations

Henceforth if $t = (t_1, \dots, t_k)$ then $\mathrm{d}t := \mathrm{d}t_1 \cdots \mathrm{d}t_k$ and

$$\Delta(t) = \prod_{1 \leqslant i < j \leqslant k} (t_i - t_j).$$

1. The Kadell integral:

$$\int_{[0,1]^k} P_\mu^{(1/\gamma)}(t) \, |\Delta(t)|^{2\gamma} \prod_{i=1}^{k} t_i^{\alpha-1} (1-t_i)^{\beta-1} \, \mathrm{d}t$$

$$= P_\mu^{(1/\gamma)}(\underbrace{1, \dots, 1}_{k \text{ times}}) \prod_{i=1}^{k} \frac{\Gamma(\alpha + (k-i)\gamma + \mu_i)\Gamma(\beta + (i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha + \beta + (2k-i-1)\gamma + \mu_i)\Gamma(1+\gamma)},$$

where $P_\mu^{(1/\gamma)}(t)$ is a Jack polynomial and $\mathrm{Re}(\alpha) > -\mu_k$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > \dots$.

This was conjectured by Macdonald and subsequently proved by Kadell in 1987.

2. The Hua–Kadell integral:

$$\int\limits_{[0,1]^k} P_\mu^{(1/\gamma)}(t) P_\nu^{(1/\gamma)}(t)\, |\Delta(t)|^{2\gamma} \prod_{i=1}^k t_i^{\alpha-1}(1-t_i)^{\gamma-1}\,\mathrm{d}t$$

$$= P_\mu^{(1/\gamma)}(\underbrace{1,\dots,1}_{k \text{ times}}) P_\nu^{(1/\gamma)}(\underbrace{1,\dots,1}_{k \text{ times}})$$

$$\times \prod_{i=1}^k \frac{\Gamma(\alpha+(k-i)\gamma+\mu_i)\Gamma(\gamma+(i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\gamma+(k-i-1)\gamma+\mu_i)\Gamma(1+\gamma)}$$

$$\times \prod_{i,j=1}^k \frac{\Gamma(\alpha+\gamma+(2k-i-j-1)\gamma+\mu_i+\nu_j)}{\Gamma(\alpha+\gamma+(2k-i-j)\gamma+\mu_i+\nu_j)},$$

which is a generalisation of Kadell's integral, but with the added restriction $\beta=\gamma$. In the above $\mathrm{Re}(\alpha) > -\mu_k$, $\mathrm{Re}(\gamma) > 0$.

Proved by Hua for $\gamma = 1$ in 1979, and in general by Kadell in 1993.

3. The Alba–Fateev–Litvinov–Tarnopolskiy (AFLT) integral:

$$\int\limits_{[0,1]^k} P^{(1/\gamma)}_{\mu}(t)P^{(1/\gamma)}_{\nu}[t+\beta/\gamma-1]\,|\Delta(t)|^{2\gamma}\prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1}\,\mathrm{d}t$$

$$= P^{(1/\gamma)}_{\mu}[k]P^{(1/\gamma)}_{\nu}[k+\beta/\gamma-1]$$

$$\times\prod_{i=1}^{k}\frac{\Gamma(\alpha+(k-i)\gamma+\mu_i)\Gamma(\beta+(i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+(2k-m-i-1)\gamma+\mu_i)\Gamma(1+\gamma)}$$

$$\times\prod_{i=1}^{k}\prod_{j=1}^{m}\frac{\Gamma(\alpha+\beta+(2k-i-j-1)\gamma+\mu_i+\nu_j)}{\Gamma(\alpha+\beta+(2k-i-j)\gamma+\mu_i+\nu_j)},$$

where m is an arbitrary integer such that $m\geqslant l(\nu)$. This is the Hua–Kadell integral with $\beta\neq\gamma$.

The AGT Conjecture

Alba, Fateev, Litvinov, and Tarnopolskiy, did not set out to prove a generalisation of the Hua–Kadell integral. Instead, they were attempting to verify the so-called AGT conjecture for SU(2).

An ingredient in the AGT conjecture is an explicit expression for the Nekrasov partition function in terms of conformal blocks in Liouville field theory. In order to verify this conjecture, AFLT had to compute the Hua–Kadell integral without the $\beta = \gamma$ restriction.

The AGT conjecture was quickly generalised to $\mathrm{SU}(n+1)$. In this case the analogue of the AFLT integral is an A_n Selberg integral with $n+1$ Jack polynomials inserted in the integrand.

For $t = (t_1, \ldots, t_k)$ and $s = (s_1, \ldots s_\ell)$ define

$$\Delta(t, s) := \prod_{i=1}^{k} \prod_{j=1}^{\ell} (t_i - s_j).$$

Then the "AGT integral" for A_2 is

$$\begin{aligned} & I^{k,\ell}_{\mu,\omega,\nu}(\alpha_1, \alpha_2, \beta; \gamma) \\ & := \int_{C^{k,\ell}_\gamma[0,1]} P^{(1/\gamma)}_\mu[t] P^{(1/\gamma)}_\omega[s-t] P^{(1/\gamma)}_\nu[s + \beta/\gamma - 1] |\Delta(t,s)|^{-\gamma} \\ & \qquad \times |\Delta(t)|^{2\gamma} |\Delta(s)|^{2\gamma} \prod_{i=1}^{k} t_i^{\alpha_1 - 1} \prod_{j=1}^{\ell} s_j^{\alpha_2 - 1} (1 - s_j)^{\beta - 1} \, \mathrm{d}t \, \mathrm{d}s \end{aligned}$$

We can evaluate this integral when either $\omega = 0$ or $\gamma = 1$.

Two techniques

The technique of Alba *et al.* of the AFLT integral mimics a proof of Kadell's integral by Warnaar which uses the Okounkov–Olshanski formula for Jack polynomials, but in this case it only works for $\alpha = N\gamma, \beta = M\gamma$.

1. The proof of the $\omega = 0$ case uses Macdonald polynomial theory, in particular higher rank analogues of the Cauchy identity

   $$\sum_{\lambda} P_{\lambda}(X; q, t) Q_{\lambda}(Y; q, t) = \prod_{x \in X} \prod_{y \in Y} \frac{(txy; q)_{\infty}}{(xy; q)_{\infty}},$$

   where $(x; q)_{\infty} = (1 - x)(1 - xq)(1 - xq^2) \cdots$

2. For the $\gamma = 1$ case we use the inverse Pieri rule for Schur functions to set up a recursion for the AGT integral. With the $\omega = 0$ case as an initial condition this has a unique solution.

The case $\gamma = 1$

For $\gamma = 1$, the Jack polynomials reduce to Schur functions

$$P_\lambda^{(1)}(X) = s_\lambda(X) := \frac{\det_{1 \leqslant i,j \leqslant k}(x_j^{\lambda_i + k - i})}{\Delta(x)}$$

where X is an alphabet of cardinality k.

The Schur functions form a basis for the algebra of symmetric functions, denoted Λ.

Some examples for $X = \{x_1, x_2, x_3\}$ are:

$$\begin{aligned}
s_{(1,1,1)}(X) &= x_1x_2x_3 \\
s_{(2,1)}(X) &= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + 2x_1x_2x_3 \\
s_{(3)}(X) &= x_1^3 + x_2^3 + x_3^2 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3 \\
&\qquad + x_1x_3^2 + x_2x_3^2 + x_1x_2x_3
\end{aligned}$$

An algebraic basis for Λ is given by the complete symmetric functions

$$h_r(X) := \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_r} x_{i_1} \cdots x_{i_r}.$$

More examples with $X = (x_1, x_2, x_3)$:

$$\begin{aligned} h_0(X) &:= 1 \\ h_1(X) &= x_1 + x_2 + x_3 \\ h_2(X) &= x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3 \end{aligned}$$

We have another definiton of s_λ in terms of the Jacobi–Trudi formula

$$s_\lambda = \det_{1 \leqslant i,j \leqslant l(\lambda)} (h_{\lambda_i + j - i}),$$

from which it is clear that

$$s_{(r)} = h_r.$$

Plethystic business

Another algebraic basis for Λ is given by the Newton power sums:

$$p_r(X) := x_1^r + x_2^r + x_3^r + \cdots = \sum_{x \in X} x^r.$$

Let X and Y be two alphabets and define the sum $X + Y$ to be their disjoint union. It follows that

$$p_r[X + Y] = p_r[X] + p_r[Y].$$

We can also subtract alphabets. Define $X - Y$ by demading

$$p_r[X - Y] := p_r[X] - p_r[Y]$$

so that

$$p_r[X + Y - Y] = p_r[X] + p_r[Y] - p_r[Y] = p_r[X]$$

as it should!

Products of alphabets are also possible. Define XY to be the Cartesian product of X and Y, then

$$p_r[XY] = p_r[X]p_r[Y].$$

This leads us to an ambiguity. Clearly by our summation convention

$$p_r[\underbrace{X + \cdots + X}_{k \text{ times}}] = k\, p_r[X],$$

which we extend to any $z \in \mathbb{C}$ by

$$p_r[zX] = z\, p_r[X].$$

But for a single-letter alphabet y we have

$$p_r[yX] = y^r p_r[X].$$

What is going on?

Single-letter alphabets are called rank-one elements of Λ, and are those that satisfy

$$p_r[yX] = y^r p_r[X].$$

Elements $z \in \mathbb{C}$ such that

$$p_r[zX] = z\, p_r[X]$$

are called binomial elements, because

$$h_r[z] = \binom{z+r-1}{r}.$$

In our Selberg integrals we always treat $\beta/\gamma - 1$ as a binomial element.

Define the Selberg average

$$\left\langle s_\mu[t] s_\omega[s-t] s_\nu[s+\beta-1] \right\rangle_{\alpha_1,\alpha_2,\beta}^{k,\ell} := \lim_{\gamma\to 1} \frac{I_{\mu,\omega,\nu}^{k,\ell}(\alpha_1,\alpha_2,\beta;1)}{I_{0,0,0}^{k,\ell}(\alpha_1,\alpha_2,\beta;1)}.$$

Taking a plethystic minus sign in a Schur function gives the transformation

$$s_\lambda[-X] = (-1)^{|\lambda|} s_{\lambda'}[X].$$

Hence we consider the equivalent integral

$$\left\langle s_\mu[t] s_\omega[s-t] s_\nu[1-\beta-s] \right\rangle_{\alpha_1,\alpha_2,\beta}^{k,\ell},$$

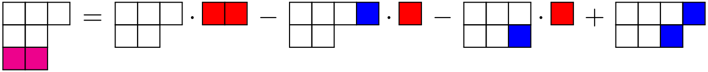
with corresponding (conjectural) evaluation. We prove this by induction on the length of ω.

Of course this requires one to guess an appropriate evaluation...

Take a partition ω and form the partition $(\omega, d) := (\omega_1, \ldots, \omega_n, d)$ where d is an integer such that $1 \leqslant d \leqslant \omega_n$. Then the inverse Pieri rule states that

$$s_{(\omega,d)}[s-t] = \sum_{\substack{\lambda \\ l(\lambda)\leqslant n}} (-1)^{|\lambda/\omega|} s_\lambda[s-t] h_{d-|\lambda/\omega|}[s-t],$$

where the skew shape λ/ω is a vertical strip. At the level of diagrams:



So we have equality between

$$\left\langle s_\mu[t] s_{(\omega,d)}[s-t] s_\nu[1-\beta-s] \right\rangle^{k,\ell}_{\alpha_1,\alpha_2,\beta}$$

and

$$\sum_{\substack{\lambda \\ l(\lambda)\leqslant n}} (-1)^{|\lambda/\omega|-d} \left\langle s_\mu[t] s_\lambda[s-t] h_{d-|\lambda/\omega|}[s-t] s_\nu[1-\beta-s] \right\rangle^{k,\ell}_{\alpha_1,\alpha_2,\beta}.$$

The next step is to use the generalised convolution formula

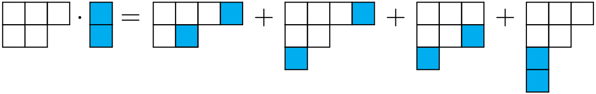
$$h_{d-|\lambda/\omega|}[s-t] = \sum_{i+j=d-|\lambda/\omega|} \binom{\beta-1}{d-|\lambda/\omega|-i-j} e_i[t] e_j[1-\beta-s]$$

where the elementary symmetric functions are given by

$$e_r[X] := (-1)^r h_r[-X].$$

Then absorb the elementary symmetric functions into $s_\mu[t]$ and $s_\nu[1-\beta-s]$ using the e-Pieri rule

$$s_\mu e_r = \sum_{\substack{\lambda \succ \mu \\ |\lambda/\mu|=r}} s_\lambda .$$



In the end we obtain an identity of the form

$$\begin{aligned}\Big\langle s_\mu[t]s_{(\omega,d)}[s-t]s_\nu[1-\beta-s]\Big\rangle^{k,\ell}_{\alpha_1,\alpha_2,\beta} \\ = \sum_{i,j,\eta,\lambda,\pi} \text{const} \times \Big\langle s_\eta[t]s_\lambda[s-t]s_\pi[1-\beta-s]\Big\rangle^{k,\ell}_{\alpha_1,\alpha_2,\beta},\end{aligned}$$

where the partition λ has length at most n.

If we can show that the conjectural evaluation of the A_2 AGT integral also satisfies this recursion, then the two expressions must be equal.

In the case of $\gamma = 1$ this boils down to the verification of a rational function identity.

The rational function identity obtained is a limiting case of Milne's A_n q-Pfaff–Saalschütz summation:

$$\sum_{\substack{k_1,\dots,k_n\geqslant 0\\ |k|\leqslant d}} \frac{(q^{-d};q)_{|k|}\,q^{|k|}}{(a_1\cdots a_n b q^{1-d}/c;q)_{|k|}} \prod_{1\leqslant r<s\leqslant n} \frac{x_r q^{k_r}-x_s q^{k_s}}{x_r-x_s}$$

$$\times \prod_{r,s=1}^{n} \frac{(a_s x_r/x_s;q)_{k_r}}{(qx_r/x_s;q)_{k_r}} \prod_{r=1}^{n} \frac{(bx_r;q)_{k_r}}{(cx_r;q)_{k_i}}$$

$$= \frac{(c/b;q)_d}{(c/a_1\cdots a_n b;q)_d} \prod_{r=1}^{n} \frac{(cx_r/a_r;q)_d}{(cx_r;q)_d}$$

where

$$(a;q)_k = \prod_{i=1}^{k}(1-aq^{i-1}).$$

For arbitrary γ define the γ-shifted factorial

$$(a;\gamma)_\lambda := \prod_{i=1}^{l(\lambda)} (a+(1-i)\gamma)_{\lambda_i}.$$

Conjecture (SA,Warnaar, 2019)

$$\left\langle \prod_{r=1}^{n+1} s_{\nu^{(r)}}\big[t^{(r)}-t^{(r-1)}\big] \right\rangle_{\alpha_1,\dots,\alpha_n,\beta;1}^{k_1,\dots,k_n}$$
$$= \prod_{r,s=1}^{n+1} \frac{(A_{r,s}+k_s-k_{s-1};1)_{\nu^{(r)}}}{(A_{r,s}+\ell_s;1)_{\nu^{(r)}}} \prod_{r=1}^{n+1} \prod_{1\leqslant i<j\leqslant \ell_r} \frac{\nu_i^{(r)}-\nu_j^{(r)}+j-i}{j-i}$$
$$\times \prod_{1\leqslant r<s\leqslant n+1} \prod_{i=1}^{\ell_r}\prod_{j=1}^{\ell_s} \frac{A_{r,s}+\nu_i^{(r)}-\nu_j^{(s)}+j-i}{A_{r,s}+j-i}$$

where for any $1\leqslant r,s\leqslant n+1$ we define

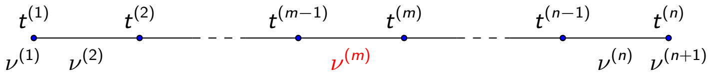
$$A_r := \alpha_r+\cdots+\alpha_n+k_{r-1}-k_r+r$$
$$A_{r,s} := A_r - A_s$$

Conclusions

Unfortunately the inductive argument does not provide a proof of the conjecture for general n. Indeed one would need to know the value of

$$\left\langle \prod_{\substack{r=1 \\ r\neq m}}^{n+1} s_{\nu^{(r)}}\left[t^{(r)} - t^{(r-1)}\right] \right\rangle^{k_1,\dots,k_n}_{\alpha_1,\dots,\alpha_n,\beta;1}$$

for every $2 \leqslant m \leqslant n-1$:



The technique using the Pieri rule does not lift to the case of general γ. The issue is not the Pieri rule for Jack polynomials, but rather that there is no γ-analogue of

$$h_r[s-t] = \sum_{i+j=r} \binom{\beta-1}{r-i-j} e_i[t] e_j[1-\beta-s].$$

The End