2-core Littlewood identities

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Classical Littlewood identities

The first "Littlewood identity" is actually due to Schur (1918)

$$\sum_{\lambda} s_\lambda(x) = \prod_{i\geqslant 1} \frac{1}{1-x_i} \prod_{i<j} \frac{1}{1-x_i x_j}.$$

Littlewood wrote down two more

$$\sum_{\substack{\lambda \\ \lambda \text{ even}}} s_\lambda(x) = \prod_{i\geqslant 1} \frac{1}{1-x_i^2} \prod_{i<j} \frac{1}{1-x_i x_j}$$

$$\sum_{\substack{\lambda \\ \lambda' \text{ even}}} s_\lambda(x) = \prod_{i<j} \frac{1}{1-x_i x_j},$$

in his 1940 text *The Theory of Group Characters and Matrix Representations of Groups*.

Littlewood identities, and particular their bounded analogues, have played important roles in the combinatorics of plane partitions and related objects, Rogers–Ramanujan identities, branching rules, and multiple elliptic hypergeometric series.

In particular Macdonald's bounded analogue of the first identity is

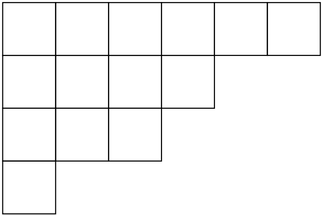
$$\sum_{\substack{\lambda \\ \lambda\subseteq(m^n)}} s_\lambda(x_1,\dots,x_n)=\frac{\det_{1\leqslant i,j\leqslant n}\big(x_i^{m+2n-j}-x_i^{j-1}\big)}{\prod_{i=1}^n(x_i-1)\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(x_ix_j-1)}.$$

He used this to deduce MacMahon's famous conjecture for the number of symmetric plane partitions in a box in the form

$$\sum_{\substack{\lambda \\ \lambda\subseteq(m^n)}} s_\lambda(q,q^3,\dots,q^{2n-1})=\prod_{i=1}^n\frac{1-q^{m+2i-1}}{1-q^{2i-1}}\prod_{1\leqslant i<j\leqslant n}\frac{1-q^{2(m+i+j-1)}}{1-q^{2(i+j-1)}}.$$

Hooks and 2-cores

Recall standard notions for partitions. In particular the Young diagram of a partition. E.g.



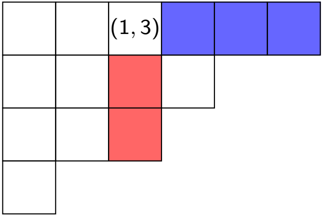
is the Young diagram of $(6, 4, 3, 1)$.

This is identified with a set of points (i, j) such that

$$1 \leqslant i \leqslant \ell(\lambda)$$
$$1 \leqslant j \leqslant \lambda_i,$$

and write s or (i, j) for a square.

For a square $s = (i, j) \in \lambda$ we have the arm and leg lengths



so for $s = (1, 3)$ we have $a(s) = 3$ and $l(s) = 2$.

The hook length for $s = (i, j)$ is then

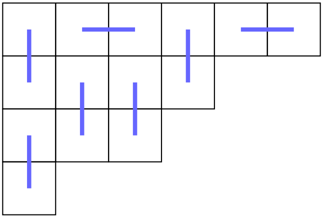
$$h(s) = a(s) + l(s) + 1 = \lambda_i + \lambda'_j - i - j + 1.$$

For a partition λ, set

$$\mathcal{H}^{\mathrm{e}}_\lambda = \{h(s) \text{ even} \mid s \in \lambda\}$$
$$\mathcal{H}^{\mathrm{o}}_\lambda = \{h(s) \text{ odd} \mid s \in \lambda\},$$

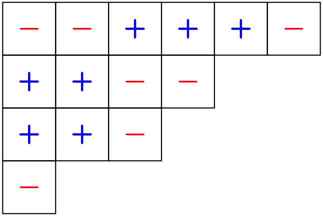
and $\mathcal{H}_\lambda = \mathcal{H}^{\mathrm{e}}_\lambda \cup \mathcal{H}^{\mathrm{o}}_\lambda$.

A partition has empty 2-core, written 2-core$(\lambda) = 0$, if it can be tiled by dominoes. For example, $\lambda = (6, 4, 3, 1)$



For us it's important to note that

$$\text{2-core}(\lambda) = 0 \quad \Longleftrightarrow \quad |\mathcal{H}_\lambda^{\mathrm{o}}| = |\mathcal{H}_\lambda^{\mathrm{e}}|.$$



Finally, we need a statistic

$$b(\lambda) := \sum_{(i,j)\in\lambda} (-1)^{\lambda_i+\lambda'_j-i-j+1}(\lambda_i - i).$$

For our running example

$$\begin{array}{r|c|c|c|c|c|c|}
\cline{2-7}
5 & - & - & + & + & + & - \\
\cline{2-7}
2 & + & + & - & - \\
\cline{2-5}
0 & + & + & - \\
\cline{2-4}
-3 & - \\
\cline{2-2}
\end{array}$$

we compute

$$b\big((6,4,3,1)\big) = 3.$$

In fact for 2-core$(\lambda) = 0$ we have $b(\lambda) \geqslant 0$ with equality if and only if λ is even.

2-core condition

In their work on the branching problem, Lee, Rains and Warnaar were led to conjecture a swathe of curious formulae including integral evaluations, Littlewood identities, branching formulae, and hypergeometric summations.

The link between all of their conjectures is the "2-core condition". For example, an integral vanishes unless 2-core$(\lambda) = 0$.

All of their conjectures are at the Macdonald, or (q, t), level. In the Schur case $q = t$, things simplify dramatically, and some of their conjectures can be resolved.

Recall the usual infinite q-shifted factorial

$$(a;q)_\infty := \prod_{i\geqslant 0}(1-aq^i).$$

Then the following conjecture of Lee, Rains and Warnaar is true.

There holds

$$\sum_{\substack{\lambda \\ \text{2-core}(\lambda)=0}} q^{b(\lambda)} \frac{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{o}}}(1-q^h)}{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{e}}}(1-q^h)} s_\lambda(x) = \prod_{i\geqslant 1}\frac{(qx_i^2;q^2)_\infty}{(x_i^2;q^2)_\infty}\prod_{i<j}\frac{1}{1-x_ix_j},$$

and

$$\sum_{\substack{\lambda \\ \text{2-core}(\lambda)=0}} q^{b(\lambda')} \frac{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{o}}}(1-q^h)}{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{e}}}(1-q^h)} s_\lambda(x) = \prod_{i\geqslant 1}\frac{(q^2x_i^2;q^2)_\infty}{(qx_i^2;q^2)_\infty}\prod_{i<j}\frac{1}{1-x_ix_j}.$$

For $q=0$ these are Littlewood's even row/even column identities respectively.

The previous identities are in the spirit of Kawanaka's 1999 formula

$$\sum_{\lambda}\left(\prod_{h\in\mathcal{H}}\frac{1+q^h}{1-q^h}\right)s_\lambda(x)=\prod_{i\geqslant 1}\frac{(-qx_i;q)_\infty}{(x_i;q)_\infty}\prod_{i<j}\frac{1}{1-x_ix_j}$$

which recovers Schur's original identity for $q=0$.

Unlike Kawanaka's identity, the 2-core identities make sense for $q\to 1$ and produce the following corollary.

$$\sum_{\substack{\lambda\\ \text{2-core}(\lambda)=0}}\frac{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{o}}}(h)}{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{e}}}(h)}s_\lambda(x)=\prod_{i\geqslant 1}\frac{1}{(1-x_i^2)^{1/2}}\prod_{i<j}\frac{1}{1-x_ix_j}$$

The proof of the previous theorem relies on some basic Koornwinder polynomial theory together with vanishing integrals.

Vanishing integrals

Fix the measure

$$\mathrm{d}T(x) = \frac{1}{2^n n! (2\pi \mathrm{i})^n} \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n}.$$

For a function $f(x)$ we define

$$f\left(x^{\pm}\right) = f(x) f(1/x)$$
$$f\left(x^{\pm} y^{\pm}\right) = f(xy) f(x/y) f(y/x) f(1/xy)$$

Lee, Rains and Warnaar prove the following.

For $a, b, q \in \mathbb{C}$ such that $|a|, |b|, |q| < 1$, the integral

$$I_\lambda^{(n)}(a, b; q)$$
$$:= \int_{\mathbb{T}^n} s_\lambda\left(x^{\pm}\right) \prod_{i=1}^{n} \frac{(x_i^{\pm 2}; q)_\infty}{(a x_i^{\pm 2}; q^2)_\infty (b x_i^{\pm 2}; q^2)_\infty} \prod_{1 \leqslant i < j \leqslant n} \left(1 - x_i^{\pm} x_j^{\pm}\right) \mathrm{d}T(x)$$

vanishes unless $2\text{-core}(\lambda) = 0$.

For 2-core$(\lambda) = 0$, they also give integral evaluations in terms of Pfaffians in two important special cases. These Pfaffians may be evaluated, and yield the following pair of evaluations.

For 2-core$(\lambda) = 0$,

$$I_\lambda^{(n)}(q, q; q) = q^{b(\lambda')} \frac{C_\lambda^{\mathrm{e}}(q^{2n}; q) H_\lambda^{\mathrm{o}}(q)}{C_\lambda^{\mathrm{o}}(q^{2n}; q) H_\lambda^{\mathrm{e}}(q)}$$

and

$$I_\lambda^{(n)}(1, q^2; q) = q^{b(\lambda)} \frac{1 + q^{n+2(b(\lambda')-b(\lambda))}}{1 + q^n} \, \frac{C_\lambda^{\mathrm{e}}(q^{2n}; q) H_\lambda^{\mathrm{o}}(q)}{C_\lambda^{\mathrm{o}}(q^{2n}; q) H_\lambda^{\mathrm{e}}(q)}.$$

Here

$$H_\lambda^{\mathrm{e}/\mathrm{o}}(q) := \prod_{\substack{s \in \lambda \\ h(s) \text{ even/odd}}} \big(1 - q^{h(s)}\big),$$

$$C_\lambda^{\mathrm{e}/\mathrm{o}}(z; q) := \prod_{\substack{(i,j) \in \lambda \\ i+j \text{ even/odd}}} \big(1 - zq^{j-i}\big).$$

The key identity is the following due to Rains and Warnaar (stated in a special case).

For nonnegative integers n, m,

$$(x_1 \cdots x_n)^m K_{(m^n)}(x; q, q; \pm a, \pm b) = \sum_{\lambda} (-1)^{|\lambda|} I_{\lambda'}^{(m)}(a, b; q) s_\lambda(x).$$

Any closed form evaluation of the integral $I_{\lambda'}^{(m)}(a, b; q)$ thus gives a bounded Littlewood-type identity.

For example with $(a, b) = (q, q)$ we obtain

$$(x_1 \cdots x_n)^m K_{(m^n)}(x; q, q; \pm q, \pm q) \\ = \sum_{\substack{\lambda \\ \text{2-core}(\lambda)=0}} q^{b(\lambda')} \frac{C_\lambda^{\mathrm{e}}(q^{-2m}; q) H_\lambda^{\mathrm{o}}(q)}{C_\lambda^{\mathrm{o}}(q^{-2m}; q) H_\lambda^{\mathrm{e}}(q)} s_\lambda(x),$$

and a similar result for $(a, b) = (1, q^2)$. Sending $m \to \infty$ gives the unbounded identity from before.

The same proof technique works at the Macdonald level. Known integral evaluations there are so-called virtual Koornwinder integrals for Macdonald polynomials due to Rains and Rains and Vazirani.

The q, t-analogues of the vanishing integrals and (bounded) Littlewood identities are still open. However, in the Hall–Littlewood case, the 2-core condition drops out and the two identities are known. For example

$$\sum_{\lambda} t^{o(\lambda)/2} \Bigg(\prod_{\substack{s \in \lambda \\ a(s)=0 \\ l(s) \text{ even}}} \big(1 - t^{l(s)+1}\big) \Bigg) P_\lambda(x;t) = \prod_{i \geqslant 1} \frac{1 - tx_i^2}{1 - x_i^2} \prod_{i<j} \frac{1 - tx_i x_j}{1 - x_i x_j},$$

where the sum is over all partitions such that odd parts have even multiplicity and $o(\lambda)$ is the sum of these multiplicities. This is due to Kawanaka.

One final curious conjecture of Lee, Rains and Warnaar is a C_n analogue of Andrews' q-analogue of Watson's ${}_3F_2$ summation

$${}_4\phi_3\left[\begin{matrix} a^{1/2}, -a^{1/2}, bq^{N-1}, q^{-N} \\ a, b^{1/2}, -b^{-1/2} \end{matrix}; q, q\right] = \begin{cases} \dfrac{a^{N/2}(q, b/a; q^2)_{N/2}}{(aq, b; q^2)_{N/2}} & \text{if } N \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Suppressing the details, it may be stated as

$$\sum_{\mu \subseteq \lambda} f_{\lambda,\mu}(a; q, t) = \begin{cases} F_\lambda(a; q, t) & \text{if 2-core}(\lambda) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $\lambda = (N)$ this is Andrews' formula.

The end