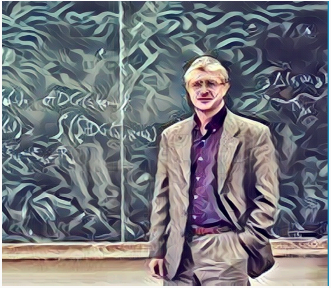
A regularized Kellerer theorem in arbitrary dimension

Benjamin A. Robinson

University of Vienna

May 30, 2023 — Conference in honor of David Nualart, Donostia



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Joint work with

Gudmund Pammer
ETH Zürich

Walter Schachermayer
University of Vienna

Problem statement

Given a family of probability measures $(\mu_t)_{t\in I}$ on $\mathbb{R}^d$, does there exist a mimicking martingale M such that

$$\mathrm{Law}(M_t) = \mu_t, \quad \forall t \in I?$$

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Necessary condition
For any convex function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\int f \mathrm{d}\mu_s \leq \int f \mathrm{d}\mu_t, \quad s \leq t.$$

Financial motivation

Fact

$$\text{Knowledge of call prices } \mathbb{E}(X_t - K)_+ \text{ for all } K \in \mathbb{R}, t \in I,$$

$$\Leftrightarrow$$

$$\text{Knowledge of marginals } \mu_t, t \in I$$

[Hirsch, Roynette '12]

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$$\Leftrightarrow$$

$$\text{Knowledge of marginals } \mu_t, t \in I$$

[Hirsch, Roynette '12]

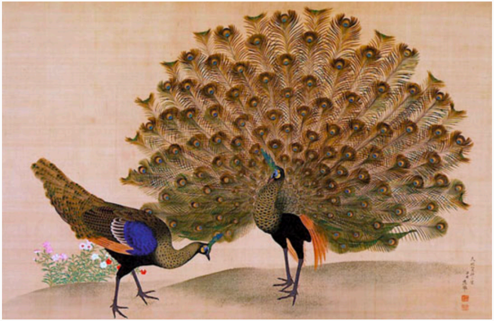
Seek a "nice" martingale model consistent with observed data.

Peacocks

Assume that μ is a peacock; i.e. for any convex function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\int f \mathrm{d}\mu_s \leq \int f \mathrm{d}\mu_t, \quad s \leq t.$$

Processus Croissant pour l'Ordre Convexe



[Hirsch, Profetta, Roynette, Yor '11]

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Yes – [Strassen '65, Doob '68]

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"Desirable" properties

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- Markovianity

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- continuity of paths
- uniqueness

Discrete time

[Strassen ’65]

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Volker Strassen *The Existence of Probability Measures with Given Marginals*, Ann. Math. Stat. 1965

Continuous time, $d = 1$

[Kellerer '72]

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Subsequent contributions (incomplete!)

Albin, Baker, Beiglböck, Brückerhoff, Boubel, Donati-Martin, Hamza, Hirsch, Huesmann, Juillet, Källblad, Klebaner, Lowther, Profetta, Roynette, Stebegg, Tan, Touzi, Yor, ...

Continuous time, $d = 1$

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[Lowther '08 – '10]

Suppose $t \mapsto \mu_t$ is weakly continuous with convex support.

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Suppose $t \mapsto \mu_t$ is weakly continuous with convex support. Then

- M has the strong Markov property;

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Suppose $t \mapsto \mu_t$ is weakly continuous with convex support. Then

- M has the strong Markov property;
- M is the unique strong Markov mimicking martingale;

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Suppose $t \mapsto \mu_t$ is weakly continuous with convex support. Then

- M has the strong Markov property;
- M is the unique strong Markov mimicking martingale;
- $t \mapsto M_t$ is continuous.

Faking Brownian motion

Do there exist stochastic processes with Brownian marginals that are not Brownian motion?

[Hamza, Klebaner '07]

There exists some fake Brownian motion.

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[Beiglböck, Lowther, Pammer, Schachermayer '21]

There exists a "very fake" Brownian motion in dimension $d = 1$.

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There exists some fake Brownian motion.

[Beiglböck, Lowther, Pammer, Schachermayer '21]

There exists a Markov process with continuous paths that mimics Brownian marginals in dimension $d = 1$.

Continuous time, $d = 1$

[Kellerer '72]

Given a continuous-time peacock $(\mu_t)_{t\in[0,1]}$ on $\mathbb{R}$, there exists a mimicking Markov martingale M.

[Lowther '08 – '10]

- M has the strong Markov property;
- M is the unique strong Markov mimicking martingale;
- $t \mapsto M_t$ is continuous under additional conditions.

Continuous time, $d \geq 2$

Existing literature ...

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... without the Markov property

Doob '68 (existence)

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Doob '68 (existence), Hirsch–Roynette '13 (right-continuity)

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Doob '68 (existence), Hirsch–Roynette '13 (right-continuity)

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no known results in dimension $d \geq 2$.

Continuous time, $d \geq 2$

Weakly continuous $\mathbb{R}^d$-valued square-integrable peacock $(\mu_t)_{t\in[0,1]}$.
Regularize with a Gaussian $\mu_t^{\mathrm{r}} := \mu_t * \gamma^{\varepsilon(t+\delta)}$

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Theorem 1 [Pammer, R., Schachermayer '22]

There exists a measurable $(t,x) \mapsto \sigma_t^{\mathrm{r}}(x)$ that is locally Lipschitz in x and non-degenerate, uniformly in $t \in [0,1]$, and a Brownian motion B such that $\mathrm{Law}(M_t^{\mathrm{r}}) = \mu_t^{\mathrm{r}}$, for all $t \in [0,1]$, where

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Key ingredients

- Stretched Brownian motion or Bass martingale in $\mathbb{R}^d$ [Backhoff, Beiglböck, Huesmann, Källblad '19], [Backhoff, Beiglböck, Schachermayer, Tschiderer '23+]

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$$\inf_{\pi \in \mathrm{MT}(\mu_0,\mu_1)} \mathbb{E}^\pi[c(X,Y)] = \inf_{M \in \mathcal{M}(\mu_0,\mu_1)} \mathbb{E}[F(M)] \qquad \text{(MOT)}$$

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$$M_t^* = \mathbb{E}[v(B_1)|\mathcal{F}_t^B]$$

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$$\mathrm{d}\hat{X}_t = \hat{b}_t(\hat{X}_t)\mathrm{d}t + \hat{\sigma}_t(\hat{X}_t)\mathrm{d}W_t, \quad \mathrm{Law}(X_t) = \mathrm{Law}(\hat{X}_t),\ t \in [0,1]$$

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  For $\mathrm{d}X_t^k = \sigma_t^k(X_t^k)\mathrm{d}B_t$ for "nice" σ^k, suppose for each (t, x)

$$\int_0^t \sigma_s^k(x)^2 \mathrm{d}s \to \int_0^t \sigma_s(x)^2 \mathrm{d}s.$$

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  Then $X^k \to X$ in f.d.d., $\mathrm{d}X_t = \sigma_t(X_t)\mathrm{d}B_t$ and σ "nice".

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Moreover:

- M^{r} is a strong Markov martingale with continuous paths;
- There is no uniqueness for $d \geq 2$;
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Non-uniqueness

Theorem 3 [Pammer, R., Schachermayer '22]

There exists a $\mathbb{R}^2$-valued strong Markov martingale diffusion with Brownian marginals, which is not a Brownian motion.

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Circular Brownian Motion
[Émery, Schachermayer '99]
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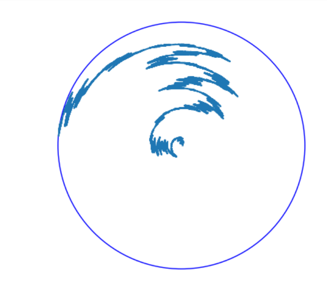

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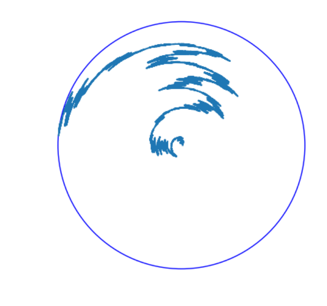
Theorem [Cox, R. '22]

There is a unique weak solution but no strong solution of

$$\mathrm{d}X_t = \frac{1}{|X_t|}\begin{bmatrix} -X_t^2 \\ X_t^1 \end{bmatrix}\mathrm{d}W_t, \quad X_0 = 0.$$



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Theorem 3 [Pammer, R., Schachermayer '22]

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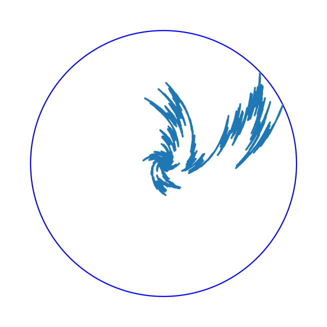
Theorem [Cox, R. '22]
Let X be a weak solution of

$$\mathrm{d}X_t = \frac{1}{|X_t|}(X_t + X_t^{\perp})\mathrm{d}W_t, \; X_0 \sim \eta.$$

Then X is a continuous strong Markov fake Brownian motion.



[Pammer, R., Schachermayer '22]

Counterexamples

Theorem 4 [Pammer, R., Schachermayer '22]

There exists a weakly continuous square-integrable peacock $(\mu_t)_{t\in[0,1]}$ on $\mathbb{R}^4$ such that, for the peacock $(\mu_t * \gamma^t)_{t\in[0,1]}$, there exists no mimicking Markov martingale.

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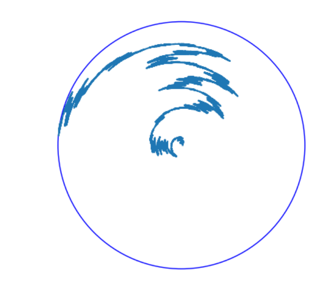

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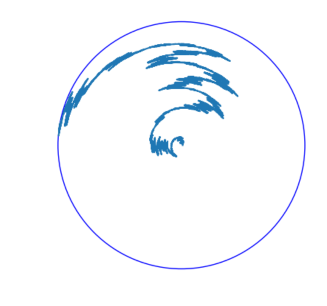

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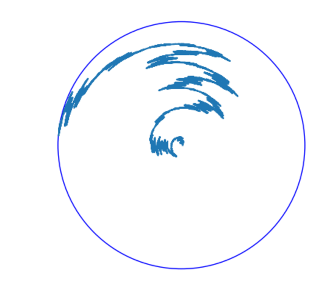

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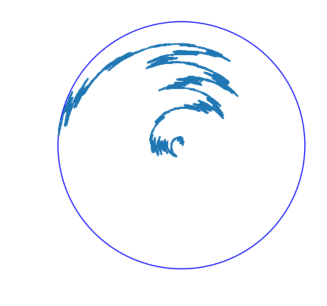
2. No Markov martingale mimicking μ;
3. No Markov martingale mimicking $(\mu * \gamma^t)_{t\in[0,1]}$.



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Summary

- We prove the first known Kellerer-type result in arbitrary dimension;
- In dimension $d \geq 2$, uniqueness fails;
- In general, the result can fail without some regularization.

arXiv:2210.13847

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David, congratulations on your career, and thank you for your contributions!