THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

Selberg integrals of Alba–Fateev–Litvinov–Tarnopolsky type

Seamus P. Albion
B.Math. (Hons)

A thesis submitted for the degree of Master of Philosophy at The University of Queensland in 2020

School of Mathematics and Physics

Abstract

In 1944 Atle Selberg published a remarkable n-dimensional analogue of Euler's beta integral which now bears his name. The Selberg integral has come to be regarded as one of the most important hypergeometric integrals, a reputation which is upheld by its uses in fields such as random matrix theory, analytic number theory, conformal field theory and enumerative and algebraic combinatorics. In their verification of the AGT conjecture for SU(2), Alba, Fateev, Litvinov and Tarnopolsky (AFLT) discovered a new generalisation of the Selberg integral over a pair of Jack polynomials. The AFLT integral unifies the well-known Kadell and Hua–Kadell integrals. The purpose of this thesis is to investigate generalisations of the AFLT integral in several directions, all based on symmetric function theory. Using new Cauchy-type identities for Macdonald polynomials we present two A_n Selberg integrals over a pair of Jack polynomials; one directly generalising the AFLT integral, and one generalising the A_2 Selberg integral of Warnaar. Following Matsuo and Zhang, we then consider A_n Selberg integrals with $n+1$ symmetric functions in the integrand. Here our results are restricted to the Schur case ($\gamma = 1$), where we use several new integral formulas for complex Schur functions to evaluate the A_n integral first considered by Matsuo and Zhang. To conclude, we discuss other recent developments including the elliptic AFLT integral, an AFLT integral for Macdonald polynomials, and the related Askey–Habseiger–Kadell-type q-AFLT integral.

Declaration by author

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Publications included in this thesis

No publications included.

Submitted manuscripts included in this thesis

1. [ARW] **Seamus P. Albion**, Eric M. Rains, and S. Ole Warnaar, *AFLT-type Selberg integrals*, submitted.

Other publications during candidature

No other publications.

Contributions by others to the thesis

All new results in this thesis have been obtained in collaboration with Eric M. Rains and S. Ole Warnaar. S. Ole Warnaar also provided many comments on preliminary versions of this thesis.

Statement of parts of the thesis submitted to qualify for the award of another degree

No works submitted towards another degree have been included in this thesis.

Research involving human or animal subjects

No animal or human subjects were involved in this research.

Acknowledgments

I am deeply thankful to Ole Warnaar for his guidance over the last few years. The many hours we have spent discussing mathematics have proved invaluable, and are something I believe I am incredibly lucky to have experienced. Thanks must also go to Masoud Kamgarpour for his encouraging me to bring this project to completion, and many other pieces of advice about mathematical life.

I would like to thank Cale Rankin and Nick Bridger for comments which helped to improve the exposition, as well as for many conversations about mathematics. Thank you also to Charles Sale, George Browning, and Jeremy Neale, whose friendship has been incredibly important to me.

To my parents, Helen Betts and Graeme Albion, thank you for all that you have done for me.

Financial support

This research was supported by an Australian Government Research Training Program Scholarship.

Keywords

AGT conjecture, (complex) Schur functions, Jack polynomials, Macdonald polynomials, Selberg integrals

Australian and New Zealand Standard Research Classifications (ANZSRC)

ANZSRC code: 010101, Algebra and Number Theory, 20%
ANZSRC code: 010104, Combinatorics and Discrete Mathematics (excl. Physical Combinatorics), 40%
ANZSRC code: 010111, Real and Complex Functions (incl. Several Variables), 20%
ANZSRC code: 010505, Mathematical Aspects of Quantum and Conformal Field Theory, Quantum Gravity and String Theory, 20%

Fields of Research (FoR) Classification

FoR code: 0101, Pure Mathematics, 80%
FoR code: 0105, Mathematical Physics, 20%

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Chapter 1

Introduction

1.1 The Selberg integral

The main object of interest in this thesis is the Selberg integral and various generalisations thereof. For k a positive integer and $\alpha, \beta, \gamma \in \mathbb{C}$ such that

$$\operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0, \quad \text{and} \quad \operatorname{Re}(\gamma) > -\min\left\{\frac{1}{k}, \frac{\operatorname{Re}(\alpha)}{k-1}, \frac{\operatorname{Re}(\beta)}{k-1}\right\}, \tag{1.1.1}$$

this is the integral evaluation

$$\begin{aligned} S_k(\alpha, \beta; \gamma) &:= \int_{[0,1]^k} \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1 \leqslant i < j \leqslant k} |t_i - t_j|^{2\gamma} \, \mathrm{d}t_1 \cdots \mathrm{d}t_k \\ &= \prod_{i=1}^{k} \frac{\Gamma(\alpha + (i-1)\gamma)\Gamma(\beta + (i-1)\gamma)\Gamma(1 + i\gamma)}{\Gamma(\alpha + \beta + (k+i-2)\gamma)\Gamma(1+\gamma)}. \end{aligned} \tag{1.1.2}$$

Selberg first discovered his formula in 1941 and used it, with the substitution $t_i = s_i/(1+s_i)$, to prove a result concerning entire functions [Sel41]. However, since he believed the result was too elementary to be new he refrained from publishing a proof. After not being able to find the formula in the literature, an opportunity arose and Selberg eventually published a proof in a journal read primarily by mathematics teachers [Sel44]; this is Selberg's only paper in Norwegian.

It is clear that the Selberg integral is a k-dimensional generalisation of Euler's beta integral. Indeed, setting $k = 1$ in the former yields

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \, \mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.1.3}$$

which is precisely Euler's formula (see [Eul38] for Euler's original paper). Here one requires that $\operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0$, and, since the parameter γ drops out, this agrees with (1.1.1). Selberg's proof of (1.1.2) hinges on (1.1.3). Assuming that γ is a positive integer one may expand

$$\prod_{1 \leqslant i < j \leqslant k} |t_i - t_j|^{2\gamma} = \sum_{0 \leqslant i_1, \ldots, i_k \leqslant 2(k-1)\gamma} c_{i_1, \ldots, i_k} t_1^{i_1} \ldots t_k^{i_k},$$

for some coefficients $c_{i_1,\dots,i_k}$, and the resulting integral evaluated by iterating (1.1.3). To obtain (1.1.2) for complex γ one may use Carlson's theorem [And86, p. 51]. For full accounts of Selberg's proof see, e.g., [And86] and [For10, Chapter 3]. Since the integrand of (1.1.2) is symmetric in the variables $t_1, \dots, t_k$, the domain of integration may be changed to the k-simplex

$$C^k_\gamma[0,1] := \{(t_1,\dots,t_k) \in \mathbb{R}^k : 0 < t_1 < \cdots < t_k < 1\}, \tag{1.1.4}$$

provided we multiply through by $k!$.

What is perhaps most fascinating about (1.1.2) is its wide range of appearances in different areas of mathematics, for which it has garnered a reputation as one of the most important hypergeometric integrals [FW08]. However, this reputation was not earned early on, and the Selberg integral had only one use outside of Selberg's original application in its first three decades (and even then only in the case $\alpha = \beta = 1$, $\gamma = 2$) [KS53].

In the fifth of his series of papers developing the statistical theory of energy levels of complex systems, Dyson, together with Mehta, conjectured an integral evaluation which is closely related to Selberg's integral [MD63, Conjecture D]. For $\gamma \in \mathbb{C}$ such that $\mathrm{Re}(\gamma) > -1/k$, their conjecture may be stated as

$$\frac{1}{(2\pi)^{k/2}} \int\limits_{\mathbb{R}^k} \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^k t_i^2} \prod_{1\leqslant i<j\leqslant k} |t_i - t_j|^{2\gamma}\, \mathrm{d}t_1 \cdots \mathrm{d}t_k = \prod_{i=1}^k \frac{\Gamma(1+i\gamma)}{\Gamma(1+\gamma)}. \tag{1.1.5}$$

This evaluation remained an open problem in random matrix theory for over ten years, and is now known as Mehta's integral due to his popularising of the conjecture [Meh67, Meh74]. A proof of (1.1.5) was unearthed by Bombieri who, after discussions with Dyson and Selberg, was able to derive Mehta's integral from (1.1.2). To obtain (1.1.5) from the Selberg integral, one fixes $\alpha = \beta$, and then makes the substitution $t_i = (1 + s_i/\sqrt{2\alpha}\,)/2$. Taking the limit $\alpha \to \infty$ with the aid of Stirling's formula then yields the desired evaluation.

In 1982 Macdonald was led to conjecture a far-reaching generalisation of Mehta's integral. Let G be a finite reflection group generated by reflections in a set of N hyperplanes in $\mathbb{R}^k$. We normalise these hyperplanes so that their normal vectors $a_i := (a_{i,1}, \dots, a_{i,k})$ satisfy (up to sign) $(a_i, a_i) = 2$ with respect to the standard inner product on $\mathbb{R}^k$. In terms of these hyperplanes we define a polynomial in $t = (t_1, \dots, t_k)$ by $P(t) := \prod_{i=1}^N (a_i, t)$. If

$$\mathrm{d}\varphi(t) := \frac{\mathrm{e}^{-\frac{1}{2}\sum_{i=1}^k t_i^2}}{(2\pi)^{k/2}} \mathrm{d}t_1 \cdots \mathrm{d}t_k$$

denotes the k-dimensional Gaußian measure on $\mathbb{R}^k$, then Macdonald conjectured that [Mac82, Conjecture 5.1],

$$\int\limits_{\mathbb{R}^k} |P(t)|^{2\gamma}\, \mathrm{d}\varphi(t) = \prod_{i=1}^n \frac{\Gamma(1+d_i\gamma)}{\Gamma(1+\gamma)}, \tag{1.1.6}$$

where the d_i are degrees of the fundamental invariants of G, which are positive integers depending on the Cartan type of the underlying root system. For $G = \mathrm{A}_{k-1}$ the polynomial $P(t)$ reduces to the

Vandermonde product

$$\Delta(t) := \prod_{1\leqslant i<j\leqslant k} (t_i - t_j), \tag{1.1.7}$$

and $d_i = i$ for $1 \leqslant i \leqslant k$ so that (1.1.6) is precisely Mehta's integral. Macdonald stated his conjecture for $\mathrm{Re}(\gamma) > 0$, but of course this range may be extended to include small but negative values of $\mathrm{Re}(\gamma)$ depending on the d_i. In [Mac82], Macdonald provides a uniform proof of his conjecture for $\gamma = 1$ and for general γ in the case that G is a dihedral group. He also communicated the result of Regev that for type B_n and D_n the integral follows from the Selberg integral. A uniform proof of (1.1.6) for all crystallographic groups was subsequently given by Opdam based on his theory of hypergeometric shift operators [Opd89]. To settle the remaining non-crystallographic cases, H_3 and H_4, Opdam was assisted by the computational tools of Garvan [Gar89, Opd93].

To round out this section we will mention some further appearances of the Selberg integral. The evaluation of the Mehta integral is the first application of the Selberg integral to random matrix theory. This is a connection which continues to this day, with integrals of Selberg type being crucial to the study of random matrices; see [For10, Meh04]. Through the theory of random matrices the Selberg integral also has a conjectural connection with the distribution of the zeros of the Riemann zeta function on the critical line. Under the assumption of the Keating–Snaith hypothesis, the distribution of these zeroes is the same as the distribution of eigenvalues of large random matrices, allowing for the moments of the zeta function to be computed using the Selberg integral [KS01]. In another direction, the Selberg integral may be used to define multivariate analogues of the Jacobi polynomials [Aom87, Las91, Maca]. The classical (single-variable) Jacobi polynomials are themselves orthogonal on $[-1, 1]$ with weight function given by $(1-t)^\alpha(1+t)^\beta$. It follows from the beta integral (1.1.3) with the change of variables $t \mapsto (1-t)/2$ that the total weight is given by

$$\int_{-1}^{1} (1-t)^\alpha (1+t)^\beta \, \mathrm{d}t = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}. \tag{1.1.8}$$

In [Sta12, Problem I.11], Stanley gives a combinatorial interpretation of the Selberg integral in terms of sequences of permutations satisfying certain conditions. Here it is necessary to assume that α, β and 2γ are nonnegative integers. This interpretation was extended by Kim and Oh who introduced Young books, which are enumerated by the evaluation of the Selberg integral [KO17]. Finally, generalisations of the Selberg integral appear in solutions to the Knizhnkik–Zamolodchikov equations, a family of partial differential equations based on Lie algebras [EFJ98, SV91, Var03]. Part of this connection is explained in Section 4.1 to motivate extension of the Selberg integral to the Lie algebra A_n.

1.2 Jack polynomials and the AFLT integral

We have already noted that the integrand of the Selberg integral is symmetric in the k integration variables, and so it is natural to consider Selberg integrals where the integrand is multiplied by a symmetric function in the k variables. Here we sometimes use the shorthand of (1.1.7) for the Vandermonde product and write $\mathrm{d}t := \mathrm{d}t_1 \cdots \mathrm{d}t_k$ for $t = (t_1, \dots, t_k)$.

In [Aom87], Aomoto provided arguably the first elementary proof of the Selberg integral by evaluating the slightly more general integral

$$\int\limits_{[0,1]^k} |\Delta(t)|^{2\gamma} \prod_{i=1}^{r} t_i \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1}\,\mathrm{d}t = S_k(\alpha,\beta;\gamma)\prod_{j=1}^{r}\frac{\alpha+(k-j)\gamma}{\alpha+\beta+(2k-j-1)\gamma},$$

where r is some nonnegative integer not exceeding k. As the domain of integration is symmetric, the extra factor may be replaced by any product of r distinct variables without changing the evaluation. Defining the elementary symmetric functions in the variables t by (see (2.2.2) below)

$$e_r(t) = \sum_{1\leqslant i_1<\cdots<i_r\leqslant k} t_{i_1}\ldots t_{i_r},$$

the evaluation of Aomoto's integral is thus equivalent to

$$\begin{aligned}\int\limits_{[0,1]^k} e_r(t)|\Delta(t)|^{2\gamma}\prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1}\mathrm{d}t \\ = \binom{k}{r} S_k(\alpha,\beta;\gamma)\prod_{j=1}^{r}\frac{\alpha+(k-j)\gamma}{\alpha+\beta+(2k-j-1)\gamma}.\end{aligned} \tag{1.2.1}$$

In [Mac87, Conjecture C5] Macdonald conjectured a generalisation of the Selberg integral where the integrand is multiplied by a Jack polynomial $P_\lambda^{(1/\gamma)}(t_1,\ldots,t_k)$, a particular one-parameter deformation of the Schur functions (defined in (2.6.11) and (2.5.1) respectively). For λ a partition with length at most k, Macdonald's formula is

$$\begin{aligned}\int\limits_{[0,1]^k} P_\lambda^{(1/\gamma)}(t_1,\ldots,t_k)\prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1}\prod_{1\leqslant i<j\leqslant k}|t_i-t_j|^{2\gamma}\,\mathrm{d}t_1\cdots\mathrm{d}t_k \\ = P_\lambda^{(1/\gamma)}(\underbrace{1,\ldots,1}_{k\text{ times}})\prod_{i=1}^{k}\frac{\Gamma(\alpha+(k-i)\gamma+\lambda_i)\Gamma(\beta+(i-1)\gamma)\Gamma(1+i\gamma)}{\Gamma(\alpha+\beta+(2k-i-1)\gamma+\lambda_i)\Gamma(1+\gamma)},\end{aligned} \tag{1.2.2}$$

where $\alpha,\beta,\gamma\in\mathbb{C}$ are such that

$$\mathrm{Re}(\alpha)>-\lambda_k,\quad \mathrm{Re}(\beta)>0,\quad \text{and}\quad \mathrm{Re}(\gamma)>-\min_{1\leqslant i\leqslant k-1}\left\{\frac{1}{k},\frac{\mathrm{Re}(\alpha)+\lambda_i}{k-i},\frac{\mathrm{Re}(\beta)}{k-1}\right\}. \tag{1.2.3}$$

The evaluation above was first proved by Kadell [Kad97], and so is known as Kadell's integral. When $\lambda=(1^r)$, a single column of height r, the Jack polynomial simplifies to the elementary symmetric function e_r. Using the fact that $e_r(1,\ldots,1)=\binom{k}{r}$ and $\Gamma(z+1)=z\Gamma(z)$, the two evaluations (1.2.1) and (1.2.2) are seen to coincide in this case. Kadell was able to extend (1.2.2) by adding a second Jack polynomial to the integrand, albeit with the additional restriction $\beta=\gamma$ [Kad93]. For $\beta=\gamma=1$, the Schur case, this same integral had already been evaluated by Hua [Hua63].

In their verification of the Adlay–Gaitto–Tachikawa (AGT) conjecture [AGT10] for SU(2), Alba, Fateev, Litvinov, and Tarnopolsky (AFLT) discovered an integral generalising the Hua–Kadell integral to the case of unequal β and γ. The AGT conjecture itself claims an intimate relationship between

Liouville field theory and $\mathcal{N} = 2$ supersymmetric gauge theory. One ingredient of the conjecture is an explicit formula for the Nekrasov instanton partition function in terms of conformal blocks in Liouville field theory. It is in verifying this expression that Alba et al. were led to consider this particular generalised Selberg integral. For a more detailed account of exactly how the integral arises in relation to the AGT conjecture see the introduction of [ARW]. What we refer to as the AFLT integral [AFLT11, Appendix B] may be stated as follows.

Theorem 1.2.1 (AFLT integral)**.** *Let k be a positive integer, $\lambda, \mu \in \mathscr{P}$ and $\alpha, \beta, \gamma \in \mathbb{C}$ such that the conditions* (1.2.3) *are satisfied. Then*

$$\int\limits_{[0,1]^k} P_\lambda^{(1/\gamma)}[t] P_\mu^{(1/\gamma)}[t + \beta/\gamma - 1] \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\leqslant i<j\leqslant k} |t_i - t_j|^{2\gamma}\, \mathrm{d}t_1 \cdots \mathrm{d}t_k \tag{1.2.4}$$

$$= P_\lambda^{(1/\gamma)}[k] P_\mu^{(1/\gamma)}[k + \beta/\gamma - 1] \prod_{i=1}^{k} \frac{\Gamma(\beta + (i-1)\gamma)\Gamma(\alpha + (k-i)\gamma + \lambda_i)\Gamma(1 + i\gamma)}{\Gamma(\alpha + \beta + (2k - m - i - 1)\gamma + \lambda_i)\Gamma(1+\gamma)}$$

$$\times \prod_{i=1}^{k}\prod_{j=1}^{m} \frac{\Gamma(\alpha + \beta + (2k - i - j - 1)\gamma + \lambda_i + \mu_j)}{\Gamma(\alpha + \beta + (2k - i - j)\gamma + \lambda_i + \mu_j)},$$

where m is an arbitrary integer such that $m \geqslant l(\mu)$.

The second Jack polynomial in the integrand and both Jack polynomials in the evaluation of (1.2.4) are expressed using plethystic notation as explained in Section 2.3. We note that for any symmetric function f,

$$f(\underbrace{1, \dots, 1}_{k \text{ times}}) = f[k],$$

where plethystic notation is employed on the right. Inside the plethystic brackets the ordinary rules of addition and multiplication still hold, so that for $\beta = \gamma$ the addition in the second Jack polynomial drops out, leaving us with the Hua–Kadell integral. Furthermore, for $\mu = 0$, the unique partition of 0, it is an easy exercise to show that the formula is equivalent to the Kadell integral (1.2.2). By (2.6.26) below both $P_\lambda^{(1/\gamma)}[k]$ and $P_\mu^{(1/\gamma)}[k + \beta/\gamma - 1]$ may be evaluated in closed form, so that the expression on the right is fully factorised.

In [And91], Anderson gave a short inductive proof of the Selberg integral. Theorem 1.2.1 is proved in [AFLT11] by generalising the Anderson-style recursive proof of Kadell's integral given in [War08b]. Key input in both these proofs is the Okounkov–Olshanski integral formula for Jack polynomials [Oko98, OO97]. Later on we will present a proof of a higher-rank generalisation of the AFLT integral based on summation formulas for Macdonald polynomials. In particular this provides an alternative to the proof of AFLT which avoids the use of the Okounkov–Olshanski formula.

1.3 A_n Selberg integrals

We have already seen that it is natural to associate the Selberg integral with the reflection group of type A_{k-1}, where k is the number of integration variables. We will, however, take a different point of view, and instead associate the ordinary Selberg integral (1.1.2) to the Lie algebra A_1 (or $\mathfrak{sl}_2(\mathbb{C})$) of 2×2 traceless matrices. This association comes about via the close relationship between solutions to Knizhnik–Zamolodchikov (KZ) equations, which are a system of partial differential equations based on Lie algebras, and hypergeometric integrals.[1] This relationship led Mukhin and Varchenko [MV00, Conjecture 1] (also stated as Conjecture 4.1.1 below) to conjecture the existence of Selberg-type integrals for each simple Lie algebra $\mathfrak{g}$. In 2009, Warnaar verified the Mukhin–Varchenko conjecture for $\mathfrak{g} = \mathrm{A}_n$ [War09], extending the A_2 Selberg integral due to Tarasov and Varchenko [TV03].

For k, ℓ nonnegative integers and $t = (t_1, \dots, t_k)$, $s = (s_1, \dots, s_\ell)$ we define the Vandermonde-type product

$$\Delta(t, s) := \prod_{i=1}^{k} \prod_{j=1}^{\ell} (t_i - s_j).$$

Let n be a positive integer, $k_1, \dots, k_n$ nonnegative integers such that $0 \leqslant k_1 \leqslant \dots \leqslant k_n$, and let $t^{(1)}, \dots, t^{(n)}$ be sets of variables (or alphabets) such that $t^{(r)}$ has cardinality k_r. Further let $\alpha_1, \dots, \alpha_n, \beta, \gamma \in \mathbb{C}$ be such that

$$\operatorname{Re}(\beta) > 0, \quad |\operatorname{Re}(\gamma)| < \frac{1}{k_n}, \quad \operatorname{Re}\big(\beta + (k_n - 1)\gamma\big) > 0, \tag{1.3.1a}$$

$$\operatorname{Re}\big(\alpha_r + \dots + \alpha_s + (r - s + i - 1)\gamma\big) \quad \text{for } 1 \leqslant r \leqslant s \leqslant n \text{ and } 1 \leqslant i \leqslant k_r - k_{r-1}, \tag{1.3.1b}$$

where $k_0 := 0$.[2] The A_n Selberg integral may thus be stated as

$$\begin{aligned} \int\limits_{C_\gamma^{k_1,\dots,k_n}[0,1]} & \prod_{r=1}^{n} \prod_{i=1}^{k_r} \big(t_i^{(r)}\big)^{\alpha_r - 1} \big(1 - t_i^{(r)}\big)^{\beta_r - 1} \prod_{r=1}^{n} \big|\Delta\big(t^{(r)}\big)\big|^{2\gamma} \\ & \qquad \times \prod_{r=1}^{n-1} \big|\Delta\big(t^{(r)}, t^{(r+1)}\big)\big|^{-\gamma} \, \mathrm{d}t^{(1)} \cdots \mathrm{d}t^{(n)} \\ = & \prod_{r=1}^{n} \prod_{i=1}^{k_r} \frac{\Gamma(\beta_r + (i - k_{r+1} - 1)\gamma)\Gamma(i\gamma)}{\Gamma(\gamma)} \\ & \times \prod_{1 \leqslant r \leqslant s \leqslant n} \prod_{i=1}^{k_r - k_{r-1}} \frac{\Gamma(\alpha_r + \dots + \alpha_s + (r - s + i - 1)\gamma)}{\Gamma(\alpha_r + \dots + \alpha_s + \beta_s + (k_s - k_{s+1} + r - s + i - 2)\gamma)}, \end{aligned} \tag{1.3.2}$$

where $k_0 = k_{n+1} := 0$,

$$\beta_1 = \dots = \beta_{n-1} := 1, \quad \beta_n := \beta \tag{1.3.3}$$

and $C_\gamma^{k_1,\dots,k_n}[0, 1]$ is a somewhat complicated real domain of integration described in Section 4.2.

In the conclusion to their paper [AFLT11], Alba et al. remark that the generalisation of their construction requires a generalisation of the A_n Selberg integral (1.3.2) with two Jack polynomials included in the integrand. For A_2 such an AFLT-type integral was considered by Fateev and Litvinov in [FL], where they again used a recursion based on the Okounkov–Olshanski integral to obtain a

[1]This relationship is described in detail in Section 4.1.

[2]The condition $\operatorname{Re}(\gamma) < 1/k_n$ may be dropped when $n = 1$.

closed-form evaluation. They also claim a more general A_n AFLT integral, but the evaluation of this integral is only implicit and, unfortunately, the stated recursion relation that needs to be solved to obtain the evaluation is incorrect for $n \geqslant 3$ (see also (7.1.4) below).

Our first main result is an explicit evaluation of the A_n AFLT integral. To compactly state this formula we introduce the notion of the Selberg average of a polynomial $\mathcal{O}(t^{(1)},\dots,t^{(n)})$, symmetric in each of the alphabets $t^{(r)}$. We denote the A_n Selberg integral augmented with the polynomial $\mathcal{O}$ by

$$
\begin{aligned}
&I^{\mathrm{A}_n}_{k_1,\dots,k_n}(\mathcal{O};\alpha_1,\dots,\alpha_n,\beta;\gamma) \qquad (1.3.4)\\
&\quad := \int\limits_{C^{k_1,\dots,k_n}_\gamma[0,1]} \mathcal{O}\big(t^{(1)},\dots,t^{(n)}\big) \prod_{r=1}^{n}\prod_{i=1}^{k_r} \big(t_i^{(r)}\big)^{\alpha_r-1}\big(1-t_i^{(r)}\big)^{\beta_r-1} \\
&\qquad\qquad \times \prod_{r=1}^{n} \big|\Delta\big(t^{(r)}\big)\big|^{2\gamma} \prod_{r=1}^{n-1}\big|\Delta\big(t^{(r)},t^{(r+1)}\big)\big|^{-\gamma}\,\mathrm{d}t^{(1)}\cdots\mathrm{d}t^{(n)},
\end{aligned}
$$

so that $I^{\mathrm{A}_n}_{k_1,\dots,k_n}(1;\alpha_1,\dots,\alpha_n,\beta;\gamma)$ corresponds to (1.3.2). Then, assuming the conditions (1.3.1) on the complex parameters, the Selberg average for $\mathcal{O}$ is

$$
\big\langle\mathcal{O}\big\rangle^{k_1,\dots,k_n}_{\alpha_1,\dots,\alpha_n,\beta;\gamma} := \frac{I^{\mathrm{A}_n}_{k_1,\dots,k_n}(\mathcal{O};\alpha_1,\dots,\alpha_n,\beta;\gamma)}{I^{\mathrm{A}_n}_{k_1,\dots,k_n}(1;\alpha_1,\dots,\alpha_n,\beta;\gamma)}. \qquad (1.3.5)
$$

For n a nonnegative integer let $(a)_n := \Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1)$ denote the Pochhammer symbol, and let $\delta_{u,v}$ be the usual Kronecker delta. Then the A_n analogue of the AFLT integral is given by the following identity for the Selberg average of the product of two Jack polynomials.

Theorem 1.3.1 (A_n AFLT integral)**.** *For n a positive integer, let $k_1,\dots,k_n$ be integers such that $0 \leqslant k_1 \leqslant \cdots \leqslant k_n$. Then for $\alpha_1,\dots,\alpha_n,\beta,\gamma \in \mathbb{C}$ such that* (1.3.1) *holds and $\lambda,\mu \in \mathscr{P}$, we have*

$$
\begin{aligned}
&\Big\langle P^{(1/\gamma)}_\lambda\big[t^{(1)}\big]P^{(1/\gamma)}_\mu\big[t^{(n)}+\beta/\gamma-1\big]\Big\rangle^{k_1,\dots,k_n}_{\alpha_1,\dots,\alpha_n,\beta;\gamma} \qquad (1.3.6)\\
&\quad = P^{(1/\gamma)}_\lambda[k_1]P^{(1/\gamma)}_\mu[k_n+\beta/\gamma-1] \\
&\qquad \times \prod_{r=1}^{n}\prod_{i=1}^{\ell} \frac{(\alpha_1+\cdots+\alpha_r+(k_1-r-i+1)\gamma)_{\lambda_i}}{(\alpha_1+\cdots+\alpha_r+\beta_r+(k_1+k_r-k_{r+1}-r-m\delta_{r,n}-i)\gamma)_{\lambda_i}} \\
&\qquad \times \prod_{r=1}^{n}\prod_{j=1}^{m} \frac{(\alpha_r+\cdots+\alpha_n+\beta+(k_n+r-n-j-1)\gamma)_{\mu_j}}{(\alpha_r+\cdots+\alpha_n+\beta+(k_r-k_{r-1}+k_n+r-n-\ell\delta_{r,1}-j-1)\gamma)_{\mu_j}} \\
&\qquad \times \prod_{i=1}^{\ell}\prod_{j=1}^{m} \frac{(\alpha_1+\cdots+\alpha_n+\beta+(k_1+k_n-n-i-j)\gamma)_{\lambda_i+\mu_j}}{(\alpha_1+\cdots+\alpha_n+\beta+(k_1+k_n-n-i-j+1)\gamma)_{\lambda_i+\mu_j}}.
\end{aligned}
$$

In the expression on the right, ℓ and m are arbitrary integers such that $\ell \geqslant l(\lambda)$, $m \geqslant l(\mu)$, $k_0 = k_{n+1} := 0$ and the β_r are as in (1.3.3).

A number of remarks about the theorem are in order. Firstly, for $n=1$ the identity is equivalent to the AFLT integral (1.2.4) after multiplying by the evaluation of the ordinary Selberg integral (1.1.2).

Implicit in this reduction is that the domain of integration reduces to the k_1-simplex $C_\gamma^{k_1}[0,1]$. The slightly larger range for the parameters α and γ can then be inferred from the product of gamma functions in the evaluation. For the explicit evaluation of the A_n AFLT integral without the normalisation by the A_n Selberg integral, see (4.3.2). Our proof of Theorem 1.3.1 is based on summation formulas for Macdonald polynomials following techniques developed by Warnaar in [War05, War08a, War09, War10] (see also [Mac95, p. 373–376]). The advantage of our approach is that it immediately leads to a second A_n integral with two Jack polynomials generalising Warnaar's A_2 Selberg integral [War10, Theorem 3.1].

One important property of the A_n Selberg integral (1.3.2) is the rank-reduction:

$$I^{\mathrm{A}_n}_{0,k_2,\dots,k_n}(1;\alpha_1,\dots,\alpha_n,\beta;\gamma) = I^{\mathrm{A}_{n-1}}_{k_2,\dots,k_n}(1;\alpha_2,\dots,\alpha_n,\beta;\gamma), \tag{1.3.7}$$

so that up to a shift in indices the two expressions are the same. The A_n AFLT integral above does not posses this property. Indeed, if we set $k_1 = 0$ in (1.3.6) we obtain

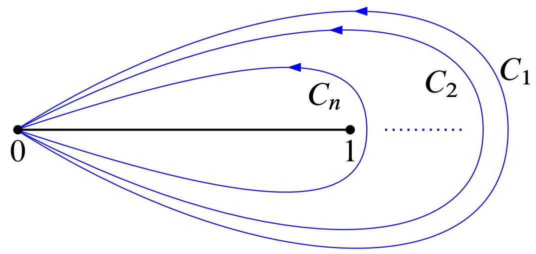
$$\begin{aligned}\Big\langle P_\lambda^{(1/\gamma)}\big[t^{(1)}\big] P_\mu^{(1/\gamma)}\big[t^{(n)} + \beta/\gamma - 1\big]\Big\rangle^{0,k_2,\dots,k_n}_{\alpha_1,\dots,\alpha_n,\beta;\gamma}& \\ = \begin{cases} \Big\langle P_\mu^{(1/\gamma)}\big[t^{(n)} + \beta/\gamma - 1\big]\Big\rangle^{k_2,\dots,k_n}_{\alpha_2,\dots,\alpha_n,\beta;\gamma} & \text{if } \lambda = 0, \\ 0 & \text{otherwise,}\end{cases}&\end{aligned}$$

where we may take either side of (1.3.6). In either case, the right-hand side is not equal to the full A_{n-1} AFLT integral. The next section contains an integral, generalising the A_n AFLT integral for $\gamma = 1$, which does possess the desired reduction property.

1.4 Complex Schur functions and the case $\gamma = 1$

In their work on the AGT conjecture for WA_n, Matsuo and Zhang [ZM] were led to consider a generalisation of the A_n AFLT integral with $n+1$ Jack polynomials in the integrand. In the appendix to their paper, the authors state a formula for such an average [ZM, Appendix C], but note that it fails some consistency checks. They were however able to conjecture a formula for $\gamma = 1$, in which case the Jack polynomials in the integrand reduce to Schur functions [ZM, Cojecture 1]. Unfortunately, as stated in their paper, this conjecture appears to be incorrect. We have managed to prove a corrected Matsuo–Zhang-type AFLT integral which contains a product of $n+1$ Schur functions in the integrand, which we now describe.

Since for reasons of convergence the A_n Selberg integral requires $\mathrm{Re}(\gamma) < 1$, we must appropriately deform the real domain of integration to an appropriate complex domain. To this end, for $1 \leqslant r \leqslant n$, let C_r denote a positively oriented simple closed curve that passes through the origin, contains $(0,1]$ in its interior, and has nonzero slope near the origin. We also require that the interior of C_r is contained in the interior of C_{r-1} for $2 \leqslant r \leqslant n$. Then a family of such contours may be visualised as



We replace the real domain $C_\gamma^{k_1,\dots,k_n}[0,1]$ by the complex contour

$$C^{k_1,\dots,k_n} := C_1^{k_1} \times \cdots \times C_n^{k_n}, \quad \text{where} \quad C_r^{k_r} = \underbrace{C_r \times \cdots \times C_r}_{k_r \text{ times}}. \tag{1.4.1}$$

For $\gamma = 1$ we now redefine the A_n Selberg average as follows. In the complex $t_i^{(r)}$-plane fix the usual principal branch of the complex logarithm, with cut along the negative real axis and argument in $(-\pi, \pi]$. Then for $0 \leqslant k_1 \leqslant \cdots \leqslant k_n$ and $\alpha_1, \dots, \alpha_n, \beta \in \mathbb{C}$ such that

$$\operatorname{Re}(\alpha_r + \cdots + \alpha_s) > s - r \quad \text{for } 1 \leqslant r \leqslant s \leqslant n \tag{1.4.2}$$

we define

$$\langle \mathcal{O} \rangle_{\alpha_1,\dots,\alpha_n,\beta}^{k_1,\dots,k_n} := \frac{I_{k_1,\dots,k_n}^{\mathrm{A}_n}(\mathcal{O}; \alpha_1, \dots, \alpha_n, \beta)}{I_{k_1,\dots,k_n}^{\mathrm{A}_n}(1; \alpha_1, \dots, \alpha_n, \beta)},$$

where, assuming (1.3.3),

$$\begin{aligned} &I_{k_1,\dots,k_n}^{\mathrm{A}_n}(\mathcal{O}; \alpha_1, \dots, \alpha_n, \beta) \\ &\quad := \frac{1}{(2\pi\mathrm{i})^{k_1+\cdots+k_n}} \int\limits_{C^{k_1,\dots,k_n}} \mathcal{O}\big(t^{(1)}, \dots, t^{(n)}\big) \prod_{r=1}^{n} \prod_{i=1}^{k_r} \big(t_i^{(r)}\big)^{\alpha_r - 1} \big(t_i^{(r)} - 1\big)^{\beta_r - 1} \\ &\qquad\qquad \times \prod_{r=1}^{n} \Delta^2\big(t^{(r)}\big) \prod_{r=1}^{n-1} \Delta^{-1}\big(t^{(r)}, t^{(r+1)}\big)\, \mathrm{d}t^{(1)} \cdots \mathrm{d}t^{(n)}. \end{aligned}$$

The integral should be understood in the sense of improper integrals since the integrand is not defined at $t_i^{(r)} = 0$, which lies on C_r. Due to the change in contour, the normalisation is now given by (see Section 5.2)

$$\begin{aligned} &I_{k_1,\dots,k_n}^{\mathrm{A}_n}(1; \alpha_1, \dots, \alpha_n, \beta) \\ &\quad = \prod_{r=1}^{n} \bigg((-1)^{\binom{k_r}{2}} \prod_{i=1}^{k_r} \frac{i!}{\Gamma(k_{r+1} - \beta_r + 2 - i)} \bigg) \\ &\qquad \times \prod_{1 \leqslant r \leqslant s \leqslant n} \prod_{i=1}^{k_r - k_{r-1}} \frac{\Gamma(\alpha_r + \cdots + \alpha_s + r - s + i - 1)}{\Gamma(\alpha_r + \cdots + \alpha_s + \beta_s + k_s - k_{s+1} + r - s + i - 2)}, \end{aligned}$$

where $k_0 = k_{n+1} := 0$.

Let $t^{(0)} := 0$. Then our next main result is a closed form evaluation of

$$\bigg\langle \bigg(\prod_{r=1}^{n} s_{\lambda^{(r)}}\big[t^{(r)} - t^{(r-1)}\big] \bigg) s_{\lambda^{(n+1)}}\big[t^{(n)} + \beta - 1\big] \bigg\rangle_{\alpha_1,\dots,\alpha_n,\beta}^{k_1,\dots,k_n}, \tag{1.4.3}$$

generalising the $\gamma = 1$ case of (1.3.6). The most concise way to state this is by using the duality (2.5.8)

$$s_{(\lambda^{(n+1)})'}\big[t^{(n)} + \beta - 1\big] = (-1)^{|\lambda^{(n+1)}|} s_{\lambda^{(n+1)}}\big[1 - \beta - t^{(n)}\big],$$

and to instead give the evaluation of

$$\bigg\langle \prod_{r=1}^{n+1} s_{\lambda^{(r)}}\big[t^{(r)} - t^{(r-1)}\big] \bigg\rangle^{k_1,\dots,k_n}_{\alpha_1,\dots,\alpha_n,\beta},$$

where $t^{(n+1)} := 1 - \beta$.[3] Before stating this evaluation we introduce the following shorthand notation. For $1 \leqslant r \leqslant n+1$ let

$$A_r := \alpha_r + \cdots + \alpha_n + k_r - k_{r-1} + r \tag{1.4.4a}$$

and $A_{r,s} := A_r - A_s$, so that $A_{r,s} = -A_{s,r}$. In particular,

$$A_{r,s} = \alpha_r + \cdots + \alpha_{s-1} + k_r - k_{r-1} - k_s + k_{s-1} + r - s \tag{1.4.4b}$$

for $1 \leqslant r \leqslant s \leqslant n+1$.

Theorem 1.4.1. *For n a positive integer, let $0 \leqslant k_1 \leqslant \cdots \leqslant k_n$ be integers, $\alpha_1, \dots, \alpha_n, \beta \in \mathbb{C}$ such that* (1.4.2) *holds, and $\lambda^{(1)}, \dots, \lambda^{(n+1)} \in \mathscr{P}$. Let $t^{(0)} := 0$ and $t^{(n+1)} := 1 - \beta$. Then*

$$\begin{aligned}
&\bigg\langle \prod_{r=1}^{n+1} s_{\lambda^{(r)}}\big[t^{(r)} - t^{(r-1)}\big] \bigg\rangle^{k_1,\dots,k_n}_{\alpha_1,\dots,\alpha_n,\beta} \\
&\quad = \prod_{r=1}^{n+1} \prod_{1\leqslant i<j\leqslant \ell_r} \frac{\lambda_i^{(r)} - \lambda_j^{(r)} + j - i}{j-i} \prod_{r,s=1}^{n+1} \prod_{i=1}^{\ell_r} \frac{(A_{r,s} - k_{s-1} + k_s - i + 1)_{\lambda_i^{(r)}}}{(A_{r,s} + \ell_s - i + 1)_{\lambda_i^{(r)}}} \\
&\quad\quad \times \prod_{1\leqslant r<s\leqslant n+1} \prod_{i=1}^{\ell_r} \prod_{j=1}^{\ell_s} \frac{\lambda_i^{(r)} - \lambda_j^{(s)} + A_{r,s} + j - i}{A_{r,s} + j - i},
\end{aligned} \tag{1.4.5}$$

where $k_0 := 0$ and $k_{n+1} := 1 - \beta$, and where ℓ_r $(1 \leqslant r \leqslant n+1)$ is an arbitrary nonnegative integer such that $\ell_r \geqslant l(\lambda^{(r)})$.

The reader is warned that in order to obtain the above compact form for the right-hand side we have used a different convention for k_{n+1} than in the previous two theorems. We also remark that (1.4.5) displays a significant amount of nontrivial cancellation. For $s = r$ the second triple product on the right becomes

$$\prod_{r=1}^{n+1} \prod_{i=1}^{\ell_r} \frac{(k_r - k_{r-1} - i + 1)_{\lambda_i^{(r)}}}{(\ell_r - i + 1)_{\lambda_i^{(r)}}}.$$

Since $\ell_r \geqslant l(\lambda^{(r)})$, this shows that the right-hand side vanishes unless $l(\lambda^{(r)}) \leqslant k_r - k_{r-1}$ for all $1 \leqslant r \leqslant n$. The integrand, however, only vanishes for $l(\lambda^{(1)}) \leqslant k_1 - k_0 = k_1$. Finally we note that (1.4.5) has the desired rank-reduction property. If we denote either side of (1.4.5) by

$$I^{k_1,\dots,k_n}_{\lambda^{(1)},\dots,\lambda^{(n+1)}}(\alpha_1, \dots, \alpha_n, \beta),$$

[3] For the evaluation of the average (1.4.3) see equation (7.1.1) below.

then it is readily verified that

$$I^{0,k_2,\dots,k_n}_{0,\lambda^{(2)},\dots,\lambda^{(n+1)}}(\alpha_1,\alpha_2,\dots,\alpha_n,\beta) = I^{k_2,\dots,k_n}_{\lambda^{(2)},\dots,\lambda^{(n+1)}}(\alpha_2,\dots,\alpha_n,\beta).$$

The evaluation of the Selberg integral (1.1.2) for $\gamma = 1$ is particularly simple, see e.g., [LT03, Ros20]. In contrast to these simple proofs, our proof of Theorem 1.4.1 is comparatively complicated. The key insight is that the first Schur function $s_{\lambda^{(1)}}[t^{(1)}]$ may be replaced by a complex Schur function $S^{(k_1)}(t^{(1)};z)$, which are indexed by sequences of complex numbers z rather than partitions. Using several novel integral formulas for such functions the Matsuo–Zhang-type AFLT integral may be evaluated recursively; see Theorem 5.2.1.

1.5 Outline

The outline for the rest of this thesis is as follows. As our approach to generalised Selberg integrals is based on symmetric function theory, Chapter 2 is devoted to an (almost) self-contained account of the theory. In particular we develop the theory of the Schur functions as an important basis for the algebra of symmetric functions. We then introduce the Macdonald polynomials, a q,t-deformation of the Schur functions, and prove many of their important properties such as the evaluation symmetry and specialisation formula.

Chapter 3 begins with some necessary preliminaries on hypergeometric notation. The bulk of the chapter is devoted to the statement and proof of a closely related pair of Cauchy-type identities associated to A_n. Along the way we prove two important identities for skew Macdonald polynomials.

In Chapter 4 we initially describe the connection between A_n Selberg integrals and solutions to the Knizhnik–Zamolodchikov equations. This motivates the existence of A_n Selberg integrals. In the rest of the chapter we prove two closely related A_n Selberg integrals with two Jack polynomials in the integrand. The first is the A_n AFLT integral of Theorem 1.3.1 above, and the second is an A_n analogue of Warnaar's A_2 Selberg integral.

Following our proof of the A_n AFLT integral the next chapter shifts gears to discussions of Matsuo–Zhang-type AFLT integrals. This begins with the introduction of the complex Schur functions, along with proofs of some of their most important properties. In view of these properties we give a proof of Theorem 1.4.1. We then give a recursion having both the integral (1.4.5) and its evaluation as solutions, which provides an alternative proof of the theorem for $n = 2$. The chapter concludes with a proof of a summation formula for A_n basic hypergeometric series used in the proof of the recursion relation.

To conclude we discuss various other analogues of the AFLT integral all associated with A_1. These include a q-AFLT integral, the elliptic AFLT integral, and the related AFLT integral for Macdonald polynomials.

Chapter 2

Symmetric functions and Macdonald polynomials

The purpose of this chapter is to discuss several families of symmetric functions that will form the basis of our approach to generalised Selberg integrals. These include the Schur functions, Jack polynomials and Macdonald polynomials. The chapter begins with a discussion of partitions and related concepts we will require. We also give an explanation of the frequently-used plethystic notation.

2.1 Partitions and tableaux

Throughout this thesis $\mathbb{N}$ denotes the set of nonnegative integers, and $\mathfrak{S}_n$ is the symmetric group on n letters with generators given by the adjacent transpositions s_i for $1 \leqslant i \leqslant n-1$.

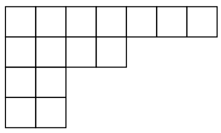
A partition is defined to be a sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ of nonnegative integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots$ and only finitely many of the λ_i are nonzero. The nonzero λ_i are called parts and the number of parts is called the length, denoted by $l(\lambda)$. Since the number of parts of a partition λ is finite the sum $|\lambda| := \lambda_1 + \lambda_2 + \lambda_3 + \cdots$ is also finite, and we sometimes call $|\lambda|$ the size of λ. If $|\lambda| = n$ for some $n \in \mathbb{N}$ then we say λ is a partition of n and write $\lambda \vdash n$. The set of all partitions is denoted by $\mathscr{P}$, and the set of all partitions with length at most n by $\mathscr{P}_n$. We ignore the number of trailing zeroes when writing down a partition, so that $(6,5,3,1,1)$ and $(6,5,3,1,1,0,0)$ are regarded as the same partition of 16 with length five. The exception to this rule is the unique partition of zero, which we denote simply by 0. Occasionally it will be useful to add the parts of two partitions. Hence we define $\lambda + \mu := (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$, which is a partition of $|\lambda| + |\mu|$ of length $\max\{l(\lambda), l(\mu)\}$. One distinguished partition is the staircase partition $\delta = (n-1, n-2, \ldots, 1)$ of length $l(\delta) = n-1$ and size $|\delta| = \binom{n}{2}$. Of course δ depends on n, however the value of n will always be implicit and so we conceal the n-dependence. If $\lambda \in \mathscr{P}_n$ and d is a nonnegative integer such that $\lambda_n \geqslant d$ we will write (λ, d) for the partition of length at most $n+1$ with d appended.

Sometimes it will be useful to extend the above notation for partitions to arbitrary infinite sequences of nonnegative integers α with finite sum. For such a sequence the length $l(\alpha)$ is defined to be the smallest integer ℓ such that $\alpha_i = 0$ for all $i > \ell$, and we use the same notation $|\alpha|$ to denote the sum of the sequence. If we let the symmetric group $\mathfrak{S}_{l(\alpha)}$ act on the indices of α in the natural way, then there is a unique partition in the orbit of this action, which we denote by α^+. For example if $\alpha = (2,4,1,0,9,1)$ then the unique partition is $(9,4,2,1,1)$, where as before we ignore trailing zeros.

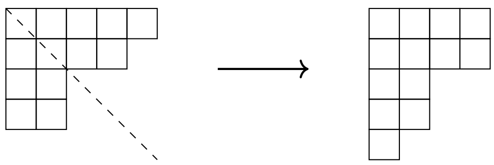
Another way of writing a partition is in terms of multiplicities. Given a partition, let m_i denote the number of occurrences of i as a part. Then we write $\lambda = (1^{m_1}2^{m_2}3^{m_3}\cdots)$. If $m_i = 0$ then we omit i and if $m_i = 1$ we suppress the superscript. Hence $(9,4,2,1,1) = (1^2 249)$. A common statistic on partitions is given by

$$n(\lambda) := \sum_{i \geqslant 1} (i-1)\lambda_i. \tag{2.1.1}$$

For example if $\lambda = (7,4,2,2)$ then $n(\lambda) = 14$. We may produce an alternative formula for $n(\lambda)$ by introducing the notion of a Young diagram. This is defined to be the array indexed by pairs of positive integers (i, j) obtained by placing λ_i squares in the ith row, left-justified, with i increasing downwards.[1] For example



is the Young diagram of the partition $(7,4,2,2)$. There is an involution on partitions which may easily be defined using the diagram. The conjugate partition λ' is obtained by reflecting the diagram of λ in the main diagonal. For example



So the conjugate of $(5,4,2,2)$ is $(4,4,2,2,1)$. Alternatively we have the more technical definition $\lambda'_i := |\{\lambda_j : \lambda_j \geqslant i\}|$. Returning to $n(\lambda)$, the definition (2.1.1) essentially assigns weight $i-1$ to row i in the Young diagram. We may represent this by populating the ith row with $i-1$ for each nonzero i. For our example this gives:

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 0&0&0&0&0&0&0\\ \hline 1&1&1&1\\ \cline{1-4} 2&2\\ \cline{1-2} 3&3\\ \cline{1-2}\end{array}$$

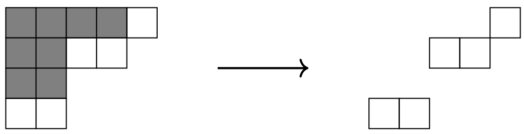
Now summing over the columns and using the formula $\sum_{i=1}^{n-1} i = \binom{n}{2}$ yields

$$n(\lambda) = \sum_{i \geqslant 1} \binom{\lambda'_i}{2}.$$

Given two partitions λ and μ we say that μ is contained in λ, written $\mu \subseteq \lambda$, if all the squares in μ are also squares in λ. This is equivalent to requiring that $\mu_i \leqslant \lambda_i$ for all $i \geqslant 1$. If $\mu \subseteq \lambda$ then we may

[1]This is the so-called English convention. Some authors favour the French convention, where i increases upwards.

form the skew diagram λ/μ by removing the boxes of μ from λ.[2] For example



represents the skew diagram $(5,4,2,2)/(4,2,2)$. Setting $\mu = 0$ tells us that $\lambda/0$ is just λ. We say that λ/μ is a horizontal strip if no two squares are in the same column, and a vertical strip if no two squares are in the same row. The example above is a horizontal strip, and its conjugate is a vertical strip. If λ/μ is a horizontal strip, then we write $\mu \prec \lambda$, so that $\mu' \prec \lambda'$ means λ/μ is a vertical strip. We will need the following identity concerning horizontal and vertical strips.

Lemma 2.1.1. *For partitions λ and ν there holds*

$$\sum_{\substack{\mu' \succ \nu' \\ \mu \prec \lambda}} (-1)^{|\mu/\nu|} = \delta_{\lambda\nu}.$$

Proof. Firstly, as the sum is over partitions μ such that $\mu' \succ \nu'$ and $\mu \prec \lambda$, both sides vanish unless $\nu \subseteq \lambda$. In the case $\lambda = \nu$ the sum contains a single term $\mu = \lambda = \nu$, so that $|\mu/\nu| = 0$ and the result follows in this case. For the rest of the proof we assume that ν is strictly contained in λ.

Since μ/ν is a vertical strip and $\mu \subseteq \lambda$, there is a μ with maximal size, say $\mu_{\max}$, satisfying these conditions. The partition $\mu_{\max}$ is obtained from ν by adding a single box to each (possibly empty) row until $\mu_{\max}$ has the same length as λ. Hence we define $\mu_{\max}$ by $(\mu_{\max})_i = \min\{\nu_i + 1, \lambda_i\}$ for $1 \leqslant i \leqslant l(\lambda)$ and zero otherwise. Similarly, since λ/μ is a horizontal strip and $\mu \supseteq \nu$, there is a smallest admissible μ, say $\mu_{\min}$. The smallest μ such that λ/μ is a horizontal strip is $(\lambda_2, \lambda_3, \ldots)$, however we also require that $\mu \supseteq \nu$ and so define $\mu_{\min}$ by $(\mu_{\min})_i = \max\{\lambda_{i+1}, \nu_i\}$. Any partition μ such that $\nu \subseteq \mu \subseteq \mu_{\max}$ (resp. $\mu_{\min} \subseteq \mu \subseteq \lambda$) has μ/ν a vertical strip (resp. λ/μ a horizontal strip). Therefore we need only consider those partitions μ for which $\mu_{\min} \subseteq \mu \subseteq \mu_{\max}$ in the sum. So if we can find an expression for

$$\sum_{\mu_{\min} \subseteq \mu \subseteq \mu_{\max}} z^{|\mu/\mu_{\min}|} \tag{2.1.2}$$

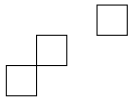
which vanishes for $z = -1$ then our proof is complete. This sum vanishes if $\mu_{\min} \not\subseteq \mu_{\max}$, which occurs if and only if $\nu_i < \lambda_{i+1}$ for any $i \geqslant 1$. Hence we may assume that $\nu_i \geqslant \lambda_{i+1}$ for each i, in other words $\nu \prec \lambda$. But by the definition of $\mu_{\min}$ this implies that $\mu_{\min} = \nu$. Now $\mu_{\max}/\nu$ has at most one box in each row by definition. If $\mu_{\max}/\nu$ has a box in row i, then since $\nu_i < \lambda_i$ this implies $\lambda_{i+1} < \lambda_i$. Then $\mu_{\max}/\nu$ cannot contain two boxes in the same column, since that would necessitate $\nu_i < \lambda_i$, $\nu_{i+1} < \lambda_{i+1}$, and $\nu_i = \nu_{i+1}$, which contradicts $\nu \prec \lambda$. From this we see that a term in the summand of (2.1.2) counts the number of ways to choose the $|\mu_{\max}/\nu|$ boxes in the skew shape, and since $\nu \prec \lambda$ with $\nu \neq \lambda$, this number is positive. It follows that

$$\sum_{\nu \subseteq \mu \subseteq \mu_{\max}} z^{|\mu/\nu|} = (1+z)^{|\mu_{\max}/\nu|},$$

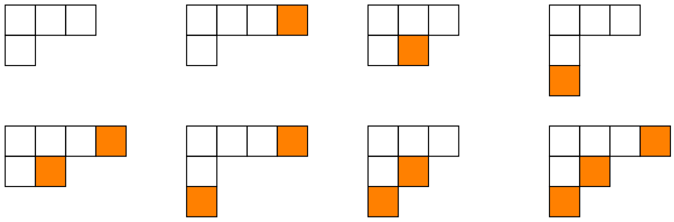
which vanishes for $z = -1$. □

[2]Note our notation differs from that of [Mac95, §1], where $\lambda - \mu$ is used for a skew shape.

Before moving on it is helpful to consider an example for the above lemma. We choose $\lambda = (5,3,1)$ and $\nu = (3,1)$. Then $\mu_{\max} = (4,2,1)$ so that $\mu_{\max}/\nu$ is the skew shape

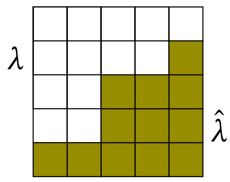


As $|\mu_{\max}/\nu| = 3$, we have $2^3 = 8$ entries in the summand, corresponding to the diagrams

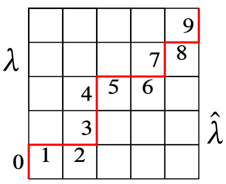


where the orange shaded squares are the boxes in μ/ν. Since the weight of a diagram is just $(-1)^{|\mu/\nu|}$, all terms cancel, and the sum vanishes.

Another way to form new partitions from old is complementation with respect to a rectangle. For positive integers n and m, let $\lambda \subseteq (m^n)$. In other words, assume that $l(\lambda) \leqslant n$ and $\lambda_1 \leqslant m$. Then the complement of λ with respect to (m^n), written $\hat{\lambda}$, is the partition for which $\hat{\lambda}_i = m - \lambda_{n-i+1}$. For example, if $\lambda = (5,4,2,2) \subseteq (5^5)$ then $\hat{\lambda} = (5,3,3,1)$:

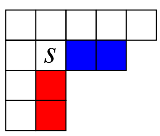


where the complement $\hat{\lambda}$ is shaded inside of (5^5). If $\lambda \subseteq (m^n)$ then the integers $\lambda_i + n - i$ for $1 \leqslant i \leqslant n$ and $n + j - 1 - \lambda'_j$ for $1 \leqslant j \leqslant m$ form a permutation of $\{0, 1, \ldots, m+n-1\}$. To see this, consider the boundary between λ and $\hat{\lambda}$ in the rectangle (m^n). Beginning in the bottom left-hand corner, we label the $n+m$ segments of this boundary with the integers $0, 1, \ldots, m+n-1$. For our previous example this is:

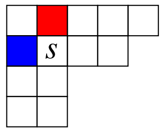


The vertical lines are labelled by the integers $\lambda_i + n - i$ while the horizontal lines are labelled by the integers $n + j - 1 - \lambda'_j$, which shows that they form a permutation of $\{0, 1, \ldots, m+n-1\}$ as desired.

Other notions we need are arm, leg, and hook lengths for partitions. Given a square s in the Young diagram of λ (written $s \in \lambda$) define the arm length $a(s)$ to be the number of squares strictly to the right of s. Similarly, define the leg length $l(s)$ as the number of squares strictly below s. As an example,



where the arm and leg of s are shaded in blue and red respectively, with $a(s) = l(s) = 2$. Further define the hook length of a square by $h(s) = a(s) + l(s) + 1$. So the hook length of s above is five. We also have the arm and leg colengths of the square s, denoted $a'(s)$ and $l'(s)$, which are the number of squares directly to the right and above s respectively. For the previous example we have



so $a'(s) = l'(s) = 1$. This leads to a third expression for $n(\lambda)$ from (2.1.1). The weighting of rows by integers $i - 1$ is the same as weighting the squares of a diagram by their leg colengths, and so

$$n(\lambda) = \sum_{s\in\lambda} l'(s).$$

We will make use of a partial order on partitions called the dominance order. For a pair of partitions λ, μ such that $|\lambda| = |\mu|$ we write $\mu \leqslant \lambda$ (read μ is less than or equal to λ in the dominance order) if for all $k \geqslant 1$,

$$\mu_1 + \cdots + \mu_k \leqslant \lambda_1 + \cdots + \lambda_k.$$

This is a total order on $\{\lambda : \lambda \vdash n\}$ for $n \leqslant 5$, but fails to be a total order for $n \geqslant 6$. For example the partitions $(3, 3)$ and $(4, 1, 1)$ are incomparable. The following lemma will prove useful, a proof of which can be found in [Mac95, p. 7].

Lemma 2.1.2. *For partitions λ, μ such that $|\lambda| = |\mu|$ we have $\lambda \geqslant \mu$ if and only if $\lambda' \leqslant \mu'$.*

Given a partition λ and its Young diagram, a tableau of shape λ is defined to be any function $T : \lambda \longrightarrow S$ from the squares of the diagram of λ to some set S. For our purposes we will always take S to be the positive integers, or some finite subset thereof. A tableau may alternatively be viewed as a filling of the squares of λ by the elements of the set S. For example,

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&2&6\\ \hline 2&2&3&4\\ \cline{1-4} 3&4&4\\ \cline{1-3} 5\\ \cline{1-1} 7\\ \cline{1-1}\end{array}$$

is a tableau of shape $\lambda = (6, 4, 3, 1, 1)$. We call a tableau a semistandard Young tableau (SSYT) if the entries in boxes weakly increase left-to-right along rows and strictly increase top-to-bottom down columns. Our above example of a tableau is semistandard. The set of all semistandard Young tableaux of a given shape is denoted $\mathrm{SSYT}(\lambda)$. Define the weight of a semistandard Young tableau T to be the sequence $\mathrm{wt}(T) = (\mathrm{wt}_1(T), \mathrm{wt}_2(T), \ldots)$ whose ith entry is the number of times i occurs in the boxes of T. For our above example we have $\mathrm{wt}(T) = (4, 3, 2, 3, 1, 1, 1)$. For a tableau T we let $T_{\geqslant j}$ denote the subtableau consisting of the columns of T with index at least j, and define $T_{<j}$, $T_{>j}$ similarly. All of these definitions are extended to skew shapes in the obvious way, and we denote the set of all semistandard Young tableau of skew shape λ/μ by $\mathrm{SSYT}(\lambda/\mu)$.

A tableau is called a Yamanouchi tableau if in its ith row all entries are equal to i. For example

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&1\\ \hline 2&2&2&2 \\ \cline{1-4} 3&3&3 \\ \cline{1-3} 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

is the Yamanouchi tableau of shape $(6, 4, 3, 1, 1)$. It is not hard to see that the Yamanouchi tableau is the unique $T \in \mathrm{SSYT}(\lambda)$ for which $\mathrm{wt}(T) = \lambda$. The following lemma contains another characterisation of the Yamanouchi tableau.

Lemma 2.1.3. *If T is a semistandard Young tableau of shape λ such that $\mathrm{wt}(T_{\geqslant j})$ is a partition for each $j \geqslant 1$, then T is Yamanouchi.*

Proof. Assume $T \in \mathrm{SSYT}(\lambda)$ is such that $\mathrm{wt}(T_{\geqslant j})$ is a partition for each j. Then $T_{\geqslant \lambda_1}$ is a column of some height h. Since $T_{\geqslant \lambda_1}$ is semistandard and $\mathrm{wt}(T_{\geqslant \lambda_1})$ is a partition we must have $\mathrm{wt}(T_{\geqslant \lambda_1}) = (1^{h_1})$. This automatically forces all entries in the ith row for $i \leqslant h$ to be equal to i. If we incrementally add columns of T to the left of the initial column $T_{\geqslant \lambda_1}$ until a box is added to row $h + 1$, then, by our assumption that the weights of these subtableaux are partitions, this first box must contain an $h + 1$. Continuing in this fashion we see that the ith row of T contains entries equal to i only, and so T is a Yamanouchi tableau. □

There is a useful family of involutions on $\mathrm{SSYT}(\lambda)$, first defined by Bender and Knuth [BK72]. Let $T \in \mathrm{SSYT}(\lambda)$ be arbitrary and fix a positive integer k. We call an entry k (resp. $k + 1$) in T free if there is no $k + 1$ (resp. k) in the same column. The free k and $k + 1$ entries in some row of T will occur between the k's and $(k + 1)$'s which are not free. Row i in T containing a_i free k's and b_i free $(k + 1)$'s will therefore locally look like:

$$\begin{array}{cccccccccc}
 & & \multicolumn{3}{c}{\overbrace{\hspace{7em}}^{a_i \text{ times}}} & & & & \multicolumn{1}{|c|}{k} & \multicolumn{1}{c|}{k} \\ \hline
\multicolumn{1}{|c|}{k} & \multicolumn{1}{c|}{k} & \multicolumn{1}{c|}{k} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{k} & \multicolumn{1}{c|}{k+1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{k+1} & \multicolumn{1}{c|}{k+1} & \multicolumn{1}{c|}{k+1} \\ \hline
\multicolumn{1}{|c|}{k+1} & \multicolumn{1}{c|}{k+1} & & & & \multicolumn{3}{c}{\underbrace{\hspace{7em}}_{b_i \text{ times}}} & & \\ \cline{1-2}
\end{array}$$

Let a_i and b_i the number of free k's and $(k + 1)$'s in row i of T respectively. Define the Bender–Knuth involution φ_k by interchanging the roles of a_i and b_i in each row, i.e., replacing the a_i entries k and b_i entries $k + 1$ with b_i entries k and a_i entries $k + 1$. The part of the tableau in the above figure thus becomes:

$$\begin{array}{cccccccccc}
 & & \multicolumn{3}{c}{\overbrace{\hspace{7em}}^{b_i \text{ times}}} & & & & \multicolumn{1}{|c|}{k} & \multicolumn{1}{c|}{k} \\ \hline
\multicolumn{1}{|c|}{k} & \multicolumn{1}{c|}{k} & \multicolumn{1}{c|}{k} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{k} & \multicolumn{1}{c|}{k+1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{k+1} & \multicolumn{1}{c|}{k+1} & \multicolumn{1}{c|}{k+1} \\ \hline
\multicolumn{1}{|c|}{k+1} & \multicolumn{1}{c|}{k+1} & & & & \multicolumn{3}{c}{\underbrace{\hspace{7em}}_{a_i \text{ times}}} & & \\ \cline{1-2}
\end{array}$$

Since we have only manipulated free entries in T, the resulting tableau $\phi_k(T)$ is still semistandard. Further it is clear from the definition that the ϕ_k are indeed involutions. The ϕ_k also have the property that $\mathrm{wt}(\phi_k(T)) = s_k \mathrm{wt}(T)$, where s_k is the adjacent transposition interchanging k and $k + 1$. To see this we need only observe that the total number of k's which are not free is equal to the number of $(k + 1)$'s which are not free. When a_i and b_i are swapped for each i we thus also swap $\mathrm{wt}_k(T)$ and $\mathrm{wt}_{k+1}(T)$.

2.2 The algebra of symmetric functions

Here we will cover the essential elements of the theory of symmetric functions. For more complete accounts see, e.g., [Mac95] and [Sta99, Chapter 7].

Let $X = \{x_1, x_2, x_3, \ldots\}$ be an infinite set of indeterminates. For a nonnegative integer k define a formal power series

$$f(X) = \sum_{\substack{\alpha \\ |\alpha|=k}} c_\alpha X^\alpha$$

where α is an infinite sequence of nonnegative integers with finite sum, $X^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$ and $c_\alpha \in \mathbb{Q}$. We say that $f(X)$ is a homogeneous symmetric function of degree k if for any permutation w of the positive integers we have $f(x_1, x_2, x_3, \ldots) = f(x_{w(1)}, x_{w(2)}, x_{w(3)}, \ldots)$. The set of all homogeneous symmetric functions of homogeneous degree k is denoted Λ^k. It is clear that if $f \in \Lambda^k$ and $g \in \Lambda^\ell$ then $fg \in \Lambda^{k+\ell}$. Hence the algebra of symmetric functions is defined as the graded $\mathbb{Q}$-algebra

$$\Lambda := \bigoplus_{k \geqslant 0} \Lambda^k,$$

so any $f \in \Lambda$ is a sum $f = \sum_{k \geqslant 0} f_k$ with $f_k \in \Lambda^k$ and only finitely many of the f_k nonzero. Later on it will be useful to adjoin further indeterminates to the coefficient field of Λ, which we define by

$$\Lambda_{\mathbb{Q}(q,t)} := \Lambda \otimes_{\mathbb{Q}} \mathbb{Q}(q,t).$$

The first important basis for Λ are the monomial symmetric functions. Let $\lambda \vdash k$, and α be a sequence of nonnegative integers with $|\alpha| = k$. Then the monomial symmetric function indexed by λ is defined to be

$$m_\lambda(X) := \sum_{\substack{\alpha \\ \alpha^+=\lambda}} X^\alpha. \tag{2.2.1}$$

The set $\{m_\lambda\}_{\lambda \vdash k}$ forms a basis for Λ^k. To see this, note that for any $f \in \Lambda^k$ by definition we may write

$$f(X) = \sum_\alpha c_\alpha X^\alpha.$$

By symmetry it follows that for any sequences α, β such that $|\alpha| = |\beta| = k$ and $\alpha^+ = \beta^+$, then $c_\alpha = c_\beta$. Hence we may express $f(X)$ as

$$f(X) = \sum_{\lambda \vdash k} c_\lambda m_\lambda(X).$$

So $\{m_\lambda\}_{\lambda \vdash k}$ is indeed a basis for Λ^k, which therefore has dimension $|\{\lambda \in \mathcal{P} : \lambda \vdash k\}|$, the number of partitions of k.

We may also define the algebra of symmetric functions in finitely many variables $X_n := \{x_1, \ldots, x_n\}$ as

$$\Lambda_n := \mathbb{Q}[x_1, \ldots, x_n]^{\mathfrak{S}_n}.$$

This is the set polynomials in n variables with rational coefficients, invariant under the natural action of $\mathfrak{S}_n$. That is, the set of $f \in \mathbb{Q}[x_1, \ldots, x_n]$ such that for any $w \in \mathfrak{S}_n$,

$$f(x_1, \ldots, x_n) = f(x_{w(1)}, \ldots, x_{w(n)}).$$

As with the case of infinitely many variables Λ_n has the structure of a graded $\mathbb{Q}$-algebra. A basis for Λ_n is given by the monomial symmetric functions on a finite alphabet. For $\lambda \in \mathscr{P}_n$ we define these by

$$m_\lambda(X_n) := \sum_{\substack{\alpha \\ \alpha^+=\lambda \\ l(\alpha)\leqslant n}} X_n^\alpha.$$

With this in mind, define a map

$$\rho_n : \Lambda \longrightarrow \Lambda_n$$

which sets $x_i = 0$ for $i > n$ and leaves the remaining indeterminates unchanged. The map ρ_n is a surjective homomorphism of $\mathbb{Q}$-algebras. It follows from (2.2.1) that

$$\rho_n\big(m_\lambda(X)\big) = \begin{cases} m_\lambda(X_n) & \text{if } l(\lambda) \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that $m_\lambda(X_n) = 0$ if $l(\lambda) > n$. Further, for a pair of positive integers $m \geqslant n$ define

$$\rho_{n,m} : \Lambda_m \longrightarrow \Lambda_n,$$

which sets $x_i = 0$ for $n + 1 \leqslant i \leqslant m$ and leaves the other indeterminates unchanged. For $n = m$ this map is the identity and for a triple of nonnegative integers $\ell \geqslant m \geqslant n$ the following diagram commutes:

$$\begin{tikzcd} \Lambda_\ell \arrow[r, "\rho_{m,\ell}"] \arrow[dr, "\rho_{n,\ell}"'] & \Lambda_m \arrow[d, "\rho_{n,m}"] \\ & \Lambda_n \end{tikzcd}$$

Note that the above diagram is still commutative if $\rho_{m,\ell}$ and $\rho_{n,\ell}$ are replaced by ρ_m and ρ_n respectively. What this is saying is that we may pass from Λ to Λ_n for any positive integer n, and it is for this reason that we primarily work with Λ in what follows.

We will now meet three more families of symmetric functions, each of which form algebraic and linear bases for Λ (here always taken over $\mathbb{Q}$). First are the elementary symmetric functions, defined for r a nonnegative integer as

$$e_r(X) := \sum_{1\leqslant i_1<\cdots<i_r} x_{i_1} \ldots x_{i_r} = m_{(1^r)}(X), \tag{2.2.2}$$

which is the sum of all square-free monomials of degree r in the variables X. For r a negative integer we define $e_r := 0$. The generating function is given by

$$\lambda_z(X) := \sum_{r=0}^{\infty} z^r e_r(X) = \prod_{i=1}^{\infty}(1 + zx_i).$$

The e_r for all $r \geqslant 1$ are algebraically independent over $\mathbb{Q}$ and generate Λ as a $\mathbb{Q}$-algebra [Mac95, p. 20]. We define the elementary symmetric function e_α indexed by a sequence α by

$$e_\alpha := e_{\alpha_1} e_{\alpha_2} e_{\alpha_3} \cdots ,$$

where we suppress the alphabets when these are irrelevant.

Next are the complete symmetric functions, defined for r a nonnegative integer as

$$h_r(X) := \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_r} x_{i_1} \ldots x_{i_r} = \sum_{\lambda \vdash r} m_\lambda(X),$$

which is the sum of all monomials of homogeneous degree r. Again we define $h_r := 0$ for r a negative integer. Their generating function is

$$\sigma_z(X) := \sum_{r=0}^{\infty} z^r h_r(X) = \prod_{i=1}^{\infty} \frac{1}{1 - z x_i}. \tag{2.2.3}$$

Like the elementary symmetric functions, the h_r are algebraically independent over $\mathbb{Q}$ and generate Λ as a $\mathbb{Q}$-algebra. As for the elementary symmetric functions we define the h_α by

$$h_\alpha := h_{\alpha_1} h_{\alpha_2} h_{\alpha_3} \cdots .$$

The next important class of symmetric functions are the power sums. These are simply defined as

$$p_r(X) := \sum_{i \geqslant 1} x_i^r = m_{(r)}(X).$$

Sometimes it will be convenient to write this in set notation as

$$p_r(X) = \sum_{x \in X} x^r.$$

The p_r are algebraically independent over $\mathbb{Q}$ and generate Λ as a $\mathbb{Q}$-algebra,

$$\Lambda \cong \mathbb{Q}[p_1, p_2, p_3, \ldots].$$

The power sums admit the generating function

$$\psi_z(X) := \sum_{r=1}^{\infty} \frac{z^r p_r(X)}{r} = \log\bigl(\sigma_z(X)\bigr) = -\log\bigl(\lambda_{-z}(X)\bigr). \tag{2.2.4}$$

Finally, we extend the p_r to sequences α as before, so that

$$p_\alpha := p_{\alpha_1} p_{\alpha_2} p_{\alpha_3} \cdots .$$

2.3 Plethystic notation

When dealing with symmetric functions it is often important to specialise or otherwise manipulate the alphabet of indeterminates. Plethystic or λ-ring notation allows for alphabets and operations on alphabets to be expressed efficiently. The use of plethystic notation simplifies much of the content that is to come, and the interested reader may consult [Hag08, Las03, RWa] for further details.

We have already noted in the previous section that the power sums p_r are algebraically independent over $\mathbb{Q}$ and generate Λ as a $\mathbb{Q}$-algebra. Let A be an algebra over a field K containing $\mathbb{Q}$. Then by assigning the power sums p_r values in A we determine a K-algebra homomorphism $\Lambda \longrightarrow A$. For example we may think of the homomorphism ρ_n as being the map on the power sums taking $p_r(X)$ to $p_r(X_n)$ for each $r \geqslant 1$. So describing how an operation on alphabets acts on the power sums is sufficient to extend this action to any symmetric function. To indicate when plethystic notation is being employed we use square parentheses instead of the usual round parentheses, so the image of some $f \in \Lambda$ under the map ρ_n above is denoted $f[X_n]$.

The first operation we would like to describe is the sum of two arbitrary alphabets. On the power sums we define

$$p_r[X + Y] := p_r[X] + p_r[Y]. \tag{2.3.1}$$

By expanding a symmetric function f on the p_r we can make sense of $f[X + Y]$ for any $f \in \Lambda$. If we can add alphabets we would also like to subtract alphabets, and so we also define the difference of alphabets by

$$p_r[X - Y] := p_r[X] - p_r[Y]. \tag{2.3.2}$$

Therefore for any pair of alphabets

$$p_r[(X + Y) - Y] = p_r[X] + p_r[Y] - p_r[Y] = p_r[X],$$

as it should. The notation (2.3.1) implies that any countable alphabet X may be written as the sum of its individual letters. So if $X = x_1 + x_2 + x_3 + \cdots$, then

$$f[X] = f[x_1 + x_2 + x_3 + \cdots] = f(x_1, x_2, x_3, \ldots).$$

Note that if X and Y are both countable alphabets, then $X + Y$ is nothing more than the (disjoint) union of X and Y as sets. The product of alphabets, again defined on the power sums, is

$$p_r[XY] := p_r[X]p_r[Y]. \tag{2.3.3}$$

If we assume that X and Y are countable then the product XY may be interpreted as the Cartesian products of X and Y as sets. For a single-letter alphabet x we have that

$$p_r[xY] = x^r p_r[Y].$$

However for any positive integer n the summation rule (2.3.1) forces

$$p_r[nX] = p_r[\underbrace{X + \cdots + X}_{n \text{ times}}] = np_r[X].$$

This is extended to any $z \in \mathbb{C}$ by

$$p_r[zX] := zp_r[X],$$

Here z is not a letter of an alphabet, but rather what we refer to as a binomial element. This terminology is justified since for z a binomial element,

$$e_r[z] = \binom{z}{r} \quad \text{and} \quad h_r[z] = \binom{z+r-1}{r}.$$

To deal with potential ambiguities arising between binomial elements and single-letter alphabets we will always point out when one should interpret an isolated symbol inside plethystic brackets as the former or the latter. From what we have developed so far

$$p_r[0 - X] = p_r[-X] = -p_r[X],$$

which is not the result of replacing each letter of X by its negative. Of course the latter is still a valid plethystic substitution, and so to distinguish between the two we write ε for the alphabet $\{-1\}$ inside of plethystic brackets. This implies that

$$p_r[\varepsilon X] = (-1)^r p_r[X].$$

We will also use plethystic notation to manipulate formal series involving symmetric functions such as the generating function for the complete symmetric functions (2.2.3). By (2.2.4) we have that

$$\sigma_z(X) = \exp\left(\sum_{r\geqslant 1} \frac{z^r p_r(X)}{r}\right).$$

Therefore by (2.3.1)

$$\sigma_z[X+Y] = \sigma_z[X]\sigma_z[Y], \tag{2.3.4a}$$

and similarly by (2.3.2)

$$\sigma_z[X-Y] = \frac{\sigma_z[X]}{\sigma_z[Y]}. \tag{2.3.4b}$$

By extracting the coefficient of z^r on both sides of (2.3.4a) we obtain the convolution formula

$$h_r[X+Y] = \sum_{i=0}^{r} h_i[X]h_{r-i}[Y]. \tag{2.3.5}$$

Using

$$\sigma_z[-X] = \frac{1}{\sigma_z[X]} = \lambda_{-z}[X]$$

and again extracting coefficients we have the reciprocity[3]

$$h_r[-X] = (-1)^r e_r[X], \tag{2.3.6}$$

which generalises the reciprocity for binomial coefficients:

$$\binom{-z}{r} = (-1)^r \binom{z+r-1}{r}.$$

[3] We should note that it is customary when dealing with symmetric functions to introduce an involution $\omega : \Lambda \longrightarrow \Lambda$ defined by $\omega(e_r) = h_r$. However, by (2.3.6), this is equivalent to the plethystic substitution $X \mapsto -\varepsilon X$, which allows for the action of ω to be readily extended to all $f \in \Lambda$. For this reason we will state any identities which ordinarily make use of ω using plethystic notation.

Using (2.3.6) we have the equivalent convolution formula

$$e_r[X+Y]=\sum_{i=0}^{r} e_i[X]e_{r-i}[Y]. \tag{2.3.7}$$

The next lemma contains a generalisation of this convolution formula.

Lemma 2.3.1. *For n a positive integer, let $Y^{(1)},\dots,Y^{(n)}$ be alphabets such that $Y^{(1)}+\cdots+Y^{(n)}=0$. Then for k a nonnegative integer and any m such that $1\leqslant m\leqslant n$ there holds*

$$h_k\big[Y^{(m)}\big]=(-1)^k\sum_{\substack{i_1,\dots,i_n\geqslant 0\\ i_1+\cdots+i_n=k\\ i_m=0}}\prod_{r=1}^{n} e_{i_r}\big[Y^{(r)}\big].$$

Proof. The identity of the lemma is nothing but a plethystically substituted version of

$$e_k\big[X^{(1)}+\cdots+X^{(n)}\big]=\sum_{\substack{i_1,\dots,i_n\\ i_1+\cdots+i_n=k}}\prod_{r=1}^{n} e_{i_r}\big[X^{(r)}\big],$$

which itself follows by iterating the convolution formula (2.3.7). Indeed, setting $X^{(m)}=0$ for some $1\leqslant m\leqslant n$, replacing $X^{(r)}\mapsto Y^{(r)}$ for all $r\neq m$ and defining $Y^{(m)}:=-\sum_{r\neq m}Y^{(r)}$, the claim follows by (2.3.6) together with the identities

$$e_{i_m}[0]=\delta_{i_m,0}\quad\text{and}\quad e_0\big[Y^{(m)}\big]=1. \qquad\square$$

If we replace $Y\mapsto -X$ in (2.3.5) then we obtain the identity

$$\sum_{i=0}^{r}(-1)^i h_i e_{r-i}=\delta_{r,0}. \tag{2.3.8}$$

Setting $r=i-j$ for $i\geqslant j$, then this is equivalent to the infinite matrices

$$H:=\big(h_{i-j}\big)_{i,j\in\mathbb{N}}\quad\text{and}\quad E:=\big((-1)^{i-j}e_{i-j}\big)_{i,j\in\mathbb{N}}$$

being inverses of one another. Note that both H and E are lower-triangular because $h_r=e_r=0$ for negative integers r, so that we need only consider $i\geqslant j$ when applying (2.3.8). Now fix a nonnegative integer n and let $\lambda,\mu\in\mathscr{P}_k$ be such that $\lambda_1,\mu_1\leqslant\ell$ and $k+\ell=n+1$. The matrices

$$H_n:=\big(h_{i-j}\big)_{0\leqslant i,j\leqslant n}\quad\text{and}\quad E_n:=\big((-1)^{i-j}e_{i-j}\big)_{0\leqslant i,j\leqslant n}$$

are mutually inverse, and so any minor of H_n is equal to the complementary cofactor in E_n^t. If we take the minor with row indices λ_i+k-i and column indices μ_j+k-j, then its complementary cofactor has row indices $k+i-1-\mu'_i$ and column indices $k+j-1-\lambda'_j$ since the integers λ_i+k-i and $k+i-1-\lambda'_i$ form a permutation of $k+\ell-1$ and similarly for μ. Together, this implies that we have the identity

$$\det_{1\leqslant i,j\leqslant k}\big(h_{\lambda_i-\mu_j+j-i}\big)=\det_{1\leqslant i,j\leqslant \ell}\big(e_{\lambda'_i-\mu'_j+j-i}\big). \tag{2.3.9}$$

One frequently occurring alphabet is $1 + q + q^2 + \cdots$ for a parameter q. Since the product of this alphabet with $(1-q)$ yields 1, we use the usual

$$\frac{1}{1-q} := 1 + q + q^2 + q^3 + \cdots \tag{2.3.10}$$

inside plethystic brackets to denote the former. For a an indeterminate define the infinite q-shifted factorial

$$(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots, \tag{2.3.11}$$

where, when viewed analytically, we require $|q| < 1$.

Lemma 2.3.2. *For a and b indeterminates there holds*

$$\sigma_z\left[\frac{a-b}{1-q}\right] = \frac{(az;q)_\infty}{(bz;q)_\infty}.$$

Proof. Using (2.3.10) we first write out the definition of the alphabet $1/(1-q)$ to obtain

$$\sigma_z\left[\frac{a-b}{1-q}\right] = \sigma_z\left[\sum_{i\geqslant 0}(a-b)q^i\right].$$

Using the sum and difference rules (2.3.4) the lemma now follows

$$\begin{aligned}
\sigma_z\left[\sum_{i\geqslant 0}(a-b)q^i\right] &= \prod_{i\geqslant 0}\sigma_z[(a-b)q^i] \\
&= \prod_{i\geqslant 0}\frac{\sigma_z[aq^i]}{\sigma_z[bq^i]} \\
&= \prod_{i\geqslant 0}\frac{1-azq^i}{1-bzq^i} \\
&= \frac{(az;q)_\infty}{(bz;q)_\infty}.
\end{aligned}$$

□

2.4 The Hall scalar product

In this section we introduce the Hall scalar product on Λ, and discuss some of its properties. This may be defined by demanding that

$$\langle p_\lambda, p_\mu\rangle = z_\lambda \delta_{\lambda\mu}, \tag{2.4.1}$$

where $z_\lambda := \prod_{i\geqslant 1} m_i(\lambda)!\, i^{m_i(\lambda)}$. The quantity z_λ may be interpreted as the size of the centraliser of any element of cycle type λ in the symmetric group [Sag01, p. 3]. Since the power sums form a basis for Λ, we may express any $f \in \Lambda$ as $f = \sum_\lambda c_\lambda p_\lambda$ for some $c_\lambda \in \mathbb{Q}$. Then

$$\langle f, f\rangle = \left\langle \sum_\lambda c_\lambda p_\lambda, \sum_\mu c_\mu p_\mu \right\rangle = \sum_\lambda c_\lambda^2 z_\lambda,$$

so that the Hall scalar product is positive-definite. Further, it is immediate from the definition (2.4.1) that the product is symmetric.

As a next step we will show that the power sums satisfy a Cauchy-type identity. To do so we will use the following lemma.

Lemma 2.4.1. *The following identities hold:*

$$\sigma_1(X) = \sum_\lambda \frac{p_\lambda(X)}{z_\lambda}, \tag{2.4.2a}$$

$$\lambda_1(X) = \sum_\lambda \frac{(-1)^{|\lambda|-l(\lambda)} p_\lambda(X)}{z_\lambda}. \tag{2.4.2b}$$

Proof. It suffices to prove (2.4.2a) since by the plethystic substitution $X \mapsto -\varepsilon X$ we have

$$\lambda_1(X) = \sigma_1[-\varepsilon X] = \sum_\lambda \frac{p_\lambda[-\varepsilon X]}{z_\lambda} = \sum_\lambda \frac{(-1)^{|\lambda|-l(\lambda)} p_\lambda(X)}{z_\lambda}.$$

Beginning with (2.2.4) and using the power series expansion for the exponential function gives

$$\sigma_1(X) = \exp\left(\sum_{r=1}^{\infty} \frac{p_r(X)}{r}\right) = \prod_{r=1}^{\infty} \exp\left(\frac{p_r(X)}{r}\right) = \prod_{r=1}^{\infty} \sum_{m_r=0}^{\infty} \frac{(p_r(X))^{m_r}}{m_r!\, r^{m_r}}.$$

For fixed r we may think of m_r as playing the role of the multiplicity of r. In the expansion of the sum each term will contain a finite list of multiplicities, one for each r, which gives a unique partition. Hence

$$\sigma_1(X) = \sum_\lambda \frac{p_\lambda(X)}{z_\lambda},$$

and the proof is complete. □

With this established we may thus claim a Cauchy identity for the power sums, which follows from (2.4.2a) under the plethystic substitution $X \mapsto XY$ and (2.3.3). For a proof that avoids plethystic notation see [Sta99, Prop. 7.7.4].

Proposition 2.4.2. *For X and Y arbitrary alphabets there holds*

$$\sum_\lambda \frac{p_\lambda(X) p_\lambda(Y)}{z_\lambda} = \sigma_1[XY]. \tag{2.4.3}$$

The orthogonality of the power sums (2.4.1) and the summation formula (2.4.3) are in fact equivalent statements. The following proposition extends this equivalence to any pair of bases for Λ which form a biorthogonal family.

Proposition 2.4.3. *Let $\{u_\lambda\}$ and $\{v_\lambda\}$ be two homogeneous bases for Λ. Then the following statements are equivalent:*

1. *For each pair of partitions λ, μ we have*

$$\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}.$$

2. *For any pair of alphabets X and Y there holds*

$$\sum_\lambda u_\lambda(X) v_\lambda(Y) = \sigma_1[XY]. \tag{2.4.4}$$

Proof. Let $\tilde{p}_\lambda := p_\lambda/z_\lambda$. Then $\{\tilde{p}_\lambda\}$ forms a basis for Λ which is dual to the basis $\{p_\lambda\}$. For arbitrary u_λ and v_μ we may expand these as

$$u_\lambda = \sum_\nu c_{\lambda,\nu} p_\nu \quad \text{and} \quad v_\mu = \sum_\omega d_{\mu,\omega} \tilde{p}_\omega,$$

for some $c_{\lambda,\nu}, d_{\mu,\omega} \in \mathbb{Q}$. Assuming that $\{u_\lambda\}$ and $\{v_\lambda\}$ are dual bases implies that

$$\sum_\nu c_{\lambda,\nu} d_{\mu,\nu} = \delta_{\lambda\mu},$$

by definition of the Hall scalar product (2.4.1). Let C and D be matrices indexed by $\mathscr{P}$ with respect to some total ordering with $C_{\lambda\nu} = c_{\lambda,\nu}$ and $D_{\mu,\nu} = d_{\mu,\nu}$ respectively. Then the above summation is equivalent to $CD^t = I$ where I is the (infinite) identity matrix and M^t denotes the transpose of the matrix M. Since $I^t = I$ this implies that $C^t D = I$, or,

$$\sum_\lambda c_{\lambda,\nu} d_{\lambda,\omega} = \delta_{\nu\omega}.$$

Now assume Cauchy-type identity (2.4.4). Taking the same expansions for u_λ and v_λ as before we have

$$\begin{aligned}\sum_\lambda u_\lambda(X) v_\lambda(Y) &= \sum_\lambda \Big(\sum_\nu c_{\lambda,\nu} p_\nu(X)\Big)\Big(\sum_\omega d_{\lambda,\omega} \tilde{p}_\omega(Y)\Big)\\ &= \sum_{\nu,\omega} \Big(\sum_\lambda c_{\lambda,\nu} d_{\lambda,\omega}\Big) p_\nu(X) \tilde{p}_\omega(Y).\end{aligned}$$

By Proposition 2.4.2 we also have that

$$\sum_\lambda u_\lambda(X) v_\lambda(Y) = \sum_\lambda p_\lambda(X) \tilde{p}_\lambda(Y).$$

For these two expressions for $\sum_\lambda u_\lambda(X) v_\lambda(Y)$ to be equal we must have that

$$\sum_\lambda c_{\lambda,\nu} d_{\lambda,\omega} = \delta_{\nu\omega}.$$

We have shown that both statements in the theorem are equivalent to the same summation for the coefficients $c_{\lambda,\nu}$ and $d_{\mu,\omega}$. Hence the two statements themselves are equivalent, and the proof is complete. □

Equipped with the above proposition it is not hard to show that the bases $\{h_\lambda\}$ and $\{m_\lambda\}$ are orthonormal under the Hall scalar product by proving the Cauchy-type identity

$$\sum_\lambda h_\lambda(X) m_\lambda(Y) = \sigma_1[XY].$$

Assuming that Y is a countable alphabet, we may write the product XY as

$$XY = \sum_{y \in Y} Xy.$$

Hence using (2.3.4a) we have

$$\sigma_1[XY] = \prod_{y \in Y} \sigma[Xy] = \prod_{y \in Y} \sum_{r=0}^{\infty} h_r[Xy] = \prod_{y \in Y} \sum_{r=0}^{\infty} h_r[X] y^r,$$

where the last equality follows by homogeneity of the complete symmetric functions. In the expansion of the product each monomial will be of the form Y^α for some sequence α with length at most the cardinality of Y. The coefficient of such a monomial will be

$$h_{\alpha_1}(X) h_{\alpha_2}(X) h_{\alpha_3}(X) \cdots ,$$

and so

$$\sigma_1[XY] = \sum_{\alpha} h_\alpha(X) Y^\alpha \tag{2.4.5}$$

For each α such that $\alpha^+ = \lambda$ for some fixed partition λ, the coefficient of Y^α will be equal to $h_\lambda(X)$. Therefore we may rewrite the above as

$$\sigma_1[XY] = \sum_{\lambda} h_\lambda(X) m_\lambda(Y).$$

By Proposition 2.4.3 we have established that

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}.$$

2.5 Schur functions

We now turn our attention to an important linear basis for Λ given by the Schur functions.[4] As we will see, the Schur functions have many interesting combinatorial properties on their own, and in relation to other families of symmetric functions. They also play an important role in representation theory as the characters of the symmetric and general linear groups, see e.g., [Sag01]. There are many ways to define the Schur functions, and in the course of this section we will see three equivalent expressions, which may all act as definitions. Our first formula for the Schur function, and the one which we will use to derive the other expressions, is the following combinatorial definition in terms of semistandard Young tableaux. For any $\lambda \in \mathscr{P}$ the Schur functions s_λ is defined as

$$s_\lambda(X) := \sum_{T \in \mathrm{SSYT}(\lambda)} X^{\mathrm{wt}(T)}. \tag{2.5.1}$$

It is immediate that s_λ is homogeneous of degree $|\lambda|$. When the alphabet X is finite of cardinality n the entries of the tableaux are taken from $\{1, \ldots, n\}$. Since the sum is over semistandard Young tableaux, $s_\lambda(X_n)$ vanishes if $l(\lambda) > n$ as the first column consists of $l(\lambda) > n$ boxes so column strictness can never be satisfied in this case. It is not at all clear from (2.5.1) that the Schur function is indeed symmetric as claimed. A proof of this fact is the content of the following proposition.

[4] These are the classical Schur functions, which differ from the complex Schur functions discussed in Chapter 5.

Proposition 2.5.1. *For any partition λ the Schur function s_λ is a symmetric function.*

Proof. We will prove that s_λ is invariant under the adjacent transposition s_k for any k, since any permutation of $\mathbb{N}$ may be written as a finite sequence of such transpositions.

Let $\mathrm{SSYT}(\lambda;\alpha)$ denote the set of semistandard Young tableaux with shape λ and weight α. We need to show that the coefficient of X^α in s_λ is equal to the coefficient of $X^{s_k\alpha}$. However, we have a bijection $\phi_k : \mathrm{SSYT}(\lambda;\alpha) \longrightarrow \mathrm{SSYT}(\lambda;s_k\alpha)$ given by the Bender–Knuth involution. Therefore $|\mathrm{SSYT}(\lambda;\alpha)| = |\mathrm{SSYT}(\lambda;s_k\alpha)|$ for any k, and so $s_\lambda(X)$ is symmetric. $\square$

Let $K_{\lambda\alpha} := |\mathrm{SSYT}(\lambda;\alpha)|$. These integers are known as the Kostka numbers, and may be used to express the Schur functions as

$$s_\lambda(X) = \sum_\alpha K_{\lambda\alpha} X^\alpha.$$

Since s_λ is symmetric, we have that $K_{\lambda\alpha} = K_{\lambda\alpha^+}$, where α^+ is the unique partition obtained by ordering the parts of α. Appealing to the definition of the monomial symmetric functions (2.2.1) we thus have

$$s_\lambda(X) = \sum_\mu K_{\lambda\mu} m_\mu(X). \tag{2.5.2}$$

The Schur functions form a basis for Λ. To see this, we will need the following lemma regarding the Kostka numbers.

Lemma 2.5.2. *If $\lambda, \mu \in \mathcal{P}$ are such that $|\lambda| = |\mu|$ and $K_{\lambda\mu} \neq 0$, then $\mu \leqslant \lambda$. Furthermore $K_{\lambda\lambda} = 1$.*

Proof. Assume that $K_{\lambda\mu} \neq 0$. Then there exists some semistandard Young tableau of shape λ and weight μ. By column strictness we can have no boxes containing an integer k below row k. Therefore

$$\mu_1 + \cdots + \mu_k \leqslant \lambda_1 + \cdots + \lambda_k$$

for all $k \geqslant 1$, and so $\mu \leqslant \lambda$. We also know that there is a unique semistandard tableau of shape and weight λ, the Yamanouchi tableau, and so $K_{\lambda\lambda} = 1$. $\square$

From the above lemma we see that the Schur function may be expanded as

$$s_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} m_\mu, \tag{2.5.3}$$

where the $K_{\lambda\mu}$ are nonnegative integers. If we replace the dominance order (which is only a partial order) by some compatible total order, then the matrix indexed by the set $\{\lambda : \lambda \vdash n\}$ and with the Kostka numbers as entries is lower triangular with 1's on the diagonal. Therefore it is invertible, and so restricting to Λ^n it follows that the Schur functions indexed over $\{\lambda : \lambda \vdash n\}$ form a basis for Λ^n for each n. The fact that the set of all Schur functions forms a basis for Λ then follows.

Our next theorem is a different expression for the Schur function when the cardinality of the alphabet is finite. Usually the following is taken as the definition of the Schur function, and this is how they were first defined by Cauchy [Cau15], but often credited to Jacobi. We will instead follow an argument of Stembridge in [Ste02] which derives Cauchy's formula from the definition in terms of Young tableaux (2.5.1).

Theorem 2.5.3 (Bialternant formula)**.** *Let $X_n = x_1 + \cdots + x_n$ be a finite set of indeterminates. Then for $\lambda \in \mathcal{P}_n$,*

$$s_\lambda(X_n) = \frac{\det_{1\leqslant i,j\leqslant n}\big(x_i^{\lambda_j+n-j}\big)}{\Delta(X_n)}. \tag{2.5.4}$$

Proof. Multiplying both sides of the equation by the Vandermonde determinant, and then expanding the left-hand side as a double sum gives

$$\Delta(X_n)s_\lambda(X_n) = \sum_{w\in\mathfrak{S}_n}\sum_{T\in\mathrm{SSYT}(\lambda)} \operatorname{sgn}(w)X_n^{w(\delta)+\mathrm{wt}(T)},$$

where we recall $\delta := (n-1, n-2, \dots, 1)$ is the staircase partition. By the symmetry of the Schur function, for any $w \in \mathfrak{S}_n$ we have

$$\sum_{T\in\mathrm{SSYT}(\lambda)} X_n^{\mathrm{wt}(T)} = \sum_{T\in\mathrm{SSYT}(\lambda)} X_n^{w(\mathrm{wt}(T))}.$$

Using this fact we see that

$$\sum_{w\in\mathfrak{S}_n}\sum_{T\in\mathrm{SSYT}(\lambda)} \operatorname{sgn}(w)X_n^{w(\delta)+\mathrm{wt}(T)} = \sum_{w\in\mathfrak{S}_n}\sum_{T\in\mathrm{SSYT}(\lambda)} \operatorname{sgn}(w)X_n^{w(\delta+\mathrm{wt}(T))}$$

Again applying the definition of the determinant we are left with

$$\Delta(X_n)s_\lambda(X_n) = \sum_{T\in\mathrm{SSYT}(\lambda)} \det_{1\leqslant i,j\leqslant n}\big(x_i^{\mathrm{wt}_j(T)+n-j}\big). \tag{2.5.5}$$

We would like to show that the only surviving term in the sum on the right-hand side is the Yamanouchi tableau of shape λ, since this tableau has weight λ. To do so we will construct a sign-reversing involution on the remaining tableaux in the sum.

Assume that $T \in \mathrm{SSYT}(\lambda)$ is not Yamanouchi. Then by Lemma 2.1.3 there exists some $j \geqslant 1$ such that $\mathrm{wt}(T_{\geqslant j})$ is not a partition. In other words, there exists a pair j, k such that

$$\mathrm{wt}_k(T_{\geqslant j}) < \mathrm{wt}_{k+1}(T_{\geqslant j}).$$

Amongst these pairs choose the largest such j, and for this j, the smallest such k. By the maximality of j, $\mathrm{wt}(T_{>j})$ is a partition, and so

$$\mathrm{wt}_k(T_{>j}) \geqslant \mathrm{wt}_{k+1}(T_{>j}).$$

The jth column of T contains either a single k, a single $k+1$, or one of each. Therefore as j varies, $\mathrm{wt}_{k+1}(T_{\geqslant j}) - \mathrm{wt}_k(T_{\geqslant j})$ changes by at most one. This implies that

$$\mathrm{wt}_k(T_{\geqslant j}) + 1 = \mathrm{wt}_{k+1}(T_{\geqslant j}),$$

i.e., column j contains a $k+1$ but no k. Let T^* be the tableau obtained from T by applying the Bender–Knuth involution φ_k to the subtableau $T_{<j}$ and fixing $T_{\geqslant j}$. Note that T^* is semistandard because there is a free $k+1$ in column j. Thus any change in k's and $(k+1)$'s occurs in the same

row as this $k+1$ and to its left, so semistandardness is preserved. Since the map φ_k is an involution, so is the map $T \mapsto T^*$. We know that the Bender–Knuth involution acts on the weight of $T_{<j}$ as $\mathrm{wt}(\phi_k(T_{<j})) = s_k \mathrm{wt}(T_{<j})$ where s_k is the adjacent transposition $(k\ k+1) \in \mathfrak{S}_n$. We also have that

$$\begin{aligned} s_k(\mathrm{wt}(T_{\geqslant j}) + \delta) &= s_k(\dots, \mathrm{wt}_k(T_{\geqslant j}) + k - 1, \mathrm{wt}_{k+1}(T_{\geqslant j}) + k - 2, \dots) \\ &= (\dots, \mathrm{wt}_{k+1}(T_{\geqslant j}) + k - 2, \mathrm{wt}_k(T_{\geqslant j})_k + k - 1, \dots) \\ &= (\dots, \mathrm{wt}_k(T_{\geqslant j}) + k - 1, \mathrm{wt}_{k+1}(T_{\geqslant j}) + k - 2, \dots), \end{aligned}$$

where the last equality follows since $\mathrm{wt}_k(T_{\geqslant j}) + 1 = \mathrm{wt}_{k+1}(T_{\geqslant j})$. So s_k fixes $\mathrm{wt}(T_{\geqslant j}) + \delta$, and we have

$$s_k(\mathrm{wt}(T) + \delta) = \mathrm{wt}(T^*) + \delta.$$

In turn this implies that

$$\det_{1 \leqslant i,j \leqslant n} \left(x_i^{\mathrm{wt}_j(T) + n - j}\right) = - \det_{1 \leqslant i,j \leqslant n} \left(x_i^{\mathrm{wt}_j(T^*) + n - j}\right).$$

All of the terms in the sum (2.5.5) will now cancel, bar the term indexed by the Yamanouchi tableau, for which $\mathrm{wt}(T) = \lambda$. Therefore,

$$\Delta(X_n) s_\lambda(X_n) = \det_{1 \leqslant i,j \leqslant n} \left(x_i^{\lambda_j + n - j}\right),$$

completing the proof. □

One advantage of the bialternant formula is that the Schur function is manifestly symmetric since it is expressed as a ratio of alternating polynomials. Another expression for the s_λ, now as a determinant of the complete symmetric functions, is the content of our next theorem.

Theorem 2.5.4 (Jacobi–Trudi identities)**.** *For $\lambda \in \mathcal{P}$ there holds*

$$s_\lambda = \det_{1 \leqslant i,j \leqslant k} (h_{\lambda_i + j - i}), \tag{2.5.6a}$$

where k is any integer such that $k \geqslant l(\lambda)$, and

$$s_\lambda = \det_{1 \leqslant i,j \leqslant \ell} (e_{\lambda'_i + j - i}), \tag{2.5.6b}$$

where ℓ is any integer such that $\ell \geqslant \lambda_1$.

Proof. We first prove (2.5.6a). Let X_n be a finite alphabet of cardinality n and α a sequence of nonnegative integers of length at most n. Observe that by (2.3.5) with $(X, Y, r) \mapsto (x_i - X_n, X_n, \alpha_j)$ for any i and j we may write

$$x_i^{\alpha_j} = h_{\alpha_j}[x_i] = \sum_{\ell=0}^{\alpha_j} h_{\alpha_j - \ell}[X_n] h_\ell[x_i - X_n].$$

By the reciprocity (2.3.6) it follows that $h_\ell[x_i - X_n] = (-1)^\ell e_\ell[X_n - x_i]$ vanishes if $\ell > n-1$. Since $h_r = 0$ for r a negative integer we may replace the upper bound in the sum with $n-1$. Relabeling $\ell \mapsto n - \ell$ we arrive at

$$x_i^{\alpha_j} = \sum_{\ell=1}^{n} h_{\alpha_j - n + \ell}[X_n] h_{n-\ell}[x_i - X_n].$$

This is equivalent to the matrix decomposition

$$\big(x_i^{\alpha_j}\big)_{1\leqslant i,j\leqslant n} = \big(h_{n-j}[x_i - X_n]\big)_{1\leqslant i,j\leqslant n}\big(h_{\alpha_j+n-i}[X_n]\big)_{1\leqslant i,j\leqslant n}. \tag{2.5.7}$$

Let us consider the case where $\alpha = \delta = (n-1, n-2, \dots, 1, 0)$ is the staircase partition. Then the above reads

$$\big(x_i^{n-j}\big)_{1\leqslant i,j\leqslant n} = \big(h_{n-j}[x_i - X_n]\big)_{1\leqslant i,j\leqslant n}\big(h_{i-j}[X_n]\big)_{1\leqslant i,j\leqslant n}.$$

Again using the fact that $h_r = 0$ if r is a negative integer and $h_0 = 1$ it follows that the second matrix on the right-hand side is lower triangular with 1's on the diagonal. The left-hand side is the Vandermonde matrix, so taking determinants tells us that

$$\Delta(X_n) = \det_{1\leqslant i,j\leqslant n}\big(h_{n-j}[x_i - X_n]\big).$$

Setting $\alpha = \lambda + \delta$ where $\lambda \in \mathscr{P}_n$ in (2.5.7) we have therefore established the identity

$$\det_{1\leqslant i,j\leqslant n}\big(x_i^{\lambda_j+n-j}\big) = \Delta(X_n)\det_{1\leqslant i,j\leqslant n}\big(h_{\lambda_i+i-j}[X_n]\big).$$

Using the bialternant formula (2.5.4), this is (2.5.6a) in the case of a finite number of variables (after taking the transpose of the right-hand side). Since n was arbitrary the identity holds more generally in Λ. To show that the determinant on the right of (2.5.6a) is independent of k as long as $k \geqslant l(\lambda)$ assume that $\lambda \in \mathscr{P}_{k-1}$. Then the $(i,j) = (k,k)$ entry of the matrix is 1 since $h_0 = 1$. Further, all of the other entries in the kth column vanish since the index of the complete symmetric function is negative. Therefore

$$\det_{1\leqslant i,j\leqslant k}\big(h_{\lambda_i+i-j}[X_n]\big) = \det_{1\leqslant i,j\leqslant k-1}\big(h_{\lambda_i+i-j}[X_n]\big).$$

By the previously established identity (2.3.9) with $\mu = 0$, the dual form (2.5.6b) also follows. □

There are several other proofs of the above theorem. Of particular note is the elegant combinatorial argument using non-intersecting lattice paths and the Lindström–Gessel–Viennot lemma [GV89]; the proof may be found in [Sag01, Theorem 4.5.1]. Again, the Schur function is manifestly symmetric if one takes the Jacobi–Trudi formula as the definition, since it is a linear combination of symmetric functions. One particular advantage of the identities (2.5.6) are that they are alphabet-independent, i.e., they hold as identities in Λ taken over any alphabet.

A useful consequence of the Jacobi–Trudi identities for our purposes is the reciprocity

$$s_\lambda[-X] = (-1)^{|\lambda|}s_{\lambda'}[X]. \tag{2.5.8}$$

To see this, take (2.5.6a) on the alphabet $-X$, giving

$$s_\lambda[-X] = \det_{1\leqslant i,j\leqslant k}\big(h_{\lambda_i+j-i}[-X]\big) = \det_{1\leqslant i,j\leqslant k}\big((-1)^{\lambda_i+j-i}e_{\lambda_i+j-i}[X]\big) = (-1)^{|\lambda|}s_{\lambda'}[X],$$

where we have made use of (2.3.6). Note that the right-hand side of (2.5.6a) makes sense for any sequence of nonnegative integers α with finite length. Hence we define the Schur function indexed by α to be

$$s_\alpha = \det_{1\leqslant i,j\leqslant k}\big(h_{\alpha_i+j-i}\big), \tag{2.5.9}$$

where k is any integer such that $k \geqslant l(\alpha)$. The determinant will vanish if the integers $\alpha_i - i$ are not distinct for all $1 \leqslant i \leqslant k$. If they are distinct, then we can place them in strictly decreasing order. Let $w \in \mathfrak{S}_k$ be such that $w(\alpha_i - i) > w(\alpha_{i+1} - (i+1))$ for $1 \leqslant i \leqslant k$. If $s_\alpha \neq 0$, it follows that

$$s_\alpha = \operatorname{sgn}(w) s_{w(\alpha+\delta)-\delta}.$$

Next, we use the Jacobi–Trudi formula to show that the Schur functions satisfy a Cauchy identity.

Proposition 2.5.5. *For any alphabets X and Y,*

$$\sum_\lambda s_\lambda(X) s_\lambda(Y) = \sigma_1[XY]. \tag{2.5.10}$$

Proof. Throughout the proof we assume that Y_n is a finite alphabet of cardinality n. We begin with the expression

$$\Delta(Y_n)\sigma_1[XY_n].$$

By (2.4.5) and the definition of the Vandermonde determinant we have

$$\Delta(Y_n)\sigma_1[XY_n] = \sum_{w \in \mathfrak{S}_n} \sum_{\substack{\alpha \\ l(\alpha) \leqslant n}} \operatorname{sgn}(w) h_\alpha(X) Y_n^{\alpha + w\delta},$$

where the restriction $l(\alpha) \leqslant n$ comes from the fact that $m_\lambda(X_n) = 0$ if $l(\lambda) > n$. For a given $w \in \mathfrak{S}_n$ define $\beta := \alpha + w\delta$. Since $h_{\beta - w\delta} = 0$ if any of the entries in $\beta - w\delta$ are negative, we are free to swap the sum over α for a sum over β, giving

$$\sum_{w \in \mathfrak{S}_n} \sum_{\substack{\alpha \\ l(\alpha) \leqslant n}} \operatorname{sgn}(w) h_\alpha(X) Y_n^{\alpha + w\delta} = \sum_{w \in \mathfrak{S}_n} \sum_{\substack{\beta \\ l(\alpha) \leqslant n}} \operatorname{sgn}(w) h_{\beta - w\delta}(X) Y_n^{\beta}.$$

The sum over $\mathfrak{S}_n$ may be evaluated using the Jacobi–Trudi identity (2.5.9) with $\alpha \mapsto \beta - \delta$, so that

$$\sum_{w \in \mathfrak{S}_n} \sum_{\substack{\beta \\ l(\alpha) \leqslant n}} \operatorname{sgn}(w) h_{\beta - w\delta}(X) Y_n^{\beta} = \sum_{\substack{\beta \\ l(\beta) \leqslant n}} s_{\beta-\delta}(X) Y_n^\beta.$$

Observe that the summand will vanish unless all of the entries of β are distinct. This allows for the the sum over sequences α of length at most n in the above to be replaced with a sum over strictly decreasing sequences of nonnegative integers of length n, at the cost of an extra sum over $\mathfrak{S}_n$ being added. We now have

$$\begin{aligned}
\sum_{\substack{\beta \\ l(\beta) \leqslant n}} s_{\beta-\delta}(X) Y_n^\beta &= \sum_{\beta_1 > \cdots > \beta_n \geqslant 0} s_{\beta-\delta}(X) \sum_{w \in \mathfrak{S}_n} \operatorname{sgn}(w) Y_n^{w\beta} \\
&= \sum_{\beta_1 > \cdots > \beta_n \geqslant 0} s_{\beta-\delta}(X) \det_{1 \leqslant i,j \leqslant n} \left(y_i^{\beta_j}\right).
\end{aligned}$$

There is a bijection between partitions of length at most n and the sequences β given by setting $\beta - \delta = \lambda$. Therefore we have shown that

$$\Delta(Y_n)\sigma_1[XY_n] = \sum_\lambda s_\lambda(X) \det_{1 \leqslant i,j \leqslant n} \left(y_i^{\lambda_j + n - j}\right),$$

which is equivalent to the theorem in the case of Y a finite alphabet by the bialternant formula (2.5.4). So the Cauchy identity has been established as an identity in $\hat{\Lambda} \otimes \hat{\Lambda}_n$, where $\hat{R}$ denotes the completion of the ring R. Since this is true for any n, we also have the corresponding identity in $\hat{\Lambda} \otimes \hat{\Lambda}$, where X and Y are arbitrary alphabets. □

A beautiful combinatorial proof of the Cauchy identity based on the Robinson–Schensted–Knuth correspondence may be found in [Sta99, Theorem 7.12.1]. We have established that the Schur functions are an orthonormal basis for Λ with respect to the Hall scalar product and so

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}.$$

In fact, this orthogonality together with the unitriangular expansion in terms of the monomial symmetric functions (2.5.3) uniquely determine the Schur functions.

Frequently we will need to evaluate Schur functions at a binomial element. Such an expression may be obtained from the following specialisation formula.

Lemma 2.5.6 (Specialisation formula)**.** *For a an indeterminate there holds*

$$s_\lambda\left[\frac{1-a}{1-q}\right] = q^{n(\lambda)} \prod_{i \geqslant 1} \frac{(aq^{1-i};q)_{\lambda_i}}{(q^{n+1-i};q)_{\lambda_i}} \prod_{1 \leqslant i<j \leqslant n} \frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{j-i}}, \tag{2.5.11}$$

where n is any integer such that $n \geqslant l(\lambda)$.

Proof. We first show that the right-hand side of (2.5.11) is independent of n, provided $n \geqslant l(\lambda)$. Assume $\lambda_n = 0$, i.e., $l(\lambda) \leqslant n-1$. Then

$$\begin{aligned}\prod_{1 \leqslant i<j \leqslant n} \frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{j-i}} &= \prod_{i=1}^{n-1} \frac{1-q^{\lambda_i+n-i}}{1-q^{n-i}} \prod_{1 \leqslant i<j \leqslant n-1} \frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{j-i}} \\ &= \prod_{i \geqslant 1} \frac{(q^{n+1-i};q)_{\lambda_i}}{(q^{n-i};q)_{\lambda_i}} \prod_{1 \leqslant i<j \leqslant n-1} \frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{j-i}}.\end{aligned}$$

Substituting this into (2.5.11) gives the same expression with n replaced by $n-1$, and so the expression is independent of n. Let k be any integer such that $k \geqslant l(\lambda)$. Both sides of (2.5.11) are polynomials in a, and thus it suffices to prove the result for $a = q^k$. Since

$$\frac{1-q^k}{1-q} = 1 + q + \cdots + q^{k-1},$$

it follows from the bialternant formula (2.5.4) that

$$s_\lambda\left[\frac{1-q^k}{1-q}\right] = \frac{\det_{1 \leqslant i,j \leqslant k}(q^{(i-1)(\lambda_j+k-j)})}{\det_{1 \leqslant i,j \leqslant k}(q^{(i-1)(k-j)})}.$$

Both the numerator and denominator may be evaluated by the Vandermonde determinant (1.1.7), giving

$$s_\lambda\left[\frac{1-q^k}{1-q}\right] = \prod_{1 \leqslant i<j \leqslant k} \frac{q^{\lambda_j+k-j} - q^{\lambda_i+k-i}}{q^{k-j} - q^{k-i}}.$$

Pulling out factors in the numerator and denominator and noting that

$$\prod_{1\leqslant i<j\leqslant k} q^{\lambda_j} = \prod_{j=1}^{k-1} q^{j\lambda_{j+1}} = q^{n(\lambda)},$$

we arrive at the formula

$$s_\lambda\left[\frac{1-q^k}{1-q}\right] = q^{n(\lambda)} \prod_{1\leqslant i<j\leqslant k} \frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{j-i}}.$$

By our considerations at the beginning of the proof we may rewrite this expression as

$$s_\lambda\left[\frac{1-q^k}{1-q}\right] = q^{n(\lambda)} \prod_{i\geqslant 1} \frac{(q^{k+1-i};q)_{\lambda_i}}{(q^{l(\lambda)+1-i};q)_{\lambda_i}} \prod_{1\leqslant i<j\leqslant l(\lambda)} \frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{j-i}},$$

which is the statement of the lemma with $a = q^k$ and $n = l(\lambda)$. As the right-hand side of (2.5.11) is independent of n as long as $n \geqslant l(\lambda)$, the proof follows. □

For $a = q^k$ where k is any integer such that $k \geqslant l(\lambda)$ the formula (2.5.11) is referred to as the principal specialisation of the Schur function s_λ. This is usually stated as

$$s_\lambda\left[\frac{1-q^k}{1-q}\right] = q^{n(\lambda)} \prod_{1\leqslant i<j\leqslant k} \frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{j-i}}, \tag{2.5.12}$$

since we are free to take $n = k$ in the right-hand side of (2.5.11). An immediate consequence of the above lemma is a formula for the Schur function evaluated at any binomial element z. We take (2.5.11) with $a \mapsto q^z$ followed by the limit $q \to 1$ which gives

$$s_\lambda[z] = \prod_{i\geqslant 1} \frac{(z-i+1)_{\lambda_i}}{(n-i+1)_{\lambda_i}} \prod_{1\leqslant i<j\leqslant n} \frac{\lambda_i-\lambda_j+j-i}{j-i}, \tag{2.5.13}$$

where n is any integer such that $n \geqslant l(\lambda)$.

As the Schur functions form an orthonormal basis for Λ, any symmetric function $f \in \Lambda$ is determined by the value of the Hall scalar product with s_λ:

$$f = \sum_\lambda \langle f, s_\lambda\rangle s_\lambda.$$

For any triple of partitions λ, μ, ν, define the Littlewood–Richardson coefficient by

$$c^\lambda_{\mu\nu} := \langle s_\lambda, s_\mu s_\nu\rangle. \tag{2.5.14}$$

It follows that

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda. \tag{2.5.15a}$$

In turn we extend the Schur function to skew shapes by

$$s_{\lambda/\mu} := \sum_\nu c^\lambda_{\mu\nu} s_\nu. \tag{2.5.15b}$$

The $s_{\lambda/\mu}$ are called skew Schur functions, and are indexed by the skew shape λ/μ. By definition there holds

$$\langle s_\lambda, s_\mu s_\nu\rangle = \langle s_{\lambda/\mu}, s_\nu\rangle. \tag{2.5.16}$$

Note that by symmetry in μ and ν in (2.5.14) we may interchange μ and ν in this last expression. Some properties of the Littlewood–Richardson coefficients and skew Schur functions are easy to see from these definitions. The left-hand side of equation (2.5.15a) is homogeneous of degree $|\mu| + |\nu|$, and so $c^\lambda_{\mu\nu} = 0$ unless $|\mu| + |\nu| = |\lambda|$. Consequently the partitions ν on the right-hand side of (2.5.15b) must satisfy $|\nu| = |\lambda/\mu|$, so that the skew Schur function is homogeneous of degree $|\lambda/\mu|$.

There is a skew analogue of the Cauchy identity (2.5.10). Using the expansions (2.5.15) we see that

$$\sum_\lambda s_{\lambda/\mu}(X)s_\lambda(Y) = \sum_{\lambda,\nu} c^\lambda_{\mu\nu} s_\nu(X)s_\lambda(Y) = \sum_\nu s_\nu(X)s_\mu(Y)s_\nu(Y) = s_\mu(Y)\sigma_1[XY]. \tag{2.5.17}$$

For $\mu = 0$ this just is the ordinary Cauchy identity for Schur functions. From here we can easily derive a skew analogue of the Jacobi–Trudi identity of Theorem 2.5.4. Assume $Y \mapsto Y_n = y_1 + \cdots + y_n$ is a finite alphabet. Then from (2.5.17) and (2.4.5) it follows that

$$\begin{aligned}\sum_\lambda s_{\lambda/\mu}(X) \det_{1\leqslant i,j\leqslant n}\big(y_i^{\lambda_j+n-j}\big) &= \det_{1\leqslant i,j\leqslant n}\big(y_i^{\mu_j+n-j}\big)\sum_\alpha h_\alpha(X)Y_n^\alpha \\ &= \sum_\alpha \sum_{w\in\mathfrak{S}_n} \operatorname{sgn}(w)h_\alpha(X)Y_n^{\alpha+w(\mu+\delta)}.\end{aligned}$$

Equating coefficients of $Y_n^{\lambda+\delta}$ implies

$$s_{\lambda/\mu}(X) = \sum_{w\in\mathfrak{S}_n} \operatorname{sgn}(w)h_{\lambda+\delta-w(\mu+\delta)}(X).$$

Written as a determinant and suppressing alphabets we therefore obtain

$$s_{\lambda/\mu} = \det_{1\leqslant i,j\leqslant n}\big(h_{\lambda_i-\mu_j+j-i}\big).$$

For $\mu = 0$ this reduces to the ordinary Jacobi–Trudi identity (2.5.6a), and indeed in the above we may assume that n is any integer such that $n \geqslant l(\lambda)$. In particular this implies that $s_{\lambda/0} = s_\lambda$. Another consequence is that $s_{\lambda/\mu} = 0$ if $\mu \not\subseteq \lambda$. To see this, assume that $\lambda_k < \mu_k$ for some fixed $k \geqslant 1$. Then for any $1 \leqslant j \leqslant k \leqslant i \leqslant n$ we have $\lambda_i \leqslant \lambda_k < \mu_k \leqslant \mu_j$. Hence $\lambda_i - \mu_j + j - i < 0$ for all pairs (i, j) satisfying the conditions. In other words, we have a $(n - k + 1) \times k$ block of zeros in the bottom left-hand corner, so that the determinant vanishes in this case. Using (2.3.9) we have the dual Jacobi–Trudi formula

$$s_{\lambda/\mu} = \det_{1\leqslant i,j\leqslant \ell}\big(e_{\lambda'_i-\mu'_j+j-i}\big), \tag{2.5.18}$$

where ℓ is any integer such that $\ell \geqslant \lambda_1$.

The skew Schur functions allow for a description of the Schur function on the sum of alphabets. To derive this formula we consider the following sum, where X, Y, Z are all distinct countable alphabets,

$$\sum_{\lambda,\mu} s_{\lambda/\mu}(X)s_\mu(Y)s_\lambda(Z).$$

First applying the skew Cauchy identity (2.5.17) with $(X, Y) \mapsto (X, Z)$, followed by the ordinary Cauchy identity (2.5.10) with $(X, Y, \lambda) \mapsto (Y, Z, \mu)$, we see that

$$\sum_{\lambda,\mu} s_{\lambda/\mu}(X)s_\mu(Y)s_\lambda(Z) = \sigma_1[XZ]\sum_\mu s_\mu(Y)s_\mu(Z) = \sigma_1[XZ]\sigma_1[YZ] = \sigma_1[(X+Y)Z],$$

where in the last equality we have applied the multiplication rule for σ_1 ((2.3.4a) with $z = 1$). Again applying the Cauchy identity (2.5.10), this time with $(X, Y) \mapsto (X + Y, Z)$, gives

$$\sum_{\lambda,\mu} s_{\lambda/\mu}(X)s_\mu(Y)s_\lambda(Z) = \sum_\lambda s_\lambda[X+Y]s_\lambda(Z).$$

The Schur functions are linearly independent over $\mathbb{Q}$ so that the coefficient of $s_\lambda(Z)$ on both sides of this expression must be equal. We therefore arrive at

$$s_\lambda[X+Y] = \sum_\mu s_{\lambda/\mu}(X)s_\mu(Y). \tag{2.5.19}$$

Observe that since $s_{\lambda/\mu}(X)$ vanishes unless $\mu \subseteq \lambda$ the sum over μ is finite. Furthermore, as the left-hand side of this expression is symmetric in X and Y, the right-hand side also possesses this symmetry. An important special case of (2.5.19) is the branching rule, which may be obtained by setting $X = \{x\}$, a single-letter alphabet. From the dual Jacobi–Trudi formula (2.5.18) it is not hard to see that for a single-letter alphabet the skew Schur function $s_{\lambda/\mu}[x]$ vanishes unless λ/μ is a horizontal strip. Indeed, assuming that λ/μ is not a horizontal strip means that $\lambda'_i - \mu'_i \geqslant 2$ for some i. However $e_r[x] = 0$ unless $r = 0$ or $r = 1$, so that there is a block of zeros in the top right-hand corner of the matrix, with one of the zeros lying on the main diagonal. Hence $s_{\lambda/\mu}[x] = 0$ unless $\mu \prec \lambda$, in which case it follows from (2.5.18) that

$$s_{\lambda/\mu}[x] = x^{|\lambda|-|\mu|}.$$

We therefore have that

$$s_\lambda[Y+x] = \sum_{\mu\prec\lambda} x^{|\lambda|-|\mu|}s_\mu[Y].$$

The formula (2.5.19) can be pushed a little further. To this end, consider the Schur function on the sum of three alphabets $s_\lambda[X+Y+Z]$. Using (2.5.19) with $(X, Y, \lambda, \mu) \mapsto (X+Y, Z, \lambda, \nu)$ as

$$s_\lambda[X+Y+Z] = \sum_\nu s_{\lambda/\nu}[X+Y]s_\nu[Z].$$

Alternatively, using the same identity twice, first with $(X, Y, \lambda, \mu) \mapsto (X, Y+Z, \lambda, \mu)$ followed by $(X, Y, \lambda, \mu) \mapsto (Y, Z, \mu, \nu)$ we have that

$$s_\lambda[X+Y+Z] = \sum_\mu s_{\lambda/\mu}[X]s_\mu[Y+Z] = \sum_{\mu,\nu} s_{\lambda/\mu}[X]s_{\mu/\nu}[Y]s_\nu[Z].$$

Hence

$$\sum_\nu s_{\lambda/\nu}[X+Y]s_\nu[Z] = \sum_{\mu,\nu} s_{\lambda/\mu}[X]s_{\mu/\nu}[Y]s_\nu[Z],$$

so that equating coefficients of $s_\nu[Z]$ on both sides yields

$$s_{\lambda/\mu}[X+Y] = \sum_\nu s_{\lambda/\nu}(X)s_{\nu/\mu}(Y), \tag{2.5.20}$$

where we have interchanged μ and ν in this last expression.

It is natural to ask if the skew Schur functions admit a nice combinatorial formula in terms of tableaux. This is indeed the case, as the following theorem asserts.

Theorem 2.5.7. *For $\lambda, \mu \in \mathcal{P}$ there holds*

$$s_{\lambda/\mu}(X) = \sum_{T\in\mathrm{SSYT}(\lambda/\mu)} X^{\mathrm{wt}(T)}.$$

Proof. Let $X^{(1)}, \dots, X^{(n)}$ be a collection of n alphabets. Then (2.5.20) may be iterated to give

$$s_{\lambda/\mu}\big[X^{(1)} + \cdots + X^{(n)}\big] = \sum_{\mu\subseteq\nu^{(1)}\subseteq\cdots\subseteq\nu^{(n-1)}\subseteq\lambda} \prod_{i=1}^n s_{\nu^{(i)}/\nu^{(i-1)}}\big[X^{(i)}\big],$$

where $\nu^{(0)} := \mu$ and $\nu^{(n)} := \lambda$. If we assume that each alphabet $X^{(i)}$ contains only a single letter, and write $X_n = x_1 + \cdots + x_n$, then we have the formula

$$s_{\lambda/\mu}[X_n] = \sum_{\mu\subseteq\nu^{(1)}\subseteq\cdots\subseteq\nu^{(n-1)}\subseteq\lambda} \prod_{i=1}^n s_{\nu^{(i)}/\nu^{(i-1)}}[x_i].$$

Recalling that $s_{\lambda/\mu}[y] = y^{|\lambda|-|\mu|}$ if $\mu \prec \lambda$ and zero otherwise, this may be rewritten as

$$s_{\lambda/\mu}[X_n] = \sum_{\mu\prec\nu^{(1)}\prec\cdots\prec\nu^{(n-1)}\prec\lambda} \prod_{i=1}^n x_i^{|\nu^{(i)}|-|\nu^{(i-1)}|}.$$

The terms in the sum are indexed by sequences of interlacing partitions $\big(\mu, \nu^{(1)}, \dots, \nu^{(n-1)}, \lambda\big)$, which are in bijection with semistandard Young tableaux of shape λ/μ on the alphabet $\{1, \dots, n\}$. Moreover, under this interpretation of the sum, the summand takes the form $X_n^{\mathrm{wt}(T)}$ for $T \in \mathrm{SSYT}(\lambda/\mu)$. We thus arrive at the statement of the theorem for X_n a finite alphabet. $\square$

With the combinatorial formula for $s_{\lambda/\mu}$ established we can immediately infer the monomial expansion

$$s_{\lambda/\mu} = \sum_\nu K_{\lambda/\mu,\nu} m_\nu$$

where $K_{\lambda/\mu,\nu} = |\mathrm{SSYT}(\lambda/\mu;\nu)|$, which generalises (2.5.2). As a consequence we have the following pair of formulas, called the e- and h-Pieri rules.

Proposition 2.5.8. *For r a nonnegative integer we have*

$$h_r s_\mu = \sum_{\substack{\lambda\succ\mu \\ |\lambda/\mu|=r}} s_\lambda, \tag{2.5.21a}$$

and

$$e_r s_\mu = \sum_{\substack{\lambda'\succ\mu' \\ |\lambda/\mu|=r}} s_\lambda. \tag{2.5.21b}$$

Proof. By (2.5.16) the coefficient of s_λ in the Schur expansion of $h_r s_\mu$ is given by

$$\langle h_r s_\mu, s_\lambda \rangle = \langle h_r, s_{\lambda/\mu} \rangle = K_{\lambda/\mu,(r)}.$$

Since the weight of λ/μ is (r), any $T \in \mathrm{SSYT}(\lambda/\mu;(r))$ contains only a single character. Therefore the shape λ/μ cannot contain two boxes in the same column as this would violate column-strictness, and so λ/μ must be a horizontal strip. Further, $|\lambda/\mu| = r$ as the weight of T is (r). It follows that $K_{\lambda/\mu,(r)}$ is either 1 if λ/μ is a horizontal r-strip, or zero otherwise.

To prove the second formula, write (2.5.21a) with alphabet $-X$ to obtain

$$h_r[-X]s_\mu[-X] = \sum_{\substack{\lambda \succ \mu \\ |\lambda/\mu| = r}} s_\lambda[-X]$$

To the complete symmetric function we apply the duality (2.3.6), and to the Schur functions the relation (2.5.8), giving

$$(-1)^{|\mu|+r} e_r[X] s_{\mu'}[X] = \sum_{\substack{\lambda \succ \mu \\ |\lambda/\mu| = r}} (-1)^{|\lambda|} s_{\lambda'}[X].$$

Since $|\lambda| - |\mu| = r$ in the sum, we may cancel the negative signs on both sides. Replacing λ and μ by their conjugates finishes the proof. $\square$

To round out this section we prove an inversion of the h-Pieri rule (2.5.21a).

Proposition 2.5.9. *For $(\mu_1, \dots, \mu_n)$ a partition of length at most n and d a nonnegative integer not exceeding μ_n,*

$$s_{(\mu,d)} = \sum_{\substack{\lambda' \succ \mu' \\ l(\lambda) \leqslant n}} (-1)^{|\lambda/\mu|} s_\lambda h_{d-|\lambda/\mu|}. \tag{2.5.22}$$

Proof. Throughout the proof we write $\mu^+ := (\mu_1, \dots, \mu_{n+1})$ where $\mu_{n+1} := d$. Note that the condition $l(\lambda) \leqslant n$ is equivalent to $\lambda_{n+1} = 0$. Applying the h-Pieri rule (2.5.21a) to the sum side of (2.5.22) gives

$$\sum_{\substack{\lambda' \succ \mu' \\ \lambda_{n+1}=0}} (-1)^{|\lambda/\mu|} s_\lambda h_{\mu_{n+1}-|\lambda/\mu|} = \sum_{\substack{\lambda' \succ \mu' \\ \lambda_{n+1}=0}} (-1)^{|\lambda/\mu|} \sum_{\substack{\nu \succ \lambda \\ |\nu| = |\mu^+|}} s_\nu = \sum_{\substack{\nu \\ |\nu|=|\mu^+| \\ l(\nu) \leqslant n+1}} s_\nu \sum_{\substack{\lambda' \succ \mu' \\ \lambda \prec \nu \\ \lambda_{n+1}=0}} (-1)^{|\lambda/\mu|},$$

where in the second equality we have interchanged the sums. The sum over λ is equal to

$$\sum_{\substack{\lambda' \succ \mu' \\ \lambda \prec \nu}} (-1)^{|\lambda/\mu|} - \sum_{\substack{\lambda' \succ \mu' \\ \lambda \prec \nu \\ \lambda_{n+1}=1}} (-1)^{|\lambda/\mu|},$$

as, since $l(\mu) \leqslant n$ and $\lambda' \succ \mu'$, we must have $\lambda_{n+1} = 1$ if $l(\lambda) = n+1$ in order for λ/μ to be a vertical strip. By Lemma 2.1.1 the first term in this sum is equal to $\delta_{\mu,\nu}$. Hence we are left with

$$\sum_{\substack{\lambda' \succ \mu' \\ \lambda_{n+1}=0}} (-1)^{|\lambda/\mu|} s_\lambda h_{\mu_{n+1}-|\lambda/\mu|} = \sum_{\substack{\nu \\ |\nu|=|\mu^+| \\ l(\nu) \leqslant n+1}} s_\nu \delta_{\mu,\nu} - \sum_{\substack{\nu \\ |\nu|=|\mu^+| \\ l(\nu) = n+1}} s_\nu \sum_{\substack{\lambda' \succ \mu' \\ \lambda \prec \nu \\ \lambda_{n+1}=1}} (-1)^{|\lambda/\mu|},$$

where in the second sum we may impose $l(\nu) = n + 1$ as else the sum over λ is empty. Since $|\nu| = |\mu^+|$, the first sum above is equal to $\chi(\mu_{n+1} = 0)s_\mu = \chi(\mu_{n+1} = 0)s_{\mu^+}$. To evaluate the second sum we observe that without loss of generality we may assume that $\mu_{n+1} \neq 0$, as if $\mu_{n+1} = 0$ then $|\nu| = |\mu|$ and $\mu \subseteq \nu$ implies that $\mu = \nu$, but $l(\nu) = n + 1$, so the sum is empty. Furthermore since $\mu_n \neq 0$ we must have $\lambda_n \neq 0$, which leads us to

$$-\sum_{\substack{\nu \\ |\nu|=|\mu^+| \\ l(\nu)=n+1}} s_\nu \sum_{\substack{\lambda'\succ\mu' \\ \lambda\prec\nu \\ \lambda_{n+1}=1}} (-1)^{|\lambda/\mu|} = \sum_{\substack{\nu \\ |\nu|=|\mu^+| \\ l(\nu)=n+1}} s_\nu \sum_{\substack{(\lambda^-)'\succ\mu' \\ \lambda^-\prec\nu^-}} (-1)^{|\lambda^-/\mu|}.$$

Again applying Lemma 2.1.1 to the interior sum we now have

$$\sum_{\substack{\lambda'\succ\mu' \\ \lambda_{n+1}=0}} (-1)^{|\lambda/\mu|} s_\lambda h_{\mu_{n+1}-|\lambda/\mu|} = \chi(\mu_{n+1} = 0)s_{\mu^+} + \sum_{\substack{\nu \\ |\nu|=|\mu^+| \\ l(\nu)=n+1}} s_\nu \delta_{\nu^-,\mu}.$$

The combination of the conditions $\nu^- = \mu$, $l(\nu) = n + 1$ and $|\nu| = |\mu^+|$ implies that

$$-\sum_{\substack{\nu \\ |\nu|=|\mu^+| \\ l(\nu)=n+1}} s_\nu \sum_{\substack{\lambda'\succ\mu' \\ \lambda\prec\nu \\ \lambda_{n+1}=1}} (-1)^{|\lambda/\mu|} = \sum_{\substack{\nu \\ |\nu|=|\mu^+| \\ l(\nu)=n+1}} s_\nu \delta_{\mu^+,\nu}.$$

Putting all of this together we see that

$$\sum_{\substack{\lambda'\succ\mu' \\ \lambda_{n+1}=0}} (-1)^{|\lambda/\mu|} s_\lambda h_{\mu_{n+1}-|\lambda/\mu|} = s_{\mu^+},$$

which is equivalent to the statement of the proposition. □

2.6 Jack and Macdonald polynomials

Throughout this section we consider Λ over the field $\mathbb{Q}(q,t)$. To extend the classical symmetric function theory developed in the previous sections to the q,t-level, we begin with the q,t-Hall scalar product. The analogue of the quantity z_λ is

$$z_\lambda(q,t) := z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

Then the q,t-deformation of the Hall scalar product is defined by imposing

$$\langle p_\lambda, p_\mu \rangle_{q,t} := z_\lambda(q,t)\delta_{\lambda\mu}.$$

It should be noted that if we take $0 < q,t < 1$ to be real numbers then the scalar product is positive-definite, with the proof being the same as for the ordinary Hall scalar product. Analogous to the classical case, we will soon see that any two bases for Λ which are orthonormal under the q,t-Hall scalar product satisfy a Cauchy identity-type identity. This requires a q,t-analogue of $\sigma_1[XY]$, which takes the form

$$\sigma_1\left[XY\frac{1-t}{1-q}\right]$$

In particular, if X and Y are both countable alphabets then by Lemma 2.3.2 it follows that

$$\sigma_1\left[XY\frac{1-t}{1-q}\right] = \prod_{x\in X}\prod_{y\in Y}\frac{(txy;q)_\infty}{(xy;q)_\infty}.$$

As before we first prove a Cauchy identity for the power sums directly.

Proposition 2.6.1. *For arbitrary alphabets X and Y,*

$$\sum_\lambda \frac{p_\lambda(X)p_\lambda(Y)}{z_\lambda(q,t)} = \sigma_1\left[XY\frac{1-t}{1-q}\right].$$

Proof. This is simply a plethystically substituted version of the first Cauchy-type identity for the p_λ found in Proposition 2.4.2. Indeed setting

$$(X,Y)\mapsto\left(X, Y\frac{1-t}{1-q}\right)$$

in (2.4.3) and using

$$\frac{1}{z_\lambda}p_\lambda\left[Y\frac{1-t}{1-q}\right] = \frac{p_\lambda(Y)}{z_\lambda(q,t)},$$

the proof follows. $\square$

We may thus claim an analogue of Proposition 2.4.3 for the q,t-Hall scalar product.

Proposition 2.6.2. *Let $\{u_\lambda\}$ and $\{v_\lambda\}$ be two homogeneous bases for Λ. Then the following statements are equivalent:*

1. *For each pair of partitions λ,μ we have*

$$\langle u_\lambda, v_\mu\rangle_{q,t} = \delta_{\lambda\mu}.$$

2. *For any pair of alphabets X and Y there holds*

$$\sum_\lambda u_\lambda(X)v_\lambda(Y) = \sigma_1\left[XY\frac{1-t}{1-q}\right].$$

Proof. The proof is identical to that of Proposition 2.4.3. $\square$

We now move towards the introduction of the Macdonald polynomials. To this end we define a family of operators $D_n : \Lambda_n \longrightarrow \Lambda_n$, but claim without proof several facts about them. For a more fleshed-out exposition, see [Mac95, §6.3]. Firstly, let T_{q,x_i} denote the shift operator which replaces x_i by qx_i in a symmetric function $f\in\Lambda_n$. Then we define D_n to be the polynomial in the T_{q,x_i},

$$D_n(z;q,t) := \frac{1}{\Delta(X_n)}\sum_{w\in\mathfrak{S}_n}\operatorname{sgn}(w)X_n^{w\delta}\prod_{i=1}^n\left(1+zt^{(w\delta)_i}T_{q,x_i}\right), \tag{2.6.1}$$

where z is an indeterminate. Let D_n^r be the coefficient of z^r when (2.6.1) is expanded out. Then it follows that

$$D_n^r = \sum_{\substack{I\subseteq\{1,\dots,n\}\\ |I|=r}} t^{\binom{r}{2}} \prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i}. \tag{2.6.2}$$

The operators D_n^r are $\mathbb{Q}(q,t)$-linear and degree-preserving. Furthermore, they are self-adjoint with respect to the q,t-Hall scalar product, i.e., for any $f, g \in \Lambda_n$ we have

$$\langle D_n^r f, g\rangle_{q,t} = \langle f, D_n^r g\rangle_{q,t}. \tag{2.6.3}$$

Unfortunately the operators D_n^r are not compatible with the restriction homomorphisms $\rho_{n,m}$ defined in Section 2.2. However, the operators [Mac95, §6.4]

$$E_n := t^{-n} D_n^1 - \sum_{i=1}^n t^{-i},$$

are compatible with the maps $\rho_{n-1,n}$. Thus we can define an operator $E : \Lambda \longrightarrow \Lambda$ which is taken to be the projective or inverse limit of the E_n. This new operator acts on the m_λ as

$$Em_\lambda = e_{\lambda\lambda} m_\lambda + \sum_{\mu<\lambda} e_{\lambda\mu} m_\mu, \tag{2.6.4}$$

where

$$e_{\lambda\lambda} = \sum_{i\geqslant 1} (q^{\lambda_i} - 1) t^{-i}. \tag{2.6.5}$$

Note that E will be self-adjoint with respect to the q,t-Hall scalar product on Λ, since by (2.6.3), E_n is self-adjoint for each n. We are now equipped with all that we need to state and prove the existence theorem for the Macdonald polynomials.

Theorem 2.6.3. *For each partition λ there exists a unique symmetric function $P_\lambda(X;q,t) \in \Lambda$ such that for some coefficients $a_{\lambda\mu}(q,t) \in \mathbb{Q}(q,t)$ there holds*

$$P_\lambda(X;q,t) = m_\lambda(X) + \sum_{\mu<\lambda} a_{\lambda\mu}(q,t) m_\mu(X), \tag{2.6.6a}$$

and we have

$$\langle P_\lambda(X;q,t), P_\mu(X;q,t)\rangle_{q,t} = 0 \quad \text{if} \quad \lambda \neq \mu. \tag{2.6.6b}$$

Proof. We first prove that functions satisfying the conditions (2.6.6) exist. Assume that $P_\lambda = P_\lambda(X;q,t)$ satisfy (2.6.6a) and are eigenfunctions of E, that is they satisfy

$$EP_\lambda = e_{\lambda\lambda} P_\lambda.$$

If we apply E to both sides of (2.6.6a) and use (2.6.4) we see that for any λ,

$$\sum_{\nu\leqslant\lambda} e_{\lambda\lambda} a_{\lambda\nu} m_\nu = \sum_{\nu\leqslant\mu\leqslant\lambda} a_{\lambda\mu} e_{\mu\nu} m_\nu,$$

where $a_{\lambda\mu} = a_{\lambda\mu}(q,t)$ and $a_{\lambda\lambda} = 1$. For any $\nu < \lambda$, extract the coefficient of m_ν on both sides of this equation to obtain

$$(e_{\lambda\lambda} - e_{\nu\nu})a_{\lambda\nu} = \sum_{\nu<\mu\leqslant\lambda} a_{\lambda\mu}e_{\mu\nu}.$$

The eigenvalues are manifestly distinct from (2.6.5) since $\lambda \neq \nu$. It follows that the coefficient $a_{\lambda\nu}$ is determined by the values of $a_{\lambda\mu}$ for $\nu < \mu \leqslant \lambda$. Hence the eigenfunctions of E satisfy the condition (2.6.6a). Furthermore, since E is self-adjoint, we readily see that

$$e_{\lambda\lambda}\langle P_\lambda, P_\mu\rangle_{q,t} = \langle EP_\lambda, P_\mu\rangle_{q,t} = \langle P_\lambda, EP_\mu\rangle_{q,t} = e_{\mu\mu}\langle P_\lambda, P_\mu\rangle_{q,t}.$$

As $e_{\lambda\lambda} \neq e_{\mu\mu}$ for $\lambda \neq \mu$ the condition (2.6.6b) also follows.

To prove uniqueness of the $\{P_\lambda\}$ we assume that P_μ is uniquely determined for all $\mu < \lambda$. Then there exist coefficients $v_{\lambda\mu} \in \mathbb{Q}(q,t)$ such that

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} v_{\lambda\mu}P_\mu.$$

Taking the inner product of both sides with P_μ for some fixed $\mu < \lambda$ we therefore see that

$$0 = \langle m_\lambda, P_\mu\rangle_{q,t} + v_{\lambda\mu}\langle P_\mu, P_\mu\rangle_{q,t}.$$

This determines $v_{\lambda\mu}$ uniquely provided $\langle P_\mu, P_\mu\rangle_{q,t}$ is nonzero. However, this is clear since the q,t-Hall scalar product is positive-definite when $0 < q,t < 1$ are real numbers. (We will in fact give an explicit formula for $\langle P_\mu, P_\mu\rangle_{q,t}$ below, from which this fact is also clear.) □

We will often write $P_\lambda(X)$ or P_λ in place of $P_\lambda(X;q,t)$ in what follows to simplify notation. Several properties of the Macdonald polynomials are immediate from the definition. Firstly, the set $\{P_\lambda\}_{\lambda\in\mathscr{P}}$ forms a basis for Λ since the monomial expansion (2.6.6a) is triangular (with respect to an appropriate total ordering compatible with the dominance ordering). This expansion also tells us that the P_λ are homogeneous polynomials of degree $|\lambda|$. If $\lambda = (1^r)$ then

$$P_{(1^r)}(X;q,t) = m_{(1^r)}(X) = e_r(X),$$

since $(1^r) \leqslant \mu$ for all partitions μ with $|\mu| = r$. Define the quantity $b_\lambda(q,t)$ by

$$b_\lambda = b_\lambda(q,t) := \frac{1}{\langle P_\lambda, P_\lambda\rangle_{q,t}}, \tag{2.6.7}$$

which we will compute explicitly later on. Denoting a scaled family of Macdonald polynomials as

$$Q_\lambda(X;q,t) := b_\lambda(q,t)P_\lambda(X;q,t)$$

it readily follows from (2.6.6b) that

$$\langle P_\lambda, Q_\mu\rangle_{q,t} = \delta_{\lambda\mu}.$$

By Proposition 2.6.2 we have the Cauchy identity

$$\sum_\lambda P_\lambda(X;q,t)Q_\lambda(Y;q,t) = \sigma_1\left[XY\frac{1-t}{1-q}\right]. \tag{2.6.8}$$

Given $f, g \in \Lambda^n$, it follows after expanding f and g in terms of the p_λ that

$$\langle f, g\rangle_{q^{-1},t^{-1}} = \frac{t^n}{q^n}\langle f, g\rangle_{q,t}.$$

This implies that the functions $P_\lambda(X; q^{-1}, t^{-1})$ also satisfy the conditions of Theorem 2.6.3, and so

$$P_\lambda(X; q, t) = P_\lambda(X; q^{-1}, t^{-1}). \tag{2.6.9}$$

Since $P_\lambda(X; q, t)$ is expressed as a sum of $m_\mu(X)$ where $\mu \leqslant \lambda$, we have that $P_\lambda(X_n; q, t) = 0$ if $l(\lambda) > n$. If we assume that $l(\lambda) > n$ then for any $\mu \leqslant \lambda$ Proposition 2.1.2 implies that $\mu' \geqslant \lambda'$. As $l(\mu) = \mu'_1$ and $l(\lambda) = \lambda'_1$ there holds $l(\mu) \geqslant \ell(\lambda) > n$. Hence each of the $m_\mu(X_n)$ on the right-hand side of (2.6.6a) vanish, and thus $P_\lambda(X_n)$ vanishes identically.

For later use we note that for X_n a finite alphabet the action of $D_n(z; q, t)$ on the $P_\lambda(X_n; q, t)$ is given by [Mac95, p. 324]

$$D_n(z; q, t)P_\lambda(X_n; q, t) = P_\lambda(X_n; q, t)\prod_{i=1}^{n}\left(1 + zq^{\lambda_i}t^{n-i}\right).$$

In particular,

$$D_n^1 P(X_n; q, t) = P_\lambda(X_n; q, t)\sum_{i=1}^{n} q^{\mu_i}t^{n-i}. \tag{2.6.10}$$

Most of the above properties of the P_λ should look familiar to the reader from the prior sections as properties of the Schur functions. In fact, setting $q = t$ in the Macdonald polynomial $P_\lambda(X; q, t)$ we recover the Schur function $s_\lambda(X)$. This follows from the fact that both $\{s_\lambda(X) : \lambda \vdash k\}$ and $\{P_\lambda(X; t, t) : \lambda \vdash k\}$ form bases for Λ^k over the field $\mathbb{Q}(t)$ which are orthogonal under the Hall scalar product and have a unitriangular expansion in terms of the monomial symmetric functions. Since these properties uniquely determine the Schur functions, we must have $P_\lambda(x; t, t) = s_\lambda(X)$. For $q = 0$ the Macdonald polynomials reduce to the Hall–Littlewood polynomials, which themselves are a one-parameter deformation of the Schur functions. We will not discuss the Hall–Littlewood polynomials in any detail, but their definition and properties may be found in [Mac95, §3]. Of crucial importance to this thesis is a different one-parameter deformation of the Schur functions known as the Jack polynomials. Assuming that $0 < q < 1$ is a real number, the Jack polynomial indexed by a parameter $\alpha \in \mathbb{C}$ is defined as [Mac95, §6.10]

$$P_\lambda^{(\alpha)}(X) := \lim_{t\to 1} P_\lambda(X; t^\alpha, t). \tag{2.6.11}$$

Setting $\alpha = 1$ in this expression, the Jack polynomial reduces to the Schur function, since this is the same as setting $q = t$ on the right. We will primarily be concerned with the Jack polynomials as an analytic object, but there is much interesting combinatorics associated with the functions $P_\lambda^{(\alpha)}$, see e.g., [Sta89]. In fact we will almost exclusively see Jack polynomials with parameter $\alpha = 1/\gamma$ for $\gamma \in \mathbb{C} \setminus \{0\}$, in which case $P_\lambda^{1/\gamma} = \lim_{q\to 1} P_\lambda(q, q^\gamma)$.

Next we will prove some results regarding Macdonald polynomials on specialised alphabets. It will be useful to introduce some additional notation, and for a partition $\lambda \in \mathscr{P}_n$ define the spectral vector

$$\begin{aligned}\langle\lambda\rangle_{n;q,t} = \langle\lambda\rangle_n &:= \left(q^{\lambda_1}t^{n-1}, q^{\lambda_2}t^{n-2}, \dots, q^{\lambda_{n-1}}t, q^{\lambda_n}t^0\right) \\ &= q^{\lambda_1}t^{n-1} + q^{\lambda_2}t^{n-2} + \cdots + q^{\lambda_{n-1}}t + q^{\lambda_n}t^0,\end{aligned} \tag{2.6.12}$$

where the second expression is the equivalent expression in plethystic notation. We refer to the Macdonald polynomial on the alphabet $\langle\mu\rangle_n$ as the specialisation of P_λ by the partition μ. In particular the principal specialisation may be written as

$$P_\lambda[\langle 0\rangle_n] = P_\lambda\left[\frac{1-t^n}{1-t}\right].$$

We note that the principal specialisation of a Macdonald polynomial is never identically zero where the Macdonald polynomial is nonzero. This follows from the explicit formula given in Proposition 2.6.6 below.

Specialisation of a Macdonald polynomial by a partition satisfies a symmetry in the indexing partition and the specialising partition. This was originally proved by Koornwinder in an unpublished manuscript [Koo88]. The first published proof appeared in Macdonald's book [Mac95, p. 332].

Theorem 2.6.4 (Koornwinder–Macdonald evaluation symmetry)**.** *Let λ, μ be partitions of length at most n. Then*

$$P_\lambda[\langle 0\rangle_n]P_\mu[\langle\lambda\rangle_n] = P_\mu[\langle 0\rangle_n]P_\lambda[\langle\mu\rangle_n]. \tag{2.6.13}$$

Our proof of this theorem hinges on the e-Pieri rule for Macdonald polynomials, which generalises the e-Pieri rule for Schur functions (2.5.21b). Unlike the Schur case, the coefficients in the expansion of $e_r P_\mu$ are not all one. For a pair of partitions $\lambda, \mu \in$ such that $\mu' \prec \lambda'$ we define

$$B_{\lambda/\mu} := t^{n(\lambda)-n(\mu)} \prod_{1\leqslant i<j\leqslant l(\lambda)} \frac{1-q^{\mu_i-\mu_j}t^{\lambda_i-\lambda_j-\mu_i+\mu_j+j-i}}{1-q^{\mu_i-\mu_j}t^{j-i}}.$$

Since λ/μ is a vertical strip, $\lambda_i - \mu_i$ is either 0 or 1 depending on whether there is a box in row i of λ/μ. Let $I \subseteq \{1, \dots, l(\lambda)\}$ be the set of indices i for which $\lambda_i - \mu_i = 1$. Then $B_{\lambda/\mu}$ admits the expression

$$B_{\lambda/\mu} := t^{n(\lambda)-n(\mu)} \prod_{\substack{i\in I\\ j\notin I\\ i<j}} \frac{1-q^{\mu_i-\mu_j}t^{j-i+1}}{1-q^{\mu_i-\mu_j}t^{j-i}} \prod_{\substack{i\notin I\\ j\in I\\ i<j}} \frac{1-q^{\mu_i-\mu_j}t^{j-i-1}}{1-q^{\mu_i-\mu_j}t^{j-i}}. \tag{2.6.14}$$

Now assume r is a nonnegative integer and $\lambda, \mu \in \mathscr{P}_n$. We require the e-Pieri rule for Macdonald polynomials in the form

$$\frac{e_r P_\mu}{P_\mu[\langle 0\rangle_n]} = \sum_{\substack{\lambda'\succ\mu'\\ |\lambda/\mu|=r}} B_{\lambda/\mu}\frac{P_\lambda}{P_\lambda[\langle 0\rangle_n]}. \tag{2.6.15}$$

For $q = t$ this is indeed equivalent to (2.5.21b) via the principal specialisation formula (2.5.12). In [Mac95, p. 332], Macdonald proves (2.6.15) simultaneously with Theorem 2.6.4. However we will have no need for the Pieri formula outside of the proof of the theorem, and so we assume the result. In addition we will need the similar identity

$$e_r[\langle\lambda\rangle_n]P_\lambda[\langle\nu\rangle_n] = \sum_{\substack{\mu'\succ\nu' \\ |\mu/\nu|=r}} B_{\mu/\nu}P_\lambda[\langle\mu\rangle_n], \tag{2.6.16}$$

where $\lambda, \nu \in \mathscr{P}_n$. This follows from the fact that the action of the operator D_n^r on $P_\lambda(X_n)$ is [Mac95, p. 324]

$$D_n^r P_\lambda(X_n) = e_r[\langle\lambda\rangle_n]P_\lambda(X_n).$$

Proof of Theorem 2.6.4. Let λ, μ be as in the statement of the theorem. We proceed by induction on $|\mu|$. When $\mu = 0$ both Macdonald polynomials indexed by μ equal 1 and the statement holds for any partition λ of length at most n.

Now assume that (2.6.13) is true for all λ with length at most n and all σ such that $|\sigma| < |\mu|$ or $|\sigma| = |\mu|$ and $\sigma < \mu$. Let ω be another partition with $|\omega| < |\mu|$. We write (2.6.15) with alphabet $\langle\lambda\rangle_n$ and $(\lambda,\mu) \mapsto (\nu,\omega)$, so

$$e_r[\langle\lambda\rangle_n]\frac{P_\omega[\langle\lambda\rangle_n]}{P_\omega[\langle 0\rangle_n]} = \sum_{\substack{\nu'\succ\omega' \\ |\nu/\omega|=r}} B_{\nu/\omega}\frac{P_\nu[\langle\lambda\rangle_n]}{P_\nu[\langle 0\rangle_n]}.$$

By the inductive hypothesis we may interchange the partitions ω and λ. We may also interchange the partitions in the sum with λ whenever $\nu \neq \omega$. This yields the expansion

$$e_r[\langle\lambda\rangle_n]\frac{P_\lambda[\langle\omega\rangle_n]}{P_\lambda[\langle 0\rangle_n]} = B_{\mu/\omega}\frac{P_\mu[\langle\lambda\rangle_n]}{P_\mu[\langle 0\rangle_n]} + \sum_{\nu<\mu} B_{\nu/\omega}\frac{P_\nu[\langle\lambda\rangle_n]}{P_\nu[\langle 0\rangle_n]}. \tag{2.6.17}$$

Here the restriction that $\nu \supset \omega$ with ν/ω a vertical r-strip on the summation $\nu < \mu$ still holds. If we instead begin with equation (2.6.16) under the substitution $(\lambda,\mu,\nu) \mapsto (\lambda,\nu,\omega)$ we obtain

$$e_r[\langle\lambda\rangle_n]\frac{P_\lambda[\langle\omega\rangle_n]}{P_\lambda[\langle 0\rangle_n]} = B_{\mu/\omega}\frac{P_\lambda[\langle\mu\rangle_n]}{P_\lambda[\langle 0\rangle_n]} + \sum_{\nu<\mu} B_{\nu/\omega}\frac{P_\lambda[\langle\nu\rangle_n]}{P_\lambda[\langle 0\rangle_n]}, \tag{2.6.18}$$

with the same restrictions on the summand as above. Again by the inductive hypothesis we are free to exchange λ and ν in the summand on the right-hand side. If we do so then the only difference between (2.6.17) and (2.6.18) is that the roles of λ and μ in the leading terms have been swapped. Equating these expressions and cancelling like terms leaves the equation

$$\frac{P_\lambda[\langle\mu\rangle_n]}{P_\lambda[\langle 0\rangle_n]} = \frac{P_\mu[\langle\lambda\rangle_n]}{P_\mu[\langle 0\rangle_n]}.$$ □

We will in fact need a more general version of this evaluation symmetry. The following lemma is a nonsymmetric version of [War10, Proposition 2.1].

Lemma 2.6.5. *For λ, μ partitions with $l(\lambda) \leqslant n$ and $l(\mu) \leqslant m$ and a an indeterminate,*

$$P_\mu\left[\frac{1-a}{1-t}\right]P_\lambda\left[at^{-m}\langle\mu\rangle_m + \frac{1-at^{-m}}{1-t}\right] = P_\lambda\left[\frac{1-a}{1-t}\right]P_\mu\left[at^{-n}\langle\lambda\rangle_n + \frac{1-at^{-n}}{1-t}\right].$$

Proof. We first prove the lemma for $n = m$. Scaling $a \mapsto at^n$ gives

$$P_\mu\left[\frac{1-at^n}{1-t}\right]P_\lambda\left[a\langle\mu\rangle_n + \frac{1-a}{1-t}\right] = P_\lambda\left[\frac{1-at^n}{1-t}\right]P_\mu\left[a\langle\lambda\rangle_n + \frac{1-a}{1-t}\right]. \tag{2.6.19}$$

This is an identity involving polynomials of degree $|\lambda| + |\mu|$ in the parameter a. Hence it suffices to prove the above identity for $a = t^k$ with k a positive integer. Setting $a = t^k$ and observing that

$$t^k\langle\lambda\rangle_n + \frac{1-t^k}{1-t} = \langle\lambda\rangle_{n+k},$$

it follows that (2.6.19) reduces to

$$P_\mu\left[\frac{1-t^{n+k}}{1-t}\right]P_\lambda[\langle\mu\rangle_{n+k}] = P_\lambda\left[\frac{1-t^{n+k}}{1-t}\right]P_\mu[\langle\lambda\rangle_{n+k}].$$

This is simply Theorem 2.6.4 for $(n+k)$-letter alphabets, and so the lemma is proved for $n = m$. Now fix m to be an integer such that $l(\mu) \leqslant m \leqslant n$. Then

$$a\langle\mu\rangle_n = at^{n-m}\langle\mu\rangle_m + a\frac{1-t^{n-m}}{1-t},$$

so we may rewrite (2.6.19) as

$$P_\mu\left[\frac{1-at^n}{1-t}\right]P_\lambda\left[at^{n-m}\langle\mu\rangle_m + \frac{1-at^{n-m}}{1-t}\right] = P_\lambda\left[\frac{1-at^n}{1-t}\right]P_\mu\left[a\langle\lambda\rangle_n + \frac{1-a}{1-t}\right].$$

Now scale a by $a \mapsto at^{-n}$ to obtain

$$P_\mu\left[\frac{1-a}{1-t}\right]P_\lambda\left[at^{-m}\langle\mu\rangle_m + \frac{1-at^{-m}}{1-t}\right] = P_\lambda\left[\frac{1-a}{1-t}\right]P_\mu\left[at^{-n}\langle\lambda\rangle_n + \frac{1-at^{-n}}{1-t}\right],$$

which is symmetric in m and n, and so the restriction $m \leqslant n$ may be dropped. □

The specialised Macdonald polynomials appearing in the evaluation symmetries stated above may in fact be evaluated explicitly. In order to state specialisation these formulas succinctly we will need some finite analogues of (2.3.11). For a nonnegative integer n and a an indeterminate we define (see also Section 3.1)

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}). \tag{2.6.20}$$

For another indeterminate z this is extended to partitions as

$$(z;q,t)_\lambda := \prod_{s\in\lambda}(1-zq^{a'(s)}t^{-l'(s)}) = \prod_{i=1}^{n}(zt^{1-i};q)_{\lambda_i}, \tag{2.6.21}$$

where the second equality follows easily from the definitions of the arm and leg colengths. Also, on the right we may choose any $n \geqslant l(\lambda)$. We also have need of the generalised hook polynomials. Define these by

$$c_\lambda(q,t) := \prod_{s\in\lambda}(1-q^{a(s)}t^{l(s)+1}) = \prod_{i=1}^{n}(t^{n-i+1};q)_{\lambda_i}\prod_{1\leqslant i<j\leqslant n}\frac{(t^{j-i};q)_{\lambda_i-\lambda_j}}{(t^{j-i+1};q)_{\lambda_i-\lambda_j}}, \tag{2.6.22a}$$

$$c'_\lambda(q,t) := \prod_{s\in\lambda}(1-q^{a(s)+1}t^{l(s)}) = \prod_{i=1}^{n}(qt^{n-i};q)_{\lambda_i}\prod_{1\leqslant i<j\leqslant n}\frac{(qt^{j-i-1};q)_{\lambda_i-\lambda_j}}{(qt^{j-i};q)_{\lambda_i-\lambda_j}}, \tag{2.6.22b}$$

where again the choice of n is irrelevant as long as $n \geqslant l(\lambda)$. Some authors use the notation $C^-_\lambda(t;q,t)$ and $C^-_\lambda(q;q,t)$ for $c_\lambda(q,t)$ and $c'_\lambda(q,t)$ respectively; see (6.2.1) below.

Proposition 2.6.6. *For λ a partition of length at most n one has*

$$P_\lambda\biggl(\biggl[\frac{1-t^n}{1-t}\biggr];q,t\biggr)=\frac{t^{n(\lambda)}(t^n;q,t)_\lambda}{c_\lambda(q,t)}. \tag{2.6.23}$$

Proof. Fix a partition $\mu\in\mathscr{P}_n$ and consider the e-Pieri rule (2.6.15) in the form

$$e_r P_\mu(X_n)=P_\mu[\langle 0\rangle_n]\sum_{\substack{\lambda'\succ\mu'\\ |\lambda/\mu|=r}}B_{\lambda/\mu}\frac{P_\lambda(X_n)}{P_\lambda[\langle 0\rangle_n]},$$

where as usual $|X_n|=n$ and r is a nonnegative integer such that $r\leqslant n$. The coefficient of the monomial $X_n^{\mu+(1^r)}$ on the left-hand side of this equation is one by (2.6.6a). Hence the coefficient of the term indexed by $\lambda=\mu+(1^r)$ on the right-hand side is also one, which means that

$$P_{\mu+(1^r)}[\langle 0\rangle_n]=B_{(\mu+(1^r))/\mu}P_\mu[\langle 0\rangle_n]. \tag{2.6.24}$$

When both $\mu=0$ and $r=0$ both sides are equal to one and thus equality holds. Since every partition in $\mathscr{P}_n$ can be obtained by successively adding columns of height $\leqslant n$, it suffices to verify (2.6.24). In turn, this boils down to verifying that

$$t^{n(\mu)-n(\mu+(1^r))}B_{(\mu+(1^r))/\mu}=\frac{(t^n;q,t)_{\mu+(1^r)}\,c_\mu(q,t)}{(t^n;q,t)_\mu\,c_{\mu+(1^r)}(q,t)}.$$

We apply the definitions (2.6.21) and (2.6.22a) to the right-hand side, which therefore admits the simplification

$$\begin{aligned}\frac{(t^n;q,t)_{\mu+(1^r)}\,c_\mu(q,t)}{(t^n;q,t)_\mu\,c_{\mu+(1^r)}(q,t)}&=\prod_{i=1}^{r}\prod_{j=r+1}^{n}\frac{(t^{j-i+1};q)_{\mu_i-\mu_j+1}(t^{j-i};q)_{\mu_i-\mu_j}}{(t^{j-i};q)_{\mu_i-\mu_j+1}(t^{j-i+1};q)_{\mu_i-\mu_j}}\\&=\prod_{i=1}^{r}\prod_{j=r+1}^{n}\frac{1-q^{\mu_i-\mu_j}t^{j-i+1}}{1-q^{\mu_i-\mu_j}t^{j-i}},\end{aligned}$$

because $\lambda_i=\mu_i$ for $r<i\leqslant n$. This is in accordance with the left-hand side by the definition of $B_{(\mu+(1^r))/\mu}$ as (2.6.14) where $I=\{1,\dots,r\}$. □

By a polynomial argument, we can also claim the following Macdonald polynomial analogue of Lemma 2.5.6.

Corollary 2.6.7. *Let λ be a partition and a an indeterminate. Then*

$$P_\lambda\biggl(\biggl[\frac{1-a}{1-t}\biggr];q,t\biggr)=\frac{t^{n(\lambda)}(a;q,t)_\lambda}{c_\lambda(q,t)}. \tag{2.6.25}$$

We can also use the corollary to obtain a formula for the Jack polynomial evaluated at a binomial element z, which generalises the Schur case (2.5.13). To do so, make the substitutions $(a,q,t)\mapsto(q^{z\gamma},q,q^\gamma)$ and take the limit $q\to 1$ in (2.6.25), which gives

$$P_\lambda^{(1/\gamma)}[z]=\prod_{i\geqslant 1}\frac{((z+1-i)\gamma)_{\lambda_i}}{((n+1-i)\gamma)_{\lambda_i}}\prod_{1\leqslant i<j\leqslant n}\frac{((j-i+1)\gamma)_{\lambda_i-\lambda_j}}{((j-i)\gamma)_{\lambda_i-\lambda_j}}, \tag{2.6.26}$$

where n is any integer such that $n \geqslant l(\lambda)$.

Our next goal is to explicitly compute the squared norm of the P_λ, which we have previously denoted by $1/b_\lambda(q,t)$. To facilitate this computation we will need the following lemma, which generalises the reciprocity of the Schur functions (2.5.8).

Lemma 2.6.8. *We have*

$$P_\lambda\left(\left[-X\frac{1-q}{1-t}\right];q,t\right) = (-1)^{|\lambda|}Q_{\lambda'}(X;t,q). \tag{2.6.27}$$

Proof. Recall that $\langle\cdot,\cdot\rangle_{q,q}$ is the ordinary Hall scalar product. First we note that for any $f,g \in \Lambda$,

$$\left\langle f\left[-\varepsilon X\frac{1-t}{1-q}\right], g[X]\right\rangle_{q,t} = \langle f[-\varepsilon X], g[X]\rangle_{q,q},$$

since on the power sums the equality is clear. The assertion of the lemma is therefore equivalent to

$$\left\langle P_{\lambda'}\left(\left[-\varepsilon X\frac{1-t}{1-q}\right];t,q\right), P_\mu(X;q,t)\right\rangle_{q,t} = \delta_{\lambda\mu},$$

which by the above is the same as

$$\langle P_{\lambda'}([-\varepsilon X];t,q), P_\mu(X;q,t)\rangle_{q,q} = \delta_{\lambda\mu}. \tag{2.6.28}$$

Assume that the Macdonald polynomial has the Schur expansion

$$P_\lambda(X;q,t) = s_\lambda(X) + \sum_{\nu<\lambda} d_{\lambda\nu}(q,t)s_\nu(X).$$

The expansion is necessarily upper unitriangular as both s_λ and P_λ can be expanded in the basis given by the m_λ in an upper unitriangular fashion. Then, since the Schur functions are orthonormal, (2.6.28) reduces to

$$\sum_\nu d_{\lambda'\nu}(t,q)d_{\mu\nu'}(q,t) = \delta_{\lambda\mu}.$$

Let $D(q,t) := \big(d_{\lambda\mu}(q,t)\big)$ and $T := \big(\delta_{\lambda'\mu}\big)$ be matrices with respect to some total order on the partitions of n which is compatible with the dominance order. Then we need to show that

$$TD(t,q)TD^t(q,t) = I, \tag{2.6.29}$$

where I is the identity matrix. To prove this we will need the matrix $S(q,t) := \big(\langle s_\lambda, s_\mu\rangle_{q,t}\big)_{\lambda,\mu\in\mathscr{P}}$ and some of its properties. Firstly we note that $S(q,t) = S^{-1}(t,q) = TS(q,t)T$; see [Mac95, p. 328] for a proof. Secondly, the matrix product $D(q,t)S(q,t)D^t(q,t)$ is diagonal, which follows from the orthogonality of the Macdonald polynomials. By Lemma 2.1.2 the matrix $TD(t,q)T$ will be lower unitriangular, and hence the product $TD(t,q)TD^t(q,t)$ is itself lower unitriangular. We now compute

$$\begin{aligned}
D(q,t)S(q,t)D^t(q,t)\big(TD(t,q)TD^t(q,t)\big)^{-1} &= D(q,t)S(q,t)TD^{-1}(t,q)T \\
&= D(q,t)TS^{-1}(t,q)D^{-1}(t,q)T \\
&= D(q,t)TD^t(t,q)\big(D(q,t)S(q,t)D^t(q,t)\big)^{-1}T.
\end{aligned}$$

The expression we started with is lower unitriangular since it is the product of a diagonal matrix and the inverse of a lower unitriangular matrix. The last line of this chain of equalities is in fact upper unitriangular, since by

$$D(q,t)TD^t(t,q) = \big(D(q,t)TD^t(t,q)T\big)^t T,$$

the expression is the product of an upper unitriangular matrix and a diagonal matrix. Thus follows (2.6.29) follows since $TD(t,q)TD^t(q,t)$ is both upper and lower unitriangular. □

The above proof is due to Garsia [Gar92, Theorem 1.2]. For a different proof see Macdonald [Mac95, p. 329].

We will now compute $b_\lambda(q,t)$ from (2.6.7) As a first step we make the simultaneous substitutions

$$(\lambda, X, q, t) \mapsto \left(\lambda', \frac{1-a}{t-1}, t, q\right)$$

in (2.6.27), which gives

$$P_{\lambda'}\left(\left[\frac{1-a}{1-q}\right]; t, q\right) = (-1)^{|\lambda|} b_\lambda(q,t) P_\lambda\left(\left[\frac{a-1}{1-t}\right]; q, t\right).$$

Recalling from (2.6.9) that $P_\lambda(X;q,t) = P_\lambda(X;q^{-1},t^{-1})$ we may express $b_\lambda(q,t)$ as

$$b_\lambda(q,t) = \frac{(-t)^{|\lambda|} P_{\lambda'}([\frac{1-a}{1-q}]; t, q)}{P_\lambda([\frac{1-a}{1-t^{-1}}]; q^{-1}, t^{-1})}.$$

Applying Corollary 2.6.7 we have

$$b_\lambda(q,t) = (-t)^{|\lambda|} q^{n(\lambda')} t^{n(\lambda)} \frac{(a;t,q)_{\lambda'} c_\lambda(q^{-1},t^{-1})}{(a;q^{-1},t^{-1})_\lambda c_{\lambda'}(t,q)},$$

which, by the definitions (2.6.21) and (2.6.22), is the same as

$$b_\lambda(q,t) = (-t)^{|\lambda|} \prod_{s\in\lambda} t^{l'(s)} \frac{1-q^{-a(s)}t^{-l(s)-1}}{1-aq^{-a'(s)}t^{l'(s)}} \prod_{s\in\lambda'} q^{l'(s)} \frac{1-aq^{-l'(s)}t^{a'(s)}}{1-q^{l(s)+1}t^{a(s)}}.$$

The product over $s \in \lambda'$ may be turned into a product over $s \in \lambda$ by interchanging arm and leg lengths with arm and leg colengths, thus giving

$$b_\lambda(q,t) = (-t)^{|\lambda|} \prod_{s\in\lambda} q^{a'(s)} t^{l'(s)} \frac{1-q^{-a(s)}t^{-l(s)-1}}{1-q^{a(s)+1}t^{l(s)}}.$$

For the final step we note that

$$\sum_{s\in\lambda} a(s) = \sum_{s\in\lambda} a'(s), \quad \text{and} \quad \sum_{s\in\lambda} l(s) = \sum_{s\in\lambda} l'(s),$$

so that the expression

$$b_\lambda(q,t) = \prod_{s\in\lambda} \frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}} = \frac{c_\lambda(q,t)}{c'_\lambda(q,t)}, \tag{2.6.30}$$

readily follows.

We have already met the skew Schur functions $s_{\lambda/\mu}$, which are indexed by skew shapes. In an analogous way we may define the skew Macdonald polynomials. First define the q, t-Littlewood–Richardson coefficients by

$$P_\mu P_\nu = \sum_\lambda f^\lambda_{\mu\nu} P_\lambda, \tag{2.6.31}$$

which is equivalent to requiring that

$$f^\lambda_{\mu\nu} = \langle P_\mu P_\nu, Q_\lambda \rangle_{q,t}.$$

Then the skew Macdonald polynomial $Q_{\lambda/\mu}$ is defined by

$$Q_{\lambda/\mu} = \sum_\nu f^\lambda_{\mu\nu} Q_\nu. \tag{2.6.32}$$

From this construction it is immediate that $f^\lambda_{\mu\nu}|_{q=t} = c^\lambda_{\mu\nu}$ and that $Q_{\lambda/\mu}(X; q, q) = s_{\lambda/\mu}(X)$. Furthermore, we have $f^\lambda_{\mu\nu} = 0$ unless $|\lambda| = |\mu| + |\nu|$ since $Q_\mu Q_\nu$ is homogeneous of degree $|\mu| + |\nu|$. In fact, we have the stronger result that $f^\lambda_{\mu\nu} = 0$ unless $\mu, \nu \subseteq \lambda$. To prove this, let I_μ be the ideal of Λ spanned by the P_λ such that $\lambda \supseteq \mu$. Then by the Pieri rule (2.6.15) it follows that $e_r I_\mu \subseteq I_\mu$. Since the e_r generate Λ as a $\mathbb{Q}(q,t)$-algebra, I_μ must be an ideal of Λ. Thus the product $P_\mu P_\nu$ lies in $I_\mu \cap I_\nu$, which means $\mu, \nu \subseteq \lambda$ for $f^\lambda_{\mu\nu}$ to be nonvanishing.

We can also define a skew function $P_{\lambda/\mu}$ by

$$\langle P_{\lambda/\mu}, Q_\nu \rangle_{q,t} = \langle P_\lambda, Q_\mu Q_\nu \rangle_{q,t},$$

from which we see that

$$Q_{\lambda/\mu} = \frac{b_\lambda(q,t)}{b_\mu(q,t)} P_{\lambda/\mu}. \tag{2.6.33}$$

The vanishing properties of $f^\lambda_{\mu\nu}$ also imply that $P_{\lambda/\mu} = Q_{\lambda/\mu} = 0$ unless $\mu \subseteq \lambda$, and both $P_{\lambda/\mu}$ and $Q_{\lambda/\mu}$ are homogeneous of degree $|\lambda| - |\mu|$. Following the same steps from (2.5.17) we may claim a skew analogue of the Cauchy identity (2.6.8)

$$\sum_\lambda P_\lambda(X; q, t) Q_{\lambda/\mu}(Y; q, t) = P_\mu(X; q, t)\sigma_1\left[XY\frac{1-t}{1-q}\right]. \tag{2.6.34}$$

Using this identity as well as the ordinary Cauchy identity leads us to

$$\begin{aligned}
\sum_{\lambda,\mu} Q_{\lambda/\mu}(X)P_\lambda(Y)Q_\mu(Z) &= \sigma\left[XY\frac{1-t}{1-q}\right]\sum_\mu P_\mu(Y)Q_\mu(Z) \\
&= \sigma\left[YZ\frac{1-t}{1-q}\right]\sigma\left[XY\frac{1-t}{1-q}\right] \\
&= \sigma\left[Y(X+Z)\frac{1-t}{1-q}\right] \\
&= \sum_\lambda P_\lambda(Y)Q_\lambda[X+Y].
\end{aligned}$$

Equating coefficients of $P_\lambda(Y)$ on both sides gives

$$Q_\lambda[X+Z] = \sum_\mu Q_{\lambda/\mu}(X)Q_\mu(Z),$$

and by (2.6.33),

$$P_\lambda[X+Z] = \sum_\mu P_{\lambda/\mu}(X)P_\mu(Z). \tag{2.6.35}$$

Both of these expressions generalise the formula for the Schur function on the sum of alphabets (2.5.19).

The skew Cauchy identity (2.6.34) allows us to prove a formula for the Macdonald polynomial on the difference of two single-letter alphabets when the indexing partition has a single row. In what follows let

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a;q)_k(b;q)_k}{(c;q)_k(q;q)_k} z^k$$

denote the usual q-analogue of the ${}_2F_1$ Gauss hypergeometric function [AAR99, GR04].

Lemma 2.6.9. *Let x and y be single-letter alphabets. Then*

$$P_{(r)}([x-y];q,t) = x^r\, {}_2\phi_1\left[\begin{matrix} t^{-1}, q^{-r} \\ q^{1-r}t^{-1} \end{matrix}; q, \frac{qy}{x}\right]. \tag{2.6.36}$$

Proof. If we set $\mu = 0$ in (2.6.34) and then replace $(X, Y) \mapsto (x - y, 1)$ we obtain

$$\sum_{r\geqslant 0} \frac{(t;q)_r}{(t;t)_r} P_{(r)}([x-y];q,t) = \sigma_1\left[\frac{1-t}{1-q}(x-y)\right] = \frac{(tx;q)_\infty(y;q)_\infty}{(x;q)_\infty(ty;q)_\infty}.$$

Using the q-binomial theorem [GR04, Equation (II.3)] (also see (3.3.1) below) to expand the right-hand side as a power series in x and y leads to

$$\sum_{r\geqslant 0} \frac{(t;q)_r}{(t;t)_r} P_{(r)}([x-y];q,t) = \sum_{k,\ell\geqslant 0} \frac{(t;q)_\ell(t^{-1};q)_k}{(q;q)_\ell(q;q)_k} x^\ell (ty)^k.$$

Equating terms of homogeneous degree r in x, y gives

$$\frac{(t;q)_r}{(t;t)_r} P_{(r)}([x-y];q,t) = \sum_{k=0}^{r} \frac{(t;q)_{r-k}(t^{-1};q)_k}{(q;q)_{r-k}(q;q)_k} x^{r-k}(ty)^k,$$

which is equivalent to (2.6.36). □

To conclude this chapter we introduce another scalar product on Λ_n, which we now take over $\mathbb{C}$. We further assume that $q, t \in \mathbb{C}$ are such that $|q|, |t| < 1$, and adopt the shorthand notation

$$(a_1, \dots, a_n; q)_\infty := \prod_{i=1}^{n} (a_i;q)_\infty.$$

Then for $f, g \in \Lambda_n$ we define the scalar product [Mac95, p. 372]

$$\langle f, g\rangle'_n := \frac{1}{n!(2\pi\mathrm{i})^n} \int_{\mathbb{T}^n} f(z)g(z^{-1}) \prod_{1\leqslant i<j\leqslant n} \frac{(z_i/z_j, z_j/z_i;q)_\infty}{(tz_i/z_j, tz_j/z_i;q)_\infty} \frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_n}{z_n}, \tag{2.6.37}$$

where $\mathbb{T}^n := \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| = 1\}$ is the complex n-torus and $z^{-1} := (1/z_1, \dots, 1/z_n)$. It is clear from the definition that for any $f \in \Lambda_n$,

$$\langle f(z)\,\cdot, \cdot\rangle'_n = \langle \cdot, f(z^{-1})\,\cdot\rangle'_n$$

Under this new scalar product the operator D_n^1 ((2.6.2) with $r = 1$) is still self-adjoint. To see this, we first note that for $u(z)$ and $v(z)$ Laurent polynomials over $\mathbb{C}$,

$$\int_{\mathbb{T}^n} T_{q,z_k}\big(u(z)\big)v\big(z^{-1}\big)\,\frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_n}{z_n} = \int_{\mathbb{T}^n} T_{q,z_k}\big(v(z)\big)u\big(z^{-1}\big)\,\frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_n}{z_n}. \tag{2.6.38}$$

This itself follows from the fact that for a Laurent polynomial $f(z)$,

$$\mathrm{CT} f(z) = \frac{1}{(2\pi\mathrm{i})^n}\int_{\mathbb{T}^n} f(z)\,\frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_n}{z_n},$$

where $\mathrm{CT} f(z)$ denotes the constant term of $f(z)$. Note that his also implies that scalar product (2.6.37) is symmetric. Thus by linearity we may assume u and v are monomials in (2.6.38), and then the result is clear. Now again by linearity we need only verify that

$$\frac{1}{n!(2\pi\mathrm{i})^n}\int_{\mathbb{T}^n} T_{q,z_k}\big(f(z)\big)g\big(z^{-1}\big)\prod_{\substack{i=1\\ i\neq k}}^{n}\frac{1-tz_k/z_j}{1-z_k/z_j}\prod_{1\leqslant i<j\leqslant n}\frac{(z_i/z_j,z_j/z_i;q)_\infty}{(tz_i/z_j,tz_j/z_i;q)_\infty}\,\frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_n}{z_n},$$

is symmetric in f and g for fixed $1 \leqslant k \leqslant n$. By the identity

$$\prod_{\substack{i=1\\ i\neq k}}^{n}\frac{1-tz_k/z_j}{1-z_k/z_j} = T_{q,z_k}\bigg(\prod_{1\leqslant i<j\leqslant n}\frac{(z_i/z_j;q)_\infty}{(tz_i/z_j;q)_\infty}\bigg)\prod_{1\leqslant i<j\leqslant n}\frac{(z_j/z_i;q)_\infty}{(tz_j/z_i;q)_\infty},$$

we may rewrite this integral as

$$\frac{1}{n!(2\pi\mathrm{i})^n}\int_{\mathbb{T}^n} T_{q,z_k}\bigg(f(z)\prod_{1\leqslant i<j\leqslant n}\frac{(z_i/z_j;q)_\infty}{(tz_i/z_j;q)_\infty}\bigg)g\big(z^{-1}\big)\prod_{1\leqslant i<j\leqslant n}\frac{(z_j/z_i;q)_\infty}{(tz_j/z_i;q)_\infty}\,\frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_n}{z_n}.$$

An application of (2.6.38) to this expression shows the desired symmetry in f and g. Since the eigenvalues of $P_\lambda(X_n;q,t)$ with respect to D_n^1 are distinct by (2.6.10), the orthogonality of the Macdonald polynomials follows:

$$\langle P_\lambda, P_\mu\rangle_n' = 0 \quad \text{if } \lambda\neq\mu. \tag{2.6.39}$$

The quadratic norm under this new scalar product may also be evaluated explicitly as [Mac95, p. 369]

$$\langle P_\lambda, Q_\mu\rangle_n' = \delta_{\lambda\mu}\frac{(t^n;q,t)_\lambda}{(qt^{n-1};q,t)_\lambda}\prod_{i=1}^{n}\frac{(t,qt^{i-1};q)_\infty}{(q,t^i;q)_\infty},$$

or, equivalently by (2.6.30),

$$\langle P_\lambda, P_\mu\rangle_n' = \delta_{\lambda\mu}\frac{(t^n;q,t)_\lambda c_\lambda'(q,t)}{(qt^{n-1};q,t)_\lambda c_\lambda(q,t)}\prod_{i=1}^{n}\frac{(t,qt^{i-1};q)_\infty}{(q,t^i;q)_\infty}. \tag{2.6.40}$$

Importantly, the P_λ may also be characterised as per Theorem 2.6.3 but with the q,t-Hall scalar product replaced by (2.6.37). We now use this fact to prove a complementation formula for the q,t-Littlewood–Richardson coefficients. Let λ be a partition contained in some rectangle (N^n). Recall that $\hat{\lambda}$ denotes the complement of λ inside this rectangle. For $z = (z_1, \dots, z_n)$ a finite alphabet, we first prove that [BF99, Equation (4.3)]

$$P_{\hat{\lambda}}(z) = (z_1 \cdots z_n)^N P_\lambda\big(z^{-1}\big). \tag{2.6.41}$$

The leading order term on both sides of this equation is given by $m_{\hat{\lambda}}(z)$. For any pair of partitions λ, μ we then have that

$$\big\langle (z_1 \cdots z_n)^N P_\lambda\big(z^{-1}\big), (z_1 \cdots z_n)^N P_\mu\big(z^{-1}\big)\big\rangle_n' = \langle P_\mu(z), P_\lambda(z)\rangle_n' = \langle P_\lambda(z), P_\mu(z)\rangle_n'.$$

So both sides of (2.6.41) satisfy the conditions which characterise the Macdonald polynomials, and therefore must be identical. Now fix another pair of partitions μ, ν such that $\mu, \nu \subseteq \lambda \subseteq (N^n)$. We wish to prove the identity

$$f_{\hat{\lambda}\nu}^{\hat{\mu}} = \frac{Q_\mu\big[\frac{1-qt^{n-1}}{1-t}\big]}{Q_\lambda\big[\frac{1-qt^{n-1}}{1-t}\big]} \frac{P_\lambda[\langle 0\rangle_n]}{P_\mu[\langle 0\rangle_n]} f_{\mu\nu}^{\lambda}. \tag{2.6.42}$$

By the orthogonality (2.6.39) the $f_{\hat{\lambda}\nu}^{\hat{\mu}}$ may be expressed as

$$f_{\hat{\lambda}\nu}^{\hat{\mu}} = \frac{\langle P_{\hat{\lambda}}(z) P_\nu(z), P_{\hat{\mu}}(z)\rangle_n'}{\langle P_{\hat{\mu}}(z), P_{\hat{\mu}}(z)\rangle_n'} = \frac{\langle P_\mu(z) P_\nu(z), P_\lambda(z)\rangle_n'}{\langle P_\mu(z), P_\mu(z)\rangle_n'} = \frac{\langle P_\lambda(z), P_\lambda(z)\rangle_n'}{\langle P_\mu(z), P_\mu(z)\rangle_n'} f_{\mu\nu}^{\lambda}.$$

All that remains is to use (2.6.40) and the specialisation formula (2.6.25) to evaluate the ratio of quadratic norms on the right to complete the proof.

Chapter 3

Cauchy-type identities

The goal of this chapter is to prove several A_n generalisations of the classical (skew) Cauchy identity (2.6.34). These formulas are based on identities for skew Macdonald polynomials, an approach that was developed by Warnaar in [War08a, War09, War10].

3.1 Hypergeometric preliminaries

We begin this chapter by introducing some of the hypergeometric functions and functional relations required for what follows. Firstly, for n a nonnegative integer define the Pochhammer symbol (or rising factorial or shifted factorial) by

$$(a)_n := (a)(a+1)\cdots(a+n-1) \tag{3.1.1}$$

for n a positive integer and $(a)_0 := 1$. Since the product here is finite we allow a to be any complex number or simply an indeterminate. For any $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$, define the Euler gamma function

$$\Gamma(z) := \int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t. \tag{3.1.2}$$

The recurrence $\Gamma(z+1) = z\Gamma(z)$ then follows by integration by parts. Using this we may extend the gamma function to a meromorphic function with simple poles at the nonpositive integers. Moreover, since the gamma function has no zeros, $1/\Gamma(z)$ is an entire function. We also note that $\Gamma(1) = 1$ automatically implies that $\Gamma(n+1) = n!$ for any nonnegative integer n. If $a \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ then we use the gamma function to extend the definition of (3.1.1) to any $z \in \mathbb{C}$ such that $a + z \neq 0, -1, -2, \dots$ by

$$(a)_z := \frac{\Gamma(a+z)}{\Gamma(a)}.$$

Notice that for $n \in \mathbb{N}$ this reduces to (3.1.1). In particular if n is a nonnegative integer (and $a+n$ is not a positive integer), then

$$\frac{1}{(a)_{-n}} = (a-n)_n.$$

We have already introduced the q-analogue of the Pochhammer symbol (2.6.20),

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

as well as its infinite analogue (2.3.11)

$$(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots .$$

This last product converges for $|q| < 1$, but in the case of the following identities for Macdonald polynomials it suffices to view the above as a formal power series in q. As for the ordinary Pochhammer symbol, we can extend the finite case to any $z \in \mathbb{C}$:

$$(a;q)_z := \frac{(a;q)_\infty}{(aq^z;q)_\infty},$$

where here, if treated analytically, we always take $0 < q < 1$ to be a real number in order to avoid fixing a branch. As before we have that this agrees with the previously-defined case when n is a nonnegative integer. In analogy with the expression for $(a)_{-n}$ we have

$$\frac{1}{(a;q)_{-n}} = (aq^{-n};q)_n.$$

In particular $1/(q;q)_{-n} = 0$ for all positive integers n. For use below we note the identity [GR04, Equation (I.11)]

$$\frac{(a;q)_{n-k}}{(b;q)_{n-k}} = \left(\frac{b}{a}\right)^k \frac{(a;q)_n}{(b;q)_n}\frac{(q^{1-n}/b;q)_k}{(q^{1-n}/a;q)_k}, \tag{3.1.3}$$

whose verification we leave to the reader. There is also a q-analogue of the gamma function (3.1.2), which is given by

$$\Gamma_q(z) := (1-q)^{1-z}(q;q)_{z-1}, \tag{3.1.4}$$

where again $0 < q < 1$. Taking the limit $q \to 1$ returns the ordinary gamma function, and a proof of this fact may be found in [Koo90, Appendix B]. When turning products q-shifted factorials into products of q-gamma functions later on it is useful to note that

$$\frac{\Gamma_q(a+z)}{\Gamma_q(a)} = \frac{(q^a;q)_z}{(1-q)^z}.$$

Throughout this chapter it will be useful to define the following oft-occurring hypergeometric term. For $\lambda, \mu \in \mathscr{P}$ and any $k \in \mathbb{N}$, $\ell \in \mathbb{N} \cup \{\infty\}$ such that $k \geqslant l(\lambda)$ and $\ell \geqslant l(\mu)$ we define [War10]

$$f^{k,\ell}_{\lambda,\mu}(a;q,t) := t^{-k|\mu|}\prod_{i=1}^{k}\prod_{j=1}^{\ell}\frac{(aqt^{j-i-1};q)_{\lambda_i-\mu_j}}{(aqt^{j-i};q)_{\lambda_i-\mu_j}}. \tag{3.1.5}$$

By (3.1.3) it follows that

$$f^{k,\ell}_{\lambda,\mu}(a;q,t) = f^{\ell,k}_{\mu,\lambda}(t/aq;q,t)$$

provided ℓ is finite. For infinite ℓ we adopt the convention $t^\ell = 0$.

For our next result we would like to specialise $a = t^{k-\ell}$ (for $k \leqslant \ell$ and ℓ finite) in (3.1.5). Potentially this could lead to problems with the double product on the right, and the following lemma serves to show that such a specialisation is in fact permitted provided the resulting double product is interpreted correctly.

Lemma 3.1.1. *Let λ and μ be partitions and $k, \ell \in \mathbb{N}$ such that $k \leqslant \ell$ and such that $k \geqslant l(\lambda)$ and $\ell \geqslant l(\mu)$. Then*

$$\lim_{b \to 1} f_{\lambda,\mu}^{k,\ell}(bt^{k-\ell}; q, t) \tag{3.1.6}$$

is well-defined. Furthermore a necessary and sufficient condition for the nonvanishing of this limit is

$$\lambda_i \geqslant \mu_{i-k+\ell} \quad \text{for } 1 \leqslant i \leqslant k. \tag{3.1.7}$$

The inequalities (3.1.7) may conveniently be visualised as:

$$\begin{array}{ccccccccccccccccc}
 & & & & \lambda_1 & \geqslant & \lambda_2 & \geqslant & \cdots & \geqslant & \lambda_k & \geqslant & \lambda_{k+1} & \geqslant & \cdots & \geqslant & 0 \\
 & & & & \veebar & & \veebar & & & & \veebar & & & & & & \\
\mu_1 & \geqslant & \cdots & \geqslant & \mu_{\ell-k+1} & \geqslant & \mu_{\ell-k+2} & \geqslant & \cdots & \geqslant & \mu_\ell & \geqslant & \mu_{\ell+1} & \geqslant & \cdots & \geqslant & 0.
\end{array} \tag{3.1.8}$$

It is assumed in Lemma 3.1.1 that q and t are generic. For the Schur case $q = t$ the equation (3.1.7) has to be replaced by

$$\lambda_i = \mu_{j_i} + i - j_i + \ell - k \quad \text{for } 1 \leqslant i \leqslant k,$$

where $1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant \ell$.

Proof of Lemma 3.1.1. To see that the limit is well-defined, note that for fixed i the powers of t in (3.1.6) are zero when $j = i - k + \ell + 1$ in the numerator and $j = i - k + \ell$ in the denominator. Since $j \leqslant \ell$ this yields $k - \ell \leqslant i \leqslant k - 1$ for the numerator and $k - \ell + 1 \leqslant i \leqslant k$ for the denominator, with both the lower bounds automatically satisfied since $k \leqslant \ell$. Therefore, taking the product of the t-independent q-shifted factorials in (3.1.6) and making a shift in the indices yields

$$\frac{\prod_{i=1}^{k-1}(bq;q)_{\lambda_i - \mu_{i-k+\ell+1}}}{\prod_{i=1}^{k}(bq;q)_{\lambda_i - \mu_{i-k+\ell}}} = \frac{1}{(bq;q)_{\lambda_k - \mu_\ell}} \prod_{i=\ell-k+1}^{\ell-1} (bq^{1+\lambda_{i+\ell-k}-\mu_i};q)_{\mu_i - \mu_{i+1}}. \tag{3.1.9}$$

Since μ is a partition, $\mu_i \geqslant \mu_{i+1}$, and hence the limit $b \to 1$ exists.

The vanishing of the limit (3.1.6) is completely determined by the vanishing of the right-hand side of (3.1.9) when $b \to 1$. Clearly the term $1/(q;q)_{\lambda_k - \mu_\ell}$ will vanish unless $\lambda_k \geqslant \mu_\ell$. In order for (3.1.9) to be nonvanishing, one of

$$\lambda_{i+k-\ell} \geqslant \mu_i, \tag{3.1.10a}$$

$$\lambda_{i+k-\ell} < \mu_i = \mu_{i+1}, \tag{3.1.10b}$$

must hold for each i such that $\ell - k + 1 \leqslant i \leqslant \ell - 1$. Now assume that $\lambda_k \geqslant \mu_\ell$ and one of (3.1.10a) and (3.1.10b) holds. Consider the largest i for which (3.1.10b) holds but (3.1.10a) does not. We cannot have $i = \ell - 1$ as this would imply

$$\lambda_k < \mu_{\ell-1} = \mu_\ell$$

contradicting $\lambda_k \geqslant \mu_\ell$. Similarly, no such maximal i exists with $i \leqslant \ell - 1$ as we then would get

$$\lambda_{i+k-\ell} < \mu_i = \mu_{i+1}.$$

However, as (3.1.10a) must now hold with $i \mapsto i + 1$, this would give $\lambda_{i+k-\ell} < \lambda_{i+k-\ell+1}$, contradicting that λ is a partition. Therefore we conclude that (3.1.10a) must hold for all $\ell - k + 1 \leqslant i \leqslant \ell - 1$. This is equivalent to the desired conditions by a shift of indices, and hence we are done. $\square$

By abuse of notation, we will write $f^{k,\ell}_{\lambda,\mu}(t^{k-\ell};q,t)$ instead of $\lim_{b\to 1} f^{k,\ell}_{\lambda,\mu}(bt^{k-\ell};q,t)$ in the following.

3.2 Identities for skew Macdonald polynomials

Here we present an identity which underpins the generalised Cauchy identity of Theorem 3.3.1 below. The following was originally proved by Warnaar in [War10, Theorem 3.4], but stated slightly differently.

Proposition 3.2.1. *For $\lambda,\mu\in\mathscr{P}$ there holds*

$$\begin{aligned}\sum_{\nu} t^{-|\nu|} P_{\mu/\nu}\bigg[\frac{1-1/a}{1-t}\bigg] Q_{\lambda/\nu}\bigg[\frac{1-aq/t}{1-t}\bigg]& \\ = P_{\mu}\bigg[\frac{1-t^k/a}{1-t}\bigg] Q_{\lambda}\bigg[\frac{1-aqt^{\ell-1}}{1-t}\bigg] f^{k,\ell}_{\lambda,\mu}(a;q,t),&\end{aligned} \tag{3.2.1}$$

where $k\in\mathbb{N}$ and $\ell\in\mathbb{N}\cup\{\infty\}$ may be chosen arbitrarily, provided that $k\geqslant l(\lambda)$ and $\ell\geqslant l(\mu)$.

In [War10] the right-hand side of (3.2.1) is stated with $\ell=k$. Of course, since the left-hand side does not depend on k and ℓ, the above form of the identity is not actually more general. Indeed,

$$\begin{aligned}&f^{k,\ell}_{\lambda,\mu}(a;q,t)\\ &\quad= t^{-(k-l(\lambda))|\mu|} f^{l(\lambda),l(\mu)}_{\lambda,\mu}(a;q,t)\prod_{i=1}^{l(\lambda)}\prod_{j=l(\mu)+1}^{\ell}\frac{(aqt^{j-i-1};q)_{\lambda_i}}{(aqt^{j-i};q)_{\lambda_i}}\prod_{i=l(\lambda)+1}^{k}\prod_{j=1}^{l(\mu)}\frac{(aqt^{j-i-1};q)_{-\mu_j}}{(aqt^{j-i};q)_{-\mu_j}}\\ &\quad= f^{l(\lambda),l(\mu)}_{\lambda,\mu}(a;q,t)\,\frac{(aqt^{l(\mu)-1};q,t)_{\lambda}}{(aqt^{\ell-1};q,t)_{\lambda}}\,\frac{(t^{l(\lambda)}/a;q,t)_{\mu}}{(t^k/a;q,t)_{\mu}}.\end{aligned}$$

Substituting this in the right-hand side of (3.2.1) and using (2.6.25) yields the identity with k and ℓ replaced by $l(\lambda)$ and $l(\mu)$. For later use we note that from the above and (2.6.25) it follows that

$$f^{k,\infty}_{\lambda,\mu}(a;q,t)Q_{\lambda}\bigg[\frac{1}{1-t}\bigg] = f^{k,\ell}_{\lambda,\mu}(a;q,t)Q_{\lambda}\bigg[\frac{1-aqt^{\ell-1}}{1-t}\bigg], \tag{3.2.2}$$

where ℓ is an arbitrary integer such that $\ell\geqslant l(\mu)$.

Proof of Proposition 3.2.1. By our previous considerations it suffices to prove the identity for $k=\ell$. Assuming this, let $\mu,\nu\in\mathscr{P}_k$ so that we have the generalised evaluation symmetry ((2.6.19) with $(n,\lambda,\mu)\mapsto(k,\mu,\nu)$), i.e.,

$$P_{\nu}\bigg[\frac{1-at^k}{1-t}\bigg]P_{\mu}\bigg[a\langle\nu\rangle_k+\frac{1-a}{1-t}\bigg] = P_{\mu}\bigg[\frac{1-at^k}{1-t}\bigg]P_{\nu}\bigg[a\langle\mu\rangle_k+\frac{1-a}{1-t}\bigg].$$

On the left-hand side we apply (2.6.35) with $(\lambda,\mu,X,Z)\mapsto(\mu,\omega,a\langle\lambda\rangle_k,(1-a)/(1-t))$, i.e.,

$$P_{\mu}\bigg[a\langle\lambda\rangle_k+\frac{1-a}{1-t}\bigg] = \sum_{\omega} P_{\mu/\omega}\bigg[\frac{1-a}{1-t}\bigg]P_{\omega}[a\langle\lambda\rangle_k].$$

For another alphabet X we multiply both sides of this new expression by $Q_\nu(X)$ and sum over ν to obtain

$$\sum_{\nu,\omega} P_\nu\left[\frac{1-at^k}{1-t}\right] Q_\nu(X) P_{\mu/\omega}\left[\frac{1-a}{1-t}\right] P_\omega[a\langle\lambda\rangle_k] = P_\mu\left[\frac{1-at^k}{1-t}\right] \sum_\nu P_\nu\left[a\langle\mu\rangle_k + \frac{1-a}{1-t}\right] Q_\nu(X).$$

By the Cauchy identity (2.6.8) this is equivalent to

$$\sum_{\nu,\omega} P_\nu\left[\frac{1-at^k}{1-t}\right] Q_\nu(X) P_{\mu/\omega}\left[\frac{1-a}{1-t}\right] P_\omega[a\langle\lambda\rangle_k] = P_\mu\left[\frac{1-at^k}{1-t}\right] \sigma_1\left[a\langle\mu\rangle_k X \frac{1-t}{1-q}\right] \sigma_1\left[X\frac{1-a}{1-q}\right],$$

where we have applied the sum rule for σ_1 on the right (2.3.4a). We now choose the alphabet X to be $X = b\langle\lambda\rangle_k$ for $\lambda \in \mathscr{P}_k$ and b a parameter, giving

$$\begin{aligned}\sum_{\nu,\omega} P_\nu\left[\frac{1-at^k}{1-t}\right] Q_\nu[b\langle\lambda\rangle_k] P_{\mu/\omega}\left[\frac{1-a}{1-t}\right] P_\omega[a\langle\nu\rangle_k] \\ = P_\mu\left[\frac{1-at^k}{1-t}\right] \sigma_1\left[ab\langle\lambda\rangle_k\langle\mu\rangle_k \frac{1-t}{1-q}\right] \sigma_1\left[b\langle\lambda\rangle_k\frac{1-a}{1-q}\right].\end{aligned} \tag{3.2.3}$$

We now turn our attention to the left-hand side, which we denote by LHS(3.2.3). By the homogeneity of the Macdonald polynomials we may pull out factors of a and b in the sum. Then by the classical evaluation symmetry (2.6.13) applied to $P_\nu[\langle\lambda\rangle_k]$ we get

$$\mathrm{LHS}(3.2.3) = \sum_{\nu,\omega} a^{|\omega|} b^{|\nu|} \frac{P_\nu[\langle 0\rangle_k]}{P_\lambda[\langle 0\rangle_k]} Q_\nu\left[\frac{1-at^k}{1-t}\right] P_{\mu/\omega}\left[\frac{1-a}{1-t}\right] P_\lambda[\langle\nu\rangle_k] P_\omega[\langle\nu\rangle_k].$$

The product $P_\lambda[\langle\nu\rangle_k] P_\omega[\langle\nu\rangle_k]$ may be expanded as in (2.6.31) with $(\lambda,\mu,\nu) \mapsto (\eta,\lambda,\omega)$, so that

$$\mathrm{LHS}(3.2.3) = \sum_{\eta,\nu,\omega} a^{|\omega|} b^{|\nu|} f^{\eta}_{\lambda\omega} \frac{P_\nu[\langle 0\rangle_k]}{P_\lambda[\langle 0\rangle_k]} Q_\nu\left[\frac{1-at^k}{1-t}\right] P_{\mu/\omega}\left[\frac{1-a}{1-t}\right] P_\eta[\langle\nu\rangle_k].$$

Again applying the evaluation symmetry, this time to $P_\eta[\langle\omega\rangle_k]$, the resulting sum over ν may be evaluated using the Cauchy identity (2.6.8) under the substitution $(\lambda, X, Y) \mapsto (\nu, b\langle\eta\rangle_k, (1-at^k)/(1-t))$. This leads us to

$$\mathrm{LHS}(3.2.3) = \sum_{\eta,\omega} a^{|\omega|} f^{\eta}_{\lambda\omega} \frac{P_\eta[\langle 0\rangle_k]}{P_\lambda[\langle 0\rangle_k]} P_{\mu/\omega}\left[\frac{1-a}{1-t}\right] \sigma_1\left[b\langle\eta\rangle_k \frac{1-at^k}{1-q}\right].$$

For now we have finishing manipulating only the left-hand side, and so return to the full identity (3.2.3). Writing out the functions σ_1 in terms of q-shifted factorials we have thus derived the formula

$$\begin{aligned}\sum_{\eta,\omega} a^{|\omega|} f^{\eta}_{\lambda\omega} P_\eta[\langle 0\rangle_k] P_{\mu/\omega}\left[\frac{1-a}{1-t}\right] \prod_{i=1}^{k} \frac{(abq^{\eta_i}t^{2k-i};q)_\infty}{(bq^{\eta_i}t^{k-i};q)_\infty} \\ = P_\lambda[\langle 0\rangle_k] P_\mu\left[\frac{1-at^k}{1-t}\right] \prod_{i,j=1}^{k} \frac{(abq^{\lambda_i+\mu_j}t^{2k-i-j+1};q)_\infty}{(abq^{\lambda_i+\mu_j}t^{2k-i-j};q)_\infty} \prod_{i=1}^{k} \frac{(abq^{\lambda_i}t^{k-i};q)_\infty}{(bq^{\lambda_i}t^{k-i};q)_\infty}.\end{aligned}$$

If we scale $b \mapsto bt^{1-k}$ and make some elementary manipulations using q-shifted factorials this reduces to the simpler expression

$$\sum_{\eta,\omega} a^{|\omega|} \frac{(b;q,t)_\eta}{(abt^k;q,t)_\eta} f^\eta_{\lambda\omega} P_\eta[\langle 0\rangle_k] P_{\mu/\omega}\left[\frac{1-a}{1-t}\right] \tag{3.2.4}$$
$$= \frac{(b;q,t)_\lambda}{(ab;q,t)_\lambda} P_\lambda[\langle 0\rangle_k] P_\mu\left[\frac{1-at^k}{1-t}\right] \prod_{i,j=1}^k \frac{(abt^{k-i-j+1};q)_{\lambda_i+\mu_j}}{(abt^{k-i-j+2};q)_{\lambda_i+\mu_j}}.$$

To complete the proof we would like to take complements of the partitions λ and η. For fixed ω in the sum there will be finitely many choices for the partition η since the q,t-Littlewood–Richardson coefficient vanishes unless $|\eta| = |\lambda| + |\omega|$. Since the sum over ω is also finite there will exist some integer N such that $\lambda, \omega, \eta \subseteq (N^k)$ for all possible choices of ω and η. Choosing such an N, we replace λ and η by their complements $\hat{\lambda}$ and $\hat{\eta}$ inside of (N^k). In order to simplify the resulting expression we will use several identities relating λ, η and $\hat{\lambda}, \hat{\eta}$. Firstly, it follows readily from (3.1.3) that

$$\frac{(a;q,t)_{\hat{\lambda}}}{(b;q,t)_{\hat{\lambda}}} = \left(\frac{b}{a}\right)^{|\lambda|} \frac{(a;q,t)_{(N^k)}}{(b;q,t)_{(N^k)}} \frac{(q^{1-n}t^{k-1}/b;q,t)_\lambda}{(q^{1-n}t^{k-1}/a;q,t)_\lambda}.$$

Also, the identity

$$P_{\hat{\lambda}}[\langle 0\rangle_k] = t^{N\binom{k}{2}-(k-1)|\lambda|} P_\lambda[\langle 0\rangle_k]$$

may be verified directly from Proposition 2.6.6 and the definition (2.6.22). Lastly, we will need (2.6.42) with $(n,\lambda,\mu,\nu) \mapsto (k,\lambda,\eta,\omega)$, which is

$$f^{\hat{\eta}}_{\hat{\lambda}\omega} = \frac{Q_\eta[\frac{1-qt^{k-1}}{1-t}]}{Q_\lambda[\frac{1-qt^{k-1}}{1-t}]} \frac{P_\lambda[\langle 0\rangle_k]}{P_\eta[\langle 0\rangle_k]} f^\lambda_{\eta\omega}.$$

Returning to (3.2.4), we specialise $b = q^{-N}$ and apply the previous three identities to obtain the expression

$$\sum_{\eta,\omega} t^{-\omega} f^\lambda_{\eta\omega} Q_\eta\left[\frac{1-q/at}{1-t}\right] P_{\mu/\omega}\left[\frac{1-a}{1-t}\right]$$
$$= t^{-k|\mu|} Q_\lambda\left[\frac{1-qt^{k-1}/a}{1-t}\right] P_\mu\left[\frac{1-at^k}{1-t}\right] \prod_{i,j=1}^k \frac{(qt^{j-i-1}/a;q)_{\lambda_i-\mu_j}}{(qt^{j-i}/a;q)_{\lambda_i-\mu_j}},$$

which is independent of N. All that remains is to evaluate the sum over η by the definition of the skew Macdonald polynomial (2.6.32) and replace $a \mapsto 1/a$ so that

$$\sum_\omega t^{-\omega} Q_{\lambda/\omega}\left[\frac{1-aq/t}{1-t}\right] P_{\mu/\omega}\left[\frac{1-1/a}{1-t}\right] = Q_\lambda\left[\frac{1-aqt^{k-1}}{1-t}\right] P_\mu\left[\frac{1-t^k/a}{1-t}\right] f^{k,k}_{\lambda,\mu}(a;q,t),$$

where we recall the definition of $f^{k,k}_{\lambda,\mu}(a;q,t)$ from (3.1.5). This is the statement of the proposition for $k = \ell$. □

For convenience we also state the following corollary, which follows directly from the proposition under the substitution $a = t^{k-\ell}$ for $k \leqslant \ell$.

Corollary 3.2.2. *Let $k, \ell \in \mathbb{N}$ such that $k \leqslant \ell$. Then, for partitions λ, μ such that $l(\lambda) \leqslant k$,*

$$\sum_{\nu} t^{-|\nu|} P_{\mu/\nu}\left[\frac{1-t^{\ell-k}}{1-t}\right] Q_{\lambda/\nu}\left[\frac{1-qt^{k-\ell-1}}{1-t}\right] = P_{\mu}\left[\frac{1-t^{\ell}}{1-t}\right] Q_{\lambda}\left[\frac{1-qt^{k-1}}{1-t}\right] f^{k,\ell}_{\lambda,\mu}(t^{k-\ell}; q, t).$$

The above corollary is essentially [War08a, Theorem 4.1, $u = 0$]. It should be noted that the condition $l(\mu) \leqslant \ell$ has been dropped in comparison with Proposition 3.2.1 and [War08a, Theorem 4.1], since both sides identically vanish when $l(\mu) > \ell$. To see this, note that the summand vanishes unless $\nu \subseteq \lambda$, $\nu \subseteq \mu$ and $\mu_{i-k+\ell} \leqslant \nu_i$ for all $i \geqslant 1$. This in particular implies that the summand vanishes unless $\mu_{i-k+\ell} \leqslant \lambda_i$ for all $i \geqslant 1$, in accordance with (3.1.7). When $i = k+1$ this yields $\mu_{\ell+1} \leqslant \lambda_{k+1}$. Now, since $l(\lambda) \leqslant k$, $\lambda_{k+1} = 0$ so that $\mu_{\ell+1} = 0$, i.e., $l(\mu) \leqslant \ell$. Since the alphabet $(1-t^{\ell})/(1-t)$ has finite cardinality ℓ, the right-hand side has this same vanishing property.

3.3 A_n Cauchy-type identities

As we will see below, the Cauchy identity for Macdonald polynomials may be viewed as a discrete analogue of the AFLT integral. This relationship may be traced back to the beta integral (1.1.3) through the well-known q-binomial theorem. For $a, q, x \in \mathbb{C}$ such that $|x|, |q| < 1$, this is [AAR99, Theorem 10.2.1]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}. \tag{3.3.1}$$

This summation may be viewed as a discrete analogue of the beta integral (1.1.3) by introducing the Jackson or q-integral. For real q such that $0 < q < 1$ the Jackson integral over the unit interval is [0, 1] defined by

$$\int_0^1 f(t)\,\mathrm{d}_q t := (1-q)\sum_{n=0}^{\infty} f(q^n) q^n, \tag{3.3.2}$$

where $f : \mathbb{R} \longrightarrow \mathbb{C}$ is any function such that the right-hand side converges. Letting q tend to 1 the Jackson integral converges to the Riemann integral on [0, 1]. If we set $(a, x) \mapsto (q^{\beta}, q^{\alpha})$ in (3.3.1) then the sum may be written as a Jackson integral and the right-hand side expressed in terms of q-gamma functions (3.1.4), so that

$$\int_0^1 t^{\alpha-1}(qt;q)_{\beta-1}\,\mathrm{d}_q t = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}. \tag{3.3.3}$$

Sending $q \to 1^-$ recovers the beta integral. This same procedure of writing a q-hypergeometric identity as a (multidimensional) q-integral, then taking the limit $q \to 1$, also applies to integrals of Selberg type. Indeed, it is this method of proof which is applied in Chapter 4 to derive the A_n AFLT integral. In the case of the Selberg integral and its generalisations, the summations we require involve Macdonald polynomials. For example, in [Mac95, p. 373–376], Macdonald uses the summation,

$$\sum_{\lambda} \frac{t^{n(\lambda)}(a;q,t)_\lambda}{c'_\lambda(q,t)} P_\lambda(X;q,t) = \prod_{x\in X} \frac{(ax;q)_\infty}{(x;q)_\infty} = \sigma_1\left[X\frac{1-a}{1-q}\right],$$

to derive a q-analogue of Kadell's integral (1.2.2). The above is known as the Kaneko–Macdonald q-binomial theorem, and was originally proved independently by Kaneko [Kan96, Theorem 3.4] and Macdonald [Macb, Equation (1.10)]. For X a single-letter alphabet this reduces to the ordinary q-binomial theorem (3.3.1). The formula itself follows from the Cauchy identity (2.6.8) under the plethystic substitution $Y \mapsto (1-a)/(1-t)$. To account for the two Jack polynomials occurring in the AFLT integral (1.2.4), the jumping-off point in the analogous proof of the AFLT integral requires the full Cauchy identity. Along the way, one picks up a q-analogue of the AFLT integral in the form of a multidimensional Jackson integral; see Theorem 6.1.1.

To evaluate the A_n Selberg and Kadell integrals, Warnaar proved an analogue of the Kaneko–Macdonald q-binomial theorem which can be thought of as an identity for A_n [War09, Theorem 3.1]. In generalising this approach to the A_n AFLT integral we require A_n summations which are of Cauchy-type. To this end, we think of (2.6.8) as an identity for A_1 in which the two alphabets X and Y are attached to the single vertex of the corresponding Dynkin diagram:

$$\begin{array}{c} X \\ \bullet \\ Y \end{array}$$

Extending this to A_n, we consider sums of the form

$$\sum_{\lambda^{(1)},\dots,\lambda^{(n)}} \prod_{r=1}^{n} P_{\lambda^{(r)}}\big[X^{(r)}\big] Q_{\lambda^{(r)}}\big[Y^{(r)}\big] \prod_{r=1}^{n-1} f^{k_r,k_{r+1}}_{\lambda^{(r)},\lambda^{(r+1)}}(a_r;q,t), \tag{3.3.4}$$

where the functions $f^{k_r,k_{r+1}}_{\lambda^{(r)},\lambda^{(r+1)}}$ represent the edges of the A_n Dynkin diagram:

$$\begin{array}{ccccccccc} X^{(1)} & & X^{(2)} & & X^{(r-1)} & & X^{(n-1)} & & X^{(n)} \\ \bullet & \text{———} & \bullet & \text{– – – ——} & \bullet & \text{—— – – –} & \bullet & \text{———} & \bullet \\ Y^{(1)} & & Y^{(2)} & & Y^{(r-1)} & & Y^{(n-1)} & & Y^{(n)} \end{array}$$

In (3.3.4) we choose $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_{n-1}$ to be nonnegative integers and $k_n \in \mathbb{N} \cup \{\infty\}$. Also, a_r for $1 \leqslant r \leqslant n-2$ will be fixed as $a_r = t^{k_r - k_{r+1}}$, whereas a_{n-1} is an indeterminate. In the sum (3.3.4) we also specialise the alphabets $X^{(2)}, \dots, X^{(n)}$ and $Y^{(1)}, \dots, Y^{(n-1)}$ as

$$X^{(r+1)} = \frac{1 - t^{k_r}/a_r}{1-t}, \qquad Y^{(r)} = z_r \frac{t - a_r q t^{k_{r+1}}}{1-t} \qquad \text{for } 1 \leqslant r \leqslant n-1, \tag{3.3.5}$$

and fix the cardinalities of $X^{(1)}$ and $Y^{(n)}$ to be

$$|X^{(1)}| = k_1, \qquad |Y^{(n)}| = k_n.$$

It should be noted that, since $a_r = t^{k_r - k_{r+1}}$ for $r \neq n-1$, we have

$$X^{(r)} = \frac{1 - t^{k_r}}{1-t} \qquad \text{for } 2 \leqslant r \leqslant n-1, \tag{3.3.6}$$

so that $|X^{(r)}| = k_r$ for all $1 \leqslant r \leqslant n-1$.

Recall the convention that $t^k := 0$ if $k = \infty$. Our first A_n Cauchy-type formula may then be stated as follows.

Theorem 3.3.1 (A_n Cauchy-type formula I)**.** *Let $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_{n-1}$ be nonnegative integers and $k_n \in \mathbb{N} \cup \{\infty\}$. Then for a_{n-1} an indeterminate, $a_r := t^{k_r - k_{r+1}}$ for $1 \leqslant r \leqslant n-2$, $X^{(1)} = x_1 + \cdots + x_{k_1}$, $Y^{(n)} = y_1 + \cdots + y_{k_n}$ and $X^{(2)}, \dots, X^{(n)}, Y^{(1)}, \dots, Y^{(n-1)}$ as in* (3.3.5)*, and*

$$W := z_1 \cdots z_{n-1} X^{(1)} + \sum_{r=1}^{n-1} z_{r+1} \cdots z_{n-1} \frac{1 - 1/a_r}{1-t},$$

we have

$$\begin{aligned}
&\sum_{\lambda^{(1)},\dots,\lambda^{(n)}} \prod_{r=1}^{n} P_{\lambda^{(r)}}\big([X^{(r)}]; q, t\big) Q_{\lambda^{(r)}/\mu^{(r)}}\big([Y^{(r)}]; q, t\big) \prod_{r=1}^{n-1} f^{k_r, k_{r+1}}_{\lambda^{(r)}, \lambda^{(r+1)}}(a_r; q, t) \qquad (3.3.7)\\
&\quad = P_{\mu^{(n)}}\big([W]; q, t\big) \prod_{r=1}^{n-1} \bigg(\prod_{i=1}^{k_1} \frac{(a_r q z_1 \cdots z_r x_i; q)_\infty}{(t z_1 \cdots z_r x_i; q)_\infty} \prod_{j=1}^{k_n} \frac{(z_{r+1} \cdots z_{n-1} y_j / a_r; q)_\infty}{(z_{r+1} \cdots z_{n-1} y_j; q)_\infty} \bigg) \\
&\quad\quad \times \prod_{i=1}^{k_1} \prod_{j=1}^{k_n} \frac{(t z_1 \cdots z_{n-1} x_i y_j; q)_\infty}{(z_1 \cdots z_{n-1} x_i y_j; q)_\infty} \prod_{1 \leqslant r < s \leqslant n-1} \prod_{i=1}^{k_{r+1} - k_r} \frac{(a_s q t^{i-1} z_{r+1} \cdots z_s; q)_\infty}{(t^i z_{r+1} \cdots z_s; q)_\infty},
\end{aligned}$$

where $\mu^{(1)}, \dots, \mu^{(n-1)} := 0$ and $\mu^{(n)}$ is an arbitrary partition.

For $n = 1$ the theorem reduces to (2.6.34) with $(X, Y, \mu) \mapsto (X^{(1)}, Y^{(1)}, \mu^{(n)})$, and for $n = 2$, k_2 finite and $\mu^{(n)} = 0$ it coincides with [War10, Theorem 1.2]. When $n \geqslant 2$ there is some mild redundancy in (3.3.7) since the substitution $X^{(1)} \mapsto z_1^{-1} X^{(1)}$ eliminates any reference to z_1. We further remark that we do not know how to evaluate the left-hand side of (3.3.7) in closed-form if one (or more) of the a_r for $1 \leqslant r \leqslant n-2$ is an indeterminate. Since $|X^{(r)}| = k_r$ for $1 \leqslant r \leqslant n-1$ the summand vanishes unless $l(\lambda^{(r)}) \leqslant k_r$ for this same range of r. If a_r for some $r \leqslant n-2$ is an indeterminate, then $\lambda^{(r+1)}$ can have an arbitrarily large length, which prevents us from applying Proposition 3.2.1 in our proof. Requiring $a_r = t^{k_r - k_{r+1}}$ for $1 \leqslant r \leqslant n-1$ allows us to use Corollary 3.2.2 in place of the proposition. We are, however, allowed to keep a_{n-1} an indeterminate since $Y^{(n)}$ is either a finite alphabet of cardinality k_n, or countably infinite, permitting us to apply Proposition 3.2.1. For more details we refer to the proof of the theorem contained in the next section.

There is a second, closely related, Cauchy-type identity, in which k_n is finite and no longer corresponds to the cardinality of $Y^{(n)}$.

Corollary 3.3.2 (A_n Cauchy-type formula II)**.** *Let $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n$ be nonnegative integers. Then for $a_r := t^{k_r - k_{r+1}}$ for $1 \leqslant r \leqslant n-1$, $X^{(1)} = x_1 + \cdots + x_{k_1}$, $Y^{(n)} = y_1 + y_2 + \cdots$ and $X^{(2)}, \dots, X^{(n)}, Y^{(1)}, \dots, Y^{(n-1)}$ as in* (3.3.5)*, and*

$$W := z_1 \cdots z_{n-1} X^{(1)} + \sum_{r=1}^{n-1} z_{r+1} \cdots z_{n-1} \frac{1 - t^{k_r - k_{r+1}}}{1-t},$$

we have

$$\begin{aligned}
&\sum_{\lambda^{(1)},\dots,\lambda^{(n)}} \prod_{r=1}^{n} P_{\lambda^{(r)}}\big([X^{(r)}];q,t\big) Q_{\lambda^{(r)}/\mu^{(r)}}\big([Y^{(r)}];q,t\big) \prod_{r=1}^{n-1} f^{k_r,k_{r+1}}_{\lambda^{(r)},\lambda^{(r+1)}}(a_r;q,t) \qquad (3.3.8)\\
&\quad = P_{\mu^{(n)}}\big([W];q,t\big) \prod_{r=1}^{n-1}\bigg(\prod_{i=1}^{k_1} \frac{(qt^{k_r-k_{r+1}}z_1\cdots z_r x_i;q)_\infty}{(tz_1\cdots z_r x_i;q)_\infty} \prod_{j\geqslant 1} \frac{(t^{k_{r+1}-k_r}z_{r+1}\cdots z_{n-1}y_j;q)_\infty}{(z_{r+1}\cdots z_{n-1}y_j;q)_\infty}\bigg)\\
&\qquad \times \prod_{i=1}^{k_1}\prod_{j\geqslant 1} \frac{(tz_1\cdots z_{n-1}x_i y_j;q)_\infty}{(z_1\cdots z_{n-1}x_i y_j;q)_\infty} \prod_{1\leqslant r<s\leqslant n-1} \prod_{i=1}^{k_{r+1}-k_r} \frac{(qt^{i+k_s-k_{s+1}-1}z_{r+1}\cdots z_s;q)_\infty}{(t^i z_{r+1}\cdots z_s;q)_\infty},
\end{aligned}$$

where $\mu^{(1)},\dots,\mu^{(n-1)} := 0$ *and* $\mu^{(n)}$ *is an arbitrary partition.*

We note that in the above corollary the range for which (3.3.6) holds includes $r = n$. In particular $|X^{(r)}| = k_r$ for all $1 \leqslant r \leqslant n$. The corollary simplifies to the A_n q-binomial theorem [War09, Theorem 3.2] if we replace $Y^{(n)} \mapsto z_n t^{k_{n-1}}(1-a)/(1-t)$ and $z_r \mapsto z_r t^{k_{r-1}-1}$ for all $1 \leqslant r \leqslant n-1$ where $k_0 := 0$.

Proof of Corollary 3.3.2. We take Theorem 3.3.1 with $k_n = \infty$. Then $Y^{(n)} = y_1 + y_2 + \cdots$ in accordance with the corollary. Moreover, by (3.2.2) and the fact that $\mu^{(n-1)} = 0$,

$$\begin{aligned}
&Q_{\lambda^{(n-1)}/\mu^{(n-1)}}\big[Y^{(n-1)}\big] f^{k_{n-1},k_n}_{\lambda^{(n-1)},\lambda^{(n)}}(a_{n-1};q,t)\\
&\quad = Q_{\lambda^{(n-1)}}\bigg[\frac{z_n t}{1-t}\bigg] f^{k_{n-1},\infty}_{\lambda^{(n-1)},\lambda^{(n)}}(a_{n-1};q,t)\\
&\quad = Q_{\lambda^{(n-1)}}\bigg[z_n \frac{t-a_{n-1}qt^{\hat{k}_n}}{1-t}\bigg] f^{k_{n-1},\hat{k}_n}_{\lambda^{(n-1)},\lambda^{(n)}}(a_{n-1};q,t)\\
&\quad = Q_{\lambda^{(n-1)}/\mu^{(n-1)}}[\hat{Y}^{(n-1)}] f^{k_{n-1},\hat{k}_n}_{\lambda^{(n-1)},\lambda^{(n)}}(a_{n-1};q,t).
\end{aligned}$$

Here $\hat{k}_n$ is an arbitrary integer such that $\hat{k}_n \geqslant l(\lambda^{(n)})$ and $\hat{Y}^{(n-1)} := z_n(t - a_{n-1}qt^{\hat{k}_n})/(1-t)$, so that $\hat{Y}^{(n-1)}$ corresponds to $Y^{(n-1)}$ in (3.3.5) except that k_n has been replaced by $\hat{k}_n$. Of course, since we are summing over all partitions $\lambda^{(n)}$ there exists no integer $\hat{k}_n$ such that $\hat{k}_n \geqslant l(\lambda^{(n)})$ for all $\lambda^{(n)}$. To get around this problem we specialise $a_{n-1} = q^{k_{n-1}-\hat{k}_n}$. Then $X^{(n)} = (1-t^{\hat{k}_n})/(1-t)$ of cardinality $|\hat{k}_n|$ so that without loss of generality we may assume that $l(\lambda^{(n)}) \leqslant \hat{k}_n$. Finally replacing $\hat{k}_n$ by k_n completes the proof. □

The proof of the A_n AFLT integral actually requires a plethystically substituted version of the $\mu^{(n)} = 0$ instance of (3.3.8) obtained by replacing $Y^{(n)} \mapsto Y^{(n)} + (c-d)/(1-t)$. For convenience we state this as a corollary.

Corollary 3.3.3. *With the same conditions as in Corollary 3.3.2,*

$$\sum_{\lambda^{(1)},\dots,\lambda^{(n)}} P_{\lambda^{(n)}}\big([X^{(n)}];q,t\big)Q_{\lambda^{(n)}}\Big(\Big[Y^{(n)}+\frac{c-d}{1-t}\Big];q,t\Big)$$
$$\times\prod_{r=1}^{n-1}\Big(P_{\lambda^{(r)}}\big([X^{(r)}];q,t\big)Q_{\lambda^{(r)}}\big([Y^{(r)}];q,t\big)f^{k_r,k_{r+1}}_{\lambda^{(r)},\lambda^{(r+1)}}(a_r;q,t)\Big)$$
$$=\prod_{r=1}^{n-1}\Big(\prod_{i=1}^{k_1}\frac{(qt^{k_r-k_{r+1}}z_1\cdots z_rx_i;q)_\infty}{(tz_1\cdots z_rx_i;q)_\infty}\prod_{j\geqslant1}\frac{(t^{k_{r+1}-k_r}z_{r+1}\cdots z_{n-1}y_j;q)_\infty}{(z_{r+1}\cdots z_{n-1}y_j;q)_\infty}\Big)$$
$$\times\prod_{i=1}^{k_1}\prod_{j\geqslant1}\frac{(tz_1\cdots z_{n-1}x_iy_j;q)_\infty}{(z_1\cdots z_{n-1}x_iy_j;q)_\infty}\prod_{i=1}^{k_1}\frac{(dz_1\cdots z_{n-1}x_i;q)_\infty}{(cz_1\cdots z_{n-1}x_i;q)_\infty}$$
$$\times\prod_{r=1}^{n-1}\prod_{i=1}^{k_{r+1}-k_r}\frac{(dz_{r+1}\cdots z_{n-1}t^{i-1};q)_\infty}{(cz_{r+1}\cdots z_{n-1}t^{i-1};q)_\infty}\prod_{1\leqslant r<s\leqslant n-1}\prod_{i=1}^{k_{r+1}-k_r}\frac{(qt^{i+k_s-k_{s+1}-1}z_{r+1}\cdots z_s;q)_\infty}{(t^iz_{r+1}\cdots z_s;q)_\infty}.$$

We will not give an explicit proof of this corollary, but note that the substitution $Y^{(n)}\mapsto Y^{(n)}+(c-d)(1-t)$ can easily be carried out noting that the right-hand side of (3.3.8) without $P_{\mu^{(n)}}[W]$ is expressible in terms of σ_1 as

$$\sigma_1\Bigg[\sum_{r=1}^{n-1}\Big(tz_1\cdots z_r\frac{1-qt^{k_r-k_{r+1}-1}}{1-q}X^{(1)}+z_{r+1}\cdots z_{n-1}\frac{1-t^{k_{r+1}-k_r}}{1-q}Y^{(n)}\Big)+$$
$$z_1\cdots z_{n-1}\frac{1-t}{1-q}X^{(1)}Y^{(n)}+\sum_{1\leqslant r<s\leqslant n-1}tz_{r+1}\cdots z_s\frac{(1-qt^{k_s-k_{s+1}-1})(1-t^{k_{r+1}-k_r})}{(1-q)(1-t)}\Bigg].$$

The expression of the corollary is then a simple computation involving the use of the addition and subtraction rules for σ_1 (2.3.4).

3.4 Proof of Theorem 3.3.1

The proof proceeds by using the identities for skew Macdonald polynomials obtained at the beginning of the chapter to inductively evaluate the sums over $\lambda^{(r)}$. We define two families of auxiliary alphabets $\{X^{(r,m)}\}_{0\leqslant m<r\leqslant n}$ and $\{Z^{(r)}\}_{r=1}^n$ as

$$X^{(r,m)}:=\begin{cases} z_1\cdots z_mX^{(1)}+\displaystyle\sum_{u=1}^m z_{u+1}\cdots z_m\frac{1-1/a_u}{1-t} & \text{if } r=m+1,\\[2ex] \dfrac{1-1/a_{r-1}}{1-t} & \text{otherwise,}\end{cases}$$

and

$$Z^{(r)}:=\begin{cases} Y^{(n)} & \text{if } r=n,\\[2ex] z_r\dfrac{t-a_rq}{1-t} & \text{otherwise.}\end{cases}$$

The first family satisfies the simple recursion

$$z_{m+1}X^{(m+1,m)}+X^{(m+2,m)}=X^{(m+2,m+1)}. \tag{3.4.1}$$

Lemma 3.4.1. *For n, m integers such that $0 \leqslant m \leqslant n-1$, and $\nu^{(n)}$ a partition, define*

$$g_m := \prod_{r=1}^{m}\prod_{i=1}^{k_1} \frac{(a_r q z_1\cdots z_r x_i;q)_\infty}{(tz_1\cdots z_r x_i;q)_\infty} \prod_{1\leqslant r<s\leqslant m} \prod_{i=1}^{k_{r+1}-k_r} \frac{(a_s q t^{i-1} z_{r+1}\cdots z_s;q)_\infty}{(t^i z_{r+1}\cdots z_s;q)_\infty} \tag{3.4.2}$$
$$\times \sum_{\nu^{(m+1)},\dots,\nu^{(n-1)}} \prod_{r=m+1}^{n} \bigg(z_r^{|\nu^{(r)}|} \sum_{\lambda^{(r)}} P_{\lambda^{(r)}/\nu^{(r-1)}}\big(\big[X^{(r,m)}\big];q,t\big) Q_{\lambda^{(r)}/\nu^{(r)}}\big(\big[Z^{(r)}\big];q,t\big)\bigg),$$

where $\nu^{(m)} := 0$. Then $g_m = g_{m+1}$ for $0 \leqslant m \leqslant n-2$.

Proof. Since $\nu^{(m)} := 0$, the sum over $\lambda^{(m+1)}$ in (3.4.2) is of the form (2.6.34) with

$$(X, Y, \lambda, \mu) \mapsto \big(X^{(m+1,m)}, Z^{(m+1)}, \lambda^{(m+1)}, \nu^{(m+1)}\big)$$

and hence equates to

$$P_{\nu^{(m+1)}}\big[X^{(m+1,m)}\big]\sigma_1\bigg[\frac{1-t}{1-q}\, X^{(m+1,m)} Z^{(m+1)}\bigg].$$

Since $0 \leqslant m \leqslant n-2$, it follows that

$$\begin{aligned}
&\sigma_1\bigg[\frac{1-t}{1-q}\, X^{(m+1,m)} Z^{(m+1)}\bigg] \\
&\quad = \sigma_1\bigg[z_1\cdots z_{m+1}\frac{t-a_{m+1}q}{1-q}\, X^{(1)} + \sum_{r=1}^{m} z_{r+1}\cdots z_{m+1}\, \frac{(1-t^{k_{r+1}-k_r})(t-a_{m+1}q)}{(1-t)(1-q)}\bigg] \\
&\quad = \prod_{i=1}^{k_1} \sigma_{z_1\cdots z_{m+1}}\bigg[\frac{t-a_{m+1}q}{1-q}\, x_i\bigg] \prod_{r=1}^{m}\prod_{i=1}^{k_{r+1}-k_r} \sigma_{z_{r+1}\cdots z_{m+1}}\bigg[\frac{t^{i-1}(t-a_{m+1}q)}{1-q}\bigg] \\
&\quad = \prod_{i=1}^{k_1} \frac{(a_{m+1}qz_1\cdots z_{m+1}x_i;q)_\infty}{(tz_1\cdots z_{m+1}x_i;q)_\infty} \prod_{r=1}^{m}\prod_{i=1}^{k_{r+1}-k_r} \frac{(a_{m+1}qt^{i-1}z_{r+1}\cdots z_{m+1};q)_\infty}{(t^i z_{r+1}\cdots z_{m+1};q)_\infty},
\end{aligned}$$

where the second equality follows from (2.3.4a) and the last equality from Lemma 2.3.2. As a result,

$$\begin{aligned}
g_m = {}& \prod_{r=1}^{m+1}\prod_{i=1}^{k_1} \frac{(a_r q z_1\cdots z_r x_i;q)_\infty}{(tz_1\cdots z_r x_i;q)_\infty} \prod_{1\leqslant r<s\leqslant m+1} \prod_{i=1}^{k_{r+1}-k_r} \frac{(a_s q t^{i-1} z_{r+1}\cdots z_s;q)_\infty}{(t^i z_{r+1}\cdots z_s;q)_\infty} \\
&\times \sum_{\nu^{(m+1)},\dots,\nu^{(n-1)}} \bigg\{ P_{\nu^{(m+1)}}\big[z_{m+1}X^{(m+1,m)}\big] \\
&\qquad \times \prod_{r=m+2}^{n} \bigg(z_r^{|\nu^{(r)}|} \sum_{\lambda^{(r)}} P_{\lambda^{(r)}/\nu^{(r-1)}}\big[X^{(r,m)}\big] Q_{\lambda^{(r)}/\nu^{(r)}}\big[Z^{(r)}\big]\bigg)\bigg\}.
\end{aligned}$$

After interchanging the order of the sum over $\nu^{(m+1)}$ with those over the $\lambda^{(r)}$, the former can be summed using (2.6.35) with

$$(X, Z, \lambda, \mu) \mapsto \big(X^{(m+2,m)}, z_{m+1}X^{(m+1,m)}, \lambda^{(m+2)}, \nu^{(m+1)}\big).$$

Thanks to the recursion (3.4.1) this yields $P_{\lambda^{(m+2)}}[X^{(m+2,m+1)}]$, resulting in $g_m = g_{m+1}$. $\square$

We are now ready to prove Theorem 3.3.1. As a first step we eliminate $f^{k_r,k_{r+1}}_{\lambda^{(r)},\lambda^{(r+1)}}(a_r;q,t)$ from the summand in (3.3.7) by applying Corollary 3.2.2 with

$$(\lambda,\mu,\nu,k,\ell)\mapsto(\lambda^{(r)},\lambda^{(r+1)},\nu^{(r)},k_r,k_{r+1})\quad\text{for } 1\leqslant r\leqslant n-2$$

and Proposition 3.2.1 with

$$(a,\lambda,\mu,\nu,k,\ell)\mapsto(a_{n-1},\lambda^{(n-1)},\lambda^{(n)},\nu^{(n-1)},k_{n-1},k_n)\quad\text{for } r=n-1.$$

Then

$$\begin{aligned}
&\sum_{\lambda^{(1)},\dots,\lambda^{(n)}}\prod_{r=1}^{n}P_{\lambda^{(r)}}\big[X^{(r)}\big]Q_{\lambda^{(r)}/\mu^{(r)}}\big[Y^{(r)}\big]\prod_{r=1}^{n-1}f^{k_r,k_{r+1}}_{\lambda^{(r)},\lambda^{(r+1)}}(a_r;q,t)\\
&\quad=\sum_{\lambda^{(1)},\dots,\lambda^{(n)}}\;\sum_{\nu^{(1)},\dots,\nu^{(n-1)}}P_{\lambda^{(1)}}\big[X^{(1)}\big]Q_{\lambda^{(n)}/\mu^{(n)}}\big[Y^{(n)}\big]\\
&\qquad\qquad\times\prod_{r=1}^{n-1}z_r^{|\lambda^{(r)}|}Q_{\lambda^{(r)}/\nu^{(r)}}\bigg[\frac{t-a_rq}{1-t}\bigg]P_{\lambda^{(r+1)}/\nu^{(r)}}\bigg[\frac{1-1/a_r}{1-t}\bigg]\\
&\quad=\sum_{\nu^{(1)},\dots,\nu^{(r-1)}}\prod_{r=1}^{n}\bigg(z_r^{|\nu^{(r)}|}\sum_{\lambda^{(r)}}P_{\lambda^{(r)}/\nu^{(r-1)}}[X^{(r,0)}]Q_{\lambda^{(r)}/\nu^{(r)}}\big[Z^{(r)}\big]\bigg)\bigg|_{z_n=1,\,\nu^{(n)}=\mu^{(n)}}\\
&\quad=g_0\big|_{z_n=1,\,\nu^{(n)}=\mu^{(n)}},
\end{aligned}$$

where in the fourth line $\nu^{(0)}:=0$, and where g_0 is defined in (3.4.2). Using Lemma 3.4.1 we may replace g_0 by g_{n-1}, leading to

$$\begin{aligned}
&\sum_{\lambda^{(1)},\dots,\lambda^{(n)}}\prod_{r=1}^{n}P_{\lambda^{(r)}}\big[X^{(r)}\big]Q_{\lambda^{(r)}/\mu^{(r)}}\big[Y^{(r)}\big]\prod_{r=1}^{n-1}f^{k_r,k_{r+1}}_{\lambda^{(r)},\lambda^{(r+1)}}(a_r;q,t)\\
&\quad=\prod_{r=1}^{n-1}\prod_{i=1}^{k_1}\frac{(a_rqz_1\cdots z_rx_i;q)_\infty}{(tz_1\cdots z_rx_i;q)_\infty}\prod_{1\leqslant r<s\leqslant n-1}\;\prod_{i=1}^{k_{r+1}-k_r}\frac{(a_sqt^{i-1}z_{r+1}\cdots z_s;q)_\infty}{(t^iz_{r+1}\cdots z_s;q)_\infty}\\
&\qquad\times\sum_{\lambda^{(n)}}P_{\lambda^{(n)}}\big[X^{(n,n-1)}\big]Q_{\lambda^{(n)}/\mu^{(n)}}\big[Y^{(n)}\big].
\end{aligned}$$

The final sum on the right can be carried out by (2.6.34) with

$$(X,Y,\lambda,\mu)\mapsto\big(X^{(n,n-1)},Y^{(n)},\lambda^{(n)},\mu^{(n)}\big).$$

Since $W=X^{(n,n-1)}$ and

$$\begin{aligned}
\sigma_1\bigg[\frac{1-t}{1-q}\,X^{(n,n-1)}Y^{(n)}\bigg]&=\sigma_1\bigg[z_1\cdots z_{n-1}\,\frac{1-t}{1-q}\,X^{(1)}Y^{(n)}+\sum_{r=1}^{n-1}z_{r+1}\cdots z_{n-1}\,\frac{1-1/a_r}{1-q}\,Y^{(n)}\bigg]\\
&=\prod_{i=1}^{k_1}\prod_{j=1}^{k_n}\frac{(tz_1\cdots z_{n-1}x_iy_j;q)_\infty}{(z_1\cdots z_{n-1}x_iy_j;q)_\infty}\prod_{r=1}^{n-1}\prod_{j=1}^{k_n}\frac{(z_{r+1}\cdots z_{n-1}y_j/a_r;q)_\infty}{(z_{r+1}\cdots z_{n-1}y_j;q)_\infty},
\end{aligned}$$

the right-hand side of the theorem results.

Chapter 4

A_n integrals

The purpose of this chapter is to use the summation formulas for Macdonald polynomials proved in the previous chapter to derive two A_n Selberg integrals. The first of these is an A_n analogue of the AFLT integral, and the second is an A_n analogue of Warnaar's $\mathfrak{sl}_3$ Selberg integral. To begin the section we describe how A_n Selberg integrals arise through the Knizhnik–Zamolodchikov equations. The domain of integration for the A_n Selberg integral is also described in detail.

4.1 $\mathfrak{g}$-Selberg integrals, Knizhnik–Zamolodchikov equations, and the Mukhin–Varchenko conjecture

Before we launch into the evaluation of the A_n AFLT integral we will explain the connection between Knizhnik–Zamolodchikov (KZ) equations and hypergeometric integrals.

The Knizhnik–Zamolodchikov equations are a family of partial differential equations based on Lie algebras, which first arose in the work of Knizhnik and Zamolodchikov in two-dimensional conformal field theory [KZ86]. Although they may be described much more generally, we restrict ourselves to the following case which is sufficient for our purposes. Let $\mathfrak{g}$ be a simple Lie algebra of rank n with associated root system Φ. We denote the simple roots of Φ by $\tilde{\alpha}_i$ for $1 \leqslant i \leqslant n$ to avoid clashes of notation with our Selberg integrals, and the Chevalley generators by e_i, f_i, and h_i. Let V_λ and V_μ be two highest weight modules for $\mathfrak{g}$ and $\Omega \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ the Casimir element. The space of singular vectors in $V_\lambda \otimes V_\mu$ of weight ν is defined to be

$$\mathrm{Sing}_{\lambda,\mu}[\nu] := \{v \in V_\lambda \otimes V_\mu : h_i v = \nu(h_i)v, e_i v = 0, 1 \leqslant i \leqslant n\}.$$

Then the KZ equations for a function $u(z,w) : \mathbb{C}^2 \longrightarrow V_\lambda \otimes V_\mu$ form the system[1]

$$\kappa \frac{\partial u}{\partial z} = \frac{\Omega}{z-w} u, \quad \text{and} \quad \kappa \frac{\partial u}{\partial w} = \frac{\Omega}{w-z} u.$$

In [SV91], Schechtman and Varchenko solve the KZ equations for an arbitrary simple Lie algebra when the function u takes values in $\mathrm{Sing}_{\lambda,\mu}[\lambda + \mu - \sum_{i=1}^n \tilde{\alpha}_i k_i]$. Their solutions may be expressed

[1] For the more general definition alluded to earlier see [EFJ98].

in terms of $k_1 + \cdots + k_n$ dimensional hypergeometric integrals. If I is an ordered multiset then we write f^I for the product $\prod_{i \in I} f_i$, where the product is taken with respect to the order. Schechtman and Varchenko express $u(z, w)$ as

$$u(z,w) = \sum u_{I,J}(z,w) f^I v_\lambda \otimes f^J v_\mu,$$

where the sum is over all ordered multisets I, J of $\{1, \ldots, n\}$ such that i appears precisely k_i times in $I \cup J$ and v_λ, v_μ are the highest weight vectors of V_λ and V_μ respectively. The coordinate functions $u_{I,J}(z, w)$ are given by

$$u_{I,J}(z,w) = \int_\Gamma \Psi(z,w;t)\omega_{I,J}(z,w;t)\,\mathrm{d}t,$$

where $\omega_{I,J}(z, w; t)$ is some explicitly known rational function and Γ is a suitable domain of integration. The function $\Psi(z, w; t)$ is known as the phase function (or master function) and is defined as follows. The first k_i integration variables are attached to $\tilde{\alpha}_i$ so that $\tilde{\alpha}_{t_j} := \tilde{\alpha}_i$ provided $k_1 + \cdots + k_{i-1} < j \leqslant k_1 + \cdots + k_i$. Then

$$\Psi(z,w;t) = (z-w)^{(\lambda,\mu)/\kappa} \prod_{i=1}^{k} (t_i - z)^{-(\lambda,\tilde{\alpha}_{t_i})/\kappa} (t_i - w)^{-(\mu,\tilde{\alpha}_{t_i})/\kappa} \prod_{1 \leqslant i < j \leqslant k} (t_i - t_j)^{(\tilde{\alpha}_{t_i},\tilde{\alpha}_{t_j})/\kappa},$$

where $(\cdot,\cdot)$ is the standard bilinear form on the dual of the Cartan subalgebra. Departing from Schechtman and Varchenko's construction, we note that the phase function is a two-parameter deformation of the integrand of the Selberg integral. In order to obtain explicit Selberg-type integrals we work with the normalised phase function

$$\Psi(t) = \prod_{i=1}^{k} t_i^{-(\lambda,\tilde{\alpha}_{t_i})/\kappa} (1-t_i)^{-(\mu,\tilde{\alpha}_{t_i})/\kappa} \prod_{1 \leqslant i < j \leqslant k} (t_i - t_j)^{(\tilde{\alpha}_{t_i},\tilde{\alpha}_{t_j})/\kappa}. \tag{4.1.1}$$

In 2000, this relationship lead Mukhin and Varchenko to make the following remarkable conjecture [MV00, Conjecture 1].

Conjecture 4.1.1. *If the space of singular vectors of weight $\lambda + \mu - \sum_{i=1}^n k_i \tilde{\alpha}_i$ is one-dimensional, then the integral*

$$\int_\Delta \Psi(t)\,\mathrm{d}t$$

evaluates as a product of gamma functions. Here $\Delta \subseteq [0,1]^k$ is a real domain of integration not specified.

For $\mathfrak{g} = \mathfrak{sl}_2 = \mathrm{A}_1$ with the single fundamental weight Λ_1, the weights may be expressed as $\lambda = \lambda_1 \Lambda_1$ and $\mu = \mu_1 \Lambda_1$. Then the function (4.1.1) becomes

$$\Psi(t) = \prod_{i=1}^{k} t_i^{-\lambda_1/\kappa} (1-t_i)^{-\mu_1/\kappa} \prod_{1 \leqslant i < j \leqslant k} (t_i - t_j)^{2/\kappa}.$$

This is simply the integrand of the Selberg integral with $(\alpha, \beta, \gamma) \mapsto (1 - \lambda_1/\kappa, 1 - \mu_2/\kappa, 1/\kappa)$. Thus the k-simplex is a suitable domain of integration, and by Selberg's formula (1.1.2) the integral evaluates as a product of gamma functions. Currently the conjecture has only been resolved for $\mathfrak{g} = \mathrm{A}_n$. The A_2 case was handled by Tarasov and Varchenko [TV03, Theorem 3.2], and the general A_n case by Warnaar [War09, Theorem 1.2]. It should be pointed out that some low-rank cases have been computed in types $\mathrm{B}_n, \mathrm{C}_n$, and D_n by Mimachi and Takamuki by iterating the Euler beta integral, or the ordinary Selberg integral [MT05].

4.2 Domains of integration

The domain $C_\gamma^{k_1,\dots,k_n}[0,1]$ of the integral (1.3.4) takes the form of a chain in the usual sense of algebraic topology. For $n = 1$ the domain $C_\gamma^{k_1}[0,1]$ is the k_1-simplex given in (1.1.4). In order to describe $C_\gamma^{k_1,\dots,k_n}[0,1]$ for $n \geqslant 2$, we first consider $D^{k_1,\dots,k_n}[0,1] \subseteq [0,1]^{k_1+\dots+k_n}$ as the set of points

$$\big(t^{(1)}, t^{(2)}, \dots, t^{(n)}\big) = \big(t_1^{(1)}, \dots, t_{k_1}^{(1)}, t_1^{(2)}, \dots, t_{k_2}^{(2)}, \dots, t_1^{(n)}, \dots, t_{k_n}^{(n)}\big) \in [0,1]^{k_1+\dots+k_n}$$

subject to

$$0 < t_1^{(r)} < \dots < t_{k_r}^{(r)} < 1 \qquad \text{for } 1 \leqslant r \leqslant n,$$

and

$$t_i^{(r)} < t_{i-k_r+k_{r+1}}^{(r+1)} \qquad \text{for } 1 \leqslant i \leqslant k_r,\ 1 \leqslant r \leqslant n-1.$$

Following (3.1.8) this may be visualised as

$$\begin{array}{ccccccccccccccc}
 & & & & 0 & < & t_1^{(r)} & < & t_2^{(r)} & < & \cdots & < & t_{k_r}^{(r)} & < & 1 \\
 & & & & & & \wedge & & \wedge & & & & \wedge & & \\
0 & < & t_1^{(r+1)} & < & \cdots & < & t_{k_{r+1}-k_r+1}^{(r+1)} & < & t_{k_{r+1}-k_r+2}^{(r+1)} & < & \cdots & < & t_{k_{r+1}}^{(r+1)} & < & 1.
\end{array}$$

We need to consider all possible total orderings between the integration variables $t^{(r)}$ and $t^{(r+1)}$ consistent with the above partial order. Each such total ordering may be described by a map

$$M_r : \{1, \dots, k_r\} \longrightarrow \{1, \dots, k_{r+1}\},$$

such that $M_r(i) \leqslant M_r(i+1)$ and $1 \leqslant M_r(i) \leqslant i + k_{r+1} - k_r$, so that

$$t_{M_r(i)-1}^{(r+1)} < t_i^{(r)} < t_{M_r(i)}^{(r+1)}, \tag{4.2.1}$$

where $t_0^{(r+1)} := 0$. In view of this we define the sets

$$D_{M_1,\dots,M_{n-1}}^{k_1,\dots,k_n} \subseteq D^{k_1,\dots,k_n}[0,1]$$

by requiring that (4.2.1) holds for fixed admissible maps $M_1, \dots, M_{n-1}$. Then $D^{k_1,\dots,k_n}$ can be written as the chain

$$D^{k_1,\dots,k_n}[0,1] = \sum_{M_1,\dots,M_{n-1}} D_{M_1,\dots,M_{n-1}}^{k_1,\dots,k_n}[0,1],$$

where the sum is over all admissible maps $M_1, \dots, M_{n-1}$. Analytically continuing the weight function

$$F_{M_1,\dots,M_{n-1}}^{k_1,\dots,k_n}(\gamma) := \prod_{r=1}^{n-1} \prod_{i=1}^{k_r} \frac{\sin(\pi(i + k_{r+1} - k_r - M_r(i) + 1)\gamma)}{\sin(\pi(i + k_{r+1} - k_r)\gamma)} \quad \text{for } \gamma \in \mathbb{C} \setminus \mathbb{Z}, \tag{4.2.2}$$

to include $\gamma = 0$, the chain $C_\gamma^{k_1,\dots,k_n}[0,1]$ is defined as

$$C_\gamma^{k_1,\dots,k_n}[0,1] := \sum_{M_1,\dots,M_{n-1}} F^{k_1,\dots,k_n}_{M_1,\dots,M_{n-1}}(\gamma) D^{k_1,\dots,k_n}_{M_1,\dots,M_{n-1}}[0,1]. \tag{4.2.3}$$

Note that it follows from the above that

$$C_\gamma^{0,k_2,\dots,k_n}[0,1] = C_\gamma^{k_2,\dots,k_n}[0,1].$$

The companion to the A_n AFLT integral, which we prove in Section 4.4, requires a deformation of the chain $C_\gamma^{k_1,\dots,k_n}[0,1]$, which we denote by $C_{\beta;\gamma}^{k_1,\dots,k_n}[0,1]$ Let

$$E^{k_1,\dots,k_n}[0,1] \subseteq [0,1]^{k_1+\dots+k_n}$$

be the set of points

$$\big(t^{(1)},\dots,t^{(n)}\big) \in [0,1]^{k_1+\dots+k_n}$$

such that

$$0 < t_1^{(r)} < \dots < t_{k_r}^{(r)} < 1 \quad \text{for } 1 \leqslant r \leqslant n$$

and

$$t_i^{(r)} < t_{i-k_r+k_{r+1}}^{(r+1)} \quad \text{for } 1 \leqslant i \leqslant k_r,\ 1 \leqslant r \leqslant n-2.$$

$E^{k_1,\dots,k_n}[0,1]$ differs from the set $D^{k_1,\dots,k_n}[0,1]$ only in that the relative ordering between the variables $t^{(n-1)}$ and $t^{(n)}$ has been removed. Accordingly we replace the sum over the maps M_{n-1} by a sum over maps

$$M'_{n-1} : \{1,\dots,k_n\} \longrightarrow \{1,\dots,k_{n+1}+1\}$$

subject to $M'_{n-1}(i) \leqslant M'_{n-1}(i+1)$ for $1 \leqslant i \leqslant k_{n-1}$ such that

$$t^{(n)}_{M'_{n-1}(i)-1} < t_i^{(n-1)} < t^{(n)}_{M'_{n-1}(i)}, \tag{4.2.4}$$

where $t_0^{(n)} := 0$ and $t_{k_n+1}^{(n)} := 1$. We then define $E^{k_1,\dots,k_n}_{M_1,\dots,M_{n-2};M'_{n-1}}[0,1] \subseteq E^{k_1,\dots,k_n}[0,1]$ by requiring that (4.2.1) holds for $M_1,\dots,M_{n-2}$ and (4.2.4) holds for M_{n-1}. Hence

$$E^{k_1,\dots,k_n}[0,1] = \sum_{M_1,\dots,M_{n-2},M'_{n-1}} E^{k_1,\dots,k_n}_{M_1,\dots,M_{n-2};M'_{n-1}}[0,1].$$

We also replace the weight function (4.2.2) by

$$G^{k_1,\dots,k_n}_{M_1,\dots,M_{n-2},M'_{n-1}}(\gamma) := F^{k_1,\dots,k_{n-1}}_{M_1,\dots,M_{n-2}}(\gamma) \prod_{i=1}^{k_{n-1}} \frac{\sin(\pi(\beta-(i+k_n-k_{n-1}-M'_{n-1}(i)+1)\gamma))}{\sin(\pi(\beta-(i+k_n-k_{n-1})\gamma))}.$$

This weight function is free of poles provided

$$\beta_{n-1} + (i-k_n-1)\gamma \notin \mathbb{Z} \quad \text{for } 1 \leqslant i \leqslant \min\{k_{n-1},k_n\},$$

see (4.4.1b) below. The new chain is then defined as

$$C_{\beta;\gamma}^{k_1,\dots,k_n}[0,1] := \sum_{M_1,\dots,M_{n-2},M'_{n-1}} G^{k_1,\dots,k_n}_{M_1,\dots,M_{n-2},M'_{n-1}}(\gamma) D^{k_1,\dots,k_n}_{M_1,\dots,M_{n-1}}[0,1].$$

4.3 The A_n AFLT integral

Here we will state the full A_n AFLT integral, which is equivalent to Theorem 1.3.1 of the introduction by the evaluation of the A_n Selberg integral (1.3.2).

Recall from the introduction that $I^{\mathrm{A}_n}_{k_1,\dots,k_n}(\mathcal{O};\alpha_1,\dots,\alpha_n,\beta;\gamma)$ denotes the A_n Selberg integral augmented with a polynomial $\mathcal{O}$ which is symmetric in the alphabets $t^{(r)}$.

Theorem 4.3.1. *Let n be a positive integer and $k_0,k_1,\dots,k_n$ integers such that $0:=k_0\leqslant k_1\leqslant\cdots\leqslant k_n$. Let $\lambda\in\mathscr{P}_{k_1}$ and $\mu\in\mathscr{P}$ and $\alpha_1,\dots,\alpha_n,\beta,\gamma\in\mathbb{C}$ such that $\mathrm{Re}(\alpha_2),\dots,\mathrm{Re}(\alpha_n),\mathrm{Re}(\beta)>0$,*

$$\mathrm{Re}(\alpha_1)>-\lambda_1,\quad -\min\left\{\frac{\mathrm{Re}(\beta)}{k_n-1},\frac{1}{k_n}\right\}<\mathrm{Re}(\gamma)<\frac{1}{k_n} \tag{4.3.1a}$$

and

$$-\frac{\mathrm{Re}(\alpha_r)+\lambda_i\delta_{r,1}}{k_r-k_{r-1}-1-(i-1)\delta_{r,1}}<\mathrm{Re}(\gamma)<\frac{\mathrm{Re}(\alpha_r+\cdots+\alpha_s)+\lambda_{k_1}\delta_{r,1}}{s-r}, \tag{4.3.1b}$$

where $1\leqslant r\leqslant s\leqslant n$ and $1\leqslant i\leqslant k_1$, and where $\pm x/0$ ($x>0$) in (4.3.1) should be interpreted as $\pm\infty$. Then

$$\begin{aligned}
&I^{\mathrm{A}_n}_{k_1,\dots,k_n}\big(P^{(1/\gamma)}_{\lambda}[t^{(1)}]P^{(1/\gamma)}_{\mu}[t^{(n)}+\beta/\gamma-1];\alpha_1,\dots,\alpha_n,\beta;\gamma\big) \\
&\quad=P^{(1/\gamma)}_{\lambda}[k_1]P^{(1/\gamma)}_{\mu}[k_n+\beta/\gamma-1]\prod_{s=1}^{n}\prod_{i=1}^{k_s}\frac{\Gamma(\beta_s+(i-k_{s+1}-1)\gamma)\Gamma_q(i\gamma)}{\Gamma(\gamma)} \\
&\qquad\times\prod_{1\leqslant r<s\leqslant n-1}\prod_{i=1}^{k_{r+1}-k_r}\frac{\Gamma(\alpha_{r+1}+\cdots+\alpha_s+(r-s+i)\gamma)}{\Gamma(1+\alpha_{r+1}+\cdots+\alpha_s+(k_s-k_{s+1}+i+r-s-1)\gamma)} \\
&\qquad\times\prod_{r=1}^{n-1}\Bigg(\prod_{i=1}^{k_{r+1}-k_r}\frac{\Gamma(\alpha_{r+1}+\cdots+\alpha_n+(r-n+i)\gamma)}{\Gamma(\alpha_{r+1}+\cdots+\alpha_n+\beta+(k_n-\ell+r-n+i-1)\gamma)} \\
&\qquad\qquad\times\prod_{i=1}^{k_1}\frac{\Gamma(\alpha_1+\cdots+\alpha_r+(k_1-r-i+1)\gamma+\lambda_i)}{\Gamma(1+\alpha_1+\cdots+\alpha_r+(k_r-k_{r+1}+k_1-r-i)\gamma+\lambda_i)} \\
&\qquad\qquad\times\prod_{j=1}^{\ell}\frac{\Gamma(\alpha_{r+1}+\cdots+\alpha_n+\beta+(k_n+r-n-j)\gamma+\mu_j)}{\Gamma(\alpha_{r+1}+\cdots+\alpha_n+\beta+(k_n+k_{r+1}-k_r+r-n-j)\gamma+\mu_j)}\Bigg) \\
&\qquad\times\prod_{i=1}^{k_1}\prod_{j=1}^{\ell}\frac{\Gamma(\alpha_1+\cdots+\alpha_n+\beta+(k_n+k_1-n-i-j)\gamma+\lambda_i+\mu_j)}{\Gamma(\alpha_1+\cdots+\alpha_n+\beta+(k_n+k_1-n-i-j+1)\gamma+\lambda_i+\mu_j)} \\
&\qquad\times\prod_{i=1}^{k_1}\frac{\Gamma(\alpha_1+\cdots+\alpha_n+(k_1-n-i+1)\gamma+\lambda_i)}{\Gamma(\alpha_1+\cdots+\alpha_n+\beta+(k_n+k_1-\ell-n-i)\gamma+\lambda_i)},
\end{aligned} \tag{4.3.2}$$

where ℓ is any integer such that $\ell\geqslant l(\mu)$ and we define $\beta_1=\cdots=\beta_{n-1}:=1$, $\beta_n:=\beta$, and $k_0=k_{n+1}:=0$.

A number of special cases are immediate. As noted in the introduction, for $n=1$ the above theorem reduces to the AFLT integral (1.2.4) up to the relabeling $(k,\alpha)\mapsto(k_1,\alpha_1)$. For $\lambda=\mu=0$ the integral reduces to Warnaar's A_n Selberg integral (1.3.2). Furthermore, setting $\mu=0$ gives the A_n analogue of Kadell's integral also due to Warnaar [War09, Theorem 6.1].

Proof of Theorems 1.3.1 and 4.3.1. We begin with the identity of Corollary 3.3.3, where we note that the alphabets $X^{(1)}$ and $Y^{(n)}$ contain k_1 variables and countably many variables respectively. We now fix a nonnegative integer m and set $Y_i^{(n)} = 0$ for $i > m$. Next we fix a pair of partitions $\lambda \in \mathscr{P}_{k_1}$ and $\mu \in \mathscr{P}_m$, and carry out the specialisation

$$\big(X^{(1)}, Y^{(n)}, c, d\big) \mapsto \big(\langle\lambda\rangle_{k_1}, bz_n t^{1-m}\langle\mu\rangle_m, z_n t, bz_n t^{1-m}\big).$$

Also replacing $\lambda^{(1)}, \dots, \lambda^{(n)}$ by $\nu^{(1)}, \dots, \nu^{(n)}$, this leads to the identity

$$\begin{aligned}
&\sum_{\nu^{(1)},\dots,\nu^{(n)}} P_{\nu^{(1)}}[\langle\lambda\rangle_{k_1}] Q_{\nu^{(n)}}\Big[bt^{-m}\langle\mu\rangle_m + \frac{1-bt^{-m}}{1-t}\Big] \\
&\qquad\times \prod_{r=1}^{n} (tz_r)^{|\nu^{(r)}|} \prod_{r=1}^{n-1} \bigg(P_{\nu^{(r+1)}}\Big[\frac{1-t^{k_{r+1}}}{1-t}\Big] Q_{\nu^{(r)}}\Big[\frac{1-qt^{k_r-1}}{1-t}\Big] f^{k_r,k_{r+1}}_{\nu^{(r)},\nu^{(r+1)}}(t^{k_r-k_{r+1}};q,t)\bigg) \\
&= \prod_{r=1}^{n-1} \bigg(\prod_{i=1}^{k_1} \frac{(z_1\cdots z_r q^{\lambda_i+1} t^{k_1+k_r-k_{r+1}-i};q)_\infty}{(z_1\cdots z_r q^{\lambda_i} t^{k_1-i+1};q)_\infty} \prod_{j=1}^{m} \frac{(bz_{r+1}\cdots z_n q^{\mu_j} t^{k_{r+1}-k_r-j};q)_\infty}{(bz_{r+1}\cdots z_n q^{\mu_j} t^{-j};q)_\infty}\bigg) \\
&\quad\times \prod_{i=1}^{k_1}\prod_{j=1}^{m} \frac{(bz_1\cdots z_n q^{\lambda_i+\mu_j} t^{k_1+1-i-j};q)_\infty}{(bz_1\cdots z_n q^{\lambda_i+\mu_j} t^{k_1-i-j};q)_\infty} \prod_{i=1}^{k_1} \frac{(bz_1\cdots z_{n-1} q^{\lambda_i} t^{k_1-m-i};q)_\infty}{(z_1\cdots z_{n-1} q^{\lambda_i} t^{k_1-i};q)_\infty} \\
&\quad\times \prod_{r=1}^{n-1}\prod_{i=1}^{k_{r+1}-k_r} \frac{(bz_{r+1}\cdots z_{n-1} t^{i-m-1};q)_\infty}{(z_{r+1}\cdots z_{n-1} t^{i-1};q)_\infty} \prod_{1\leqslant r<s\leqslant n-1} \prod_{i=1}^{k_{r+1}-k_r} \frac{(qt^{k_s-k_{s+1}+i-1} z_{r+1}\cdots z_s;q)_\infty}{(t^i z_{r+1}\cdots z_s;q)_\infty},
\end{aligned}$$

where we have dropped a_r in favour of $t^{k_r-k_{r+1}}$ in comparison with Corollary 3.3.3. By virtue of the evaluation symmetry Theorem 2.6.4 and the generalised evaluation symmetry of Lemma 2.6.5, we have

$$P_{\nu^{(1)}}\big[\langle\lambda\rangle_{k_1}\big] = \frac{P_{\nu^{(1)}}[\frac{1-t^{k_1}}{1-t}]}{P_\lambda[\frac{1-t^{k_1}}{1-t}]} P_\lambda\big[\langle\nu^{(1)}\rangle_{k_1}\big]$$

and

$$Q_{\nu^{(n)}}\Big[bt^{-m}\langle\mu\rangle_m + \frac{1-bt^{-m}}{1-t}\Big] = \frac{Q_{\nu^{(n)}}[\frac{1-b}{1-t}]}{P_\mu[\frac{1-b}{1-t}]} P_\mu\Big[bt^{-k_n}\langle\nu^{(n)}\rangle_{k_n} + \frac{1-bt^{-k_n}}{1-t}\Big],$$

effectively allowing us to interchange the roles of $\nu^{(1)}, \nu^{(n)}$ and λ, μ in the summand. Carrying this out and multiplying both sides by

$$P_\lambda\Big[\frac{1-t^{k_1}}{1-t}\Big] P_\mu\Big[\frac{1-b}{1-t}\Big], \tag{4.3.3}$$

the left-hand side of the above identity becomes

$$\begin{aligned}
&\sum_{\nu^{(1)},\dots,\nu^{(n)}} P_\lambda\big[\langle\nu^{(1)}\rangle_{k_1}\big] P_\mu\Big[bt^{-k_n}\langle\nu^{(n)}\rangle_{k_n} + \frac{1-bt^{-k_n}}{1-t}\Big](b;q,t)_{\nu^{(n)}} \\
&\qquad\times \prod_{r=1}^{n} \frac{(tz_r)^{|\nu^{(r)}|} t^{2n(\nu^{(r)})} (t^{k_r};q,t)_{\nu^{(r)}}}{c_{\nu^{(r)}}(q,t) c'_{\nu^{(r)}}(q,t)} \prod_{r=1}^{n-1} (qt^{k_r-1};q,t)_{\nu^{(r)}} f^{k_r,k_{r+1}}_{\nu^{(r)},\nu^{(r+1)}}(t^{k_r-k_{r+1}};q,t),
\end{aligned}$$

where we have also used the specialisation formulas (2.6.23) and (2.6.25). The corresponding right-hand side is as before, except for the additional factor (4.3.3). Next we use (2.6.21) and (2.6.22) in the summand, make the further substitutions

$$b \mapsto q^{\beta+(k_n-1)\gamma}, \quad t \mapsto q^{\gamma} \quad \text{and} \quad z_r \mapsto q^{\alpha_r-\gamma} \quad \text{for } 1 \leqslant r \leqslant n,$$

and introduce auxiliary variables $(t^{(1)},\dots,t^{(n)})$ as $t_i^{(r)} := q^{\nu_i^{(r)}+(k_r-i)\gamma}$. To more simply express the resulting identity we introduce some additional notation, and for alphabets $t = (t_1,\dots,t_k)$, $s = (s_1,\dots,s_\ell)$ define

$$\Delta_\gamma(t;q) := \prod_{1\leqslant i<j\leqslant k} t_j^{2\gamma}\big(1-t_i/t_j\big)\big(q^{1-\gamma}t_i/t_j;q\big)_{2\gamma-1}$$

and

$$\Delta_\gamma(t,s;q) := \prod_{i=1}^{k}\prod_{j=1}^{\ell} s_j^{-\gamma}\big(qt_i/s_j;q\big)_{-\gamma}.$$

After multiplying through by $(1-q)^{k_1+\cdots+k_n}$, and using the definition of the q-gamma function (3.1.4) this yields

$$\begin{aligned}
(1-q)^{k_1+\cdots+k_n} &\sum_{\nu^{(1)},\dots,\nu^{(n)}} P_\lambda\big(t^{(1)};q,q^\gamma\big) P_\mu\bigg(\bigg[q^{\beta-\gamma}t^{(n)}+\frac{1-q^{\beta-\gamma}}{1-q^\gamma}\bigg];q,q^\gamma\bigg) \qquad (4.3.4)\\
&\qquad\times\prod_{r=1}^{n}\bigg(\Delta_\gamma\big(t^{(r)};q\big)\prod_{i=1}^{k_r}\big(t_i^{(r)}\big)^{\alpha_r}\big(qt_i^{(r)};q\big)_{\beta_r-1}\bigg)\prod_{r=1}^{n-1}\Delta_\gamma\big(t^{(r)},t^{(r+1)};q\big)\\
&= q^{\gamma\sum_{r=1}^{n}\left(\alpha_r\binom{k_r}{2}+2\gamma\binom{k_r}{3}\right)-\gamma^2\sum_{r=1}^{n-1}k_r\binom{k_r+1}{2}}\\
&\times P_\lambda\bigg(\bigg[\frac{1-q^{k_1\gamma}}{1-q^\gamma}\bigg];q,q^\gamma\bigg)P_\mu\bigg(\bigg[\frac{1-q^{\beta+(k_n-1)\gamma}}{1-q^\gamma}\bigg];q,q^\gamma\bigg)\\
&\times\prod_{r=1}^{n-1}\prod_{i=1}^{k_r}\frac{\Gamma_q(1+(i-k_{r+1}-1)\gamma)\Gamma_q(i\gamma)}{\Gamma_q(\gamma)}\prod_{i=1}^{k_n}\frac{\Gamma_q(\beta+(i-1)\gamma)\Gamma_q(i\gamma)}{\Gamma_q(\gamma)}\\
&\times\prod_{1\leqslant r<s\leqslant n-1}\prod_{i=1}^{k_{r+1}-k_r}\frac{\Gamma_q(\alpha_{r+1}+\cdots+\alpha_s+(r-s+i)\gamma)}{\Gamma_q(1+\alpha_{r+1}+\cdots+\alpha_s+(k_s-k_{s+1}+i+r-s-1)\gamma)}\\
&\times\prod_{r=1}^{n-1}\Bigg(\prod_{i=1}^{k_{r+1}-k_r}\frac{\Gamma_q(\alpha_{r+1}+\cdots+\alpha_n+(r-n+i)\gamma)}{\Gamma_q(\alpha_{r+1}+\cdots+\alpha_n+\beta+(k_n-m+r-n+i-1)\gamma)}\\
&\qquad\times\prod_{i=1}^{k_1}\frac{\Gamma_q(\alpha_1+\cdots+\alpha_r+(k_1-r-i+1)\gamma+\lambda_i)}{\Gamma_q(1+\alpha_1+\cdots+\alpha_r+(k_1+k_r-k_{r+1}-r-i)\gamma+\lambda_i)}\\
&\qquad\times\prod_{j=1}^{m}\frac{\Gamma_q(\alpha_{r+1}+\cdots+\alpha_n+\beta+(k_n+r-n-j)\gamma+\mu_j)}{\Gamma_q(\alpha_{r+1}+\cdots+\alpha_n+\beta+(k_{r+1}-k_r+k_n+r-n-j)\gamma+\mu_j)}\Bigg)\\
&\times\prod_{i=1}^{k_1}\frac{\Gamma_q(\alpha_1+\cdots+\alpha_n+(k_1-n-i+1)\gamma+\lambda_i)}{\Gamma_q(\alpha_1+\cdots+\alpha_n+\beta+(k_1+k_n-m-n-i)\gamma+\lambda_i)}\\
&\times\prod_{i=1}^{k_1}\prod_{j=1}^{m}\frac{\Gamma_q(\alpha_1+\cdots+\alpha_n+\beta+(k_1+k_n-n-i-j)\gamma+\lambda_i+\mu_j)}{\Gamma_q(\alpha_1+\cdots+\alpha_n+\beta+(k_1+k_n-n-i-j+1)\gamma+\lambda_i+\mu_j)},
\end{aligned}$$

where $\beta_1 = \cdots = \beta_{n-1} := 1$ and $\beta_n := \beta$. The above is a restricted q-integral over the domain $D^{k_1,\dots,k_n}[0,1]$. If we impose the additional assumptions that

$$t_i^{(r)} < t_j^{(r)}$$

for $1 \leqslant i < j \leqslant k_r$ and

$$t_i^{(r)} < t_j^{(r+1)} \tag{4.3.5}$$

for $1 \leqslant i \leqslant k_r$ and $1 \leqslant j \leqslant k_{r+1}$, then we are free to take the limits of the Vandermonde-type products to obtain

$$\lim_{q\to 1^-} \Delta_\gamma\big(t^{(r)};q\big) = \Delta\big(-t^{(r)}\big)^{2\gamma},$$
$$\lim_{q\to 1^-} \Delta_\gamma\big(t^{(r)},t^{(r+1)};q\big) = \Delta\big(-t^{(r)},-t^{(r+1)}\big)^{-\gamma}.$$

The limit of the integrand in this case is thus

$$\begin{aligned} &P_\mu^{(1/\gamma)}[k_1]P_\nu^{(1/\gamma)}[k_n+\beta/\gamma-1]\prod_{i=1}^{k_n}\big(1-t_i^{(n)}\big)^{\beta-1} \\ &\quad\times\prod_{r=1}^{n}\Bigg(\Delta\big(-t^{(r)}\big)^{2\gamma}\prod_{i=1}^{k_r}\big(t_i^{(r)}\big)^{\alpha_r-1}\Bigg)\prod_{r=1}^{n-1}\Delta\big(-t^{(r)},-t^{(r+1)}\big)^{-\gamma}. \end{aligned} \tag{4.3.6}$$

Unfortunately this is wishful thinking, and the integration variables do not necessarily satisfy such a total ordering as in (4.3.5). Hence we must consider the limit in the case that $t_i^{(r)} > t_j^{(r+1)}$. Observe that if $t_i^{(r)} < t_j^{(r+1)}$ we have

$$\lim_{q\to 1^-}\big(q^{1-(i-j-k_r-k_{r+1})\gamma}t_i^{(r)}/t_j^{(r+1)};q\big)_{-\gamma} = \big(1-t_i^{(r)}/t_j^{(r+1)}\big)^{-\gamma}. \tag{4.3.7}$$

Here the t variables occurring on the right-hand side represent integration variables, rather than powers of q. The limit in (4.3.7) is invalid if $t_i^{(r)} > t_j^{(r+1)}$ as the ratio of t variables becomes too large. In order to resolve this we make use of the q-reflection formula [War09, p. 294]

$$\Gamma_q(z)\Gamma_q(1-z) = \frac{2\sqrt{-1}\,q^{z/2}\theta_1(\sqrt{-1}\ \log q^{z/2};q^{1/2})}{(1-q)\theta_1'(0;q^{1/2})},$$

where θ_1 is a theta function as defined in [AAR99, §10.7]. Now define the quantity

$$R_{ij}^{(r)}(\gamma) = \frac{\sin(\pi(i-j-k_r-k_{r+1})\gamma)}{\sin(\pi(i-j-k_r-k_{r+1}+1)\gamma)}.$$

Following the calculations of [War09, p. 294–295] we see that in the case $t_i^{(r)} > t_j^{(r+1)}$ the limit is

$$\begin{aligned} \lim_{q\to 1^-}\big(q^{1-(i-j-k_r-k_{r+1})\gamma}t_i^{(r)}/t_j^{(r+1)};q\big)_{-\gamma} &= \big(t_i^{(r)}/t_j^{(r+1)}-1\big)^{-\gamma}R_{ij}^{(r)}(\gamma) \\ &= \big|1-t_i^{(r)}/t_j^{(r+1)}\big|^{-\gamma}R_{ij}^{(r)}(\gamma). \end{aligned}$$

Therefore we may conclude that in the $q \to 1^-$ limit the integrand is exactly (4.3.6) with $\Delta(-t^{(r)}, -t^{(r+1)})^{-\gamma}$ replaced by

$$\big|\Delta\big(-t^{(r)}, -t^{(r+1)}\big)^{-\gamma}\big| = \big|\Delta\big(t^{(r)}, t^{(r+1)}\big)^{-\gamma}\big|$$

multiplied by a factor of $R_{ij}^{(r)}(\gamma)$ for each occurrence of $t_i^{(r)} > t_j^{(r+1)}$.

With this established we need only determine the chain of integration. This is precisely the chain $C_\gamma^{k_1,\dots,k_n}[0,1]$ of Section 4.2. To see this note that

$$D^{k_1,\dots,k_n}[0,1] = \sum_{M_1,\dots,M_{n-1}} D^{k_1,\dots,k_n}_{M_1,\dots,M_{n-1}}[0,1].$$

From this and the conditions (4.2.1) we pick up a factor of $R_{ij}^{(r)}(\gamma)$ for $1 \leqslant j \leqslant M_r(i) - 1$ as in this case $t_j^{(r+1)} \leqslant t_i^{(r)}$. The product over these factors gives

$$\prod_{i=1}^{M_r(i)-1} R_{ij}^{(r)}(\gamma) = \frac{\sin(\pi(i + k_{r+1} - k_r - M_r(i) + 1)\gamma)}{\sin(\pi(i + k_{r+1} - k_r)\gamma)}.$$

Therefore, when sending $q \to 1^{-1}$ we obtain the chain $C_\gamma^{k_1,\dots,k_n}[0,1]$ in the form (4.2.3).

This completes the proof of Theorem 4.3.1. To complete the proof we divide the above identity by its $\lambda = \mu = 0$ case and then take the $q \to 1^-$ limit. In this limit $(1-q^z)/(1-q^\gamma)$ becomes the binomial element z/γ and the domain of integration becomes $C_\gamma^{k_1,\dots,k_n}[0,1]$, exactly as in the proof of the A_n Selberg integral (cf. [War09, §5]). The resulting A_n Selberg average is the $\ell = k_1$ case of Theorem 1.3.1. It is a trivial exercise to verify that the right-hand side of (1.3.6) is independent of the choice of ℓ, as long as $\ell \geqslant l(\lambda)$. □

4.4 A companion to the A_n AFLT integral

In [War10] the $n = 2$ case of Theorem 3.3.1 with $\mu^{(2)} = 0$ is employed to prove an A_2 Selberg integral with two Jack polynomials in the integrand, but in a different form to that of the A_2 AFLT integral (Theorem 1.3.1 for $n = 2$). The two integrals differ in that the argument of the second Jack polynomial in [War10, Theorem 3.1] is simply the alphabet $t^{(2)}$ with cardinality k_2 and there is an additional parameter β_1 subject to $\beta_1 + \beta_2 = \gamma + 1$ (here β_2 is the β of the A_2 AFLT integral). By the rank-n case of Theorem 3.3.1 we obtain an A_n analogue of [War10, Theorem 3.1] described below.

For $\alpha_1, \dots, \alpha_n, \beta_{n-1}, \beta_n, \gamma \in \mathbb{C}$ such that

$$\beta_{n-1} + \beta_n = \gamma + 1, \tag{4.4.1a}$$

$$\beta_{n-1} + (i - k_n - 1)\gamma \notin \mathbb{Z} \quad \text{for } 1 \leqslant i \leqslant \min\{k_{n-1}, k_n\}, \tag{4.4.1b}$$

and

$$\mathrm{Re}(\gamma) > -\frac{1}{\max\{k_{n-1},k_n\}}, \quad \mathrm{Re}\big(\beta_r + (i-k_{r+1}-1)\gamma\big) > 0 \quad \text{for } 1 \leqslant r \leqslant n \text{ and } 1 \leqslant i \leqslant k_r, \tag{4.4.1c}$$

$$\mathrm{Re}\big(\alpha_r + \cdots + \alpha_s + (r-s+i-1)\gamma\big) > 0 \tag{4.4.1d}$$
$$\text{for } \begin{cases} 1 \leqslant r \leqslant s \leqslant n-1 \text{ and } 1 \leqslant i \leqslant k_r - k_{r-1}, \\ 1 \leqslant r \leqslant n-1,\, s = n \text{ and } 1 \leqslant i \leqslant \min\{k_n, k_r - k_{r-1}\}, \\ r = s = n \text{ and } 1 \leqslant i \leqslant k_n, \end{cases}$$

where $k_{n+1} := 0$ and $\beta_1 = \cdots = \beta_{n-2} := 1$, we modify the A_n Selberg average (1.3.5) to

$$\big\langle \mathcal{O} \big\rangle^{k_1,\dots,k_n}_{\alpha_1,\dots,\alpha_n,\beta_{n-1},\beta_n;\gamma} := \frac{I^{\mathrm{A}_n}_{k_1,\dots,k_n}(\mathcal{O};\alpha_1,\dots,\alpha_n,\beta_{n-1},\beta_n;\gamma)}{I^{\mathrm{A}_n}_{k_1,\dots,k_n}(1;\alpha_1,\dots,\alpha_n,\beta_{n-1},\beta_n;\gamma)}. \tag{4.4.2}$$

Here

$$\begin{aligned} &I^{\mathrm{A}_n}_{k_1,\dots,k_n}(\mathcal{O};\alpha_1,\dots,\alpha_n,\beta_{n-1},\beta_n;\gamma) \\ &\quad := \int\limits_{C^{k_1,\dots,k_n}_{\beta_{n-1},\gamma}[0,1]} \mathcal{O}\big(t^{(1)},\dots,t^{(n)}\big) \prod_{r=1}^{n}\prod_{i=1}^{k_r} \big(t_i^{(r)}\big)^{\alpha_r-1}\big(1-t_i^{(r)}\big)^{\beta_r-1} \\ &\qquad\qquad \times \prod_{r=1}^{n} \big|\Delta\big(t^{(r)}\big)\big|^{2\gamma} \prod_{r=1}^{n-1} \big|\Delta\big(t^{(r)},t^{(r+1)}\big)\big|^{-\gamma}\,\mathrm{d}t^{(1)}\cdots\mathrm{d}t^{(n)} \end{aligned}$$

and $C^{k_1,\dots,k_n}_{\beta;\gamma}[0,1]$ is the β-deformation of the chain defined in Section 4.2. We are now ready to state the counterpart to Theorem 1.3.1.

Theorem 4.4.1. *For $n \geqslant 2$, let $k_1,\dots,k_n$ be nonnegative integers such that $k_1 \leqslant \cdots \leqslant k_{n-1}$. Then for $\alpha_1,\dots,\alpha_n,\beta_{n-1},\beta_n,\gamma \in \mathbb{C}$ such that* (4.4.1) *holds and $\lambda,\mu \in \mathscr{P}$, we have*

$$\begin{aligned} &\Big\langle P^{(1/\gamma)}_{\lambda}\big(t^{(1)}\big) P^{(1/\gamma)}_{\mu}\big(t^{(n)}\big)\Big\rangle^{k_1,\dots,k_n}_{\alpha_1,\dots,\alpha_n,\beta_{n-1},\beta_n;\gamma} \qquad (4.4.3) \\ &\quad = P^{(1/\gamma)}_{\lambda}[k_1] P^{(1/\gamma)}_{\mu}[k_n] \prod_{r=1}^{n-1} \frac{(\alpha_1+\cdots+\alpha_r+(k_1-r)\gamma;\gamma)_\lambda}{(\alpha_1+\cdots+\alpha_r+\beta_r+(k_1+k_r-k_{r+1}-r-1)\gamma;\gamma)_\lambda} \\ &\qquad \times \prod_{r=2}^{n} \frac{(\alpha_r+\cdots+\alpha_n+(k_n+r-n-1)\gamma;\gamma)_\mu}{(1+\alpha_r+\cdots+\alpha_n-\beta_{r-1}+(k_r-k_{r-1}+k_n+r-n-1)\gamma;\gamma)_\mu} \\ &\qquad \times \prod_{i=1}^{k_1}\prod_{j=1}^{k_n} \frac{(\alpha_1+\cdots+\alpha_n+(k_1+k_n-n-i-j+1)\gamma)_{\lambda_i+\mu_j}}{(\alpha_1+\cdots+\alpha_n+(k_1+k_n-n-i-j+2)\gamma)_{\lambda_i+\mu_j}} \end{aligned}$$

and

$$\begin{aligned}
&I^{\mathrm{A}_n}_{k_1,\dots,k_n}(1;\alpha_1,\dots,\alpha_n,\beta_{n-1},\beta_n;\gamma) \qquad (4.4.4)\\
&\quad=\prod_{r=1}^{n}\prod_{i=1}^{k_r}\frac{\Gamma(\beta_r+(i-k_{r+1}-1)\gamma)\Gamma(i\gamma)}{\Gamma(\gamma)}\\
&\qquad\times\prod_{1\leqslant r\leqslant s\leqslant n-1}\prod_{i=1}^{k_r-k_{r-1}}\frac{\Gamma(\alpha_r+\dots+\alpha_s+(r-s+i-1)\gamma)}{\Gamma(\alpha_r+\dots+\alpha_s+\beta_s+(k_s-k_{s+1}+r-s+i-2)\gamma)}\\
&\qquad\times\prod_{r=1}^{n}\prod_{i=1}^{k_n}\frac{\Gamma(\alpha_r+\dots+\alpha_n+(r-n+i-1)\gamma)}{\Gamma(1+\alpha_r+\dots+\alpha_n-\beta_{r-1}+(k_r-k_{r-1}+r-n+i-1)\gamma)},
\end{aligned}$$

where $k_0=k_{n+1}:=0$ *and* $\beta_0=\dots=\beta_{n-2}:=1$.

It should be noted that the final product in (4.4.4) may alternatively be expressed as

$$\prod_{r=1}^{n-1}\prod_{i=1}^{k_r-k_{r-1}}\frac{\Gamma(\alpha_r+\dots+\alpha_n+(r-n+i-1)\gamma)}{\Gamma(\alpha_r+\dots+\alpha_n+(k_n+r-n+i-1)\gamma)}\prod_{i=1}^{k_n}\frac{\Gamma(\alpha_n+(i-1)\gamma)}{\Gamma(\alpha_n+\beta_n+(k_n-k_{n-1}+i-2)\gamma)}.$$

When $\beta=\gamma$ in (1.3.6) and $(\beta_{n-1},\beta_n)=(1,\gamma)$ in (4.4.3) both integral evaluations coincide. For $k_n=0$ equation (4.4.3) simplifies to the A_{n-1} analogue of Kadell's integral [War09, Theorem 6.1].

Proof. We start with (3.3.7) with k_n finite and, for $\lambda,\mu\in\mathscr{P}$ with $l(\lambda)\leqslant k_1$ and $l(\mu)\leqslant k_n$, make the substitutions

$$\big(X^{(1)},Y^{(n)},a_{n-1},t,\mu^{(n)}\big)\mapsto\big(\langle\lambda\rangle_{k_1},q^{\alpha_n}\langle\mu\rangle_{k_n},q^{\beta_{n-1}+(k_{n-1}-k_n)\gamma},q^{\gamma},0\big)$$

and

$$z_r\mapsto q^{\alpha_r-\gamma}\qquad\text{for } 1\leqslant r\leqslant n-1.$$

Then the resulting sum may be turned into an integral following the steps outlined in Section 4.3. For $\lambda=\mu=0$ this yields (4.4.4) and for general λ and μ it gives

$$\Big\langle P^{(1/\gamma)}_{\lambda}\big(t^{(1)}\big)P^{(1/\gamma)}_{\mu}\big(t^{(n)}\big)\Big\rangle^{k_1,\dots,k_n}_{\alpha_1,\dots,\alpha_n,\beta_{n-1},\beta_n;\gamma}\times I^{\mathrm{A}_n}_{k_1,\dots,k_n}(1;\alpha_1,\dots,\alpha_n,\beta_{n-1},\beta_n;\gamma). \qquad \square$$

Chapter 5

A_n AFLT integrals and complex Schur functions

As noted in the introduction, the A_n AFLT integral does not satisfy the same rank-reduction property (1.3.7) as the A_n Selberg integral. In fact, it is not hard to write down an analogue of the A_n AFLT integral that has the desired reduction property. One may replace the pair of Jack polynomials in the integrand of (1.3.6) or (4.3.2) with a function

$$g_{\lambda^{(1)},\dots,\lambda^{(n+1)}}\big(t^{(1)},\dots,t^{(n)},\beta;\gamma\big)$$

such that

$$g^{k_1,\dots,k_n}_{\lambda^{(1)},0,\dots,0,\lambda^{(n+1)}}\big(t^{(1)},\dots,t^{(n)},\beta;\gamma\big) = P^{(1/\gamma)}_{\lambda^{(1)}}\big[t^{(1)}\big]P^{(1/\gamma)}_{\lambda^{(n+1)}}\big[t^{(n)}+\beta/\gamma-1\big] \tag{5.0.1a}$$

and

$$g^{0,k_2,\dots,k_n}_{\lambda^{(1)},\dots,\lambda^{(n+1)}}\big(t^{(1)},\dots,t^{(n)},\beta;\gamma\big) = \begin{cases} g^{k_2,\dots,k_n}_{\lambda^{(2)},\dots,\lambda^{(n+1)}}\big(t^{(1)},\dots,t^{(n)},\beta;\gamma\big) & \text{if } \lambda^{(1)}=0,\\ 0 & \text{otherwise.}\end{cases} \tag{5.0.1b}$$

The generalisations of the Selberg integral conjectured by Matsuo and Zhang in [ZM] do possess the rank-reduction property. For general γ they choose the function g to be

$$g_{\lambda^{(1)},\dots,\lambda^{(n+1)}}\big(t^{(1)},\dots,t^{(n)},\beta;\gamma\big) = \bigg(\prod_{r=1}^{n} P^{(1/\gamma)}_{\lambda^{(r)}}\big[t^{(r)}-t^{(r-1)}\big]\bigg)P^{(1/\gamma)}_{\lambda^{(n+1)}}\big[t^{(n)}+\beta/\gamma-1\big],$$

where $t^{(0)}:=0$. This is the simplest product of Jack polynomials satisfying the conditions (5.0.1). Matsuo and Zhang also considered A_n Selberg integrals with $\gamma=1$ (we delay discussions of convergence until the next section). For $\gamma=1$ the Jack polynomials reduce to Schur functions, and so we have

$$g_{\lambda^{(1)},\dots,\lambda^{(n+1)}}\big(t^{(1)},\dots,t^{(n)},\beta;1\big) = \bigg(\prod_{r=1}^{n} s_{\lambda^{(r)}}\big[t^{(r)}-t^{(r-1)}\big]\bigg)s_{\lambda^{(n+1)}}\big[t^{(n)}+\beta-1\big].$$

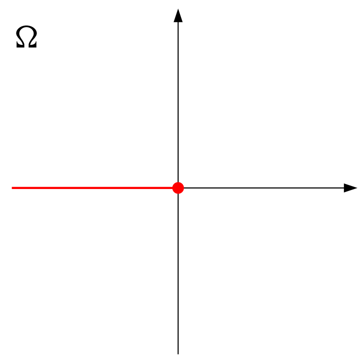
As in the introduction, by the reciprocity for Schur functions (2.5.8), it is more convenient to consider the average

$$\left\langle \prod_{r=1}^{n+1} s_{\lambda^{(r)}}\big[t^{(r)} - t^{(r-1)}\big] \right\rangle_{\alpha_1,\dots,\alpha_n,\beta;1}^{k_1,\dots,k_n},$$

where $t^{(0)} := 0$ and $t^{(n+1)} := 1-\beta$. The main result of this Chapter is a proof of Theorem 1.4.1. In fact we can do slightly better than the above average by replacing the first Schur function $s_{\lambda^{(1)}}(t^{(1)})$ with a complex Schur function $S^{(k_1)}(t^{(1)};z)$, which were first introduced by Kadell [Kad00]. Along the way we prove several integral formulas for complex Schur functions.

5.1 Complex Schur functions

Throughout this chapter we denote by Ω the slit complex plane with $|\mathrm{Im}(\log(x))| < \pi$, where we take the usual principal branch of the complex logarithm with branch cut along the negative real axis and argument in $(-\pi,\pi]$:



The complex Schur functions were originally introduced by Kadell in [Kad00]. For an alphabet x of cardinality n he replaces the partition $\lambda = (\lambda_1,\dots,\lambda_n)$ with a sequence of complex numbers $z = (z_1,\dots,z_n)$ in the definition of $s_\lambda(x)$ as a ratio of alternants (2.5.1). He also observed that the evaluations of the Kadell and Hua–Kadell integrals hold for $\gamma = 1$ with the Schur functions in the integrand replaced by his complex Schur functions. For our purposes it is convenient to slightly alter Kadell's definition. To this end, for $x \in \Omega^n$ and $z \in \mathbb{C}^n$ we define the complex Schur function by

$$S^{(n)}(x;z) := \frac{\det_{1\leqslant i,j\leqslant n}\big(x_i^{z_j}\big)}{\Delta(x)}, \tag{5.1.1}$$

where the denominator is the Vandermonde determinant (1.1.7). It is immediate from this definition that

$$s_\lambda(x) = S^{(n)}(x;\lambda_1+n-1,\lambda_2+n-2,\dots,\lambda_n). \tag{5.1.2}$$

We also have that $S^{(n)}(x;z)$ is a symmetric function in x and so sometimes write $S^{(n)}([x];z)$. Here it is important that x is an alphabet of cardinality n. Further, the function $S^{(n)}(x;z)$ has removable singularities at $x_i = x_j$ and by removing these we may extend the complex Schur function to a holomorphic function on Ω^n.

In view of (5.1.2) the complex Schur function satisfies a simple specialisation formula.

Lemma 5.1.1. *For n a positive integer and $(z_1, \ldots, z_n) \in \mathbb{C}^n$ there holds*

$$S^{(n)}(\underbrace{1, \ldots, 1}_{n \text{ times}}; z_1, \ldots, z_n) = S^{(n)}\big([n]; z_1, \ldots, z_n\big) = \prod_{1 \leqslant i < j \leqslant n} \frac{z_i - z_j}{j - i}.$$

Proof. First observe that $S^{(n)}\big([n]; z_1, \ldots, z_n\big)$ is a polynomial in the z_i. By (5.1.2) and the ordinary specialisation formula (2.5.13) (with $k = n$) we see that

$$S^{(n)}\big([n]; \lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_n\big) = \prod_{1 \leqslant i < j \leqslant n} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

for any $\lambda \in \mathscr{P}_n$, and so the claim follows. □

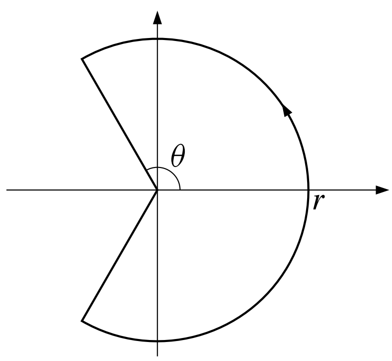
We also require the following simple decomposition for the complex Schur functions.

Lemma 5.1.2. *For any integer m such that $0 \leqslant m \leqslant n$ there holds*

$$S^{(n)}(x_1, \ldots, x_n; z_1, \ldots, z_n) = \sum_{\substack{I \subseteq \{1, \ldots, n\} \\ |I| = m}} \frac{S^{(m)}\big(\big[\sum_{i \in I} x_i\big]; z_1, \ldots, z_m\big) S^{(n-m)}\big(\big[\sum_{i \notin I} x_i\big]; z_{m+1}, \ldots, z_n\big)}{\prod_{i \in I} \prod_{j \notin I} (x_i - x_j)}.$$

Proof. Since the complex Schur function (5.1.1) is a ratio of determinants, the lemma follows by taking the Laplace expansion of both the numerator and denominator. □

Before stating our first theorem of this section we give a description of a particular contour $C_{\theta,r}$ which is used throughout the rest of the chapter. For fixed real $r > 0$ and $0 < \theta < \pi$ the contour $C_{\theta,r}$ is defined to be the positively-oriented boundary of the region $x \in \mathbb{C}$ such that $|x| \leqslant r$ and $|\mathrm{Im}(\log(x))| \leqslant \theta$. This may be visualised as follows.



In the following we let $C_{\theta,r}^k$ denote the k-fold product $C_{\theta,r} \times \cdots \times C_{\theta,r}$.

Theorem 5.1.3. *For ℓ a nonnegative integer, let $y = (y_1, \ldots, y_\ell) \in \Omega^\ell$, and let $0 < \theta < \pi$, $r > 0$ be such that $y_i \in \mathrm{int}(C_{\theta,r})$ for all $1 \leqslant i \leqslant \ell$. Also, for k a nonnegative integer, let $z = (z_1, \ldots, z_k) \in \Omega^k$*

be such that $\mathrm{Re}(z_i) > -1$ *for all* $1 \leqslant i \leqslant k$. *Then, for* $\lambda \in \mathcal{P}$,

$$\frac{1}{k!(2\pi\mathrm{i})^k} \int_{C_{\theta,r}^k} S^{(k)}(x;z) s_\lambda[y-x] \prod_{1 \leqslant i<j \leqslant k} (x_i - x_j)^2 \prod_{i=1}^{k} \prod_{j=1}^{\ell} (x_i - y_j)^{-1} \, \mathrm{d}x_1 \cdots \mathrm{d}x_k \tag{5.1.3}$$
$$= \begin{cases} (-1)^{\binom{k}{2}} S^{(\ell)}\big(y;(z,\lambda_1+\ell-k-1,\dots,\lambda_{\ell-k-1}+1,\lambda_{\ell-k})\big) & \textit{if } l(\lambda) \leqslant \ell-k, \\ 0 & \textit{otherwise}. \end{cases}$$

It is important to remark that the integrand of (5.1.3) is not defined at the origin. Hence the above integral should be interpreted in the sense of improper integrals. For some small $\varepsilon > 0$ and fixed θ, r we let $C_{\theta,r}(\varepsilon)$ be the contour $C_{\theta,r}$ with points inside the ball of radius ε centred at the origin removed. Then the above integral should be thought of as the limit

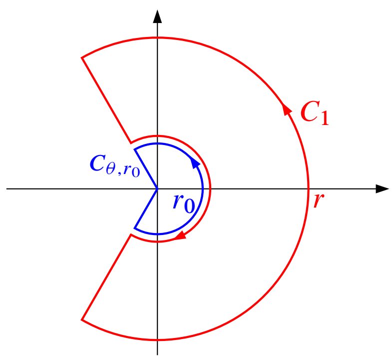
$$\int_{C_{\theta,r}} = \lim_{\varepsilon \to 0} \int_{C_{\theta,r}(\varepsilon)} .$$

Proof of Theorem 5.1.3. First we note that since the integral (5.1.3) is continuous in y, it suffices to prove the result on the dense subset of $\mathrm{int}(C_{\theta,r}^k)$ for which the y_i are all distinct. Define the function

$$\mathcal{I}_m^{(k,\ell)}(y) := \frac{(-1)^{\binom{m}{2}} m!}{k!} \sum_{\substack{I \subseteq \{1,\dots,\ell\} \\ |I|=m}} \Bigg(\prod_{i \in I} \prod_{j \notin I} (y_i - y_j)^{-1}$$
$$\times \frac{1}{(2\pi\mathrm{i})^{k-m}} \int_{C_{\theta,r}^{k-m}} S^{(k)}\bigg(\Big[\sum_{i \in I} y_i + \sum_{i=m+1}^{k} x_i \Big]; z \bigg) s_\lambda \Big[\sum_{i \notin I} y_i - \sum_{i=m+1}^{k} x_i \Big]$$
$$\times \prod_{m+1 \leqslant i<j \leqslant k} (x_i - x_j)^2 \prod_{i=m+1}^{k} \frac{\prod_{j \in I}(x_i - y_j)}{\prod_{j \notin I}(x_i - y_j)} \, \mathrm{d}x_{m+1} \cdots \mathrm{d}x_k \Bigg),$$

where we have suppressed the dependence on z and λ. It is immediate that $\mathcal{I}_0^{(k,\ell)}(y)$ gives the left-hand side of (5.1.3), and that $\mathcal{I}_m^{(k,\ell)}(y) = 0$ if $m > \ell$.

We wish to show that $\mathcal{I}_0^{(k,\ell)}(y) = \mathcal{I}_k^{(k,\ell)}(y)$. To this end we fix an m such that $0 \leqslant m \leqslant k-1$ and, for a fixed term in the summand, carry out the integral over x_{m+1}. Observe that the integrand of $\mathcal{I}_m^{(k,\ell)}(y)$ has only simple poles at the $\ell - m$ points y_j for $j \notin I$, which by our assumption are all distinct. Therefore we decompose the contour $C_{\theta,r}$ corresponding to x_{m+1} as $C_{\theta,r} = C_{\theta,r_0} + C_1$ where $r_0 > 0$ is sufficiently small such that $y_j \in \mathrm{ext}(C_{\theta,r_0})$ for all $j \notin I$ and C_1 is the positively-oriented boundary of the annular sector with inner radius r_0, outer radius r, and forming angle θ with the real axis:



To compute the integral over C_{θ,r_0} note that the arc length of the contour is $2r_0(\theta+1)$ and the integral grows slower than $1/|x_{m+1}|$ as $|x_{m+1}|$ tends to 0. Since the integral is independent of r_0 and vanishes as $r_0 \to 0$, it is identically zero. By the residue theorem we may express the integral over C_1 as a sum over the residues at the $\ell - r$ distinct simple poles. For any $r \notin I$ the residue at $x_{m+1} = y_r$ is given by

$$S^{(k)}\Big(\Big[\sum_{i\in I\cup\{r\}} y_i + \sum_{i=m+2}^{k} x_i\Big]; z\Big) s_\lambda\Big[\sum_{i\notin I\cup\{r\}} y_i - \sum_{i=k+2}^{k} x_i\Big] \frac{\prod_{j\in I}(y_r - y_j)}{\prod_{j\notin I\cup\{r\}}(y_r - y_j)} \prod_{j=m+2}^{k} (y_r - x_j)^2.$$

Therefore, after some elementary manipulations, we may express $\mathcal{I}_m^{(k,\ell)}(y)$ as

$$\begin{aligned}\mathcal{I}_m^{(k,\ell)}(y) = \frac{(-1)^{\binom{m+1}{2}} m!}{k!} \sum_{\substack{I\subseteq\{1,\dots,\ell\}\\ |I|=m}} \sum_{r\notin I} \Bigg(\prod_{i\in I\cup\{r\}} \prod_{j\notin I\cup\{r\}} (y_i - y_j)^{-1} \\ \times \frac{1}{(2\pi \mathrm{i})^{k-m-1}} \int_{C_{\theta,r}^{k-m-1}} S^{(k)}\Big(\Big[\sum_{i\in I\cup\{r\}} y_i + \sum_{i=m+2}^{k} x_i\Big]; z\Big) s_\lambda\Big(\Big[\sum_{i\notin I\cup\{r\}} y_i - \sum_{i=m+2}^{k} x_i\Big]\Big) \\ \times \prod_{m+2\leqslant i<j\leqslant k} (x_i - x_j)^2 \prod_{i=m+2}^{k} \frac{\prod_{j\in I\cup\{r\}}(x_i - y_j)}{\prod_{j\notin I\setminus\{r\}}(x_i - y_j)} \, \mathrm{d}x_{m+2}\cdots \mathrm{d}x_k \Bigg).\end{aligned}$$

Thanks to the sum over $r \notin I$ this vanishes for $m \geqslant \ell$, and so possesses the same vanishing as $\mathcal{I}_{m+1}^{(k,\ell)}(y)$. Defining $J := I \cup \{r\}$ we observe that each J occurs $m+1$ times in the sum as there are $m+1$ choices for the singleton r. Therefore we may combine the above sums into a single sum over subsets of size $m+1$ provided we multiply by $m+1$. This gives $\mathcal{I}_m^{(k,\ell)}(y) = \mathcal{I}_{m+1}^{(k,\ell)}(y)$ which, implies that $\mathcal{I}_0^{(k,\ell)}(y) = \mathcal{I}_k^{(k,\ell)}(y)$. We are left with

$$\mathcal{I}_0^{(k,\ell)}(y) = (-1)^{\binom{k}{2}} \sum_{\substack{I\subseteq\{1,\dots,\ell\}\\ |I|=k}} \frac{S^{(k)}([\sum_{i\in I} y_i]; z) s_\lambda[\sum_{i\notin I} y_i]}{\prod_{i\in I}\prod_{j\notin I}(y_i - y_j)}.$$

If $l(\lambda) > \ell - k$ then for each term in the summand, $|\sum_{i\notin I} y_i| = \ell - k$, implying $s_\lambda[\sum_{i\notin I} y_i] = 0$ so that the sum vanishes identically. Assuming $l(\lambda) \leqslant \ell - k$ then we may combine Lemma 5.1.1 and Lemma 5.1.2 to obtain

$$\mathcal{I}_0^{(k,\ell)}(y) = (-1)^{\binom{k}{2}} S^{(\ell)}\big(y; (z, \lambda_1 + \ell - k - 1, \dots, \lambda_{\ell-k})\big),$$

completing the proof. □

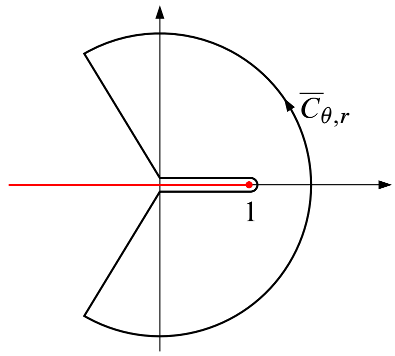
Our next result is a variant of the Euler beta integral (1.1.3).

Lemma 5.1.4. *Let $\alpha, \beta \in \mathbb{C}$ be such that $\mathrm{Re}(\alpha) > 0$. Then for $r > 1$ and $0 < \theta < \pi$ there holds*

$$\frac{1}{2\pi \mathrm{i}} \int_{C_{\theta,r}} x^{\alpha-1}(x-1)^{\beta-1}\, \mathrm{d}x = \frac{\Gamma(\alpha)}{\Gamma(1-\beta)\Gamma(\alpha+\beta)}.$$

Proof. We may assume throughout the proof that $\mathrm{Re}(\beta) > 0$, since by holomorphicity in β it suffices to prove the result on a connected open subset of the complex β-plane. The complex-valued function

$x^{\alpha-1}(x-1)^{\beta-1}$ is multivalued with branch points at 0 and 1. As 1 lies inside $C_{\theta,r}$, we take a branch cut along the real axis from 1 to $-\infty$. This requires us to deform the contour $C_{\theta,r}$ to the contour pictured below:



We may now express the integral as

$$\begin{aligned}\int_{C_{\theta,r}} x^{\alpha-1}(x-1)^{\beta-1}\,\mathrm{d}x &= \int_{\overline{C}_{\theta,r}} x^{\alpha-1}(x-1)^{\beta-1}\,\mathrm{d}x + \lim_{r_0\to 0}\int_{\partial B_{1,r_0}} x^{\alpha-1}(x-1)^{\beta-1}\,\mathrm{d}x \\ &\quad - \mathrm{e}^{\pi\mathrm{i}(\beta-1)}\int_0^1 x^{\alpha-1}|x-1|^{\beta-1}\,\mathrm{d}x + \mathrm{e}^{-\pi\mathrm{i}(\beta-1)}\int_0^1 x^{\alpha-1}|x-1|^{\beta-1}\,\mathrm{d}x,\end{aligned}$$

where $\partial B_{1,r_0}$ is the positively oriented boundary of the closed disk of radius r_0 centred at 1. The integral over $\overline{C}_{\theta,r}$ vanishes by the Cauchy–Goursat theorem. Since $\mathrm{Re}(\beta) > 0$, the integral over $\partial B_{1,r_0}$ also vanishes in the limit as $r_0 \to 0$. To compute the remaining integrals we apply the Euler beta integral (1.1.3) and simplify to arrive at

$$\int_{C_{\theta,r}} x^{\alpha-1}(x-1)^{\beta-1}\,\mathrm{d}x = 2\mathrm{i}\sin(\pi\beta)\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

The proof now follows from the reflection formula [AAR99, Theorem 1.2.1]

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \qquad \square$$

Our main theorem in this chapter requires a generalisation of Theorem 5.1.3 in the case $y = (1,\dots,1) \in \mathbb{C}^\ell$ (or plethystically $y = \ell$).

Theorem 5.1.5. *Let $\beta \in \mathbb{C}$ and $z = (z_1,\dots,z_k) \in \mathbb{C}^k$ such that $\mathrm{Re}(z_i) > -1$ for all $1 \leqslant i \leqslant k$. Then, for $r > 1$, $0 < \theta < \pi$ and $\lambda \in \mathscr{P}$ there holds*

$$\begin{aligned}\frac{1}{(2\pi\mathrm{i})^k}\int_{C_{\theta,r}^k} S^{(k)}(x;z)s_\lambda[1-\beta-x]\prod_{1\leqslant i<j\leqslant k}(x_i-x_j)^2\prod_{i=1}^k(x_i-1)^{\beta-1}\,\mathrm{d}x_1\cdots\mathrm{d}x_k \quad (5.1.4)\\ = (-1)^{\binom{k}{2}}S^{(k)}([k];z)s_\lambda[1-\beta-k] \\ \times\prod_{i=1}^k\biggl(\frac{i!\,\Gamma(z_i+1)}{\Gamma(2-i-\beta)\Gamma(z_i+\beta+k)}\prod_{j\geqslant 1}\frac{z_i-\lambda_j+\beta+k+j-1}{z_i+\beta+k+j-1}\biggr).\end{aligned}$$

Proof. First assume that $\beta = 1 - \ell$ for some integer ℓ. Then (5.1.4) becomes

$$\frac{1}{(2\pi\mathrm{i})^k} \int\limits_{C_{\theta,r}^k} S^{(k)}(x;z) s_\lambda[\ell - x] \prod_{1 \leqslant i < j \leqslant k} (x_i - x_j)^2 \prod_{i=1}^k (x_i - 1)^{-\ell} \, \mathrm{d}x_1 \cdots \mathrm{d}x_k$$
$$= (-1)^{\binom{k}{2}} S^{(k)}([k]; z) s_\lambda[\ell - k]$$
$$\times \prod_{i=1}^k \biggl(\frac{i!\,\Gamma(z_i + 1)}{\Gamma(\ell - i + 1)\Gamma(z_i + k - \ell + 1)} \prod_{j \geqslant 1} \frac{z_i - \lambda_j + k - \ell + j}{z_i + k - \ell + j} \biggr).$$

If $\ell < k$ then the product

$$\prod_{i=1}^k \frac{1}{\Gamma(\ell - i + 1)}$$

occurring on the right-hand side vanishes.[1] If $\ell \geqslant k$ then the right-hand side vanishes unless $l(\lambda) \leqslant \ell - k$ thanks to the Schur function $s_\lambda[\ell - k]$. Assuming that $\ell \geqslant k$ we may write the above as

$$\frac{1}{(2\pi\mathrm{i})^k} \int\limits_{C_{\theta,r}^k} S^{(k)}(x;z) s_\lambda[\ell - x] \prod_{1 \leqslant i < j \leqslant k} (x_i - x_j)^2 \prod_{i=1}^k (x_i - 1)^{-\ell} \, \mathrm{d}x_1 \cdots \mathrm{d}x_k$$
$$= (-1)^{\binom{k}{2}} k!\, S^{(k)}([k]; z) s_\lambda[\ell - k] \prod_{i=1}^k \prod_{j=1}^{\ell-k} \frac{z_i - \lambda_j + k - \ell + j}{k + j - i}.$$

With the aid of Lemma 5.1.1 to combine the expression on the right-hand side we see that

$$\frac{1}{k!(2\pi\mathrm{i})^k} \int\limits_{C_{\theta,r}^k} S^{(k)}(x;z) s_\lambda[\ell - x] \prod_{1 \leqslant i < j \leqslant k} (x_i - x_j)^2 \prod_{i=1}^k (x_i - 1)^{-\ell} \, \mathrm{d}x_1 \cdots \mathrm{d}x_k$$
$$= \begin{cases} (-1)^{\binom{k}{2}} S^{(\ell)}\bigl([\ell]; z, \lambda_1 + \ell - k - 1, \dots, \lambda_{\ell-k}\bigr) & \text{if } l(\lambda) \leqslant \ell - k, \\ 0 & \text{otherwise,} \end{cases}$$

which is precisely Theorem 5.1.3 for $y = \ell$.

To extend the integral to complex β we will show that, after normalisation, both sides become rational functions in β. Since we know the result holds for all integral β this will complete the proof. To this end we replace λ by its conjugate λ' and make use of (2.5.8) with $X \mapsto x + \beta - 1$, i.e.,

$$s_{\lambda'}[1 - \beta - x] = (-1)^{|\lambda|} s_\lambda[x + \beta - 1]$$

together with

$$\prod_{j \geqslant 1} \frac{w - \lambda'_j + j - 1}{w + j - 1} = \prod_{j \geqslant 1} \frac{w - j}{w + \lambda_j - j}.$$

[1] For negative ℓ the vanishing of the integral immediately follows from the Cauchy–Goursat theorem.

The resulting identity is

$$\begin{aligned}\frac{1}{(2\pi \mathrm{i})^k}\int_{C_{\theta,r}^k} S^{(k)}(x;z)s_\lambda[x+\beta-1]\prod_{1\leqslant i<j\leqslant k}(x_i-x_j)^2\prod_{i=1}^k(x_i-1)^{\beta-1}\,\mathrm{d}x_1\cdots\mathrm{d}x_k \qquad (5.1.5)\\ =(-1)^{\binom{k}{2}}S^{(k)}([k];z)s_\lambda[k+\beta-1]\\ \times\prod_{i=1}^k\left(\frac{i!\,\Gamma(z_i+1)}{\Gamma(2-i-\beta)\Gamma(z_i+\beta+k)}\prod_{j\geqslant 1}\frac{z_i+\beta+k-j}{z_i+\lambda_j+\beta+k-j}\right).\end{aligned}$$

In the integral we use the formula for the Schur function on the sum of alphabets (2.5.19) in the form

$$s_\lambda[x+\beta-1]=\sum_\mu s_{\lambda/\mu}[\beta-1]s_\mu(x).$$

Then if LHS(5.1.5) denotes the left-hand side of (5.1.5),

$$\mathrm{LHS}(5.1.5)=\sum_\mu\frac{s_{\lambda/\mu}[\beta-1]}{(2\pi\mathrm{i})^k}\int_{C_{\theta,r}^k}S^{(k)}(x;z)s_\mu[x]\prod_{1\leqslant i<j\leqslant k}(x_i-x_j)^2\prod_{i=1}^k(x_i-1)^{\beta-1}\,\mathrm{d}x_1\cdots\mathrm{d}x_k.$$

The next step is to use the determinantal expression both kinds of Schur functions, (2.5.4) and (5.1.1), to simplify the above to

$$\begin{aligned}\mathrm{LHS}(5.1.5)&=\sum_\mu\sum_{w\in\mathfrak{S}_k}s_{\lambda/\mu}[\beta-1]\frac{\mathrm{sgn}(w)}{(2\pi\mathrm{i})^k}\int_{C_{\theta,r}^k}\det_{1\leqslant i,j\leqslant k}\big(x_i^{\mu_j+k-j}\big)\prod_{i=1}^k x_i^{z_{w(i)}}(x_i-1)^{\beta-1}\mathrm{d}x_1\cdots\mathrm{d}x_k\\ &=k!\sum_\mu s_{\lambda/\mu}[\beta-1]\det_{1\leqslant i,j\leqslant k}\left(\frac{1}{2\pi\mathrm{i}}\int_{C_{\theta,r}}x^{z_i+\mu_j+k-j}(x-1)^{\beta-1}\mathrm{d}x\right).\end{aligned}$$

We may evaluate each integral in the determinant by Lemma 5.1.4, resulting in

$$\begin{aligned}\mathrm{LHS}(5.1.5)&=k!\sum_\mu s_{\lambda/\mu}[\beta-1]\det_{1\leqslant i,j\leqslant k}\left(\frac{\Gamma(z_i+\mu_j+k-j+1)}{\Gamma(1-\beta)\Gamma(z_i+\mu_j+\beta+k-j+1)}\right)\\ &=k!\sum_\mu s_{\lambda/\mu}[\beta-1]\det_{1\leqslant i,j\leqslant k}\left(\frac{(z_i+1)_{\mu_j+k-j}}{(z_i+\beta+1)_{\mu_j+k-j}}\right)\prod_{i=1}^k\frac{\Gamma(z_i+1)}{\Gamma(1-\beta)\Gamma(z_i+\beta+1)}.\end{aligned}$$

Dividing this by the right-hand side of (5.1.5) we obtain

$$\begin{aligned}\frac{(-1)^{\binom{k}{2}}}{S^{(k)}([k];z)}\sum_\mu\frac{s_{\lambda/\mu}[\beta-1]}{s_\lambda[k+\beta-1]}\prod_{i=1}^k\left(\frac{(z_i+\beta+1)_{k-1}}{(i-1)!(2-i-\beta)_{i-1}}\prod_{j\geqslant1}\frac{z_i+\beta+k-j}{z_i+\lambda_j+\beta+k-j}\right)\\ \times\det_{1\leqslant i,j\leqslant k}\left(\frac{(z_i+1)_{\mu_j+k-j}}{(z_i+\beta+1)_{\mu_j+k-j}}\right).\end{aligned}$$

This is manifestly a rational function in β which, by our prior argument, is equal to one for $1-\beta=\ell$ an integer. This completes the proof. □

Of course, if one could more simply show that the final expression above is equal to one for arbitrary β, then the proof is complete and the case of integral β does not need to be considered separately. For $\lambda = 0$ the above rational function reduces to

$$\frac{(-1)^{\binom{k}{2}}}{S^{(k)}([k];z)} \det_{1\leqslant i,j\leqslant k}\left(\frac{(z_i+1)_{k-j}}{(z_i+\beta+1)_{k-j}}\right)\prod_{i=1}^{k}\frac{(z_i+\beta+1)_{k-1}}{(i-1)!(2-i-\beta)_{i-1}}. \tag{5.1.6}$$

Clearing the denominator in the determinant gives

$$\frac{(-1)^{\binom{k}{2}}}{S^{(k)}([k];z)} \det_{1\leqslant i,j\leqslant k}\big((z_i+1)_{k-j}(z_i+\beta+k-j+1)_{j-1}\big)\prod_{i=1}^{k}\frac{1}{(i-1)!(2-i-\beta)_{i-1}}.$$

To evaluate this determinant one may use the following general evaluation, where $X_1,\dots,X_n$, $A_2,\dots,A_n$ and $B_2,\dots,B_n$ are indeterminates [Kra99, Lemma 3]

$$\det_{1\leqslant i,j\leqslant n}\big((X_i+A_n)\cdots(X_i+A_{j+1})(X_i+B_j)\cdots(X_i+B_2)\big)=\prod_{1\leqslant i<j\leqslant n}(X_i-X_j)\prod_{2\leqslant i\leqslant j\leqslant n}(B_i-A_j).$$

Applying this formula with $(n,X_i,A_i,B_i)\mapsto(k,z_i,k-i+1,\beta+k-i+1)$ leads to

$$\det_{1\leqslant i,j\leqslant k}\big((z_i+1)_{k-j}(z_i+\beta+k-j+1)_{j-1}\big)=\prod_{1\leqslant i<j\leqslant k}(z_i-z_j)\prod_{2\leqslant i\leqslant j\leqslant k}(\beta+j-i).$$

The specialisation formula of Lemma 5.1.1 and some elementary manipulations then show that (5.1.6) simplifies to one. Unfortunately, it does not seem to be as simple an exercise to show that the full rational function without $\lambda = 0$ is equal to one.

5.2 A generalisation of the A_n AFLT integral for $\gamma = 1$

In this section we will put the results of the previous section together in order to prove an A_n AFLT-type Selberg integral with $n+1$ Schur functions in the integrand. The first of these functions is a complex Schur function, so that the theorem generalises Theorem 1.4.1.

Theorem 5.2.1. *For n a positive integer, let $k_1,\dots,k_n$ be nonnegative integers, $\alpha_1,\dots,\alpha_n,\beta\in\mathbb{C}$, $z\in\mathbb{C}^{k_1}$ and $\lambda^{(2)},\dots,\lambda^{(n+1)}\in\mathscr{P}$, such that*

$$\begin{aligned}\operatorname{Re}(z_i+\alpha_1+\cdots+\alpha_s)&>s-1 \quad \text{for } 1\leqslant s\leqslant n \text{ and } 1\leqslant i\leqslant k_1,\\ \operatorname{Re}\big(\lambda^{(r)}_{k_r-k_{r-1}}+\alpha_r+\cdots+\alpha_s\big)&>s-r \quad \text{for } 2\leqslant r\leqslant s\leqslant n.\end{aligned}$$

Further let $t^{(1)},\dots,t^{(n)}$ be alphabets of cardinality $k_1,\dots,k_n$, set $\lambda^{(1)}_i := z_i-k_1+i$ for $1\leqslant i\leqslant k_1$

and $t^{(n+1)} := 1-\beta$. *Then,*

$$\int_{C^{k_1}_{\theta_1,r_1}\times\cdots\times C^{k_n}_{\theta_n,r_n}} S^{(k_1)}\big(t^{(1)};z\big)\prod_{r=2}^{n+1} s_{\lambda^{(r)}}\big[t^{(r)}-t^{(r-1)}\big]\prod_{r=1}^{n}\prod_{i=1}^{k_r}\big(t_i^{(r)}\big)^{\alpha_r-1}\big(t_i^{(r)}-1\big)^{\beta_r-1} \tag{5.2.1}$$

$$\times\prod_{r=1}^{n}\Delta^2\big(t^{(r)}\big)\prod_{r=1}^{n-1}\Delta^{-1}\big(t^{(r)},t^{(r+1)}\big)\frac{\mathrm{d}t^{(1)}}{(2\pi\mathrm{i})^{k_1}}\cdots\frac{\mathrm{d}t^{(n)}}{(2\pi\mathrm{i})^{k_n}}$$

$$= S^{(k_1)}([k_1];z)\prod_{r=1}^{n}\bigg((-1)^{\binom{k_r}{2}}s_{\lambda^{(r+1)}}[k_{r+1}-k_r]\prod_{i=1}^{k_r}\frac{i!}{\Gamma(k_{r+1}-i+1)}\bigg)$$

$$\times\prod_{1\leqslant r<s\leqslant n}\prod_{i=1}^{k_r-k_{r-1}}\prod_{j=1}^{k_s-k_{s-1}}\big(\lambda_i^{(r)}-\lambda_j^{(s)}+A_{r,s}+j-i\big)$$

$$\times\prod_{r=1}^{n}\prod_{i=1}^{k_r-k_{r-1}}\bigg(\frac{\Gamma(\lambda_i^{(r)}+A_r-i-n)}{\Gamma(\lambda_i^{(r)}+A_{r,n+1}-i+1)}\prod_{j\geqslant 1}\frac{\lambda_i^{(r)}-\lambda_j^{(n+1)}+A_{r,n+1}+j-i}{\lambda_i^{(r)}+A_{r,n+1}+j-i}\bigg),$$

where $\beta_1=\cdots=\beta_{n-1}:=1$, $\beta_n:=\beta$, $k_0:=0$ *and* $k_{n+1}:=1-\beta$. *In the above we may take* $r_1,\dots,r_n$ *and* $\theta_1,\dots,\theta_n$ *to be arbitrary real numbers so long as* $r_1>\cdots>r_n>1$ *and* $\pi>\theta_1>\cdots>\theta_n>0$.

Before providing a proof of the theorem we note a few important special cases. Firstly, setting $\lambda^{(1)}=\cdots=\lambda^{(n+1)}=0$ so that $z_i=k_1-i$ we obtain the evaluation

$$\int_{C^{k_1}_{\theta_1,r_1}\times\cdots\times C^{k_n}_{\theta_n,r_n}}\prod_{r=1}^{n}\bigg(\Delta^2\big(t^{(r)}\big)\prod_{i=1}^{k_r}\big(t_i^{(r)}\big)^{\alpha_r-1}\big(t_i^{(r)}-1\big)^{\beta_r-1}\bigg)\prod_{r=1}^{n-1}\Delta^{-1}\big(t^{(r)},t^{(r+1)}\big)\frac{\mathrm{d}t^{(1)}}{(2\pi\mathrm{i})^{k_1}}\cdots\frac{\mathrm{d}t^{(n)}}{(2\pi\mathrm{i})^{k_n}}$$

$$=\prod_{r=1}^{n}\bigg((-1)^{\binom{k_r}{2}}\prod_{i=1}^{k_r}\frac{i!}{\Gamma(k_{r+1}-i+1)}\bigg)\prod_{1\leqslant r<s\leqslant n+1}\prod_{i=1}^{k_r-k_{r-1}}\frac{\Gamma(A_{r,s}+k_s-k_{s-1}-i+1)}{\Gamma(A_{r,s}-i+1)},$$

which is a complex variant of the ordinary A_n Selberg integral with $\gamma=1$ (in the notation of the introduction this is $I^{\mathrm{A}_n}_{k_1,\dots,k_n}(1;\alpha_1,\dots,\alpha_n,\beta)$). If we set $z_i=\lambda_i^{(1)}+k_1-i$ for some $\lambda^{(1)}\in\mathscr{P}_{k_1}$ and $\lambda^{(2)}=\cdots=\lambda^{(n)}=0$ then (5.2.1) reduces to a complex version of Theorem 1.3.1 for $\gamma=1$. Still fixing $z_i=\lambda_i^{(1)}+k_1-i$ and leaving the remaining parameters arbitrary, the theorem reduces to the integral of Theorem 1.4.1 of the introduction. In particular the domain of integration reduces to the contour $C^{k_1,\dots,k_n}$ in this case (1.4.1).

Unlike with our previous A_n Selberg integrals we do not impose any ordering on $k_1,\dots,k_n$ in the theorem. However, due to the factor

$$\prod_{r=1}^{n}\prod_{i=1}^{k_n}\frac{1}{\Gamma(k_{r+1}-i+1)}$$

the integral vanishes unless $k_{r+1}-k_r\geqslant 0$ for $1\leqslant r\leqslant n-1$. In other words, unless $k_1\leqslant k_2\leqslant\ldots\leqslant k_n$. If we assume such an ordering then the factor

$$\prod_{r=2}^{n}s_{\lambda^{(r)}}[k_r-k_{r-1}]$$

on the right-hand side vanishes unless $l\big(\lambda^{(r)}\big) \leqslant k_r - k_{r-1}$ for each $2 \leqslant r \leqslant n$.

Proof of Theorem 5.2.1. Throughout the proof we denote the left-hand side of (5.2.1) by

$$\mathcal{L}^{k_1,\dots,k_n}_{\lambda^{(2)},\dots,\lambda^{(n+1)}}(z;\alpha_1,\dots,\alpha_n;\beta).$$

We proceed by induction on the rank n. The base case $n=1$ corresponds to Theorem 5.1.5 with $z_i \mapsto z_i + \alpha_1 - 1$ for all $1 \leqslant i \leqslant k_1$ since

$$S^{(k_1)}\big(t^{(1)}; z_1+\alpha_1-1,\dots,z_{k_1}+\alpha_1-1\big) = S^{(k_1)}\big(t^{(1)};z\big)\prod_{i=1}^{k_1}\big(t_i^{(1)}\big)^{\alpha_1-1}.$$

For the inductive step, assume that the Theorem holds for some $n \geqslant 2$. Then we may use Theorem 5.1.3 with $(k,\ell,x,y,\lambda) \mapsto (k_1,k_2,t^{(1)},t^{(2)},\lambda^{(2)})$ and $z_i \mapsto z_i+\alpha_1-1$ for $1 \leqslant i \leqslant k_1$ to give

$$\begin{aligned}
&\mathcal{L}^{k_1,\dots,k_n}_{\lambda^{(2)},\dots,\lambda^{(n+1)}}(z;\alpha_1,\dots,\alpha_n;\beta)\\
&\qquad = \begin{cases} k_1!(-1)^{\binom{k_1}{2}}\mathcal{L}^{k_2,\dots,k_n}_{\lambda^{(3)},\dots,\lambda^{(n+1)}}(z';\alpha_2,\dots,\alpha_n;\beta) & \text{if } l\big(\lambda^{(2)}\big) \leqslant k_2-k_1\\ 0 & \text{otherwise,}\end{cases}
\end{aligned}$$

where we define

$$z' := \big(z_1+\alpha_1-1,\dots,z_{k_1}+\alpha_1-1,\lambda_1^{(2)}+k_2-k_1-1,\dots,\lambda^{(2)}_{k_2-k_1}\big).$$

By the inductive hypothesis we know the value of

$$\mathcal{L}^{k_2,\dots,k_n}_{\lambda^{(3)},\dots,\lambda^{(n+1)}}(z';\alpha_2,\dots,\alpha_n;\beta),$$

and so assuming $l\big(\lambda^{(2)}\big) \leqslant k_2-k_1$ we have

$$\begin{aligned}
&\mathcal{L}^{k_1,\dots,k_n}_{\lambda^{(2)},\dots,\lambda^{(n+1)}}(z;\alpha_1,\dots,\alpha_n,\beta)\\
&= S^{(k_2)}\big([k_2];z'\big)k_1!(-1)^{\binom{k_1}{2}}\prod_{r=2}^{n}\left(k_r!(-1)^{\binom{k_r}{2}}s_{\lambda^{(r+1)}}[k_{r+1}-k_r]\prod_{i=1}^{k_r}\frac{i!}{\Gamma(k_{r+1}-i+1)}\right)\\
&\times\prod_{s=3}^{n}\prod_{i=1}^{k_2}\prod_{j=1}^{k_s-k_{s-1}}\big(z_i'-\lambda_j^{(s)}+\alpha_2+\dots+\alpha_{s-1}+k_{s-1}-k_s-s+j+2\big)\\
&\times\prod_{3\leqslant r<s\leqslant n}\prod_{i=1}^{k_r-k_{r-1}}\prod_{j=1}^{k_s-k_{s-1}}\big(\lambda_i^{(r)}-\lambda_j^{(s)}+A_{r,s}+j-i\big)\\
&\times\prod_{i=1}^{k_2}\left(\frac{\Gamma(z_i'+\alpha_2+\dots+\alpha_n-n+2)}{\Gamma(z_i'+\alpha_2+\dots+\alpha_n+k_n-k_{n+1}-n+2)}\right.\\
&\qquad\qquad\left.\times\prod_{j\geqslant 1}\frac{z_i'-\lambda_j^{(n+1)}+\alpha_2+\dots+\alpha_n+k_n-k_{n+1}+j-n+1}{z_i'+\alpha_2+\dots+\alpha_n+k_n-k_{n+1}+j-n+1}\right)\\
&\times\prod_{r=3}^{n}\prod_{i=1}^{k_r-k_{r-1}}\left(\frac{\Gamma(\lambda_i^{(r)}+A_r-i-n)}{\Gamma(\lambda_i^{(r)}+A_{r,n+1}-i+1)}\prod_{j\geqslant 1}\frac{\lambda_i^{(r)}-\lambda_j^{(n+1)}+A_{r,n+1}+j-i}{\lambda_i^{(r)}+A_{r,n+1}+j-i}\right).
\end{aligned}$$

To massage this into the desired form we define $\lambda_i^{(1)} := z_i - k_1 + i$ (note that $\lambda^{(1)}$ is not necessarily a partition). This facilitates a great simplification of the above to

$$\begin{aligned}
&\mathcal{L}^{k_1,\dots,k_n}_{\lambda^{(2)},\dots,\lambda^{(n+1)}}(z;\alpha_1,\dots,\alpha_n,\beta) \\
&= S^{(k_1)}([k_1];z)\prod_{r=1}^{n}\left((-1)^{\binom{k_r}{2}} s_{\lambda^{(r+1)}}[k_{r+1}-k_r]\prod_{i=1}^{k_r}\frac{i!}{\Gamma(k_{r+1}-i+1)}\right) \\
&\quad\times\prod_{1\leqslant r<s\leqslant n}\prod_{i=1}^{k_r-k_{r-1}}\prod_{j=1}^{k_s-k_{s-1}}\left(\lambda_i^{(r)}-\lambda_j^{(s)}+A_{r,s}+j-i\right) \\
&\quad\times\prod_{r=1}^{n}\prod_{i=1}^{k_r-k_{r-1}}\left(\frac{\Gamma(\lambda_i^{(r)}+A_r+r-i-n)}{\Gamma(\lambda_i^{(r)}+A_{r,n+1}+r-i+1)}\prod_{j\geqslant 1}\frac{\lambda_i^{(r)}-\lambda_j^{(n+1)}+A_{r,n+1}+j-i}{\lambda_i^{(r)}+A_{r,n+1}+j-i}\right).
\end{aligned}$$

Earlier we assumed that $l\big(\lambda^{(2)}\big) \leqslant k_2 - k_1$. However, the final expression holds regardless of the length of $\lambda^{(2)}$ or the relative ordering between k_1 and k_2, as per our discussion of vanishing preceding the proof. □

5.3 A recursion for Theorem 5.2.1

To conclude this chapter we discuss a system of recurrence relations which have Theorem 1.4.1 as a solution. The system is based on the inverse Pieri rule for Schur functions contained in Proposition 2.5.9.

For $t^{(0)} := 0$, $t^{(n+1)} := 1-\beta$ a binomial element and $t^{(1)},\dots,t^{(n)}$ alphabets of cardinality $k_1,\dots,k_n$, define

$$\mathcal{J}\big(\lambda^{(1)},\dots,\lambda^{(n+1)}\big) := \left\langle\prod_{r=1}^{n+1} s_{\lambda^{(r)}}\big[t^{(r)}-t^{(r-1)}\big]\right\rangle^{k_1,\dots,k_n}_{\alpha_1,\dots,\alpha_n,\beta}, \tag{5.3.1}$$

where $\lambda^{(1)},\dots,\lambda^{(n+1)} \in \mathscr{P}$. Since the parameters $k_1,\dots,k_n,\alpha_1,\dots,\alpha_n,\beta$ are all fixed in the next proposition we suppress these from the notation.

Proposition 5.3.1. *Let m be any integer such that $1 \leqslant m \leqslant n+1$, and $\mu^{(1)},\dots,\mu^{(n+1)} \in \mathscr{P}$. Then for d a nonnegative integer not exceeding the smallest part of $\mu^{(m)}$,*

$$\begin{aligned}
&\mathcal{J}\big(\mu^{(1)},\dots,\mu^{(m-1)},\big(\mu^{(m)},d\big),\mu^{(m+1)},\dots,\mu^{(n+1)}\big) \\
&\qquad = (-1)^d\sum_{\substack{(\lambda^{(r)})'\succ(\mu^{(r)})' \\ 1\leqslant r\leqslant n+1 \\ l(\lambda^{(m)})=l(\mu^{(m)})}}\binom{\beta-1}{d-\sum_{r=1}^{n+1}|\lambda^{(r)}/\mu^{(r)}|}\mathcal{J}\big(\lambda^{(1)},\dots,\lambda^{(n+1)}\big).
\end{aligned}\tag{5.3.2}$$

Before we prove this proposition it should be noted that for $n=3$ the above leads to an alternative proof of Theorem 1.4.1 for $n=2$. With the A_n AFLT integral, which tells us the value of $\mathcal{J}(\lambda,0^{n-1},\mu)$, as initial condition, (5.3.2) has a unique solution provided $n=2$. For $n\geqslant 3$ assuming only the AFLT does not allow for sufficiently many initial conditions, however as we will show below both sides of the theorem are solutions to the system of recursions.

Proof of Proposition 5.3.1. To simplify our notation we define $Y^{(r)} := t^{(r)} - t^{(r-1)}$ for $1 \leqslant r \leqslant n+2$ where $t^{(0)} = t^{(n+2)} := 0$ and $t^{(n+1)} := 1 - \beta$ is a binomial element. For the fixed choice of m we apply the inverse Pieri rule to the Schur function $s_{(\mu^{(m)},d)}$ with $(\mu, \lambda) \mapsto (\mu^{(m)}, \lambda^{(m)})$, which gives

$$\begin{aligned}&\mathcal{J}\big(\mu^{(1)}, \dots, \mu^{(m-1)}, \big(\mu^{(m)}, d\big), \mu^{(m+1)}, \dots, \mu^{(n+1)}\big)\\ &\quad= \sum_{\substack{(\lambda^{(m)})' \succ (\mu^{(m)})' \\ l(\lambda^{(m)}) = l(\mu^{(m)})}} (-1)^{|\lambda^{(m)}/\mu^{(m)}|} \Big\langle s_{\lambda^{(m)}}\big[Y^{(m)}\big] h_{d - |\lambda^{(m)}/\mu^{(m)}|}\big[Y^{(m)}\big] \prod_{\substack{r=1 \\ r \neq m}}^{n+1} s_{\mu^{(r)}}\big[Y^{(r)}\big] \Big\rangle,\end{aligned}$$

where we have used the definition (5.3.1) but suppress the parameters on the right-hand side for notational convenience. Since $Y^{(1)} + \cdots + Y^{(n+2)} = 0$ we may apply Lemma 2.3.1 with $(n, k) \mapsto (n+2, d - |\lambda^{(m)}/\mu^{(m)}|)$ yielding

$$\begin{aligned}&\mathcal{J}\big(\mu^{(1)}, \dots, \mu^{(m-1)}, \big(\mu^{(m)}, d\big), \mu^{(m+1)}, \dots, \mu^{(n+1)}\big)\\ &\quad= (-1)^d \sum_{\substack{i_1, \dots, i_{n+1} \geqslant 0 \\ i_m = 0}} \sum_{\substack{(\lambda^{(m)})' \succ (\mu^{(m)})' \\ l(\lambda^{(m)}) = l(\mu^{(m)})}} e_{d - |i| - |\lambda^{(m)}/\mu^{(m)}|}\big[Y^{(n+2)}\big]\\ &\qquad\qquad \times \Big\langle s_{\lambda^{(m)}}\big[Y^{(m)}\big] \prod_{\substack{r=1 \\ r \neq m}}^{n+1} s_{\mu^{(r)}}\big[Y^{(r)}\big] e_{i_r}\big[Y^{(r)}\big] \Big\rangle,\end{aligned}$$

where $|i| := i_1 + \cdots + i_{n+1}$ and we recall that $e_r := 0$ for $r < 0$, so that the sum is in fact finite. The next step is to apply the e-Pieri rule (2.5.21b) with $(\mu, \lambda, k) \mapsto (\mu^{(r)}, \lambda^{(r)}, i_r)$ for each $r \neq m$. This leads to the expression

$$\begin{aligned}&\mathcal{J}\big(\mu^{(1)}, \dots, \mu^{(m-1)}, \big(\mu^{(m)}, d\big), \mu^{(m+1)}, \dots, \mu^{(n+1)}\big)\\ &\quad= (-1)^d \sum_{\substack{(\lambda^{(r)})' \succ (\mu^{(r)})' \\ 1 \leqslant r \leqslant n+1 \\ l(\lambda^{(m)}) = l(\mu^{(m)})}} e_{d - \sum_{r=1}^{n+1} |\lambda^{(r)}/\mu^{(r)}|}\big[Y^{(n+2)}\big] \Big\langle \prod_{r=1}^{n+1} s_{\lambda^{(r)}}\big[Y^{(r)}\big] \Big\rangle.\end{aligned}$$

All that is left is to note that

$$e_k\big[Y^{(2)}\big] = \binom{\beta - 1}{k}$$

and the proof is complete. $\square$

Proposition 5.3.2. *With the same conditions as Proposition 5.3.1, the evaluation in Theorem 1.4.1 is a solution to the system of recurrence relations* (5.3.2).

Proof. Let $u^{(1)}, \dots, u^{(n+1)}$ be infinite sequences of nonnegative integers such that $|u^{(r)}| := u_1^{(r)} + u_2^{(r)} + \cdots$ is finite for each r, and define

$$\begin{aligned}\mathcal{K}\big(u^{(1)}, \dots, u^{(n+1)}\big) := \prod_{r=1}^{n+1} \prod_{1 \leqslant i < j \leqslant \ell_r} \frac{u_i^{(r)} - u_j^{(r)} + j - i}{j - i} \prod_{r,s=1}^{n+1} \prod_{i=1}^{\ell_r} \frac{(A_{r,s} + k_s - k_{s-1} - i + 1)_{u_i^{(r)}}}{(A_{r,s} + \ell_s - i + 1)_{u_i^{(r)}}} \\ \times \prod_{1 \leqslant r < s \leqslant n+1} \prod_{i=1}^{\ell_r} \prod_{j=1}^{\ell_s} \frac{A_{r,s} + u_i^{(r)} - u_j^{(s)} + j - i}{A_{r,s} + j - i}.\end{aligned} \tag{5.3.3}$$

Here ℓ_r for each r is an arbitrary nonnegative integer such that $u_i^{(r)} = 0$ for $i > \ell_r$. It is not hard to check that $\mathcal{K}(u^{(1)}, \dots, u^{(n+1)})$ is independent of the choice of the ℓ_r. We also note that if $u_{i+1}^{(r)} = u_i^{(r)} + 1$ for some fixed $1 \leqslant r \leqslant n+1$ and $i \geqslant 1$ then $\mathcal{K}(u^{(1)}, \dots, u^{(n+1)})$ vanishes. Since the right-hand side of (1.4.5) corresponds to $\mathcal{K}(\lambda^{(1)}, \dots, \lambda^{(n+1)})$, our task is to show that the system of recursion relations (5.3.2) holds with $\mathcal{J}$ replaced by $\mathcal{K}$.

Let λ, μ be a pair of partitions such that λ/μ is a vertical strip. Then λ may be encoded in terms of μ and a finite subset I of the positive integers as follows: $i \in I$ if $\lambda_i - \mu_i = 1$ and $i \notin I$ if $\lambda_i = \mu_i$. Equivalently,

$$\lambda_i = \begin{cases} \mu_i + 1 & \text{if } i \in I, \\ \mu_i & \text{if } i \notin I, \end{cases}$$

for all $i \geqslant 1$. In what follows we will write the above partition λ as μ_I. It should be noted, however, that given a partition μ and an arbitrary finite subset I of the positive integers, the sequence μ_I defined by

$$(\mu_I)_i := \begin{cases} \mu_i + 1 & \text{if } i \in I, \\ \mu_i & \text{if } i \notin I, \end{cases}$$

for all $i \geqslant 1$ is not necessarily a partition. More precisely, μ_I is not a partition if and only if there exists an $i \geqslant 1$ such that $\mu_i = \mu_{i+1}$ and $i \notin I$ but $i+1 \in I$. By our earlier remark about the vanishing of $\mathcal{K}$, for such a μ_I,

$$\mathcal{K}\big(\dots, \mu_I, \dots\big) = 0$$

since $(\mu_I)_{i+1} = (\mu_I)_i + 1$. Consequently, we may rewrite (5.3.2) with $\mathcal{J}$ replaced by $\mathcal{K}$ as

$$\begin{aligned} &\mathcal{K}\big(\mu^{(1)}, \dots, \mu^{(m-1)}, \big(\mu^{(m)}, d\big), \mu^{(m+1)}, \dots, \mu^{(n+1)}\big) \\ &\quad = (-1)^d \sum_{\substack{I_r \subseteq \{1,\dots,L_r\} \\ 1 \leqslant r \leqslant n+1}} \binom{\beta - 1}{d - \sum_{r=1}^{n+1} |I_r|} \mathcal{K}\Big(\mu^{(1)}_{I_1}, \dots, \mu^{(n+1)}_{I_{n+1}}\Big), \end{aligned} \tag{5.3.4}$$

where $L_r := l(\mu^{(r)}) + d\,\chi(r \neq m)$ with χ the truth or indicator function. Here we note that the replacement of $I_r \subseteq \{1, 2, \dots\}$ for $r \neq m$ by $I_r \subseteq \{1, \dots, l(\mu^{(r)}) + d\}$ is justified since, for nonvanishing of the summand, $\mu^{(r)}_{I_r}$ must be a partition and $|I_r| \leqslant d$. These two conditions combined imply that no $i \in I_r$ can exceed $l(\mu^{(r)}) + d$. The condition that $l(\lambda^{(m)}) = l(\mu^{(m)})$ in (5.3.2) of course translates to $I_m \subseteq \{1, \dots, l(\mu^{(m)})\}$.

Let $u^{(1)}, \dots, u^{(n+1)}$ all be partitions. Then, due to the factor

$$\prod_{r=1}^{n} \prod_{i=1}^{\ell_r} (A_{r,r} + k_r - k_{r-1} - i + 1)_{u_i^{(r)}} = \prod_{r=1}^{n} \prod_{i=1}^{\ell_r} (k_r - k_{r-1} - i + 1)_{u_i^{(r)}}$$

(where ℓ_r may be chosen as $\ell_r = l(u^{(r)})$) in the numerator of (5.3.3), it follows that

$$\mathcal{K}(u^{(1)}, \dots, u^{(n+1)}) = 0$$

if there exists an r such $l(u^{(r)}) > k_r - k_{r-1}$. In (5.3.4) we may thus assume without loss of generality that $l(\mu^{(r)}) \leqslant k_r - k_{r-1}$ for all $1 \leqslant r \leqslant n+1$.[2] Assuming these conditions on the lengths of $\mu^{(1)}, \dots, \mu^{(n)}$ we may divide both sides of (5.3.4) by $\mathcal{K}(\mu^{(1)}, \dots, \mu^{(n+1)})$. After some simplifications this yields[3]

$$\begin{aligned}(-1)^d \sum_{\substack{I_r \subseteq \{1,\dots,L_r\} \\ 1\leqslant r\leqslant n+1}} \binom{-k_{n+1}}{d - \sum_{r=1}^{n+1}|I_r|} \prod_{r,s=1}^{n+1} \Bigg(\prod_{i\in I_r} \frac{A_{r,s} + k_s - k_{s-1} - i + 1 + \mu_i^{(r)}}{A_{r,s} + L_s - i + 1 + \mu_i^{(r)}} & \\ \times \prod_{i\in I_r} \prod_{j\notin I_s} \frac{A_{r,s} + \mu_i^{(r)} - \mu_j^{(s)} + j - i + 1}{A_{r,s} + \mu_i^{(r)} - \mu_j^{(s)} + j - i} \Bigg) & \qquad (5.3.5) \\ = \prod_{r=1}^{n+1} \frac{(A_{m,r} + k_r - k_{r-1} - l_m)_d}{(A_{m,r} + l_r - l_m + \delta_{r,m})_d} \prod_{r=1}^{n+1} \prod_{i=1}^{l_r} \frac{A_{r,m} + l_m - d - i + 1 + \mu_i^{(r)}}{A_{r,m} + l_m - i + 1 + \mu_i^{(r)}}, &\end{aligned}$$

where $l_r := l(\mu^{(r)})$ and where we have eliminated β in favour of k_{n+1}. Since the above identity is unchanged upon interchanging m with any other index between 1 and $n+1$, we fix $m = n+1$ in the following.

Since $\mu_i^{(r)} = 0$ for $i > l_r$, many terms in the summand of (5.3.5) are identically zero. To see this, consider the factor

$$\prod_{i\in I_r} \prod_{j\notin I_r} \frac{A_{r,r} + \mu_i^{(r)} - \mu_j^{(r)} + j - i + 1}{A_{r,r} + \mu_i^{(r)} - \mu_j^{(r)} + j - i} = \prod_{i\in I_r} \prod_{j\notin I_r} \frac{\mu_i^{(r)} - \mu_j^{(r)} + j - i + 1}{\mu_i^{(r)} - \mu_j^{(r)} + j - i}.$$

If the indices i and j both exceed l_r then this vanishes for $j = i-1$. Hence, for the above expression to be nonvanishing, the elements of I_r exceeding l_r must form a progression of the form $l_r+1, \dots, l_r+j_r$ for some integer $0 \leqslant j_r \leqslant d$. This leads to the rewriting of (5.3.5) as

$$\begin{aligned}(-1)^d \sum_{j_1,\dots,j_n\geqslant 0} \sum_{\substack{I_r \subseteq \{1,\dots,l_r\} \\ 1\leqslant r\leqslant n+1}} \binom{-k_{n+1}}{d - |j| - \sum_{r=1}^{n+1}|I_r|} S_{I_1,\dots,I_{n+1}} & \qquad (5.3.6) \\ = \prod_{r=1}^{n+1} \frac{(A_{n+1,r} + k_r - k_{r-1} - l_{n+1})_d}{(A_{n+1,r} + l_r - l_{n+1} + \delta_{r,n+1})_d} \prod_{r=1}^{n+1} \prod_{i=1}^{l_r} \frac{A_{r,n+1} + l_{n+1} - d - i + 1 + \mu_i^{(r)}}{A_{r,n+1} + l_{n+1} - i + 1 + \mu_i^{(r)}}, &\end{aligned}$$

where $|j| := j_1 + \cdots + j_n$ and

$$\begin{aligned}S_{I_1,\dots,I_{n+1}} := \prod_{r,s=1}^{n+1} \Bigg(\frac{(-A_{r,s} - k_s + k_{s-1} + l_r)_{j_r}}{(-A_{r,s} - l_s - j_s + l_r)_{j_r}} \prod_{i\in I_r} \frac{A_{r,s} + k_s - k_{s-1} - i + 1 + \mu_i^{(r)}}{A_{r,s} + l_s + j_s - i + 1 + \mu_i^{(r)}} & \\ \times \prod_{i\notin I_s} \frac{-A_{r,s} + l_r - i + \mu_i^{(s)}}{-A_{r,s} + l_r + j_r - i + \mu_i^{(s)}} \prod_{i\in I_r} \prod_{j\notin I_s} \frac{A_{r,s} + \mu_i^{(r)} - \mu_j^{(s)} + j - i + 1}{A_{r,s} + \mu_i^{(r)} - \mu_j^{(s)} + j - i} \Bigg), &\end{aligned}$$

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[2]The the left-hand side of (5.3.4) trivially vanishes if $l(\mu^{(m)}) = k_m - k_{m-1}$ and $d > 0$. However, the corresponding vanishing of the right-hand side is not obvious and arises after cancellation of terms.

[3]To conveniently carry out these simplifications one may uniformly choose all ℓ_r arising in the various functions $\mathcal{K}$ as $\ell_r = L_r + \delta_{r,m}$.

with $j_{n+1} := 0$. We will prove the rational function identity (5.3.6) for each $\mu^{(r)}$ an arbitrary sequence of l_r indeterminates and each k_r an indeterminate. To this end we replace

$$\mu_i^{(r)} \mapsto x_{l_1+\cdots+l_{r-1}+i} + A_{n+1,r} + i - 1 \quad \text{for } 1 \leqslant r \leqslant n+1 \text{ and } 1 \leqslant i \leqslant l_r,$$
$$k_r \mapsto \sum_{s=1}^{r} (a_s + l_s) \quad \text{for } 1 \leqslant r \leqslant n+1,$$

where the $x_1, \dots, x_l$ for $l := l_1 + \cdots + l_{n+1}$ and the $a_1, \dots, a_{n+1}$ are indeterminates. Given $I_r \subseteq \{1, \dots, l_r\}$ for all $1 \leqslant r \leqslant n+1$, we define the set $I \subseteq \{1, \dots, l\}$ as

$$I := \bigcup_{r=1}^{n+1} (I_r + l_1 + \cdots + l_{r-1}),$$

where $I + z := \{i + z | i \in I\}$. Defining $a := a_1 + \cdots + a_{n+1}$, this leads to the much simpler identity

$$\sum_{j_1,\dots,j_n \geqslant 0} \sum_{I \subseteq \{1,\dots,l\}} (-1)^{|I|+|j|} \frac{(-d)_{|I|+|j|}}{(1-a-d-l)_{|I|+|j|}} \prod_{i \in I} \prod_{j \notin I} \frac{x_i - x_j + 1}{x_i - x_j} \prod_{r,s=1}^{n+1} \frac{(b_r - b_s - a_s)_{j_r}}{(b_r - b_s - j_s)_{j_r}}$$
$$\times \prod_{r=1}^{n+1} \left(\prod_{i \in I} \frac{a_r + b_r + x_i}{b_r + j_r + x_i} \prod_{i \notin I} \frac{b_r + x_i - 1}{b_r + j_r + x_i - 1} \right)$$
$$= \frac{(a_{n+1})_d}{(a+l)_d} \prod_{r=1}^{n} \frac{(a_r + b_r)_d}{(b_r)_d} \prod_{i=1}^{l} \frac{x_i - d}{x_i},$$

where we have also replaced $A_r \mapsto l_r - b_r$ for $1 \leqslant r \leqslant n$ and fixed $A_{n+1} = 0$.[4] The above identity is the $q \to 1$ limit of Lemma 5.4.1 with $a_r \mapsto q^{a_r}$, $b_r \mapsto q^{b_r}$ and $x_i \mapsto q^{x_i}$. The proof of the lemma is contained in the next section. □

5.4 A summation formula for A_n basic hypergeometric series

Over the course of this thesis we have already implicitly met many basic hypergeometric functions and series implicitly. We say that a series $\sum_n c_n$ is a basic hypergeometric series if the ratio c_{n+1}/c_n is a rational function of q^n where q is usually some complex number such that $|q| < 1$. The summation we are concerned with in this section is a particular basic hypergeometric summation of type A, meaning it contains the factor

$$\prod_{1 \leqslant i < j \leqslant n} \left(x_i q^{k_i} - x_j q^{k_j} \right),$$

which is nothing but the Vandermonde product on the alphabet $(x_1 q^{k_1}, \dots, x_n q^{k_n})$. Indeed, the summation which we will use below is the multivariate A_n q-Pfaff–Saalschütz summation due to Milne [Mil97, Theorem 4.15] (see also [Sch, Equation (2.15)]),

$$\sum_{k_1,\dots,k_n \geqslant 0} \frac{(q^{-d};q)_{|k|} q^{|k|}}{(a_1 \cdots a_n b q^{1-d}/c;q)_{|k|}} \prod_{1 \leqslant r < s \leqslant n} \frac{x_r q^{k_r} - x_s q^{k_s}}{x_r - x_s} \prod_{r,s=1}^{n} \frac{(a_s x_r/x_s;q)_{k_r}}{(q x_r/x_s;q)_{k_r}} \prod_{r=1}^{n} \frac{(b x_r;q)_{k_r}}{(c x_r;q)_{k_i}}$$
$$= \frac{(c/b;q)_d}{(c/a_1 \cdots a_n b;q)_d} \prod_{r=1}^{n} \frac{(c x_r/a_r;q)_d}{(c x_r;q)_d}. \tag{5.4.1}$$

[4]Since (5.3.6) only depends on differences of the A_r, there is no loss of generality in choosing $A_{n+1} = 0$.

Since $(q^{-d};q)_{|k|}=0$ for $|k|>d$, the series on the left terminates. In the previous section we used a slightly modified version of this sum, which now state.

Lemma 5.4.1. *Let d,l,n be nonnegative integers and $a_1,\dots,a_{n+1}$, $b_1,\dots,b_n$, $x_1,\dots,x_l$ indeterminates. Then*

$$\sum_{k_1,\dots,k_n\geqslant 0}\sum_{I\subseteq\{1,\dots,l\}}(-1)^{|I|+|k|}q^{\binom{|I|}{2}-\binom{|k|}{2}}\frac{(q^{-d};q)_{|I|+|k|}}{(q^{1-d-l}/a_1\cdots a_{n+1};q)_{|I|+|k|}}\Big(\frac{q^{1-l}}{a_1\cdots a_{n+1}}\Big)^{|I|} \tag{5.4.2}$$
$$\times\prod_{i\in I}\prod_{j\notin I}\frac{x_j-qx_i}{x_j-x_i}\prod_{r=1}^{n+1}\Big(\prod_{i\in I}\frac{1-a_rb_rx_i}{1-q^{k_r}b_rx_i}\prod_{j\notin I}\frac{1-q^{-1}b_rx_j}{1-q^{k_r-1}b_rx_j}\prod_{s=1}^{n+1}\frac{(b_r/a_sb_s;q)_{k_r}}{(q^{-k_s}b_r/b_s;q)_{k_r}}\Big)$$
$$=\frac{q^{dl}(a_{n+1};q)_d}{(a_1\cdots a_{n+1}q^l;q)_d}\prod_{r=1}^{n}\frac{(a_rb_r;q)_d}{(b_r;q)_d}\prod_{i=1}^{l}\frac{1-q^{-d}x_i}{1-x_i},$$

where $|k|:=k_1+\dots+k_n$, $k_{n+1}:=0$ and $b_{n+1}:=1$.

Proof. In the identity (5.4.1) we first apply

$$\prod_{1\leqslant r<s\leqslant n}\frac{x_rq^{k_r}-x_sq^{k_s}}{x_r-x_s}=(-1)^{|k|}q^{-\binom{|k|+1}{2}}\prod_{r,s=1}^{n}\frac{(qx_r/x_s;q)_{k_r}}{(q^{-k_s}x_r/x_s;q)_{k_r}},$$

which leads to

$$\sum_{k_1,\dots,k_n\geqslant 0}(-1)^{|k|}q^{-\binom{|k|}{2}}\frac{(q^{-d};q)_{|k|}}{(a_1\cdots a_nbq^{1-d}/c;q)_{|k|}}\prod_{r,s=1}^{n}\frac{(a_sx_r/x_s;q)_{k_r}}{(q^{-k_s}x_r/x_s;q)_{k_r}}\prod_{r=1}^{n}\frac{(bx_r;q)_{k_r}}{(cx_r;q)_{k_r}}$$
$$=\frac{(c/b;q)_d}{(c/a_1\cdots a_nb;q)_d}\prod_{r=1}^{n}\frac{(cx_r/a_r;q)_d}{(cx_r;q)_d},$$

which is (5.4.2) for $l=0$ with $b_r\mapsto cx_r$, $a_r\mapsto 1/a_r$ for $1\leqslant r\leqslant n$ and $a_{n+1}\mapsto c/b$. To obtain (5.4.2) for all l we scale $n\mapsto n+l$ and split the set of $n+l$ variables x_r in two, relabeling so that

$$x_r\mapsto\begin{cases}b_r & \text{if } 1\leqslant r\leqslant n,\\ x_{r-n} & \text{if } n+1\leqslant r\leqslant n+l.\end{cases}$$

We wish to specialise $a_r\mapsto 1/q$ for $n+1\leqslant r\leqslant n+l$. This ensures the summand vanishes unless $0\leqslant k_{n+1},\dots,k_{n+l}\leqslant 1$. We encode each term in the sum over $k_{n+1},\dots,k_{n+l}$ as a subset $I\subseteq\{1,\dots,l\}$ where $i\in I$ if $k_{n+i}=1$ and $i\notin I$ if $k_{n+i}=0$. This allows the sum over $k_{n+1},\dots,k_{k+d}$ to be replaced by a sum over I. Making these changes and relabeling $b\mapsto a_{n+1}$

(recall that the "old" a_{n+1} has been specialised), the identity becomes

$$\begin{aligned}
&\sum_{k_1,\dots,k_n\geqslant 0}\sum_{I\subseteq\{1,\dots,l\}}(-1)^{|I|+|k|}q^{-\binom{|I|+|k|}{2}}\frac{(q^{-d};q)_{|I|+|k|}}{(a_1\cdots a_{n+1}q^{1-d-l}/c;q)_{|I|+|k|}}\\
&\qquad\times\prod_{r,s=1}^{n}\frac{(a_sb_r/b_s;q)_{k_r}}{(q^{-k_s}b_r/b_s;q)_{k_r}}\prod_{r=1}^{n}\biggl(\prod_{i\in I}\frac{1-a_rx_i/b_r}{1-q^{-k_r}x_i/b_r}\prod_{i\notin I}\frac{1-q^{-1}b_r/x_i}{1-q^{k_r-1}b_r/x_i}\biggr)\\
&\qquad\times\prod_{i\in I}\prod_{j\notin I}\frac{1-q^{-1}x_i/x_j}{1-x_i/x_j}\prod_{r=1}^{n}\frac{(a_{n+1}b_r;q)_{k_r}}{(b_r;q)_{k_r}}\prod_{i\in I}\frac{1-a_{n+1}x_i}{1-cx_i}\\
&\quad=\frac{(c/a_{n+1};q)_d}{(q^lc/a_1\cdots a_{n+1};q)_d}\prod_{r=1}^{n}\frac{(cb_r/a_r;q)_d}{(cb_r;q)_d}\prod_{i=1}^{l}\frac{1-q^dx_i}{1-cx_i}.
\end{aligned}$$

The claim now follows after the substitution

$$(a_r,c,x_i)\mapsto(1/a_r,1,1/x_i)\quad\text{or}\quad(b_r,c,q)\mapsto(1/b_r,1,1/q)$$

and defining $k_{n+1}:=0$ and $b_{n+1}:=1$. □

Chapter 6

Further AFLT-type Selberg integrals

This chapter is devoted to several further AFLT-type Selberg integrals. We first discuss an AFLT-type analogue of the Askey–Habsieger–Kadell integral, which takes the form of a multiple q-integral. Next we introduce the elliptic analogue of the Selberg integral, and state without proof the elliptic analogue of the AFLT integral obtained in [ARW]. Here the role of the Jack polynomials is played by a pair of elliptic interpolation functions. As a limiting case, an AFLT-integral for Macdonald polynomials is also obtained. The thesis concludes with a discussion of several open problems.

6.1 A q-analogue of the AFLT integral

In [Ask80] Askey conjectured several q-analogues of the Selberg integral (1.1.2). The first of his conjectures, which takes the form of a multiple Jackson or q-integral, was proved independently by Habsieger [Hab88] and Kadell [Kad88] and is now known as the Askey–Habsieger–Kadell integral. In a similar vein, Jackson-type q-analogues of the Kadell and Hua–Kadell integrals were obtained in [Kan96, Proposition 5.2] and [War05, Theorem 1.4] respectively. By a minor adaptation of the proof of Theorem 1.3.1 in the case of A_1 and positive integral γ, we obtain a q-analogue of the AFLT integral (1.2.4) which generalises all of the previously mentioned integral evaluations.[1] Since (positive) integral values of γ are problematic for A_n with $n \geqslant 2$, our result is restricted to the rank-one case only.

In the following we assume q to be real such that $0 < q < 1$. Recall the 1-dimensional Jackson integral (3.3.2). The k-dimensional Jackson-integral over $[0,1]^k$ may then be defined as

$$\int\limits_{[0,1]^k} f(t_1,\dots,t_k)\,\mathrm{d}_q t_1\cdots\mathrm{d}_q t_k := (1-q)^k \sum_{v_1,\dots,v_k\geqslant 0} f\big(q^{v_1},\dots,q^{v_k}\big)q^{v_1+\dots+v_k}, \tag{6.1.1}$$

where $f:\mathbb{R}^k \longrightarrow \mathbb{C}$ is a function such that the sum on the right is absolutely convergent. For $\alpha,\beta\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$ such that $\mathrm{Re}(\alpha)>0$ and γ a positive integer, the Askey–Habsieger–Kadell

[1] Although closely related, our q-analogue of the AFLT integral is slightly different to [MMSS12, Equation (101)] by Mironov et al. (or the integral arising in [IOY13, Equation (4.4.5)]) which is essentially our (6.1.3) below and which does not include the Askey–Habsieger–Kadell integral as a special case.

integral may be stated as

$$\begin{aligned}\int\limits_{[0,1]^k} \prod_{i=1}^{k} t_i^{\alpha-1}(qt_i;q)_{\beta-1} \prod_{1\leqslant i<j\leqslant k} t_j^{2\gamma}(q^{1-\gamma}t_i/t_j;q)_{2\gamma}\,\mathrm{d}_q t_1\cdots\mathrm{d}_q t_k \\ = q^{\alpha\gamma\binom{k}{2}+2\gamma^2\binom{k}{3}} \prod_{i=1}^{k} \frac{\Gamma_q(\alpha+(i-1)\gamma)\Gamma_q(\beta+(i-1)\gamma)\Gamma_q(1+i\gamma)}{\Gamma(\alpha+\beta+(2k-i-1)\gamma)\Gamma_q(1+\gamma)}.\end{aligned}$$

For $k=1$ this reduces to the q-analogue of Euler's beta integral (3.3.3). As previously remarked, one may derive the Askey–Habsieger–Kadell integral from the Kaneko–Macdonald q-binomial theorem. However, to prove our q-analogue of the AFLT integral we require the full Cauchy identity for Macdonald polynomials (2.6.8).

Theorem 6.1.1. *Let k be a positive integer and $\lambda \in \mathscr{P}_k$ and $\mu \in \mathscr{P}$. Then for γ a positive integer and $\alpha, \beta \in \mathbb{C}$ such that* $\mathrm{Re}(\alpha) > -\mu_k$ *and $-\beta \notin \mathbb{N}$ there holds*

$$\begin{aligned}\int\limits_{[0,1]^k} P_\lambda(t;q,q^\gamma) P_\mu\Big(\Big[q^{\beta-\gamma}t+\frac{1-q^{\beta-\gamma}}{1-q^\gamma}\Big];q,q^\gamma\Big) \\ \times \prod_{i=1}^{k} t_i^{\alpha-1}(qt_i;q)_{\beta-1} \prod_{1\leqslant i<j\leqslant k} t_j^{2\gamma}(q^{1-\gamma}t_i/t_j;q)_{2\gamma}\,\mathrm{d}_q t_1\cdots\mathrm{d}_q t_k \\ = q^{\alpha\gamma\binom{k}{2}+2\gamma^2\binom{k}{3}} P_\lambda\Big(\Big[\frac{1-q^{k\gamma}}{1-q^\gamma}\Big];q,q^\gamma\Big) P_\mu\Big(\Big[\frac{1-q^{\beta+(k-1)\gamma}}{1-q^\gamma}\Big];q,q^\gamma\Big) \\ \times \prod_{i=1}^{k} \frac{\Gamma_q(\alpha+(k-i)\gamma+\lambda_i)\Gamma_q(\beta+(i-1)\gamma)\Gamma_q(1+i\gamma)}{\Gamma(\alpha+\beta+(2k-m-i-1)\gamma+\lambda_i)\Gamma_q(1+\gamma)} \\ \times \prod_{i=1}^{k}\prod_{j=1}^{m} \frac{\Gamma_q(\alpha+\beta+(2k-i-j-1)\gamma+\lambda_i+\mu_j)}{\Gamma_q(\alpha+\beta+(2k-i-j)\gamma+\lambda_i+\mu_j)},\end{aligned}$$

where m is an arbitrary integer such that $m \geqslant l(\mu)$.

The proof of the theorem uses the following simple lemma [War05, Lemma 3.1].

Lemma 6.1.2. *Let γ be a positive integer and*

$$f(t_1,\dots,t_k) \prod_{1\leqslant i<j\leqslant k} (1-t_i/t_j)$$

a symmetric function such that $f(t_1,\dots,t_k)$ vanishes if $t_i/t_j \in \{q^{1-\gamma},q^{2-\gamma},\dots,q^{\gamma-1}\}$ for any $1\leqslant i<j\leqslant k$. Then

$$\begin{aligned}(1-q)^k \sum_{\lambda\in\mathscr{P}_k} f(t_1,\dots,t_k) \prod_{1\leqslant i<j\leqslant k} (1-t_i/t_j) \\ = \frac{(1-q^\gamma)^k}{(q^\gamma;q^\gamma)_k} \int\limits_{[0,1]^k} f(t_1,\dots,t_k) \prod_{1\leqslant i<j\leqslant k} (1-q^\gamma t_i/t_j)\,\frac{\mathrm{d}_q t_1}{t_1}\cdots\frac{\mathrm{d}_q t_k}{t_k},\end{aligned}$$

where on the left $t_i := q^{\lambda_i+(k-i)\gamma}$.

Proof. Since $t_i := q^{\lambda_i+(k-i)\gamma}$ in the sum on the left-hand side of the lemma, the sum is essentially over all sequences $v = (v_1, \dots, v_k)$ of nonnegative integers of length at most k such that $v_j - v_i \geqslant \gamma$ for $i < j$. Taking v to be such a sequence, we are free to replace $v_i \mapsto v_i - (k-i)\gamma$ since if $0 \leqslant v_i - v_j < \gamma$ the summand will vanish. Indeed, for any $i < j$ if $1 \leqslant v_i - v_j < \gamma$ the function f will vanish by the assumption of the lemma, and if $v_i = v_j$ then the product $\prod_{1\leqslant i<j\leqslant k}(1 - t_i/t_j)$ gives the desired vanishing. In fact these vanishing conditions still hold for all $1 \leqslant i \neq j \leqslant k$, and so we may transform the sum on the left into

$$\frac{(1-q)^k}{k!} \sum_{v_1,\dots,v_k\geqslant 0} f(q^{v_1}, \dots, q^{v_k}) \prod_{1\leqslant i<j\leqslant k} (1 - q^{v_i - v_j}).$$

To proceed we will need the identity

$$\sum_{w\in\mathfrak{S}_k} w\Bigg(\prod_{1\leqslant i<j\leqslant k} \frac{x_i - tx_j}{x_i - x_j}\Bigg) = \frac{(t;t)_k}{(1-t)^k}, \tag{6.1.2}$$

for $X_k = (x_1, \dots, x_k)$ a finite alphabet. To prove this, note that the denominator is the Vandermonde product up to a sign, and thus

$$\sum_{w\in\mathfrak{S}_k} w\Bigg(\prod_{1\leqslant i<j\leqslant k} \frac{x_i - tx_j}{x_i - x_j}\Bigg) = \frac{1}{\Delta(X_k)} \sum_{w\in\mathfrak{S}_k} \operatorname{sgn}(w) w\Bigg(\prod_{1\leqslant i<j\leqslant k} (x_i - tx_j)\Bigg).$$

From this, both numerator and denominator are polynomials of degree $k-1$ which vanish for $x_i = x_j$. Therefore the left-hand side is independent of X_k, and so we may choose $X_k = (1, t, \dots, t^{k-1})$. However in this case there is only one surviving term in the sum, corresponding to the permutation which reverses the order of the word $12\cdots k$, from which the evaluation follows.

With this established, choose $x_i = q^{v_i}$ and $t = q^\gamma$ in (6.1.2). Then we have

$$\begin{aligned}
&\frac{\big((1-q)(1-q^\gamma)\big)^k}{k!(q^\gamma;q^\gamma)_k} \sum_{v_1,\dots,v_k\geqslant 0} f(q^{v_1}, \dots, q^{v_k}) \prod_{1\leqslant i<j\leqslant k} (1 - q^{v_i - v_j}) \sum_{w\in\mathfrak{S}_k} w\Bigg(\prod_{1\leqslant i<j\leqslant k} \frac{q^{v_i} - q^{v_j+\gamma}}{q^{v_i} - q^{v_j}}\Bigg) \\
&= \frac{\big((1-q)(1-q^\gamma)\big)^k}{k!(q^\gamma;q^\gamma)_k} \sum_{v_1,\dots,v_k\geqslant 0} \sum_{w\in\mathfrak{S}_k} w\Bigg(f(q^{v_1}, \dots, q^{v_k}) \prod_{1\leqslant i<j\leqslant k} (1 - q^{v_j - v_i + \gamma})\Bigg) \\
&= \frac{\big((1-q)(1-q^\gamma)\big)^k}{(q^\gamma;q^\gamma)_k} \sum_{v_1,\dots,v_k\geqslant 0} f(q^{v_1}, \dots, q^{v_k}) \prod_{1\leqslant i<j\leqslant k} (1 - q^{v_j - v_i + \gamma}),
\end{aligned}$$

where the last equality follows by the symmetry of the summand. This sum is precisely as in the definition of the Jackson integral (6.1.1) after introducing $q^{v_1+\dots+v_k}$, and so the proof is complete. □

Proof of Theorem 6.1.1. Assume that γ is a positive integer. For $n = 1$ the identity (4.3.4) states that

$$(1-q)^k \sum_{\nu} P_\lambda(t;q,q^\gamma) P_\mu\Big(\Big[q^{\beta-\gamma}t + \frac{1-q^{\beta-\gamma}}{1-q^\gamma}\Big];q,q^\gamma\Big)\Delta_\gamma(t;q)\prod_{i=1}^k t_i^\alpha(qt_i;q)_{\beta-1} \tag{6.1.3}$$
$$= q^{\alpha\binom{k}{2}\gamma+2\binom{k}{3}\gamma^2} P_\lambda\Big(\Big[\frac{1-q^{k\gamma}}{1-q^\gamma}\Big];q,q^\gamma\Big) P_\mu\Big(\Big[\frac{1-q^{\beta+(k-1)\gamma}}{1-q^\gamma}\Big];q,q^\gamma\Big)$$
$$\times \prod_{i=1}^k \frac{\Gamma_q(\alpha_1+(k-i)\gamma+\lambda_i)\Gamma_q(\beta+(i-1)\gamma)\Gamma_q(i\gamma)}{\Gamma_q(\alpha_1+\beta+(2k_1-m-i-1)\gamma+\lambda_i)\Gamma_q(\gamma)}$$
$$\times \prod_{i=1}^k\prod_{j=1}^m \frac{\Gamma_q(\alpha_1+\beta+(2k-i-j-1)\gamma+\lambda_i+\mu_j)}{\Gamma_q(\alpha_1+\beta+(2k-i-j)\gamma+\lambda_i+\mu_j)}.$$

Since

$$\Delta_\gamma(t;q) = \prod_{1\leqslant i<j\leqslant k} t_j^{2\gamma}(1-t_i/t_j)(q^{1-\gamma}t_i/t_j;q)_{2\gamma-1} = \prod_{1\leqslant i<j\leqslant k}(-t_it_j)^\gamma q^{-\binom{\gamma}{2}}(t_i/t_j;q)_\gamma(t_j/t_i;q)_\gamma,$$

the summand of (6.1.3) is symmetric in t. Moreover, $\Delta_\gamma(t;q)$ vanishes if $t_i/t_j \in \{q^{1-\gamma}, q^{2-\gamma}, \dots, q^{\gamma-1}\}$ for $1 \leqslant i < j \leqslant k$ due to the presence of the factor $(q^{1-\gamma}t_i/t_j;q)_{2\gamma-1}$. Hence the claim follows by Lemma 6.1.2 upon noting that

$$\prod_{i=1}^k \frac{\Gamma_q(i\gamma)}{\Gamma_q(\gamma)} = \frac{(1-q^\gamma)^k}{(q^\gamma;q^\gamma)_k}\prod_{i=1}^k\frac{\Gamma_q(1+i\gamma)}{\Gamma_q(1+\gamma)}. \qquad \square$$

6.2 The elliptic AFLT integral

Here we will state without proof the elliptic analogue of the AFLT integral and an AFLT integral for Macdonald polynomials. For more details we refer the reader to [ARW]. Throughout this section let $p, q \in \mathbb{C}$ be complex numbers such that $|p|, |q| < 1$, and we use n rather than k to denote the number of integration variables.

6.2.1 Elliptic preliminaries

In the previous section we proved a q-extension of the AFLT integral. Much of the theory of q-hypergeometric series and integrals extends naturally to the elliptic level. For an introduction to the theory of elliptic hypergeometric functions see [GR04, Ros, Spi]. The primary building block of the elliptic theory is the modified theta function

$$\theta(z;q,p) := (z;p)_\infty(p/z;p)_\infty,$$

where $z \in \mathbb{C}^*$ and $p \in \mathbb{C}$ is such that $|p| < 1$. The elliptic shifted factorial is then given by

$$(a;q,p)_n := \prod_{i=1}^{n-1}\theta(aq^i;p),$$

so that $(a;q,0)_n = (a;q)_n$, the ordinary q-shifted factorial. We also require the elliptic gamma function [Rui97]

$$\Gamma(z;p,q) := \prod_{i,j=0}^{\infty} \frac{1-p^{i+1}q^{j+i}/z}{1-zp^iq^i},$$

which by definition satisfies

$$\Gamma(z;p,q) = \frac{1}{\Gamma(pq/z;p,q)}.$$

For the elliptic gamma function we adopt the usual plus-minus conventions

$$\begin{aligned}
\Gamma(z^{\pm};p,q) &:= \Gamma(z;p,q)\Gamma(z^{-1};p,q)\\
\Gamma(z^{\pm}w^{\pm};p,q) &:= \Gamma(zw;p,q)\Gamma(zw^{-1};p,q)\Gamma(z^{-1}w;p,q)\Gamma(z^{-1}w^{-1};p,q).
\end{aligned}$$

We also have an elliptic generalisation of (2.6.21)

$$(b;q,t;p)_\lambda := \prod_{s\in\lambda} \theta\big(bq^{a'(s)}t^{-l'(s)};p\big) = \prod_{i\geqslant 1}(bt^{1-i};q,p)_{\lambda_i}.$$

It is also convenient to define the following shorthand for products of elliptic shifted factorials:

$$\Delta^0_\lambda(a|b_1,\dots,b_n;q,t;p) := \prod_{i=1}^{n} \frac{(b_i;q,t;p)_\lambda}{(pqa/b_i;q,t;p)_\lambda}.$$

For $\boldsymbol{\lambda} = (\lambda^{(1)},\lambda^{(2)}) \in \mathscr{P}^2$ a bipartition we also use the notation

$$\Delta^0_{\boldsymbol{\lambda}}(a|b_1,\dots,b_n;q,t;p) := \Delta^0_{\lambda^{(1)}}(a|b_1,\dots,b_n;q,t;p)\Delta^0_{\lambda^{(2)}}(a|b_1,\dots,b_n;q,t;p).$$

The generalised hook polynomials (2.6.22) also admit elliptic analogues,

$$C^-_\lambda(b;q,t;p) := \prod_{s\in\lambda} \theta\big(bq^{a_\lambda(s)}t^{l_\lambda(s)};p\big) \tag{6.2.1a}$$

$$C^+_\lambda(b;q,t;p) := \prod_{(i,j)\in\lambda} \theta\big(bq^{\lambda_i+j-1}t^{2-\lambda'_j-i};p\big), \tag{6.2.1b}$$

so that, in particular, $c_\lambda(q,t) = C^-_\lambda(t;q,t;0)$ and $c'_\lambda(q,t) = C^-_\lambda(q;q,t;0)$.

6.2.2 The elliptic beta, Selberg, and AFLT integrals

We will now state the elliptic beta, Selberg, and AFLT integrals, and give some of the necessary background in order to understand these formulas. The elliptic beta integral was discovered by Spiridonov [Spi01], and may be stated as

$$\int_{\mathbb{T}} \frac{\prod_{r=1}^{6}\Gamma(t_rz^{\pm};p,q)}{\Gamma(z^{\pm 2};p,q)} \frac{\mathrm{d}z}{2\pi\mathrm{i}z} = \frac{2}{(p;p)_\infty(q;q)_\infty} \prod_{1\leqslant r<s\leqslant 6} \Gamma(t_rt_s;p,q), \tag{6.2.2}$$

where $|t_r| < 1$ for each $1 \leqslant r \leqslant 6$ and the parameters satisfy $t_1\cdots t_6 = pq$. The reduction of this integral to the Euler beta integral (1.1.3) is not entirely straightforward. Sending $p \to 0$ one obtains the Rahman integral [Rah86]. This is itself a generalisation of the well-known Askey–Wilson integral, which reduces to the beta integral in the form (1.1.8) (this is worked out in [GR04, §6.1]).

To state our next formula we introduce the elliptic factor

$$\kappa_n := \frac{(p;p)_\infty^n (q;q)_\infty^n \Gamma^n(t;p,q)}{2^n n! (2\pi \mathrm{i})^n}.$$

Then the elliptic Selberg integral is an n-dimensional analogue of the integral (6.2.2),

$$\begin{aligned} &S_n(t_1,\dots,t_6;t;p,q) \qquad\qquad (6.2.3)\\ &:= \kappa_n \int_{\mathbb{T}^n} \prod_{1\leqslant i<j\leqslant n} \frac{\Gamma(tz_i^{\pm}z_j^{\pm};p,q)}{\Gamma(z_i^{\pm}z_j^{\pm};p,q)} \prod_{i=1}^n \frac{\prod_{r=1}^6 \Gamma(t_r z_i^{\pm};p,q)}{\Gamma(z_i^{\pm 2};p,q)} \frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_n}{z_n} \\ &= \prod_{i=1}^n \biggl(\Gamma(t^i;p,q) \prod_{1\leqslant r<s\leqslant 6} \Gamma(t^{i-1}t_r t_s;p,q)\biggr), \end{aligned}$$

where again $|t_r| < 1$ and the balancing condition $t^{2n-2}t_1\cdots t_6 = pq$ holds. The above, which is also referred to as the type II C_n elliptic beta integral, was conjectured by van Diejen and Spiridonov in [vDS00, vDS01]. The first proof is due to Rains [Rai10], with subsequent alternative proofs being found by Spiridonov [Spi07] and Ito and Noumi [IN17]. Again, the reduction to the classical case is highly nontrivial; see [ARW, §6.1] for the details.

In order to lift (6.2.3) to an analogue of the AFLT integral, we need to introduce a family of elliptic skew interpolation functions. It is these functions that will play the role of the pair of Jack polynomials in the AFLT integral. Our treatment of these functions is quite terse, and we refer the reader to [ARW, CG06, Rai06, Rai10, Rai12, RWb]. In the following we say that a function $f(x_1,\dots,x_n)$ is BC_n-symmetric if it is invariant under the action of the hyperoctahedral group $\mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_n$, the group of signed permutations on n letters.[2] We begin with Rains' BC_n-symmetric elliptic interpolation functions, which for $\mu \in \mathscr{P}_n$ we denote by

$$\mathcal{R}_\mu^*(x_1,\dots,x_n;a,b;q,t;p).$$

For any $\lambda \in \mathscr{P}_n$ such that $\mu \not\subseteq \lambda$ these satisfy the vanishing conditions

$$\mathcal{R}_\mu^*\bigl(a\langle\lambda\rangle_{n;q,t};a,b;q,t;p\bigr) = 0,$$

where $\langle\lambda\rangle_{n;q,t}$ is the usual spectral vector (2.6.12). Suppressing the variables x_i, the functions $\mathcal{R}_\mu^*(a,b;q,t;p)$ generalise the Okounkov BC_n-symmetric interpolation Macdonald polynomials $P_\mu^*(q,t,s)$ [Oko98]. While the $P_\mu^*(q,t,s)$ are inhomogeneous, they contain the ordinary Macdonald polynomial $P_\mu(q,t)$ as the homogeneous term of maximal degree $|\mu|$. Like the Macdonald polynomials, the $\mathcal{R}_\mu^*$ satisfy an evaluation symmetry and have a nice principal specialisation formula, which may be found in the aforementioned references.

[2] Here $\mathbb{Z}/2\mathbb{Z}$ acts as $x_j \mapsto 1/x_j$.

Using the elliptic interpolation functions we can define the elliptic binomial coefficients

$$\binom{\lambda}{\mu}_{[a,b];q,t;p} := \frac{(pqa;q,t;p)_{2\lambda^2}}{C^-_\lambda(pq;q,t;p)C^-_\lambda(t;q,t;p)C^+_\lambda(a;q,t;p)C^+_\lambda(pqa/t;q,t;p)} \times \Delta^0_\mu(a/b|t^n,1/b;q,t;p)\,\mathcal{R}^*_\mu\big(t^{1-n}a^{1/2}\langle\lambda\rangle_{n;q,t};t^{1-n}a^{1/2},ba^{-1/2};q,t;p\big),$$

where $2\lambda^2$ is shorthand for the partition $(2\lambda_1,2\lambda_1,2\lambda_2,2\lambda_2,\dots)$ and n is an integer such that $n \geqslant l(\lambda), l(\mu)$. This definition is independent of the choice of $a^{1/2}$ and n. The elliptic binomial coefficients vanish unless $\mu \subseteq \lambda$ and trivialise to 1 when $\mu = 0$ (but not when $\mu = \lambda$). In fact we will use the normalised elliptic binomial coefficients

$$\left\langle{\lambda \atop \mu}\right\rangle_{[a,b](v_1,\dots,v_k);q,t;p} := \frac{\Delta^0_\lambda(a|b,v_1,\dots,v_k;q,t;p)}{\Delta^0_\mu(a/b|1/b,v_1,\dots,v_k;q,t;p)}\binom{\lambda}{\mu}_{[a,b];q,t;p}.$$

Putting all of the above together we may define the elliptic skew interpolation functions by

$$\mathcal{R}^*_{\lambda/\nu}([v_1,\dots,v_{2n}];a,b;q,t;p) := \sum_\mu \Delta^0_\mu(pq/b^2|pq/bv_1,\dots,pq/bv_{2n};q,t;p) \times \left\langle{\lambda \atop \mu}\right\rangle_{[a/b,ab/pq];q,t;p}\left\langle{\mu \atop \nu}\right\rangle_{[pq/b^2,pqV/ab];q,t;p},$$

where $V := v_1\cdots v_{2n}$. Note that unlike the BC_n-symmetric interpolation functions the skew interpolation functions are $\mathfrak{S}_{2n}$-symmetric. On the left the square brackets should be interpreted as a kind of plethystic bracket, since in the $p \to 0$ limit the variables v_i are related to the variables x_i of the BC_n-symmetric interpolation functions via plethystic substitution (see [ARW, §6.2]).

For $\nu = 0$ the skew functions generalise the BC_n-symmetric interpolation functions

$$\mathcal{R}^*_{\lambda/0}([t^{1/2}x_1^\pm,\dots,t^{1/2}x_n^\pm];t^{n-1/2}a,t^{1/2}b;q,t;p) = \begin{cases} \Delta^0_\lambda(t^{n-1}a/b|t^n;q,t;p)\,\mathcal{R}^*_\lambda(x_1,\dots,x_n;a,b;q,t;p) & \text{for } \lambda\in\mathscr{P}_n,\\ 0 & \text{otherwise,}\end{cases}$$

where we adopt the convention

$$\mathcal{R}^*_{\lambda/\nu}\big([uz_1^\pm,\dots,uz_n^\pm,v_1,\dots,v_{2m}];a,b;t;p,q\big) := \mathcal{R}^*_{\lambda/\nu}\big([uz_1,uz_1^{-1},\dots,uz_n,uz_n^{-1},v_1,\dots,v_{2m}];a,b;t;p,q\big).$$

Finally, we must extend all of the previous functions depending on partitions to bipartitions as defined in Section 2.1. For $\boldsymbol{\lambda} = (\lambda^{(1)},\lambda^{(2)})$ and $\boldsymbol{\mu} = (\mu^{(1)},\mu^{(2)})$ be bipartitions,

$$\mathcal{R}^*_{\boldsymbol{\mu}}(a,b;t;p,q) := \mathcal{R}^*_{\mu^{(1)}}(a,b;q,t;p)\mathcal{R}^*_{\mu^{(2)}}(a,b;p,t;q)$$

$$\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(a,b;t;p,q) := \mathcal{R}^*_{\lambda^{(1)}/\mu^{(1)}}(a,b;q,t;p)\mathcal{R}^*_{\lambda^{(2)}/\mu^{(2)}}(a,b;p,t;q).$$

These definitions are extended to $\Delta^0_{\boldsymbol{\lambda}}(a|b_1,\dots,b_k;t;p,q)$ and $\left\langle{\boldsymbol{\lambda} \atop \boldsymbol{\mu}}\right\rangle_{[a,b](v_1,\dots,v_k);t;p,q}$ in the obvious way. Finally, we extend the spectral vector (2.6.12) to bipartitions as

$$\langle\boldsymbol{\lambda}\rangle_{n;t;p,q} := \big(q^{\lambda_1^{(1)}}p^{\lambda_1^{(2)}}t^{n-1}, q^{\lambda_2^{(1)}}p^{\lambda_2^{(2)}}t^{n-2},\dots,q^{\lambda_{n-1}^{(1)}}p^{\lambda_{n-1}^{(2)}}t, q^{\lambda_n^{(1)}}p^{\lambda_n^{(2)}}\big).$$

We are now ready to state the elliptic AFLT integral [ARW, Theorem 1.4].

Theorem 6.2.1. *Let n be a positive integer, $p,q,t,t_1,t_2,t_3,t_4,t_5,t_6 \in \mathbb{C}$ such that the elliptic balancing condition $t^{2n-2}t_1\cdots t_6 = pq$ holds and such that $|p|,|q|<1$. Then, for $\boldsymbol{\lambda}\in\mathscr{P}^2_n$ and $\boldsymbol{\mu}\in\mathscr{P}^2$,*

$$\begin{aligned}
&\kappa_n \int\limits_{C_{\boldsymbol{\lambda}\boldsymbol{\mu}}} \mathcal{R}^*_{\boldsymbol{\lambda}/\mathbf{0}}\big([t^{1/2}z_1^{\pm},\dots,t^{1/2}z_n^{\pm}];t^{n-1/2}t_1,t^{1/2}t_6;t;p,q\big)\\
&\qquad\times \mathcal{R}^*_{\boldsymbol{\mu}/\mathbf{0}}\big([t^{1/2}z_1^{\pm},\dots,t^{1/2}z_n^{\pm},t^{-1/2}t_3,t^{-1/2}t_4];t^{n-3/2}t_2t_3t_4,t^{1/2}t_5;t;p,q\big)\\
&\qquad\times \prod_{1\leqslant i<j\leqslant n}\frac{\Gamma(tz_i^{\pm}z_j^{\pm};p,q)}{\Gamma(z_i^{\pm}z_j^{\pm};p,q)}\prod_{i=1}^n\frac{\prod_{r=1}^6\Gamma(t_rz_i^{\pm};p,q)}{\Gamma(z_i^{\pm 2};p,q)}\frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_n}{z_n}\\
&=\prod_{i=1}^n\bigg(\Gamma(t^i;p,q)\prod_{1\leqslant r<s\leqslant 6}\Gamma(t^{i-1}t_rt_s;p,q)\bigg)\\
&\qquad\times\Delta^0_{\boldsymbol{\lambda}}(t^{n-1}t_1/t_6|t^n,t^{n-1}t_1t_2,t^{n-1}t_1t_3,t^{n-1}t_1t_4,t^{n-1}t_1t_5;t;p,q)\\
&\qquad\times\Delta^0_{\boldsymbol{\mu}}(t^{n-2}t_2t_3t_4/t_5|t^{n-1}t_2t_3,t^{n-1}t_2t_4,t^{n-1}t_3t_4;t;p,q)\\
&\qquad\times\frac{\Delta^0_{\boldsymbol{\mu}}(t^{n-2}t_2t_3t_4/t_5|t^{n-2}t_1t_2t_3t_4\langle\boldsymbol{\lambda}\rangle_{n;t;p,q})}{\Delta^0_{\boldsymbol{\mu}}(t^{n-2}t_2t_3t_4/t_5|t^{n-1}t_1t_2t_3t_4\langle\boldsymbol{\lambda}\rangle_{n;t;p,q})},
\end{aligned}$$

where $C_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ is a deformation of $\mathbb{T}^n$ (with $\mathbb{T}$ the positively oriented unit circle) separating sequences of poles of the integrand tending to zero from sequences of poles tending to infinity.

In [ARW], the proof of the theorem relies on an integral formula for the product of two non-skew BC_n-symmetric elliptic interpolation functions due to Rains [Rai10, Theorem 9.2]. By taking an appropriate $p\to 0$ limit Theorem 6.2.1 reduces to the following AFLT integral for a pair of Macdonald polynomials [ARW, Corollary 1.5].

Corollary 6.2.2. *For $\lambda\in\mathscr{P}_n$, $\mu\in\mathscr{P}$, and $a,b,q,t\in\mathbb{C}$ such that $|b|,|q|,|t|<1$,*

$$\begin{aligned}
&\frac{1}{n!(2\pi\mathrm{i})^n}\int\limits_{\mathbb{T}^n} P_\lambda(z;q,t)P_\mu\Big(\Big[z+\frac{t-b}{1-t}\Big];q,t\Big)\\
&\qquad\times\prod_{i=1}^n\frac{(a/z_i,qz_i/a;q)_\infty}{(b/z_i,z_i;q)_\infty}\prod_{1\leqslant i<j\leqslant n}\frac{(z_i/z_j,z_j/z_i;q)_\infty}{(tz_i/z_j,tz_j/z_i;q)_\infty}\frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_n}{z_n}\\
&=b^{|\lambda|}t^{|\mu|}P_\lambda\Big(\Big[\frac{1-t^n}{1-t}\Big];q,t\Big)P_\mu\Big(\Big[\frac{1-bt^{n-1}}{1-t}\Big];q,t\Big)\\
&\qquad\times\prod_{i=1}^n\frac{(t,at^{n-m-i}q^{\lambda_i},at^{1-i}/b,qt^{i-1}b/a;q)_\infty}{(q,t^i,bt^{i-1},at^{1-i}q^{\lambda_i}/b;q)_\infty}\prod_{i=1}^n\prod_{j=1}^m\frac{(at^{n-i-j+1}q^{\lambda_i+\mu_j};q)_\infty}{(at^{n-i-j}q^{\lambda_i+\mu_j};q)_\infty},
\end{aligned}$$

where m is an arbitrary integer such that $m\geqslant l(\mu)$.

Alternatively, as shown in [ARW, Corollary 6.1], this formula be expressed as a generalisation of the orthogonality of the Macdonald polynomials under the modified scalar product (2.6.37).

Chapter 7

Open problems

To conclude we discuss a number of open problems.

7.1 Generalising Theorem 1.4.1 for $\gamma \neq 1$

The main open problem is to generalise Theorem 1.4.1 to the case of Jack polynomials. Using the reciprocity for Schur functions (2.5.8) this theorem can be rewritten as

$$\left\langle \left(\prod_{r=1}^{n} s_{\lambda^{(r)}}\big[t^{(r)} - t^{(r-1)}\big] \right) s_{\lambda^{(n+1)}}\big[t^{(n)} + \beta - 1\big] \right\rangle_{\alpha_1,\dots,\alpha_n,\beta}^{k_1,\dots,k_n} \tag{7.1.1}$$

$$= \left(\prod_{r=1}^{n} s_{\lambda^{(r)}}[k_r - k_{r-1}] \right) s_{\lambda^{(n+1)}}[k_n + \beta - 1] \prod_{\substack{r,s=1 \\ r \neq s}}^{n+1} \prod_{i=1}^{\ell_r} \frac{\big(\varepsilon_r (A_{r,s} - k_{s-1} + k_s) - i + 1\big)_{\lambda_i^{(r)}}}{\big(\varepsilon_r (A_{r,s} + \varepsilon_s \ell_s) - i + 1\big)_{\lambda_i^{(r)}}}$$

$$\times \prod_{1 \leqslant r < s \leqslant n+1} \prod_{i=1}^{\ell_r} \prod_{j=1}^{\ell_s} \frac{(A_{r,s} - i + \varepsilon_s j + 1)_{\lambda_i^{(r)} - \varepsilon_s \lambda_j^{(s)}}}{(A_{r,s} - i + \varepsilon_s (j-1) + 1)_{\lambda_i^{(r)} - \varepsilon_s \lambda_j^{(s)}}},$$

where $\ell_1, \dots, \ell_{n+1}$ are arbitrary integers such that $\ell_r \geqslant l(\lambda^{(r)})$ for $1 \leqslant r \leqslant n+1$, $\varepsilon_1 = \cdots = \varepsilon_n = 1$, $\varepsilon_{n+1} = -1$, $k_0 := 0$ and $k_{n+1} := 1 - \beta$. It is not difficult to define a function, say

$$R_{\lambda^{(1)},\dots,\lambda^{(n+1)}}^{k_1,\dots,k_n}(\alpha_1, \dots, \alpha_n, \beta; \gamma),$$

such that for $\gamma = 1$ it gives the right-hand side of (7.1.1) and such that for $\lambda^{(1)} = \lambda$, $\lambda^{(2)} = \cdots = \lambda^{(n)} = 0$ and $\lambda^{(n+1)} = \mu$ it yields the right-hand side of (1.3.6). To describe this function, we generalise our earlier definition (1.4.4) of A_r and $A_{r,s}$ to include γ:

$$A_r := \alpha_r + \cdots + \alpha_n + (k_r - k_{r-1} + r)\gamma \quad \text{and} \quad A_{r,s} := A_r - A_s,$$

for $1 \leqslant r, s \leqslant n + 1$. Hence $A_{r,s} = -A_{r,s}$ and

$$A_{r,s} = \alpha_r + \cdots + \alpha_{s-1} + (k_r - k_{r-1} - k_s + k_{s-1} + r - s)\gamma \tag{7.1.2}$$

for $1 \leqslant r \leqslant s \leqslant n+1$. Finally, we define the γ-shifted factorial indexed by a partition λ by

$$(a;\gamma)_\lambda := \prod_{i \geqslant 1} (a + (1-i)\gamma)_{\lambda_i}.$$

Lemma 7.1.1. *Let $A_{r,s}$ be as in* (7.1.2)*, where $0 = k_0 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n$ are integers and $k_{n+1} := 1 - \beta/\gamma$. Set $\varepsilon_1 = \cdots = \varepsilon_n = 1$ and $\varepsilon_{n+1} = -1$, and define*

$$\begin{aligned}
&R^{k_1,\dots,k_n}_{\lambda^{(1)},\dots,\lambda^{(n+1)}}(\alpha_1,\dots,\alpha_n,\beta;\gamma) \qquad (7.1.3)\\
&\quad := \bigg(\prod_{r=1}^{n} P^{(1/\gamma)}_{\lambda^{(r)}}[k_r - k_{r-1}]\bigg) P^{(1/\gamma)}_{\lambda^{(n+1)}}[k_n + \beta/\gamma - 1]\\
&\qquad \times \prod_{1 \leqslant r < s \leqslant n+1} \frac{(-\varepsilon_s A_{r,s} - \varepsilon_s (k_{r-1} - k_r)\gamma;\gamma)_{\lambda^{(s)}}}{(-\varepsilon_s A_{r,s} + \varepsilon_s \ell_r \gamma;\gamma)_{\lambda^{(s)}}}\\
&\qquad \times \prod_{1 \leqslant r < s \leqslant n} \Bigg(\frac{(A_{r,s} - (k_{s-1} - k_s)\gamma;\gamma)_{\lambda^{(r)}}}{(1 + A_{r,s} + (\varepsilon_s \ell_s - 1)\gamma;\gamma)_{\lambda^{(r)}}} \prod_{i=1}^{\ell_r}\prod_{j=1}^{\ell_s} \frac{(1 + A_{r,s} + (j-i)\gamma)_{\lambda^{(r)}_i - \lambda^{(s)}_j}}{(1 + A_{r,s} + (j-i-1)\gamma)_{\lambda^{(r)}_i - \lambda^{(s)}_j}} \Bigg)\\
&\qquad \times \prod_{r=1}^{n} \Bigg(\frac{(A_{r,n+1} - (k_n - k_{n+1})\gamma;\gamma)_{\lambda^{(r)}}}{(A_{r,n+1} - \ell_{n+1}\gamma;\gamma)_{\lambda^{(r)}}} \prod_{i=1}^{\ell_r}\prod_{j=1}^{\ell_{n+1}} \frac{(A_{r,n+1} - (i+j-1)\gamma)_{\lambda^{(r)}_i + \lambda^{(n+1)}_j}}{(A_{r,n+1} - (i+j-2)\gamma)_{\lambda^{(r)}_i + \lambda^{(n+1)}_j}} \Bigg),
\end{aligned}$$

where $\ell_1,\dots,\ell_{n+1}$ are arbitrary integers such that $\ell_r \geqslant l(\lambda^{(r)})$ and $\ell_1 \leqslant k_1$. Then

$$R^{k_1,\dots,k_n}_{\lambda^{(1)},\dots,\lambda^{(n+1)}}(\alpha_1,\dots,\alpha_n,\beta;\gamma)$$

is well-defined (*i.e., independent of the choice of the ℓ_r*)*,*

$$R^{0,k_2,\dots,k_n}_{\lambda^{(1)},\dots,\lambda^{(n+1)}}(\alpha_1,\dots,\alpha_n,\beta;\gamma) = \begin{cases} R^{k_2,\dots,k_n}_{\lambda^{(2)},\dots,\lambda^{(n+1)}}(\alpha_2,\dots,\alpha_n,\beta;\gamma) & \textit{if } \lambda^{(1)} = 0,\\ 0 & \textit{otherwise} \end{cases}$$

and

$$R^{k_1,\dots,k_n}_{\lambda,\underbrace{0,\dots,0}_{n-1 \text{ times}},\mu}(\alpha_1,\dots,\alpha_n,\beta;\gamma) \quad \textit{and} \quad R^{k_1,\dots,k_n}_{\lambda^{(1)},\dots,\lambda^{(n+1)}}(\alpha_1,\dots,\alpha_n,\beta;1)$$

agree with the right-hand side of the A_n *AFLT integral* (1.3.6) *and the right-hand side of* (7.1.1) *respectively. Moreover, when $l(\lambda^{(1)}) < k_1$ and $k_1,\dots,k_n \geqslant 1$,*

$$\begin{aligned}
&R^{k_1,\dots,k_n}_{\lambda^{(1)},\dots,\lambda^{(n+1)}}(\alpha_1,\alpha_2,\dots,\alpha_n,\beta;\gamma) \qquad (7.1.4)\\
&\quad = R^{k_1-1,\dots,k_n-1}_{\lambda^{(1)},\dots,\lambda^{(n+1)}}(\alpha_1+\gamma,\alpha_2,\dots,\alpha_n,\beta+\gamma,\gamma) \prod_{s=1}^{n+1} \frac{(-\varepsilon_s A_{1,s} + \varepsilon_s k_1 \gamma;\gamma)_{\lambda^{(s)}}}{(-\varepsilon_s A_{1,s} + \varepsilon_s (k_1 - 1)\gamma;\gamma)_{\lambda^{(s)}}}.
\end{aligned}$$

We note that the assumption that $l(\lambda^{(1)}) < k_1$ is not actually a restriction since

$$R^{k_1,\dots,k_n}_{\lambda^{(1)},\dots,\lambda^{(n+1)}}(\alpha_1,\dots,\alpha_n,\beta;\gamma)$$

depends on $\lambda^{(1)} + \alpha_1$ only, where, for m a scalar, $\lambda + m := (\lambda_1 + m, \lambda_2 + m, \dots)$. For $n = 2$ and $\lambda^{(2)} = 0$ the recursion (7.1.4) agrees with [FL, Equation (A.15)] (provided $[k_2\gamma]/[(k_2-1)\gamma]$ in the latter is corrected to $[k_2\gamma]_\lambda/[(k_2-1)\gamma]_\lambda$). For $n > 2$ and $\lambda^{(2)} = \cdots = \lambda^{(n)} = 0$, however, (7.1.4) and the recursion at the bottom of page 36 of [FL] are inconsistent.

Proof of Lemma 7.1.1. To see that the right-hand side of (7.1.3) is independent of the ℓ_r, fix a t such that $1 \leqslant t \leqslant n+1$. Then, assuming that $\lambda^{(t)}_{\ell_t}=0$, it follows from elementary manipulations and the use of

$$\frac{(a)_{-n}}{(b)_{-n}} = \frac{(1-b)_n}{(1-a)_n}$$

that the right-hand side of (7.1.3) reduces to the same expression with $\ell_t \mapsto \ell_t - 1$.

For (7.1.4), write

$$k_{n+1} = k_{n+1}(\beta;\gamma) \quad \text{and} \quad A_{r,s} = A^{k_1,\dots,k_n}_{r,s}(\alpha_1,\dots,\alpha_n,\beta;\gamma).$$

It is then readily checked that

$$\begin{aligned} k_{n+1}(\beta+\gamma;\gamma) &= k_{n+1}(\beta;\gamma) - 1 \\ A^{k_1-1,\dots,k_n-1}_{r,s}(\alpha_1+\gamma,\alpha_2,\dots,\alpha_n,\beta+\gamma;\gamma) &= A^{k_1,\dots,k_n}_{r,s}(\alpha_1,\dots,\alpha_n,\beta;\gamma). \end{aligned}$$

Hence, for $l(\lambda^{(1)}) \leqslant k_1 - 1$,

$$\frac{R^{k_1,\dots,k_n}_{\lambda^{(1)},\dots,\lambda^{(n+1)}}(\alpha_1,\alpha_2,\dots,\alpha_n,\beta;\gamma)}{R^{k_1-1,\dots,k_n-1}_{\lambda^{(1)},\dots,\lambda^{(n+1)}}(\alpha_1+\gamma,\alpha_2,\dots,\alpha_n,\beta+\gamma,\gamma)} = \frac{P^{(1/\gamma)}_{\lambda^{(1)}}[k_1]}{P^{(1/\gamma)}_{\lambda^{(1)}}[k_1-1]} \prod_{s=2}^{n+1} \frac{(-\varepsilon_s A_{1,s} + \varepsilon_s k_1\gamma;\gamma)_{\lambda^{(s)}}}{(-\varepsilon_s A_{1,s} + \varepsilon_s (k_1-1)\gamma;\gamma)_{\lambda^{(s)}}}.$$

By the specialisation formula (2.6.26) the recursion (7.1.4) follows.

The remaining claims of the lemma are immediate and left to the reader. $\square$

An obvious guess would be that

$$\begin{aligned} \Biggl\langle \biggl(\prod_{r=1}^{n} P^{(1/\gamma)}_{\lambda^{(r)}}\bigl[t^{(r)} - t^{(r-1)}\bigr] \biggr) P^{(1/\gamma)}_{\lambda^{(n+1)}}\bigl[t^{(n)} + \beta/\gamma - 1\bigr] \Biggr\rangle^{k_1,\dots,k_n}_{\alpha_1,\dots,\alpha_n,\beta;\gamma} \qquad (7.1.5) \\ = R^{k_1,\dots,k_n}_{\lambda^{(1)},\dots,\lambda^{(n+1)}}(\alpha_1,\dots,\alpha_n,\beta;\gamma), \end{aligned}$$

but this is easily shown to be false unless $\lambda^{(2)} = \cdots = \lambda^{(n)} = 0$. To do so, we introduce the generalised hypergeometric function

$${}_{r+1}F_r\left(\begin{matrix} a_1,\dots,a_{r+1} \\ b_1,\dots,b_r \end{matrix};z\right) = \sum_{k=0}^{\infty} \frac{(a_1,\dots,a_{r+1})_k}{(b_1,\dots,b_r)_k}\frac{z^k}{k!},$$

where we have used the usual condensed notation for products of shifted factorials. By a direct computation using Theorem 1.3.1 and the Jack polynomial limit of (2.6.36) given by

$$P^{(1/\gamma)}_{(r)}[x-y] = x^r\, {}_2F_1\left(\begin{matrix} -\gamma,-r \\ 1-\gamma-r \end{matrix};\frac{y}{x}\right),$$

where ${}_2F_1$ is the Gauss hypergeometric function [AAR99, Definition 2.1.5], it follows that for $k_1 = \cdots = k_n = 1$ and

$$\bigl(\lambda^{(1)},\dots,\lambda^{(n)},\lambda^{(n+1)}\bigr) = \bigl((u_1),\dots,(u_n),\mu\bigr)$$

the left-hand side of (7.1.5) evaluates as a product of $n-1$ terminating ${}_3F_2$ series. Specifically,

$$\begin{aligned}
&\biggl\langle \biggl(\prod_{r=1}^{n} P_{(u_r)}^{(1/\gamma)}\bigl[t_r - t_{r-1}\bigr]\biggr) P_{\mu}^{(1/\gamma)}\bigl[t_n+\beta/\gamma-1\bigr]\biggr\rangle_{\alpha_1-u_1,\dots,\alpha_n-u_n,\beta;\gamma}^{1,\dots,1} \\
&\quad = P_{\mu}^{(1/\gamma)}[\beta/\gamma]\, \frac{(\alpha_1+\cdots+\alpha_n+\beta-n\gamma;\gamma)_\mu}{(\alpha_1+\cdots+\alpha_n+\beta-(n-1)\gamma;\gamma)_\mu} \\
&\qquad \times \prod_{r=1}^{n} \frac{(1-\alpha_1-\cdots-\alpha_r+(r-1)\gamma)_{u_1+\cdots+u_r}}{(1-\alpha_1-\cdots-\alpha_r-\beta_r+(r-\delta_{r,n})\gamma)_{u_1+\cdots+u_r}} \\
&\qquad \times \prod_{r=1}^{n-1} {}_3F_2\biggl(\begin{matrix} -\gamma, \alpha_1+\cdots+\alpha_r-(r-1)\gamma, -u_{r+1} \\ 1-\gamma-u_{r+1}, 1+\alpha_1+\cdots+\alpha_r-r\gamma \end{matrix}; 1\biggr),
\end{aligned}$$

where $t_0 := 0$ and $\beta_1 = \cdots = \beta_{n-1} := 1$. For $\gamma = 1$ the rth ${}_3F_2$ series simplifies to $\delta_{u_{r+1},0}$ in accordance with the $k_1 = \cdots = k_n$ case of (7.1.1). We do not know how to modify the product of Jack polynomials on the left of (7.1.5) so that equality holds.

7.2 A complex Schur function analogue of Theorem 4.4.1

Another open problem is to generalise the $\gamma = 1$ case of (4.4.3) to include a product of n Schur functions. For $\beta_{n-1} + \beta_n = 2$, denote by

$$\langle \mathcal{O} \rangle_{\alpha_1,\dots,\alpha_n,\beta_{n-1},\beta_n}^{k_1,\dots,k_n}$$

the $\gamma = 1$ case of the A_n Selberg average (4.4.2) (this again requires a complex integration contour). Then the problem is to extend the method of Chapter 5 to prove that

$$\begin{aligned}
&\biggl\langle \biggl(\prod_{r=1}^{n-1} s_{\lambda^{(r)}}\bigl[t^{(r)} - t^{(r-1)}\bigr]\biggr) s_{\lambda^{(n)}}\bigl[t^{(n)}\bigr]\biggr\rangle_{\alpha_1,\dots,\alpha_n,\beta_{n-1},\beta_n}^{k_1,\dots,k_n} \\
&\stackrel{?}{=} \prod_{r=1}^{n} \prod_{1\leqslant i<j\leqslant \ell_r} \frac{\lambda_i^{(r)}-\lambda_j^{(r)}+j-i}{j-i} \prod_{1\leqslant r<s\leqslant n-1} \prod_{i=1}^{\ell_r}\prod_{j=1}^{\ell_s} \frac{\lambda_i^{(r)}-\lambda_j^{(s)}+A_{r,s}+j-i}{A_{r,s}+j-i} \\
&\quad \times \prod_{r=1}^{n-1}\prod_{i=1}^{\ell_r}\prod_{j=1}^{k_n} \frac{A_{r,n+1}-i-j+1}{\lambda_i^{(r)}+\lambda_j^{(n)}+A_{r,n+1}-i-j+1} \\
&\quad \times \prod_{r,s=1}^{n-1}\prod_{i=1}^{\ell_r} \frac{(A_{r,s}-k_{s-1}+k_s-i+1)_{\lambda_i^{(r)}}}{(A_{r,s}+\ell_s-i+1)_{\lambda_i^{(r)}}} \prod_{r=1}^{n-1}\prod_{i=1}^{\ell_r} \frac{(A_{r,n}-k_{n-1}+k_n-i+1)_{\lambda_i^{(r)}}}{(A_{r,n}+\beta_{n-1}-i)_{\lambda_i^{(r)}}} \\
&\quad \times \prod_{r=2}^{n-1}\prod_{i=1}^{k_n} \frac{(A_{r,n+1}+k_{r-1}-k_r-i+1)_{\lambda_i^{(n)}}}{(A_{r,n+1}-\ell_r-i+1)_{\lambda_i^{(n)}}} \prod_{i=1}^{k_n} \frac{(A_{n,n+1}+k_{n-1}-k_n-i+1)_{\lambda_i^{(n)}}}{(A_{n,n+1}+\beta_n-i)_{\lambda_i^{(n)}}},
\end{aligned}$$

where $t^{(0)} := 0$, $k_0 = k_{n+1} := 0$, $\ell_1 = k_1$, $\ell_n = k_n$, ℓ_r for $2 \leqslant r \leqslant n-1$ are arbitrary nonnegative integers such that $\ell_r \geqslant l(\lambda^{(r)})$, and the $A_{r,s}$ are defined as in (1.4.4). The more general average

$$\biggl\langle \biggl(\prod_{r=1}^{n} s_{\lambda^{(r)}}\bigl[t^{(r)} - t^{(r-1)}\bigr]\biggr) s_{\lambda^{(n)}}\bigl[t^{(n)}\bigr]\biggr\rangle_{\alpha_1,\dots,\alpha_n,\beta_{n-1},\beta_n}^{k_1,\dots,k_n}$$

appears not to have a similarly simple evaluation. For example, for $n=2$ and $\beta_1+\beta_2=\gamma+1$ it follows that

$$\Big\langle P^{(1/\gamma)}_{(u_1)}\big[t_1\big]P^{(1/\gamma)}_{(u_2)}\big[t_2-t_1\big]P^{(1/\gamma)}_{(u_3)}\big[t_2\big]\Big\rangle^{1,1}_{\alpha_1,\alpha_2,\beta_1,\beta_2;\gamma}$$
$$=\frac{(\alpha_1)_{u_1}(\alpha_2)_{u_2+u_3}(\alpha_1+\alpha_2-\gamma)_{u_1+u_2+u_3}}{(\alpha_1+\beta_1-\gamma)_{u_1}(\alpha_2+\beta_2-\gamma)_{u_2+u_3}(\alpha_1+\alpha_2)_{u_1+u_2+u_3}}$$
$$\times\,{}_4F_3\left(\begin{matrix}-\gamma,\alpha_1+u_1,-\alpha_2+\beta_1-u_2-u_3,-u_2\\ 1-\gamma-u_2,\alpha_1+\beta_1-\gamma+u_1,1-\alpha_2-u_2-u_3\end{matrix};1\right).$$

For $\gamma=1$ this does not vanish when $u_2=0$, but instead yields the non-uniform expression

$$\Big\langle s_{(u_1)}\big[t_1\big]s_{(u_2)}\big[t_2-t_1\big]s_{(u_3)}\big[t_2\big]\Big\rangle^{1,1}_{\alpha_1,\alpha_2,\beta_1,\beta_2}$$
$$=\begin{cases}\dfrac{(\alpha_1)_{u_1}(\alpha_2)_{u_3}(\alpha_1+\alpha_2-1)}{(\alpha_1+\beta_1-1)_{u_1}(\alpha_2+\beta_2-1)(\alpha_1+\alpha_2-1+u_1+u_3)} & \text{if } u_2=0,\\[3mm] \dfrac{(\alpha_1)_{u_1}(\alpha_2)_{u_2+u_3-1}(\alpha_1+\alpha_2-1)(\beta_1-1)}{(\alpha_1+\beta_1-1)_{u_1+1}(\alpha_2+\beta_2-1)_{u_2+u_3}} & \text{if } u_2\geqslant 1.\end{cases}$$

7.3 Elliptic A_n integrals

Both the ordinary Selberg integral (1.1.2) and AFLT integral (1.2.4) have elliptic analogues. Since both of these integral formulas also admit extensions to A_n, it is natural to ask whether there exist A_n analogues of the elliptic Selberg integral and the elliptic AFLT integral.

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