Adapted Wasserstein distance between the laws of SDEs

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Goal

For one-dimensional SDEs

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \quad \rightsquigarrow \quad \mu = \text{Law}\left((X_t)_{t\in[0,1]}\right)$$
$$d\bar{X}_t = \bar{b}(\bar{X}_t)dt + \bar{\sigma}(\bar{X}_t)d\bar{W}_t \quad \rightsquigarrow \quad \nu = \text{Law}\left((\bar{X}_t)_{t\in[0,1]}\right)$$

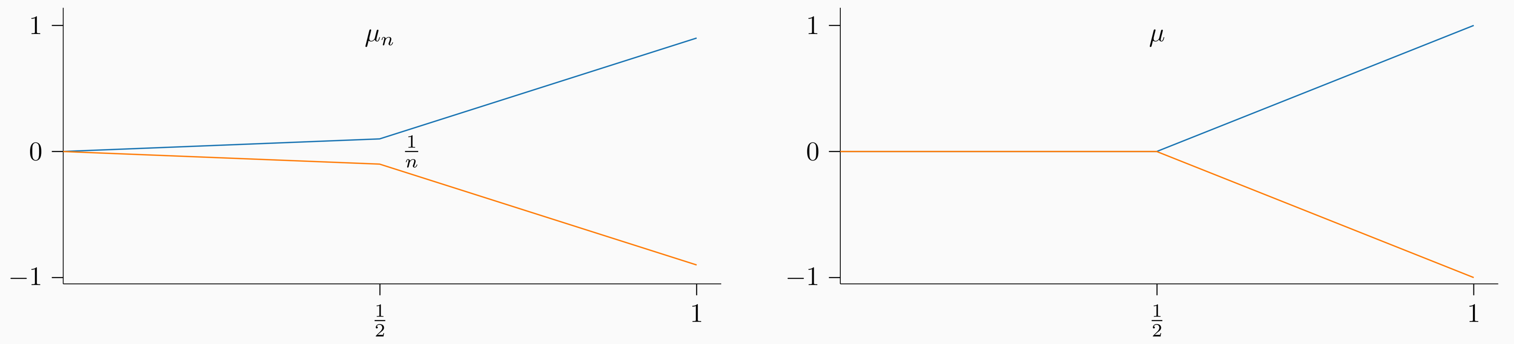
- Find a suitable distance between μ and ν
- Show convergence of discrete to continuous distances

Adapted Wasserstein

Canonical distance between probability measures is the Wasserstein distance

$$\mathcal{W}_p^p(\mu,\nu) = \inf_{\pi\in\text{Cpl}(\mu,\nu)} \int_\Omega \int_0^1 |\omega_s - \bar{\omega}_s|^p \, dt \, d\pi(\omega,\bar{\omega})$$

Example:



- Optimal coupling is to match the blue paths and the orange paths, and so $\mathcal{W}_p(\mu_n,\mu) \to 0$
- A sequence of deterministic processes converging to a martingale on $[0, 1/2]$
- This is not a good distance between stochastic processes

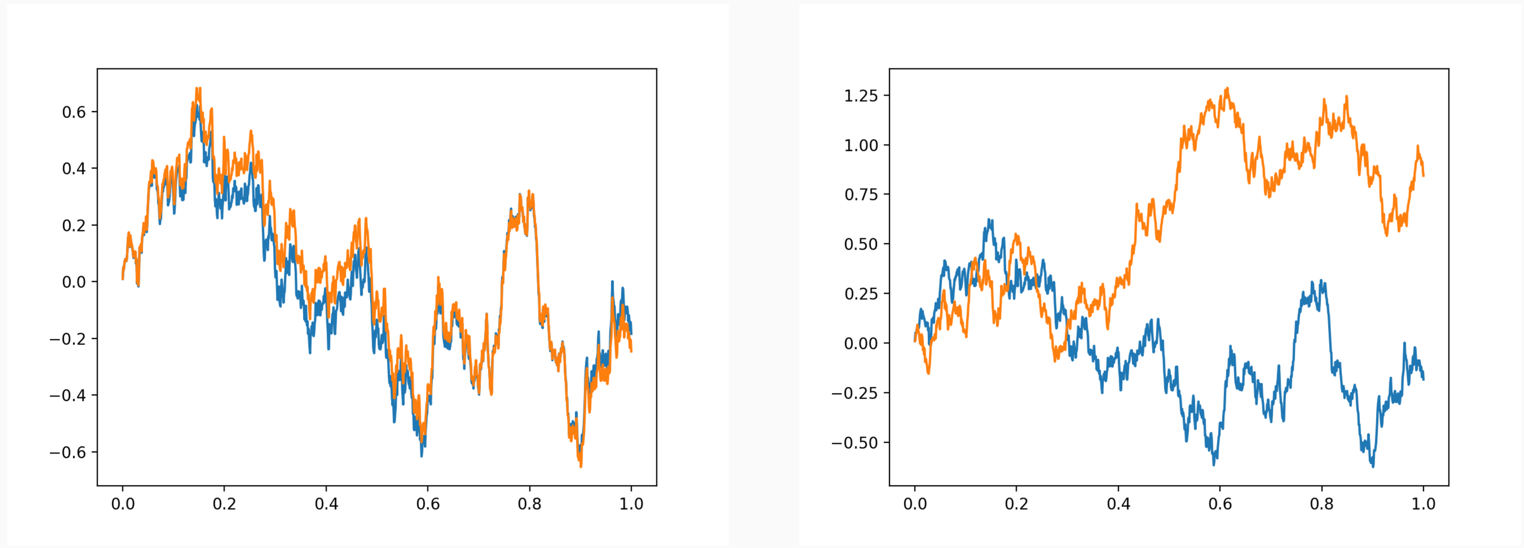
Define the adapted Wasserstein distance $\mathcal{AW}_p(\mu,\nu)$ by replacing $\text{Cpl}(\mu,\nu)$ with

$$\text{Cpl}_{\text{bc}}(\mu,\nu) = \{\pi \in \text{Cpl}(\mu,\nu) : \ \pi \text{ is bi-causal}\}$$

- $\pi = \text{Law}(X,\bar{X})$ is bi-causal if *given the past of X, the future of $\bar{X}$ is independent of X*, and vice-versa
- The optimal coupling above is no longer admissible
- $\mathcal{AW}_2(\mu_n,\mu) \approx \frac{1}{2}$ for all n large

Results

Theorem. *Optimising over bi-causal couplings is equivalent to optimising over correlations between $W, \bar{W}$.*

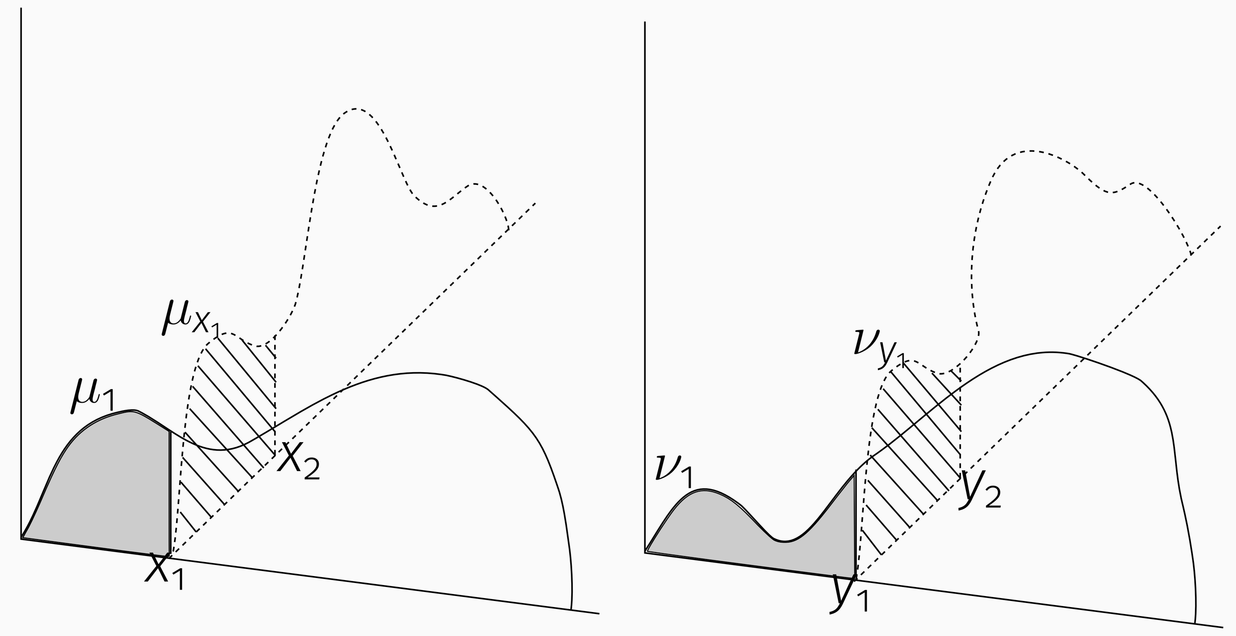


Synchronous coupling: $W = \bar{W}$ **Product coupling:** $W, \bar{W}$ independent

Theorem. *Suppose that the coefficients are continuous with linear growth and that pathwise uniqueness holds. Then the synchronous coupling attains $\mathcal{AW}_p(\mu,\nu)$.*

Knothe–Rosenblatt

- For μ,ν on $\mathbb{R}^n$, $U_1,\dots,U_n \overset{iid}{\sim} \mathcal{U}[0,1]$, $X_1 = F_{\mu_1}^{-1}(U_1)$, $Y_1 = F_{\nu_1}^{-1}(U_1)$,
$$X_k = F^{-1}_{\mu_{X_1,\dots,X_{k-1}}}(U_k), \quad Y_k = F^{-1}_{\nu_{Y_1,\dots,Y_{k-1}}}(U_k)$$
- The Knothe–Rosenblatt rearrangement between μ,ν is
$$\pi^{\text{KR}}_{\mu,\nu} = \text{Law}(X_1,\dots,X_n,Y_1,\dots,Y_n)$$



Knothe–Rosenblatt rearrangemnt for $n = 2$

Discretisation

Monotone Euler–Maruyama scheme: $n \in \mathbb{N}$, $h = 1/n$

$$X_0^h = X_0,$$
$$X_t^h = X_{kh}^h + b(X_{kh}^h)(t-kh) + \sigma(X_{kh}^h)(W_t^h - W_{kh}^h), \quad t \in (kh,(k+1)h]$$

- The process W^h is a martingale with Brownian increments stopped when hitting some barrier

Theorem. *Convergence in L^p holds for all $p \geq 1$ and, for $h \ll 1$,*

$$\mathbb{E}\left[\sup_{t\in[0,1]} |X_t^h - X_t|^2\right] \leq Ch.$$

Theorem (Rüschendorf '85). *For μ_n,ν_n on $\mathbb{R}^n$ stochastically co-monotone, the Knothe–Rosenblatt rearrangement attains $\mathcal{AW}_p(\mu_n,\nu_n)$.*

- If b,σ are Lipschitz and $h \ll 1$, then $\mu^h = \text{Law}((X_{kh}^h)_{k=1,\dots,n})$ is stochastically increasing
- Therefore
$$\mathcal{AW}_p(\mu^h,\nu^h) \to \mathcal{AW}_p(\mu,\nu)$$
- The Knothe–Rosenblatt rearrangement attains the *LHS* for each h and the synchronous coupling attains the *RHS*
- By a stability argument, we arrive at the main theorem

Reference

[1] J. Backhoff-Veraguas, S. Källblad, and B. A. Robinson. Adapted Wasserstein distance between the laws of SDEs. *arXiv:2209.03243 [math]*, Sep. 2022.

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