

BICAUSAL OPTIMAL TRANSPORT FOR SDES WITH IRREGULAR COEFFICIENTS

BENJAMIN A. ROBINSON

Faculty of Mathematics, University of Vienna, Austria

MICHAELA SZÖLGYENYI

Department of Statistics, University of Klagenfurt, Austria

ABSTRACT. We solve constrained optimal transport problems between the laws of solutions of stochastic differential equations (SDEs). We consider SDEs with irregular coefficients, making only minimal regularity assumptions. We show that the so-called synchronous coupling is optimal among bicausal couplings, that is couplings that respect the flow of information encoded in the stochastic processes. Our results provide a method to numerically compute the adapted Wasserstein distance between laws of SDEs with irregular coefficients. Moreover, we introduce a transformation-based semi-implicit numerical scheme and establish the first strong convergence result for SDEs with exponentially growing and discontinuous drift.

1. INTRODUCTION

A fundamental question in stochastic modelling is that of quantifying the effects of model uncertainty. In this context it is of interest to compute an appropriate distance between different stochastic models. In particular, we focus on models that are described by stochastic differential equations (SDEs), potentially having irregular coefficients. The main contribution of this paper is to provide a methodology to efficiently compute such a distance between the laws of SDEs, by solving an optimal transport problem. Our approach successfully brings together optimal transport and numerical analysis of SDEs. Moreover, for SDEs with exponentially growing and discontinuous drift, we prove strong well-posedness and strong convergence of a novel numerical scheme.

In order to compare stochastic models, a reasonable choice of distance is a modification of the Wasserstein distance on the space of probability measures. The Wasserstein distance arises from an optimal transport problem and is defined as the infimum of an expected cost over the set of couplings $\text{Cpl}(\mu, \nu)$ between probability measures μ and ν , that is the set of probability measures on the product space having marginals μ and ν . This distance metrises the weak topology. However, in many situations the weak topology is unsuitable for comparing stochastic processes, since it does not take into account the information structure that is encoded in the filtrations generated by the processes. As a remedy, the adapted Wasserstein distance and associated bicausal optimal transport problem has been proposed. This constitutes the main object of study in this paper.

E-mail addresses: ben.robinson@univie.ac.at, michaela.szolgyenyi@aau.at.

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We consider stochastic models described by scalar SDEs of the form

$$(1.1) \quad dX_t = b_t(X_t) dt + \sigma_t(X_t) dW_t, \quad t \in [0, T],$$

with $X_0 = x_0 \in \mathbb{R}$, where $T \in (0, \infty)$, $b: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma: [0, T] \times \mathbb{R} \rightarrow [0, \infty)$ are measurable functions, and $W = (W_t)_{t \in [0, T]}$ is a standard scalar Brownian motion. Now consider two different models with data (b, σ, x_0, W) , respectively $(\bar{b}, \bar{\sigma}, x_0, \bar{W})$. For $\Omega = C([0, T], \mathbb{R})$, $p \geq 1$, and $\mathcal{P}_p(\Omega)$ being the set of all probability measures on Ω with finite p^{th} moment, denote the law of the solutions of these SDEs by $\mu \in \mathcal{P}_p(\Omega)$, respectively $\nu \in \mathcal{P}_p(\Omega)$. Let $(\omega, \bar{\omega})$ denote the canonical process on $\Omega \times \Omega$. Then the adapted Wasserstein distance between μ and ν is defined by

$$\mathcal{AW}_p^p(\mu, \nu) := \inf_{\pi \in \text{Cpl}_{\text{bc}}(\mu, \nu)} \mathbb{E}^\pi \left[\int_0^T |\omega_t - \bar{\omega}_t|^p dt \right],$$

where we optimise over the set $\text{Cpl}_{\text{bc}}(\mu, \nu)$ of bicausal couplings between μ and ν , that is the set of couplings that satisfy an additional constraint that encodes the filtrations of the processes. Informally, this bicausality condition imposes that for any $t \in [0, T]$, X_t is conditionally independent of the process \bar{X} given the value \bar{X}_t , with the symmetric condition when exchanging the roles of X and \bar{X} . In the context of optimisation problems for stochastic processes, such as those that arise in mathematical finance and mathematical biology, the causality condition appears natural. One is typically only able to make decisions based on information contained in the filtration of the process at the present time.

We aim to solve such a bicausal transport problem under minimal regularity assumptions on the coefficients of the SDE (1.1). For one-dimensional SDEs (2.6) with smooth coefficients, Bion-Nadal and Talay [10, §2.1] exploit PDE methods to show that the so-called synchronous coupling attains the adapted Wasserstein distance \mathcal{AW}_2 . This coupling is the joint law of the SDEs when they are driven by a common Brownian motion. Following this work, Backhoff-Veraguas, Källblad, and Robinson [7, Theorem 1.3] show that the synchronous coupling attains \mathcal{AW}_p , $p \geq 1$, between the laws of SDEs with globally Lipschitz coefficients. The result of [7] holds also for continuous linearly growing coefficients, provided that pathwise uniqueness holds. However, no explicit conditions for this are given in [7].

We show that optimality of the synchronous coupling holds in much greater generality, considering also SDEs with irregular coefficients. We consider two classes of irregularity:

- discontinuous drift that grows up to exponentially, degenerate diffusion;
- bounded measurable drift, bounded diffusion that is α -Hölder, with $\alpha \in [1/2, 1]$.

Our main result can be stated informally as follows:

Main result. *For models with data (b, σ, x_0, W) , respectively $(\bar{b}, \bar{\sigma}, x_0, \bar{W})$, each potentially having any one of the above irregularities, the synchronous coupling is optimal. Furthermore, one can compute the adapted Wasserstein distance by approximating each scalar SDE with any numerical scheme for which strong convergent rates are known.*

Given that the synchronous coupling attains the adapted Wasserstein distance, the problem of computing this distance reduces to that of simulating two one-dimensional SDEs, using the same noise for each, and applying (multilevel-)Monte Carlo methods. Such computational efficiency is not usually expected, even in the case of classical optimal transport, without some regularisation.

The methods of proof in our paper are based on transformations of SDEs. In the first case, for SDEs with discontinuous and exponentially growing drift, we introduce a transformation-based semi-implicit Euler–Maruyama scheme for which we prove strong convergence. We design this scheme such that it is stochastically monotone. For measures on \mathbb{R}^n satisfying such a monotonicity condition, it is known that the well-studied Knothe–Rosenblatt rearrangement is an optimiser for

the bicausal transport problem between them. Applying the strong convergence of the scheme and a stability result for bicausal optimal transport, we prove the optimality of the synchronous coupling. In the case of bounded and measurable drift, we apply the Zvonkin transformation to obtain SDEs with sufficiently regular coefficients, for which optimality results are already known. We show that such a transformation preserves the optimality property of the synchronous coupling. The novel strong convergence result in the first case can be stated informally as follows:

Auxiliary result. *For the SDE (1.1) with drift b that has exponential growth and satisfies mild piecewise continuity conditions, and with diffusion σ that may be degenerate except at the points of discontinuity of b ,*

- *there exists a unique strong solution with bounded moments of all orders;*
- *a transformation-based drift-implicit Euler–Maruyama scheme converges strongly;*
- *we provide convergence rates.*

1.1. Structure of the article. Since this work combines optimal transport and numerical analysis of SDEs, we keep it self-contained such that it is accessible to readers from both fields. We provide the necessary background on optimal transport theory in Section 2.1 and its applications in the theory of stochastic processes in Section 2.2. Readers familiar with classical optimal transport can skip Section 2.1, and those familiar with adapted or causal optimal transport can also skip Section 2.2. In Section 2.3, we discuss numerical approximation schemes for SDEs and their application to bicausal optimal transport. We present some preliminary stability and optimality results in Section 3.

Our first theorem is presented in Section 4, where we consider the first class of irregularity of the coefficients stated above, that is discontinuous drift with exponential growth and degenerate diffusion. We use the transformation method developed in [42] to handle the discontinuities and combine it with a semi-implicit scheme to handle the super-linear growth. With these ingredients we define a novel numerical scheme. We prove strong convergence of this scheme and provide convergence rates in Section 4.2. We prove optimality of the synchronous coupling for this class of coefficients in Section 4.3.

In Section 5 we consider the second class of irregularity of the coefficients stated above, that is bounded measurable drift and Hölder continuous diffusion. Applying the Zvonkin transformation, we also prove optimality of the synchronous coupling for this class of coefficients.

We combine the results of the preceding sections in Section 6 to prove the main theorem of this paper. In Section 7 we apply our results to compute adapted Wasserstein distances, and finally we discuss potential applications to robust optimisation in Section 8.

The proofs of several results from Section 3 and Section 4 are collected in Appendix A and Appendix B, respectively.

The relevant literature can be found in the respective sections.

2. OPTIMAL TRANSPORT AND APPLICATIONS TO STOCHASTIC PROCESSES

Here we introduce the concepts from optimal transport that will be needed for our applications. The field has a long history, originating with Monge [49] in 1781 and taken up again by Kantorovich [35] in 1942. For the interested reader, modern accounts can be found, for example, in Villani [77], Ambrosio, Gigli, and Savaré [2], and Santambrogio [71], with the latter taking a more applied point of view.

2.1. Introduction to optimal transport. Given Polish spaces \mathcal{X} and \mathcal{Y} supporting probability measures μ and ν , respectively, and a convex lower semicontinuous function $c: \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ the

optimal transport problem [35] is to find

$$(OT) \quad \inf_{\pi \in \text{Cpl}(\mu, \nu)} \mathbb{E}^\pi [c(X, Y)],$$

where the set of couplings between μ and ν is defined by

$$\text{Cpl}(\mu, \nu) := \{\pi \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}) : \pi(A, \mathcal{Y}) = \mu(A), \pi(\mathcal{X}, B) = \nu(B), \text{ for all Borel } A \subseteq \mathcal{X}, B \subseteq \mathcal{Y}\}.$$

2.1.1. *Brenier's theorem.* Central to the theory of optimal transport is the following key result of Brenier [13]: *an optimiser for the quadratic cost is given by the gradient of a convex function.* More precisely, let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^n$ and $c(x, y) := |x - y|^2$, and suppose that μ is absolutely continuous with respect to the Lebesgue measure. Then there exists a unique convex function $v: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\pi^{\text{mono}} := (\text{id}, \nabla v)_\# \mu$ is optimal for (OT). In the one-dimensional case, $\nabla v = F_\nu^{-1} \circ F_\mu$, where F_η, F_η^{-1} denote the cumulative distribution function and quantile function of a measure η , and the coupling π^{mono} is known as the *monotone rearrangement* (or the *Hoeffding–Fréchet coupling*).

2.1.2. *Wasserstein distance.* Given $\mathcal{X} = \mathcal{Y}$ equipped with some metric d , a metric is induced on the space of probability measures $\mathcal{P}_p(\mathcal{X})$, for any $p \geq 1$, as follows. Taking the cost function $c(x, y) = d(x, y)^p$ in (OT), for measures $\mu, \nu \in \mathcal{P}_p(\mathcal{X})$ the p -Wasserstein distance is defined by

$$(2.1) \quad \mathcal{W}_p^p(\mu, \nu) := \inf_{\pi \in \text{Cpl}(\mu, \nu)} \mathbb{E}^\pi [d(X, Y)^p].$$

The p -Wasserstein distance metrises the weak topology on $\mathcal{P}_p(\mathcal{X})$ and is thus a natural candidate to compare probability measures on \mathcal{X} .

2.2. **Optimal transport applied to stochastic processes.** We now consider discrete-time real-valued stochastic processes. Set $\mathcal{X} = \mathcal{Y} = \mathbb{R}^n$, for some $n \in \mathbb{N}$, and equip this space with the L^p norm, defined by $\|x\|_p^p := \sum_{i=1}^n |x_i|^p$ for $x \in \mathbb{R}^n$, for some $p \geq 1$. Let $\mathcal{F} = (\mathcal{F}_k)_{k \in \{1, \dots, n\}}$ denote the canonical filtration on \mathbb{R}^n , $x = (x_1, \dots, x_n)$ the canonical process on \mathbb{R}^n , and (x, \bar{x}) the canonical process on the product space $\mathbb{R}^n \times \mathbb{R}^n$. Given a random variable X on $(\mathbb{R}^n, \mathcal{F}_n)$ with law η , let $\mathcal{F}^X = (\mathcal{F}_k^X)_{k \in \{1, \dots, n\}}$ denote the completion of its natural filtration with respect to η .

As stated in Section 2.1.2, the Wasserstein distance is a natural candidate to measure the distance between two probability measures on $(\mathbb{R}^n, \mathcal{F})$. However, in the case that such probability measures are the laws of stochastic processes, the usual Wasserstein distance (and associated weak topology) lacks some desirable properties. For example, Aldous [1, Section 11] already noted that deterministic processes can converge weakly to martingales. We illustrate this with the following example from Backhoff-Veraguas, Bartl, Beiglböck, and Eder [5]; cf. Aldous [1, Example 11.4]. A continuous-time analogue of this example also appears in Backhoff-Veraguas et al. [7, Example 6.1].

Example 2.1. Consider the two-step real-valued processes X, X^ε , with laws μ, μ^ε , for each $\varepsilon > 0$, defined by

$$\begin{aligned} \mathbb{P}[(X_1^\varepsilon, X_2^\varepsilon) = (\varepsilon, 1)] &= \mathbb{P}[(X_1^\varepsilon, X_2^\varepsilon) = (-\varepsilon, -1)] = \frac{1}{2}, \\ \mathbb{P}[(X_1, X_2) = (0, 1)] &= \mathbb{P}[(X_1, X_2) = (0, -1)] = \frac{1}{2}. \end{aligned}$$

The paths of the processes are shown in Figure 1. Define π^* such that positive (resp. negative) values of X^ε are coupled with positive (resp. negative) values of X ; that is the solid blue paths in Figure 1 are coupled with each other, and the dashed orange paths are coupled with each other. Then we find that

$$\mathcal{W}_p^p(\mu^\varepsilon, \mu) \leq \sum_{i=1}^2 \mathbb{E}^{\pi^*} [|X_i^\varepsilon - X_i|^p] = \varepsilon^p,$$

implying weak convergence $\mu^\varepsilon \rightharpoonup \mu$ as $\varepsilon \rightarrow 0$.

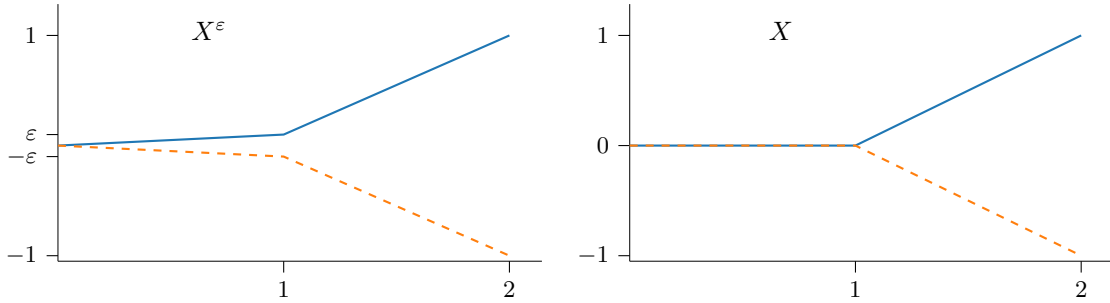


FIGURE 1. A sequence of two-step deterministic processes X^ε converges weakly as $\varepsilon \rightarrow 0$ to a martingale X [5, Figure 1].

On the other hand, consider the values of optimal stopping problems $V^\varepsilon := \sup_\tau \mathbb{E}[X_\tau^\varepsilon]$ and $V := \sup_\tau \mathbb{E}[X_\tau]$, where the suprema are taken over stopping times taking values in $\{1, 2\}$. Since X is a martingale centred at 0, we immediately see that $V = 0$. However, in the case of X^ε , the optimal stopping time is 2 in the case that $X_1^\varepsilon > 0$, and 1 otherwise, and so we obtain

$$V^\varepsilon = \frac{1}{2}(1 - \varepsilon) \rightarrow \frac{1}{2}.$$

Thus we see that the processes X^ε and X have a very different information structure, but nevertheless their laws are close in the usual Wasserstein distance. This example motivates the introduction of the *adapted Wasserstein distance* and the associated *bicausal optimal transport* problem.

2.2.1. Bicausal optimal transport. We have seen that the drawback of the Wasserstein distance for comparing stochastic processes is that this distance is not able to distinguish different information structures. To rectify this shortcoming, we consider Wasserstein distances with the additional constraint that the couplings must respect the flow of information. This constraint has been formalised in various ways, including the *Markov constructions* of Rüschemdorf [69], the *nested distances* of Pflug and Pichler [60, 61], a stochastic control problem of Bion-Nadal and Talay [10], the *causal transport plans* of Lassalle [40], and the *adapted Wasserstein distance* introduced by Backhoff-Veraguas et al. [5]. Here we will use the terminology *adapted Wasserstein distance*, which we define below. Although there are other candidates for distances between stochastic processes, many such distances have been shown to be topologically equivalent to the adapted Wasserstein distance; see, for example, [6, 12].

Analogously to the classical Wasserstein distance, the adapted Wasserstein distance is induced by a *bicausal optimal transport* problem.

Definition 2.2. Let μ, ν be probability measures on $(\mathbb{R}^n, \mathcal{F})$ and let $\pi \in \text{Cpl}(\mu, \nu)$. The coupling π is *causal* if, for any random variables X, Y with $\text{Law}(X, Y) = \pi$, we have the conditional independence,

$$(2.2) \quad \mathcal{F}_k^Y \text{ is independent of } \mathcal{F}_n^X \text{ under } \pi \text{ conditional on } \mathcal{F}_k^X,$$

for all $k \in \{1, \dots, n\}$. If $\pi \in \text{Cpl}(\mu, \nu)$ is causal, and the coupling $\theta_\# \pi \in \text{Cpl}(\nu, \mu)$ is causal, where $\theta(x, y) = (y, x)$, then we say that π is *bicausal* and we write $\text{Cpl}_{\text{bc}}(\mu, \nu)$ for the set of all such couplings. The *bicausal optimal transport problem* is then to find (OT) with the usual set of couplings $\text{Cpl}(\mu, \nu)$ replaced by the set of bicausal couplings $\text{Cpl}_{\text{bc}}(\mu, \nu)$.

In words, $\pi = \text{Law}(X, Y)$ is causal if the present value of Y is independent of the future of the process X , conditional on the past of X .

From now on, we make the following assumptions on the cost functional c in (OT). For each $k \in \{1, \dots, n\}$, let $c_k: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous with polynomial growth of order p ; that is there

exists $K \geq 0$ such that, for all $x, y \in \mathbb{R}$, $k \in \{0, \dots, n\}$,

$$(2.3) \quad |c_k(x, y)| \leq K[1 + |x|^p + |y|^p].$$

Suppose also that, for each $k \in \{0, \dots, n\}$, the function c_k is *quasi-monotone*¹ in the sense that

$$(2.4) \quad c_k(x, y) + c_k(x', y') - c_k(x, y') - c_k(x', y) \geq 0, \quad \text{for all } x \leq x', y \leq y'.$$

For $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^n)$, the *bicausal optimal transport problem* is then to find

$$(2.5) \quad \inf_{\pi \in \text{Cpl}_{\text{bc}}(\mu, \nu)} \mathbb{E}^\pi \left[\sum_{k=1}^n c_k(x_k, \bar{x}_k) \right].$$

Taking each cost function c_k in (2.5) to be the p^{th} power of the Euclidean distance, we define an adapted analogue of the Wasserstein distance (2.1) as follows.

Definition 2.3 (adapted Wasserstein distance – discrete time). Let $p \geq 1$ and $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^n)$. Then the *adapted Wasserstein distance* is defined by

$$\mathcal{AW}_p^p(\mu, \nu) := \inf_{\pi \in \text{Cpl}_{\text{bc}}(\mu, \nu)} \mathbb{E}^\pi \left[\sum_{k=1}^n |x_k - \bar{x}_k|^p \right].$$

Remark 2.4. The term *adapted* appears natural here, since Definition 2.2 constrains the usual set of couplings to those that satisfy a condition analogous to adaptedness of stochastic processes. The term *causal*, introduced in [40], does not necessarily correspond to the notion of causality in statistics; for example, the product coupling is bicausal. On the other hand, Cheridito and Eckstein [14] relate causal optimal transport to the causal structure of graphical models.

Example 2.5 (Example 2.1 revisited). For the laws given in Example 2.1, the adapted Wasserstein distance $\mathcal{AW}_p^p(\mu^\varepsilon, \mu)$ is bounded away from 0, for any $p \geq 1$. Indeed, under any $\pi \in \text{Cpl}_{\text{bc}}(\mu, \mu^\varepsilon)$, $\mathcal{F}_1^{X^\varepsilon}$ is independent of \mathcal{F}_2^X conditional on \mathcal{F}_1^X . Since \mathcal{F}_1^X is trivial, $\mathcal{F}_1^{X^\varepsilon}$ and \mathcal{F}_2^X are independent under π , without any conditioning. Moreover, X_2^ε is fully determined by X_1^ε , so $\mathcal{F}_2^{X^\varepsilon} = \mathcal{F}_1^{X^\varepsilon}$. Thus $\{\mathcal{F}_k^{X^\varepsilon}\}_{k \in \{1, 2\}}$ is independent of \mathcal{F}_2^X under π and, using again that \mathcal{F}_1^X is trivial, is independent of $\{\mathcal{F}_k^X\}_{k \in \{1, 2\}}$. Hence the product coupling is the only element of $\text{Cpl}_{\text{bc}}(\mu, \mu^\varepsilon)$, and so $\mathcal{AW}_p^p(\mu^\varepsilon, \mu) = \varepsilon^p + 2^{p-1} > 2^{p-1}$. Recalling that, for the optimal stopping problems in Example 2.1, $V^\varepsilon \not\rightarrow V$ as $\varepsilon \rightarrow 0$, the adapted Wasserstein distance appears to be better suited to comparing stochastic processes than the usual Wasserstein distance.

Stability of various optimisation problems with respect to the adapted Wasserstein distance is shown in Backhoff-Veraguas et al. [5]. Moreover, optimal stopping problems are in fact continuous with respect to the adapted Wasserstein distance by [6, Lemma 7.1]; cf. [1, Theorem 17.2]. This confirms our intuition from Examples 2.1 and 2.5.

2.2.2. Optimality of bicausal couplings. Under the following condition of stochastic monotonicity, Rüschendorf [69] identified an optimiser for a bicausal transport problem between the laws of discrete-time Markov processes. For a measure η on \mathbb{R}^n , denote by η_1 the first one-dimensional marginal of η and, for each $k \in \{2, \dots, n\}$, denote by $\eta_{x_1, \dots, x_{k-1}}$ the k^{th} conditional marginal of η , given that the first $k-1$ coordinates are equal to (x_1, \dots, x_{k-1}) .

Definition 2.6 (stochastic monotonicity [16]). Let $(X_n)_{n \in \{1, \dots, N\}}$ be a real-valued Markov process with law $\mu \in \mathcal{P}$. The process X , or the measure μ , is said to be *stochastically increasing* (resp. *decreasing*) if $x_k \mapsto \mu_{x_1, \dots, x_{k-1}, x_k}$ is increasing (resp. decreasing) in first order stochastic dominance, for each $k \in \{1, \dots, N-1\}$.

¹also called *L-superadditive* or *supermodular*; see Block, Griffith, and Savits [11, Remark 1.1]

Two such processes or measures are *stochastically co-monotone* if they are both stochastically increasing, both stochastically decreasing, or one is stochastically increasing and decreasing while the other is arbitrary.

As we shall see below, the optimal bicausal coupling under this monotonicity condition is the Knothe–Rosenblatt rearrangement, introduced independently by Rosenblatt [68] and Knothe [38]. This coupling is a multi-dimensional extension of the monotone rearrangement described in Section 2.1.1. Under the Knothe–Rosenblatt rearrangement, the first marginals are coupled monotonically and then, inductively, the conditional marginals are coupled monotonically; see [7, Figure 1] for an illustration.

Definition 2.7 (Knothe–Rosenblatt rearrangement). Given probability measures μ, ν on \mathbb{R}^n , let U_1, \dots, U_n be independent uniform random variables on $[0, 1]$, define $X_1 = F_{\mu_1}^{-1}(U_1)$, $Y_1 = F_{\nu_1}^{-1}(U_1)$ and, for $k \in \{2, \dots, n\}$, define inductively the random variables

$$X_k = F_{\mu_{X_1, \dots, X_{k-1}}}^{-1}(U_k), \quad Y_k = F_{\nu_{Y_1, \dots, Y_{k-1}}}^{-1}(U_k).$$

The Knothe–Rosenblatt rearrangement between the marginals μ and ν is then given by

$$\pi_{\mu, \nu}^{\text{KR}} = \text{Law}(X_1, \dots, X_n, Y_1, \dots, Y_n).$$

We will make use of the following equivalent definition, which is proved in Backhoff-Veraguas, Beiglböck, Lin, and Zalashko [4, Proposition 5.8].

Proposition 2.8. *A coupling $\pi \in \text{Cpl}(\mu, \nu)$ is equal to $\pi_{\mu, \nu}^{\text{KR}}$ if and only if $\pi = \text{Law}(X, Y)$, where*

$$X = (T_1(U_1), T_2(U_2; X_1), \dots, T_n(U_n; X_1, \dots, X_{n-1})),$$

$$Y = (S_1(U_1), S_2(U_2; Y_1), \dots, S_n(U_n; Y_1, \dots, Y_{n-1})),$$

for U_1, \dots, U_n independent uniform random variables on $[0, 1]$, and T_i, S_i co-monotone in U_i , for each $i \in \{1, \dots, n\}$.

The following optimality result is a straightforward extension of [7, Proposition 3.5], which generalises earlier results from Backhoff-Veraguas et al. [4], Rüschemdorf [69].

Proposition 2.9. *For $p \geq 1$, $n \in \mathbb{N}$, let $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^n)$ be the laws of Markov processes. If μ and ν are stochastically co-monotone, then the Knothe–Rosenblatt rearrangement is the unique optimiser of the bicausal optimal transport problem (2.5), with continuous cost functions $c_k: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $k \in \{1, \dots, n\}$, satisfying (2.3) and (2.4).*

Remark 2.10. We show in Example 5.11 that, when first order stochastic dominance is replaced by second order stochastic dominance for one of the marginals in the definition of stochastic co-monotonicity, optimality of the Knothe–Rosenblatt rearrangement may fail.

2.2.3. Continuous-time bicausal optimal transport. The notions of bicausality and adapted Wasserstein distance extend to continuous-time processes. Fix a finite time horizon $T > 0$, let $\Omega := C([0, T], \mathbb{R})$ be the space of real-valued continuous paths on the time interval $[0, T]$, and equip this space with the canonical filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in [0, T]}$. We consider the product space $\Omega \times \Omega$ equipped with the product filtration, and let $(\omega, \bar{\omega}) = ((\omega_t, \bar{\omega}_t))_{t \in [0, T]}$ denote the canonical process on the product space. As in the discrete-time case, for a random variable X on (Ω, \mathcal{F}_T) with law η , the completion of its natural filtration under η is denoted by $\mathcal{F}^X = (\mathcal{F}_t^X)_{t \in [0, T]}$.

Let $\mu, \nu \in \mathcal{P}_p(\Omega)$, for some $p \geq 1$. We say that a coupling $\pi \in \text{Cpl}(\mu, \nu)$ is bicausal if it satisfies the continuous-time analogue of Definition 2.2, again writing $\text{Cpl}_{\text{bc}}(\mu, \nu)$ for the set of bicausal couplings. Note that, by [40, Example 1(v), Lemma 4, and Proposition 2], our definition is equivalent to Backhoff-Veraguas et al. [5, Definition 1.2] and Lassalle [40, Definition 1].

Let $c: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function for which there exists $K \geq 0$ and $p \geq 1$ such that, for each fixed $t \in [0, T]$, $c_t(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies (2.3) and (2.4). Then the continuous-time *bicausal transport problem* is to find the value

$$(BCOT) \quad V_c(\mu, \nu) := \inf_{\pi \in \text{Cpl}_{bc}(\mu, \nu)} \mathbb{E}^\pi \left[\int_0^T c_t(\omega_t, \bar{\omega}_t) dt \right].$$

Analogously to the discrete-time setting, we equip the space Ω with the L^p norm, defined by $\|\omega\|_p^p := \int_0^T |\omega_t|^p dt$ for $\omega \in \Omega$, and we define the adapted Wasserstein distance in the continuous-time setting as follows.

Definition 2.11 (adapted Wasserstein distance – continuous time). For $p \geq 1$, define the adapted Wasserstein distance between $\mu, \nu \in \mathcal{P}_p(\Omega)$ as

$$\mathcal{AW}_p^p(\mu, \nu) := \inf_{\pi \in \text{Cpl}_{bc}(\mu, \nu)} \mathbb{E}^\pi \left[\int_0^T |\omega_t - \bar{\omega}_t|^p dt \right].$$

Remark 2.12. The adapted Wasserstein distance \mathcal{AW}_2 , in essentially the form of Definition 2.11, was first studied by Bion-Nadal and Talay [10] via a stochastic control formulation and associated Hamilton-Jacobi-Bellman equation, and Backhoff-Veraguas et al. [7] studied \mathcal{AW}_p as defined in Definition 2.11, for general $p \geq 1$. We will see in Section 7 that this distance is amenable to efficient computation. One could also define the adapted Wasserstein distance with respect to the L^∞ norm on Ω ; in Theorem 8.1 we apply this distance to robust optimisation. A further alternative definition of adapted Wasserstein distance between semimartingales is studied in Backhoff-Veraguas et al. [5], where the authors show stability of various stochastic optimisation problems. This is in contrast to the discrete-time setting, where there is no loss of generality in choosing the L^p norm in Definition 2.3, since all norms are equivalent on \mathbb{R}^n .

We consider the problem (BCOT) between the laws of stochastic processes that are described by SDEs. To the best of our knowledge, for such continuous-time problems, Backhoff-Veraguas et al. [5], Bion-Nadal and Talay [10] were the first to find an explicit optimiser, the so-called *synchronous coupling*, which we define after the following preparations.

Definition 2.13. For an adapted process $\rho = (\rho_t)_{t \in [0, T]}$ taking values in $[-1, 1]$, we call (W, \bar{W}) a ρ -correlated Brownian motion if W, \bar{W} are each one-dimensional Brownian motions and, for each $t \in [0, T]$, their correlation is given by $\text{corr}(W_t, \bar{W}_t) = \rho_t$.

Let $b, \bar{b}: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma, \bar{\sigma}: [0, T] \times \mathbb{R} \rightarrow [0, \infty)$ be measurable functions, and let $x_0 \in \mathbb{R}$. Suppose that there exists a unique strong solution (X, \bar{X}) of the system of SDEs

$$(2.6) \quad \begin{cases} dX_t = b_t(X_t) dt + \sigma_t(X_t) dW_t, & X_0 = x_0, \\ d\bar{X}_t = \bar{b}_t(\bar{X}_t) dt + \bar{\sigma}_t(\bar{X}_t) d\bar{W}_t, & \bar{X}_0 = x_0, \end{cases}$$

on the time interval $[0, T]$, where (W, \bar{W}) is a ρ -correlated Brownian motion, and write μ, ν for the laws of X, \bar{X} , respectively. It is shown in [7, Proposition 2.2] that the set of bicausal couplings $\text{Cpl}_{bc}(\mu, \nu)$ is equal to the set of all couplings of the form $\pi = \text{Law}(X, \bar{X})$, where (2.6) is driven by some correlated Brownian motion. In particular, the synchronous coupling is defined as follows.

Definition 2.14. Let (X, \bar{X}) be the unique strong solution of (2.6) with $W = \bar{W}$. Then the *synchronous coupling* is defined as $\pi_{\mu, \nu}^{\text{sync}} := \text{Law}(X, \bar{X})$, the joint law of the solutions when the SDEs are driven by a common Brownian motion.

2.3. Numerical methods for SDEs. In order to solve SDEs of the form (1.1) and (2.6), one usually has to resort to numerical methods. The simplest of these is the *Euler–Maruyama scheme*.

Let $N \in \mathbb{N}$, $h = T/N$, and set $X_0^{h,\text{EM}} = x_0$. Then the Euler–Maruyama scheme $(X_{kh}^{h,\text{EM}})_{k \in \{0, \dots, N\}}$ is defined for all $k \in \{0, \dots, N-1\}$ by

$$(2.7) \quad X_{(k+1)h}^{h,\text{EM}} = X_{kh}^{h,\text{EM}} + b_{kh}(X_{kh}^{h,\text{EM}})h + \sigma_{kh}(X_{kh}^{h,\text{EM}})\Delta W_{k+1},$$

where $\Delta W_{k+1} := W_{(k+1)h} - W_{kh} \sim \mathcal{N}(0, h)$. Under global Lipschitz conditions on the coefficients, this scheme converges in the strong L^p -sense to the true solution of the SDE; see, for example, Kloeden and Platen [37]. In recent years, numerical methods for SDEs with irregular (non-globally Lipschitz continuous) coefficients have been studied intensively. We refer to Remarks 4.4 and 5.2 and Examples 7.3 and 7.4 for references to relevant results.

The *monotone Euler–Maruyama scheme* is introduced in [7, Definition 3.11]. This scheme is defined analogously to (2.7) with the Brownian increments ΔW_{k+1} replaced by *truncated Brownian increments* ΔW_{k+1}^h , which are defined as follows.

Definition 2.15 (truncated Brownian motion; cf. [48, §2.1]). Let $(W_t)_{t \in [0, T]}$ be a standard Brownian motion, let $N \in \mathbb{N}$, $h = \frac{T}{N}$, $A_h = 4\sqrt{-h \log h}$, and for each $k \in \{0, \dots, N-1\}$, define the stopping time $\tau_k^h := \inf\{t \geq kh : |W_t - W_{kh}| > A_h\}$. The *truncated Brownian motion* $(W_t^h)_{t \in [0, T]}$ is defined by $W_0^h = 0$ and $W_t^h := W_{kh}^h + W_{t \wedge \tau_k^h} - W_{kh}$, for $t \in (kh, (k+1)h]$, $k \in \{0, \dots, N-1\}$. The *truncated Brownian increments* are defined by $\Delta W_{k+1}^h := W_{(k+1)h}^h - W_{kh}^h = W_{(k+1)h \wedge \tau_k^h} - W_{kh}$, $k \in \{0, \dots, N-1\}$.

If the coefficients b and σ of the SDE (1.1) are time-homogeneous and globally Lipschitz continuous, [7, Lemma 3.15] shows that, in contrast to (2.7), the monotone Euler–Maruyama scheme is stochastically increasing for sufficiently small step size, and [7, Lemma 3.17] proves strong L^p convergence of the scheme. Combined with a stability argument, these properties are key to the proof of optimality of the synchronous coupling in [7, Proposition 3.24].

3. PRELIMINARY RESULTS

We present two general stability results for bicausal optimal transport, which we will use in our optimality proofs. The first allows us to pass from discrete to continuous time, and the second allows us to approximate the coefficients of SDEs. As an immediate consequence, we obtain a first generalisation of [7, Theorem 1.3], which we will later apply in combination with a transformation of SDEs in order to prove Theorem 5.8. The proofs of the results in this preliminary section are presented in Appendix A.

Assumption 3.1. For each $N \in \mathbb{N}$, $h = T/N$, suppose that there exist measurable maps $\phi^h, \bar{\phi}^h : \{0, h, \dots, T\} \times C([0, h], \mathbb{R}) \times \mathbb{R}$ and stochastic processes $X^h = (X_{kh}^h)_{k \in \{0, \dots, N\}}$, $\bar{X}^h = (\bar{X}_{kh}^h)_{k \in \{0, \dots, N\}}$ such that, for any correlated Brownian motion (W, \bar{W}) , and each $k \in \{0, \dots, N-1\}$,

$$X_{(k+1)h}^h = \phi^h(kh, X_{kh}, (W_{kh+t} - W_{kh})_{t \in [0, h]}), \quad \bar{X}_{(k+1)h}^h = \bar{\phi}^h(kh, \bar{X}_{kh}, (\bar{W}_{kh+t} - \bar{W}_{kh})_{t \in [0, h]}).$$

Suppose further that

- (i) the processes $(X_{kh}^h)_{k \in \{1, \dots, N\}}$ and $(\bar{X}_{kh}^h)_{k \in \{1, \dots, N\}}$ are stochastically co-monotone (see Definition 2.6);
- (ii) for each $k \in \{1, \dots, N\}$, X_{kh}^h and \bar{X}_{kh}^h are co-monotone functions of some common random variable when $\bar{W} = W$;
- (iii) for some $p \geq 1$, the L^p -convergence $\lim_{h \rightarrow 0} \mathbb{E} \left[\sum_{k=1}^N \int_{(k-1)h}^{kh} |X_{kh}^h - X_s|^p ds \right] = 0$ and $\lim_{h \rightarrow 0} \mathbb{E} \left[\sum_{k=1}^N \int_{(k-1)h}^{kh} |\bar{X}_{kh}^h - \bar{X}_s|^p ds \right] = 0$ holds.

Proposition 3.2. *Suppose that there exists a unique strong solution (X, \bar{X}) to the system of SDEs (2.6) with laws $\mu = \text{Law}(X)$, $\nu = \text{Law}(\bar{X})$. For some $p \geq 1$, suppose that there exist X^h, \bar{X}^h satisfying Assumption 3.1. Let $c : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function for which there*

exists $K \geq 0$ such that, for each fixed $t \in [0, T]$, $c_t(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies (2.3) and (2.4). Then

$$(3.1) \quad \lim_{h \rightarrow 0} \inf_{\pi^h \in \text{CPl}_{bc}(\mu^h, \nu^h)} \mathbb{E}^{\pi^h} \left[h \sum_{k=1}^N c_k(x_k, y_k) \right] = \inf_{\pi \in \text{CPl}_{bc}(\mu, \nu)} \mathbb{E}^{\pi} \left[\int_0^T c_t(\omega_t, \bar{\omega}_t) dt \right],$$

and the synchronous coupling attains the infimum on the right hand side.

Lemma 3.3. *Let $b, \bar{b}, b^N, \bar{b}^N: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma, \bar{\sigma}, \sigma^N, \bar{\sigma}^N: [0, T] \times \mathbb{R} \rightarrow [0, \infty)$ be measurable functions with uniform linear growth in space. For each $t \in [0, T]$, suppose that $b_t, \bar{b}_t, \sigma_t, \bar{\sigma}_t$ are continuous in space and that the following convergence holds uniformly on compact sets as $N \rightarrow \infty$:*

$$b_t^N \rightarrow b_t, \bar{b}_t^N \rightarrow \bar{b}_t, \sigma_t^N \rightarrow \sigma_t, \bar{\sigma}_t^N \rightarrow \bar{\sigma}_t.$$

Suppose moreover that, for coefficients (b, σ) , $(\bar{b}, \bar{\sigma})$, there exist unique strong solutions (X, \bar{X}) of the SDEs (2.6), and that for coefficients (b^N, σ^N) , $(\bar{b}^N, \bar{\sigma}^N)$, $N \in \mathbb{N}$, there exist unique strong solutions (X^N, \bar{X}^N) of the SDEs (2.6), for each $N \in \mathbb{N}$. Write $\mu = \text{Law}(X)$, $\nu = \text{Law}(\bar{X})$ and $\mu^N = \text{Law}(X^N)$, $\nu^N = \text{Law}(\bar{X}^N)$.

Let $c: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function for which there exists $K \geq 0$ and $p \geq 1$ such that, for each fixed $t \in [0, T]$, $c_t(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies (2.3). Suppose that, for each $N \in \mathbb{N}$, $\pi_{\mu^N, \nu^N}^{\text{sync}}$ attains the infimum

$$\inf_{\pi^N \in \text{CPl}_{bc}(\mu^N, \nu^N)} \mathbb{E}^{\pi^N} \left[\int_0^T c_t(\omega_t, \bar{\omega}_t) dt \right].$$

Then

$$\lim_{N \rightarrow \infty} \inf_{\pi^N \in \text{CPl}_{bc}(\mu^N, \nu^N)} \mathbb{E}^{\pi^N} \left[\int_0^T c_t(\omega_t, \bar{\omega}_t) dt \right] = \inf_{\pi \in \text{CPl}_{bc}(\mu, \nu)} \mathbb{E}^{\pi} \left[\int_0^T c_t(\omega_t, \bar{\omega}_t) dt \right],$$

and $\pi_{\mu, \nu}^{\text{sync}}$ attains the infimum on the right-hand side.

Under the following assumption on both (b, σ) and $(\bar{b}, \bar{\sigma})$, the monotone Euler–Maruyama scheme satisfies Assumption 3.1 and we can apply Proposition 3.2.

Assumption 3.4. Suppose that (b, σ) satisfy the following:

- (1) $t \mapsto b_t(x), t \mapsto \sigma_t(x)$ are Lipschitz uniformly in $x \in \mathbb{R}$;
- (2) $x \mapsto b_t(x), x \mapsto \sigma_t(x)$ are Lipschitz uniformly in $t \in [0, T]$.

Proposition 3.5. *Suppose that the coefficients (b, σ) and $(\bar{b}, \bar{\sigma})$ of (2.6) satisfy Assumption 3.4 and write μ, ν for the laws of the respective strong solutions X, \bar{X} . Let $c: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function for which there exists $K \geq 0$ and $p \geq 1$ such that, for each fixed $t \in [0, T]$, $c_t(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies (2.3) and (2.4). Then the synchronous coupling attains the value $V_c(\mu, \nu)$ of the bicausal optimal transport problem (BCOT).*

Combining Proposition 3.5 with the stability result Lemma 3.3 leads to the following generalisation of [7, Theorem 1.3, Remark 4.12] to the case of both time-dependent coefficients and cost functional.

Assumption 3.6. Suppose that (b, σ) satisfy the following:

- (1) b, σ are continuous in t and x ;
- (2) b, σ have linear growth;
- (3) pathwise uniqueness holds for the SDE (1.1) with coefficients (b, σ) .

Corollary 3.7. *Suppose that the coefficients (b, σ) and $(\bar{b}, \bar{\sigma})$ of (2.6) satisfy Assumption 3.6. Then there exist unique strong solutions X, \bar{X} of the SDEs (2.6), and we let μ, ν denote their*

laws. Further, let $c: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function for which there exists $K \geq 0$ and $p \geq 1$ such that, for each fixed $t \in [0, T]$, $c_t(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies (2.3) and (2.4).. Then the synchronous coupling attains the value $V_c(\mu, \nu)$ of the bicausal optimal transport problem (BCOT).

4. DISCONTINUOUS DRIFT WITH EXPONENTIAL GROWTH

Consider the SDE (1.1) with time-homogeneous coefficients. We allow the drift coefficient b to exhibit both *discontinuities* and *exponential growth*. We will impose only mild piecewise continuity assumptions on b , and we will assume that the diffusion coefficient σ is globally Lipschitz continuous and that σ is non-zero only at the points of discontinuity of b . In this setting we will prove:

- existence and uniqueness of strong solutions of (1.1);
- strong L^p -convergence of a transformed semi-implicit Euler–Maruyama scheme;
- optimality of the synchronous coupling between the laws of two such SDEs for the problem (BCOT).

To the best of our knowledge, this is the first work to treat SDEs with both discontinuous and exponentially growing drift. Precisely, we make the following assumptions on the coefficients (b, σ) .

Assumption 4.1. Let (b, σ) be time-homogeneous; i.e. $b_t \equiv b: \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma_t \equiv \sigma: \mathbb{R} \rightarrow [0, \infty)$, for all $t \in [0, T]$. Suppose that there exists $m \in \mathbb{N} \cup \{0\}$ and $\xi_1, \dots, \xi_m \in \mathbb{R}$, with $-\infty = \xi_0 < \xi_1 < \xi_1 < \dots, \xi_m < \xi_{m+1} = \infty$, and there exist constants $L_b, K_b, L_\sigma, \gamma, \eta \in (0, \infty)$ such that (b, σ) satisfy the following:

- (A1) on each interval (ξ_k, ξ_{k+1}) , $k \in \{0, \dots, m\}$,
 - (i) b is absolutely continuous,
 - (ii) b is one-sided Lipschitz; i.e. $(x - y)(b(x) - b(y)) \leq L_b|x - y|^2$, for all $x, y \in (\xi_k, \xi_{k+1})$,
 - (iii) b is locally Lipschitz with exponential growth; i.e. $|b(x) - b(y)| \leq K_b(\exp\{\gamma|x|^\eta\} + \exp\{\gamma|y|^\eta\})|x - y|$ for all $x, y \in (\xi_k, \xi_{k+1})$;
- (A2) σ is globally Lipschitz; i.e. $|\sigma(x) - \sigma(y)| \leq L_\sigma|x - y|$ for all $x, y \in \mathbb{R}$;
- (A3) $\sigma(\xi_k) \neq 0$, for all $k \in \{1, \dots, m\}$.

Remark 4.2. We note that we do not impose any uniform non-degeneracy condition on the diffusion coefficient σ . This is in contrast to the setting of Dareiotis, Gerencsér, and Lê [18], for example.

Example 4.3. Consider the introductory example from [32]:

$$(4.1) \quad dX_t = -X_t^3 + dW_t, \quad X_0 = x_0 \in \mathbb{R},$$

and a modification of the example in [75, Section 2]:

$$(4.2) \quad dX_t = \left(\frac{1}{2} - 2\text{sign}(X_t - 1) \right) dt + |X_t| dW_t, \quad X_0 = x_0 \in \mathbb{R}.$$

The coefficients of the SDEs (4.1) and (4.2) each satisfy Assumption 4.1.

Remark 4.4.

- (i) In the case that $m = 0$, the drift coefficient b is continuous, and there is no non-degeneracy condition on σ . In this setting, convergence in probability and almost sure convergence of the explicit Euler–Maruyama scheme were shown by Gyöngy and Krylov [25] and Gyöngy [24], respectively. However, Hutzenthaler et al. [32, Theorem 1] shows that, for the SDE (4.1), this scheme has unbounded moments and diverges in the strong L^p sense. Two approaches to recover moment bounds and strong convergence have been taken in the literature.

- (a) Tamed numerical schemes, in which polynomially growing coefficients are appropriately rescaled, have been studied, for example, by Hutzenthaler, Jentzen, and Kloeden [33], Sabanis [70]. These are explicit schemes, which are computationally efficient. However, the technique of taming does not readily yield a scheme satisfying the monotonicity property Assumption 3.1 (i) and to our knowledge has not been applied to exponentially growing coefficients.
 - (b) For a semi-implicit Euler–Maruyama scheme, Hu [30] proves L^2 -convergence of order $1/2$ at the terminal time. When b has at most polynomial growth, Higham, Mao, and Stuart [28] prove uniform-in-time L^2 -convergence at the same rate. Under additional assumptions, Hutzenthaler and Jentzen [31], Mao and Szpruch [45, 46] also study strong convergence of this scheme. However, under Assumption 4.1 with $m = 0$, we are not aware of any existing results on uniform-in-time L^p -convergence.
- (ii) Strengthening Assumption 4.1 (A1) to piecewise Lipschitz continuity, we are in the setting of Leobacher and Szölgényi [41, 42], who were the first to consider transformation-based schemes for SDEs with piecewise-Lipschitz drift. Under these assumptions, the authors proved strong well-posedness of the SDEs and strong L^p -convergence of a transformation-based Euler–Maruyama scheme. Leobacher and Szölgényi [44], Müller-Gronbach and Yaroslavtseva [50] also prove strong convergence rates for the classical Euler–Maruyama scheme. Higher-order transformation-based schemes and adaptive schemes have been introduced in Müller-Gronbach and Yaroslavtseva [51], Neuenkirch, Szölgényi, and Szpruch [55], Przybyłowicz, Schwarz, and Szölgényi [66], Yaroslavtseva [80]. Przybyłowicz and Szölgényi [63], Przybyłowicz, Szölgényi, and Xu [64] additionally allow for processes with jumps. Overviews of this line of research can be found in Leobacher and Szölgényi [43], Szölgényi [75].
- (iii) Recent results of Müller-Gronbach, Sabanis, and Yaroslavtseva [53], Spendier and Szölgényi [73], as well as Hu and Gan [29], prove existence and uniqueness of strong solutions and convergence of tamed Euler–Maruyama schemes for SDEs with drift coefficients that are both discontinuous and superlinearly growing. However, these results impose a polynomial growth condition on the drift coefficient, whereas we allow for exponential growth. We note, however, that [53] allows for both the drift and diffusion coefficient to have polynomial growth.

4.1. A transformed SDE. In order to remove the discontinuities from the drift coefficient of the SDE (1.1), we define a transformation of space $G: \mathbb{R} \rightarrow \mathbb{R}$ as in Leobacher and Szölgényi [42].

First we define $\phi: \mathbb{R} \rightarrow \mathbb{R}$ by

$$\phi(u) = \begin{cases} (1+u)^3(1-u)^3 & |u| \leq 1, \\ 0 & |u| > 1. \end{cases}$$

Then, for constants $c_0 \in (0, \infty)$ and $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{R} \setminus \{0\}$, define $\bar{\phi}_k: \mathbb{R} \rightarrow \mathbb{R}$, for each $k \in \{1, \dots, m\}$, by

$$(4.3) \quad \bar{\phi}_k(x) := \phi\left(\frac{x - \xi_k}{c_0}\right)(x - \xi_k)|x - \xi_k|, \quad \text{for } x \in \mathbb{R},$$

and

$$(4.4) \quad G(x) := x + \sum_{k=1}^m \alpha_k \bar{\phi}_k(x), \quad \text{for } x \in \mathbb{R}.$$

Before fixing the constants in the definition of G , we make the following remark.

Remark 4.5. For any $k \in \{1, \dots, m-1\}$, the drift b is absolutely continuous on the open interval (ξ_k, ξ_{k+1}) , by Assumption 4.1 (A1) (i). Thus, b has an absolutely continuous extension to the closed interval $[\xi_k, \xi_{k+1}]$ and, in particular, there exist finite one-sided limits $b(\xi_k+) := \lim_{x \searrow \xi_k} b(x)$ and $b(\xi_{k+1}-) := \lim_{x \nearrow \xi_{k+1}} b(x)$. Similarly, the one-sided limits $b(\xi_1-), b(\xi_m+)$ exist and are finite.

By Remark 4.5 and Assumption 4.1 (A3), for each $k \in \{1, \dots, m\}$ we can now define

$$(4.5) \quad \alpha_k := \frac{1}{2} \sigma(\xi_k)^{-2} (b(\xi_k-) - b(\xi_k+)),$$

and choose

$$(4.6) \quad c_0 < \min_{k \in \{1, \dots, m\}} \left\{ \frac{1}{6|\alpha_k|} \right\} \wedge \min_{k \in \{1, \dots, m-1\}} \left\{ \frac{1}{2} (\xi_{k+1} - \xi_k) \right\}.$$

We will see that this choice of constants ensures that the transformation G acts only locally at each discontinuity, is strictly increasing, and the drift of the transformed SDE is continuous. We prove the following properties of the transformation G in Appendix B.

Lemma 4.6. *For each $k \in \{1, \dots, m\}$, $\bar{\phi}_k$ defined in (4.3) with c_0 chosen according to (4.6) satisfies the following:*

- $\bar{\phi}_k(x) = \bar{\phi}'_k(x) = \bar{\phi}''_k(x) = 0$ for $|x - \xi_k| \geq c_0$,
- $\bar{\phi}_k(\xi_k) = \bar{\phi}'_k(\xi_k) = 0$, $\bar{\phi}''_k(\xi_k-) = -2$, $\bar{\phi}''_k(\xi_k+) = 2$,
- $\bar{\phi}'_k(x) \in [-6c_0, 6c_0]$, for all $x \in \mathbb{R}$,
- $\bar{\phi}_k: \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz with Lipschitz derivative and piecewise Lipschitz almost-everywhere second derivative.

Lemma 4.7 (Cf. [42, Lemma 2.2]). *The transformation G defined by (4.4), (4.5), and (4.6) satisfies the following:*

- $G = \text{id}$ on the set $\mathbb{R} \setminus \bigcup_{k \in \{1, \dots, m\}} (\xi_k - c_0, \xi_k + c_0)$,
- G is strictly increasing with strictly increasing inverse $G^{-1}: \mathbb{R} \rightarrow \mathbb{R}$,
- G, G^{-1} are bounded and Lipschitz with Lipschitz constants $L_G, L_{G^{-1}} \in (0, \infty)$,
- G, G^{-1} have bounded Lipschitz first derivatives $G', (G^{-1})'$ and bounded piecewise Lipschitz almost-everywhere second derivatives $G'', (G^{-1})''$.

Example 4.8. Suppose that $m = 1$; i.e. there is only a single exceptional point $\xi \in \mathbb{R}$ at which b is not Lipschitz. Figure 2 illustrates that G is close to linear, increasing, and has Lipschitz first derivative and Lipschitz-almost-everywhere second derivative.

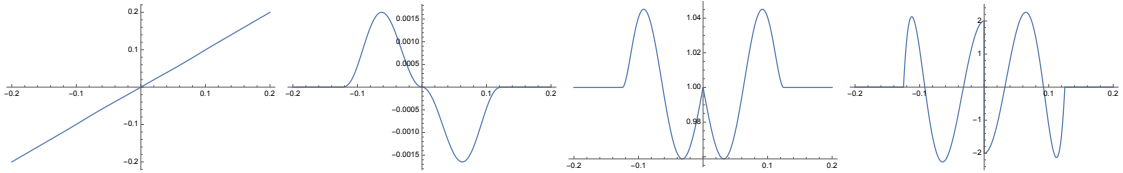


FIGURE 2. From left to right: the functions G , $G - \text{id}$, G' , and G'' , for $\xi = 0$.

Supposing that there exists a strong solution X of (1.1), we define $Z_t := G(X_t)$, for all $t \in [0, T]$. Applying Itô's formula, we find that Z solves the SDE

$$(4.7) \quad dZ_t = \tilde{b}(Z_t) dt + \tilde{\sigma}(Z_t) dW_t, \quad Z_0 = G(x_0),$$

where $\tilde{b}: \mathbb{R} \rightarrow \mathbb{R}$, $\tilde{\sigma}: \mathbb{R} \rightarrow [0, \infty)$ are defined by

$$(4.8) \quad \begin{aligned} \tilde{b} &:= (bG') \circ G^{-1} + \frac{1}{2} (\sigma^2 G'') \circ G^{-1}, \\ \tilde{\sigma} &:= (\sigma G') \circ G^{-1} \end{aligned}$$

Note that, by Lemma 4.7, $\tilde{b} = b$ and $\tilde{\sigma} = \sigma$ on the set $\mathbb{R} \setminus \bigcup_{k \in \{1, \dots, m\}} (\xi_k - c_0, \xi_k + c_0)$. That is, the transformation only affects the coefficients on a neighbourhood of each discontinuity point.

To prove regularity of the transformed coefficients, we will make use of the following two results. The first is analogous to [42, Lemma 2.5], the second to [41, Lemma 2.6]. Again we postpone the proofs to Appendix B.

Lemma 4.9. *Suppose that $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfies Assumption 4.1 (A1) and $g: \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz and bounded. Then the product $(f \cdot g): \mathbb{R} \rightarrow \mathbb{R}$ also satisfies Assumption 4.1 (A1).*

Lemma 4.10. *Suppose that a function $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfies Assumption 4.1 (A1) and f is continuous. Then f satisfies Assumption 4.1 (A1) (i)–(iii) globally on \mathbb{R} .*

We say that coefficients (b, σ) satisfy the *monotonicity condition* if there exists $\alpha, \beta > 0$ such that, for all $x \in \mathbb{R}$,

$$(4.9) \quad xb(x) \vee |\sigma(x)|^2 \leq \alpha + \beta|x|^2.$$

The proof of the following properties of the transformed coefficients is once again found in Appendix B.

Lemma 4.11. *Suppose that (b, σ) satisfy Assumption 4.1. Then there exist constants $L_{\tilde{b}}, K_{\tilde{b}}, L_{\tilde{\sigma}}, \tilde{\gamma}, \tilde{\eta} \in (0, \infty)$ such that the transformed coefficients $(\tilde{b}, \tilde{\sigma})$ defined in (4.8) satisfy the following:*

- (i) \tilde{b} is absolutely continuous;
- (ii) \tilde{b} is one-sided Lipschitz; i.e. $(x - y)(\tilde{b}(x) - \tilde{b}(y)) \leq L_{\tilde{b}}|x - y|^2$, for all $x, y \in \mathbb{R}$;
- (iii) \tilde{b} satisfies the following exponential growth and local Lipschitz condition: for all $z_0, z_1 \in \mathbb{R}$,

$$(4.10) \quad |\tilde{b}(z_0) - \tilde{b}(z_1)| \leq K_{\tilde{b}}(\exp\{\tilde{\gamma}|z_0|^{\tilde{\eta}}\} + \exp\{\tilde{\gamma}|z_1|^{\tilde{\eta}}\})|z_0 - z_1|;$$

- (iv) $\tilde{\sigma}$ is globally Lipschitz; i.e. $|\tilde{\sigma}(x) - \tilde{\sigma}(y)| \leq L_{\tilde{\sigma}}|x - y|$, for all $x, y \in \mathbb{R}$;
- (v) $(\tilde{b}, \tilde{\sigma})$ satisfy the monotonicity condition (4.9).

The following result gives well-posedness of the transformed SDE, as well as moment bounds and one-step error bounds that will be used in the convergence analysis of the numerical scheme.

Proposition 4.12. *Suppose that (b, σ) satisfy Assumption 4.1. Then the SDE (4.7) admits a unique strong solution $(Z_t)_{t \in [0, T]}$. Moreover, for any $p \geq 1$, there exist constants $C_0(p, T), C_1(p, T), C_2(p, T) > 0$ such that*

$$(4.11) \quad \mathbb{E} \left[\sup_{t \in [0, T]} |Z_t|^p \right] \leq C_0(p, T)$$

and, for all $s, t \in [0, T]$ with $s < t$,

$$(4.12) \quad \mathbb{E} \left[\sup_{r \in [s, t]} |Z_r - Z_s|^p \right] \leq C_1(p, T)(t - s)^{\frac{p}{2}}$$

and

$$(4.13) \quad \mathbb{E} \left[\sup_{r \in [s, t]} |\tilde{b}(Z_r) - \tilde{b}(Z_s)|^p \right] \leq C_2(p, T)(t - s)^{\frac{p}{2}}.$$

Proof. By Lemma 4.11, $(\tilde{b}, \tilde{\sigma})$ satisfy (4.9), \tilde{b} is one-sided Lipschitz, and $\tilde{\sigma}$ is Lipschitz. Under these conditions, the classical result of Krylov [39, Theorem 1] shows existence and uniqueness of a strong solution Z of (4.7). The monotonicity condition (4.9) also implies moment bounds (4.11) on the solution, as shown, for example, in [28, Lemma 3.2].

Given that \tilde{b} also satisfies the exponential growth condition (4.10), we can follow the proofs of Hu [30, Lemma 3.4, Theorem 3.5] to prove the bounds (4.12) and (4.13). We note that Hu [30]

assumes that the coefficients $(\tilde{b}, \tilde{\sigma})$ are continuously differentiable, but an inspection of the proofs reveals that no such assumptions are necessary; in particular, absolute continuity of \tilde{b} is sufficient for the fundamental theorem of calculus to hold. \square

We now prove our existence and uniqueness result, following the proof of [42, Theorem 2.6].

Theorem 4.13. *Suppose that the coefficients (b, σ) satisfy Assumption 4.1. Then there exists a unique strong solution X of (1.1) with coefficients (b, σ) and initial condition $X_0 = x_0 \in \mathbb{R}$. Moreover, for any $p \geq 1$, there exists a constant $C(p, T) > 0$ such that*

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t|^p \right] \leq C(p, T).$$

Proof. By Proposition 4.12, there exists a unique strong solution Z of the SDE (4.7). Now define $X_t = G^{-1}(Z_t)$, for all $t \in [0, T]$. By Lemma 4.7, G^{-1} is Lipschitz with Lipschitz constant $L_{G^{-1}}$, and by Proposition 4.12, Z satisfies the one-step error bound (4.12). Hence, for $p \geq 1$,

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t|^p \right] \leq 2^{p-1} L_{G^{-1}}^p \mathbb{E} \left[\sup_{t \in [0, T]} |Z_t - Z_0|^p \right] + 2^{p-1} |x_0|^p \leq C(p, T),$$

where $C(p, T) := 2^{p-1} L_{G^{-1}}^p C_1(p, T) T^{\frac{p}{2}} + 2^{p-1} |x_0|^p$.

By Lemma 4.7, $G^{-1}: \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable and has a second derivative almost everywhere, and so we may apply Itô's formula (see, e.g. [67, §VI Theorem 1.5 and Remarks]). Hence X satisfies the SDE (1.1) with coefficients (b, σ) , and $X_0 = G(Z_0) = x_0$. Since Z is a strong solution, and X is a deterministic transformation of Z at each time, X is also a strong solution. Conversely, for any strong solution X' of (1.1) with coefficients (b, σ) and with $X'_0 = x_0$, the process Z' defined by taking $Z'_t = G(X'_t)$ for $t \in [0, T]$ is a strong solution of (4.7). By pathwise uniqueness, $Z'_t = Z_t$ for all $t \in [0, T]$ almost surely. Thus, $X'_t = G^{-1}(Z'_t) = G^{-1}(Z_t) = X_t$ for all $t \in [0, T]$ almost surely. We conclude that strong existence and pathwise uniqueness hold for (1.1) with coefficients (b, σ) . \square

4.2. Transformed semi-implicit Euler–Maruyama scheme. We now define a numerical scheme via the transformation introduced in the previous section.

Definition 4.14 (transformed (monotone) semi-implicit Euler–Maruyama scheme). Fix $N \in \mathbb{N}$, $h = T/N$. For the SDE (1.1) with coefficients (b, σ) satisfying Assumption 4.1, let G be defined as in (4.4) and $(\tilde{b}, \tilde{\sigma})$ as in (4.8).

Define the (discrete time) *transformed monotone semi-implicit Euler–Maruyama scheme* X^h by $X_0^h = x_0$ and for $k \in \{0, \dots, N-1\}$,

$$(4.14) \quad X_{(k+1)h}^h := G^{-1} \left(G(X_{kh}^h) + (\tilde{b} \circ G)(X_{(k+1)h}^h) \cdot h + (\tilde{\sigma} \circ G)(X_k^h) \Delta W_{k+1}^h \right),$$

where the ΔW_k^h are the truncated Brownian increments from Definition 2.15.

We may define the (discrete time) *transformed semi-implicit Euler–Maruyama scheme* \tilde{X}^h analogously, replacing the truncated increments with the usual Brownian increments ΔW_k .

Remark 4.15.

- (i) Similarly to the monotone Euler–Maruyama scheme discussed in Section 2.3, the term *monotone* in Definition 4.14 is motivated by the fact that the scheme will be shown to be stochastically monotone under certain conditions; see Lemma 4.22.
- (ii) If b is continuous, then the transformation G is the identity and \tilde{X}^h is the classical semi-implicit (or backward) Euler–Maruyama scheme, as studied for example in [28, 30, 45, 46].

For the scheme defined in Definition 4.14, we will prove moment bounds and strong L^p -convergence for $p \geq 1$. In particular, for the semi-implicit Euler–Maruyama scheme, we generalise the moment

bounds from Higham et al. [28] and the L^2 convergence from Hu [30]. We note that [28, 30] also assume continuous differentiability of b and σ , but [30] uses this condition only for existence and uniqueness of solutions, which we have established under Assumption 4.1 in Theorem 4.13. Similarly, the result of [28] can be established without the continuous differentiability assumption in our setting.

We will repeatedly make use of the following lemma.

Lemma 4.16. *Suppose that b is one-sided Lipschitz with coefficient L_b and let $h \in (0, 1/L_b)$. Then the map $(\text{id} - hb)$ is strictly increasing and invertible with strictly increasing inverse.*

Proof. Suppose that $x, y \in \mathbb{R}$, $x < y$. By the one-sided Lipschitz condition,

$$h(x - y)(b(x) - b(y)) \leq hL_b|x - y|^2 < |x - y|^2.$$

Then, since $x - y < 0$,

$$h(b(x) - b(y)) > (x - y)^{-1}|x - y|^2 = x - y,$$

or equivalently

$$(\text{id} - hb)(x) < (\text{id} - hb)(y).$$

Hence $(\text{id} - hb)$ is strictly increasing, and thus invertible with strictly increasing inverse. \square

Lemma 4.17. *Suppose that the coefficients of (1.1) satisfy Assumption 4.1. Then for all $h < 1/L_b$, the transformed (monotone) semi-implicit Euler–Maruyama scheme (4.14) is well-defined.*

Proof. By Lemma 4.11, \tilde{b} is one-sided Lipschitz with coefficient $L_{\tilde{b}} \in (0, \infty)$. Then, for $h < 1/L_{\tilde{b}}$, the map $(\text{id} - h\tilde{b})$ is invertible by Lemma 4.16. We can rewrite (4.14) as

$$(\text{id} - h\tilde{b})(G(X_{(k+1)h}^h)) = G(X_{kh}^h) + (\tilde{\sigma} \circ G)(X_{kh}^h)\Delta W_{k+1}^h,$$

and similarly in the case of non-truncated Brownian increments. Since G is also invertible by Lemma 4.7, we may conclude. \square

Proposition 4.18. *Suppose that b is one-sided Lipschitz with coefficient L_b , and that the monotonicity condition (4.9) holds with constants α, β . Let $N > 4\beta T \vee 2L_b T$ and $h = T/N$. Then, for any $p \geq 1$, there exists a constant $C(p, T) > 0$ such that*

$$(4.15) \quad \mathbb{E} \left[\sup_{k \in \{0, \dots, N\}} |X_{kh}^h|^p \right] \leq C(p, T),$$

and likewise for \tilde{X}^h .

Proof. Supposing that (4.15) holds for $p = 2$, it also holds for $p \in [1, 2)$ with $C(p, T) = C(2, T)^{\frac{p}{2}}$, by Jensen's inequality. Now let $p \geq 2$. For $k \in \{0, \dots, N\}$, define $Z_{kh}^h := G(X_{kh}^h)$. By Lemma 4.7, G^{-1} is Lipschitz with constant $L_{G^{-1}}$, and so

$$(4.16) \quad \mathbb{E} \left[\sup_{k \in \{0, \dots, N\}} |X_{kh}^h|^p \right] \leq 2^{2(p-1)} L_{G^{-1}}^p \mathbb{E} \left[\sup_{k \in \{0, \dots, N\}} |Z_{kh}^h|^p \right] + 2^{2(p-1)} L_{G^{-1}}^p |G(x_0)|^p + 2^{p-1} |x_0|^p,$$

and so we aim to bound the moments of $\sup_{k \in \{0, \dots, N\}} |Z_{kh}^h|$.

Define $K := (1 - 2\beta h)^{-1} \leq 2$. Then, as in [46, Lemma 3.2], the monotonicity condition (4.9) implies that, for any $x \in \mathbb{R}$,

$$(4.17) \quad |x|^2 \leq K(|(\text{id} - hb)(x)|^2 + 2\alpha h).$$

Therefore it suffices to bound the moments of $\sup_{k \in \{0, \dots, N\}} |Y_{kh}^h|$, where $Y_{kh}^h := (\text{id} - hb)(Z_{kh}^h)$, for $k \in \{0, \dots, N\}$. Observe that, for any $k \in \{0, \dots, N - 1\}$,

$$(4.18) \quad Y_{(k+1)h}^h = Z_{kh}^h + \tilde{\sigma}(Z_{kh}^h)\Delta W_{k+1}^h,$$

with

$$Z_{kh}^h = Y_{kh}^h + h\tilde{b}(Z_{kh}^h),$$

and so $(Y_{kh}^h)_{k \in \{0, \dots, N\}}$ coincides with the so-called *split-step backward Euler method* described in [28, Section 3.3], started from $Y_0^h = x_0 - h\tilde{b}(x_0)$ and with the increments $(\Delta W_{k+1})_{k \in \{0, \dots, N-1\}}$ replaced by $(\Delta W_{k+1}^h)_{k \in \{0, \dots, N-1\}}$. We now closely follow the proof of [28, Lemma 3.7] to obtain moment bounds. To ensure finite moments, for $R > 0$, define the stopping time

$$\theta_R := \inf\{k \in \{0, \dots, N\} : |Y_{kh}^h| > R\},$$

with the convention that $\inf \emptyset := \infty$. For $k \in \{0, \dots, \theta_R - 1\}$, we have the bound $|Y_{kh}^h| \leq R$. To bound the moments of $|Y_{\theta_R h}^h|$ we first note that, by (4.17),

$$|Z_{(\theta_R-1)h}^h|^p \leq 2^{\frac{p}{2}-1} K^{\frac{p}{2}} (|Y_{(\theta_R-1)h}^h|^p + (2\alpha h)^{\frac{p}{2}}) \leq 2^{\frac{p}{2}-1} K^{\frac{p}{2}} (R^p + (2\alpha)^{\frac{p}{2}}) =: C(p, R).$$

Then, combining (4.18) with Definition 2.15 and the monotonicity condition (4.9) proved in Lemma 4.11, we have

$$\begin{aligned} \mathbb{E}[|Y_{\theta_R h}^h|^p] &\leq 2^{p-1} \mathbb{E}[|Z_{(\theta_R-1)h}^h|^p] + 2^{p-1} A_h^p \mathbb{E}[|\tilde{\sigma}(Z_{(\theta_R-1)h}^h)|^p] \\ &\leq 2^{p-1} \mathbb{E}[|Z_{(\theta_R-1)h}^h|^p] + 2^{p-1} 2^{\frac{p}{2}-1} A_h^p (\alpha^{\frac{p}{2}} + \beta^{\frac{p}{2}} \mathbb{E}[|Z_{(\theta_R-1)h}^h|^p]) \\ &\leq 2^{p-1} C(p, R)^p (1 + 2^{\frac{p}{2}-1} \beta^{\frac{p}{2}} A_h^p) + 2^{\frac{3p}{2}-2} \alpha^{\frac{p}{2}} A_h^p =: \tilde{C}(p, R). \end{aligned}$$

Hence

$$(4.19) \quad \sup_{k \in \{0, \dots, N\}} \mathbb{E}[|Y_{(k \wedge \theta_R)h}^h|^p] \leq \max\{R^p, \tilde{C}(p, R)\} < \infty.$$

Now, combining (4.17) with (4.18) gives

$$\begin{aligned} |Y_{(k+1)h}^h|^2 &= |Z_{kh}^h|^2 + 2Z_{kh}^h \tilde{\sigma}(Z_{kh}^h) \Delta W_{k+1}^h + |\tilde{\sigma}(Z_{kh}^h) \Delta W_{k+1}^h|^2 \\ &\leq K |Y_{kh}^h|^2 + 2\alpha h K + 2Z_{kh}^h \tilde{\sigma}(Z_{kh}^h) \Delta W_{k+1}^h + |\tilde{\sigma}(Z_{kh}^h) \Delta W_{k+1}^h|^2, \end{aligned}$$

for any $k \in \{0, \dots, N-1\}$. For all $\ell \in \{1, \dots, N\}$, noting that $K = 1 + 2\beta h K$, and summing the above expressions, we have

$$\begin{aligned} |Y_{(\ell \wedge \theta_R)h}^h|^2 &\leq |Y_0^h|^2 + 2\beta h K \sum_{k=0}^{\ell \wedge \theta_R - 1} |Y_{kh}^h|^2 + 2\alpha T K + 2 \sum_{k=0}^{\ell \wedge \theta_R - 1} Z_{kh}^h \tilde{\sigma}(Z_{kh}^h) \Delta W_{k+1}^h \\ &\quad + \sum_{k=0}^{\ell \wedge \theta_R - 1} |\tilde{\sigma}(Z_{kh}^h) \Delta W_{k+1}^h|^2. \end{aligned}$$

We now raise each side to the power $p/2$ and use the inequality $(a_1 + a_2 + a_3 + a_4)^{\frac{p}{2}} \leq 4^{\frac{p}{2}-1} (|a_1|^{\frac{p}{2}} + |a_2|^{\frac{p}{2}} + |a_3|^{\frac{p}{2}} + |a_4|^{\frac{p}{2}})$, for any $a_1, a_2, a_3, a_4 \in \mathbb{R}$. Setting $c_0(p, T) := 4^{\frac{p}{2}-1} (|x_0 - h\tilde{b}(x_0)|^2 + 2\alpha T K)^{\frac{p}{2}}$, we have

$$(4.20) \quad \begin{aligned} |Y_{(\ell \wedge \theta_R)h}^h|^p &\leq c_0(p, T) + 2^{\frac{3p}{2}-2} \left(\beta h K \sum_{k=0}^{\ell \wedge \theta_R - 1} |Y_{kh}^h|^2 \right)^{\frac{p}{2}} + 2^{\frac{3p}{2}-2} \left| \sum_{k=0}^{\ell \wedge \theta_R - 1} Z_{kh}^h \tilde{\sigma}(Z_{kh}^h) \Delta W_{k+1}^h \right|^{\frac{p}{2}} \\ &\quad + 2^{p-2} \left(\sum_{k=0}^{\ell \wedge \theta_R - 1} |\tilde{\sigma}(Z_{kh}^h) \Delta W_{k+1}^h|^2 \right)^{\frac{p}{2}}. \end{aligned}$$

For all $M \in \{1, \dots, N\}$, we seek a bound for $\mathbb{E}[\sup_{\ell \in \{0, \dots, M\}} |Y_{(\ell \wedge \theta_R)h}^h|^p]$. We will treat each term in the sum (4.20) in turn. First, by Hölder's inequality with $q > 1$ satisfying $\frac{1}{q} + \frac{2}{p} = 1$,

$$(4.21) \quad 2^{\frac{3p}{2}-2} \left(\beta h K \sum_{k=0}^{\ell \wedge \theta_R - 1} |Y_{kh}^h|^2 \right)^{\frac{p}{2}} \leq c_1(p, T) (h^{\frac{q-1}{q}})^{\frac{p}{2}} \sum_{k=0}^{\ell \wedge \theta_R - 1} |Y_{kh}^h|^p \leq c_1(p, T) h \sum_{k=0}^{\ell-1} |Y_{(k \wedge \theta_R)h}^h|^p,$$

where $c_1(p, T) := 2^{\frac{3p}{2}-2} (\beta K)^{\frac{p}{2}} T^{\frac{p}{2}-1}$.

To bound the second term in (4.20), we will apply the Burkholder–Davis–Gundy inequality, noting that the quadratic variation $\langle W^h \rangle$ of the truncated Brownian motion W^h satisfies the bound $\langle W^h \rangle_t \leq \langle W \rangle_t = t$, for all $t \geq 0$. Thus there exists a constant $c_2(p, T) > 0$ such that

$$\begin{aligned} \mathbb{E} \left[\sup_{\ell \in \{0, \dots, M\}} \left| \sum_{k=0}^{\ell \wedge \theta_R - 1} Z_{kh}^h \tilde{\sigma}(Z_{kh}^h) \Delta W_{k+1}^h \right|^{\frac{p}{2}} \right] &= \mathbb{E} \left[\sup_{\ell \in \{0, \dots, M\}} \left| \sum_{k=0}^{\ell \wedge \theta_R - 1} \int_{kh}^{(k+1)h} Z_{kh}^h \tilde{\sigma}(Z_{kh}^h) dW_s^h \right|^{\frac{p}{2}} \right] \\ &\leq c_2(p, T) \mathbb{E} \left[\left(\sum_{k=0}^{M \wedge \theta_R - 1} h |Z_{kh}^h \tilde{\sigma}(Z_{kh}^h)|^2 \right)^{\frac{p}{4}} \right]. \end{aligned}$$

Then, by Hölder's inequality with $q > 1$ satisfying $\frac{1}{q} + \frac{4}{p} = 1$,

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{k=0}^{M \wedge \theta_R - 1} h |Z_{kh}^h \tilde{\sigma}(Z_{kh}^h)|^2 \right)^{\frac{p}{4}} \right] &\leq T^{\frac{p}{4} - 1} (h^{\frac{q-1}{4}})^{\frac{p}{4}} \sum_{k=0}^{M \wedge \theta_R - 1} \mathbb{E} \left[|Z_{kh}^h \tilde{\sigma}(Z_{kh}^h)|^{\frac{p}{2}} \right] \\ &\leq T^{\frac{p}{4} - 1} h \sum_{k=0}^{M-1} \mathbb{E} \left[|Z_{(k \wedge \theta_R)h}^h \tilde{\sigma}(Z_{(k \wedge \theta_R)h}^h)|^{\frac{p}{2}} \right]. \end{aligned}$$

Using the monotonicity condition (4.9) and the inequality $a \leq \frac{1}{2}(1 + a^2)$ for any $a \in \mathbb{R}$, we find a constant $c_3(p, T) > 0$ such that

$$\begin{aligned} |Z_{(k \wedge \theta_R)h}^h \tilde{\sigma}(Z_{(k \wedge \theta_R)h}^h)|^{\frac{p}{2}} &\leq |Z_{(k \wedge \theta_R)h}^h|^{\frac{p}{2}} (\alpha + \beta |Z_{(k \wedge \theta_R)h}^h|^2)^{\frac{p}{4}} \\ &\leq c_3(p, T) |Z_{(k \wedge \theta_R)h}^h|^{\frac{p}{2}} (\alpha^{\frac{p}{4}} + \beta^{\frac{p}{4}} |Z_{(k \wedge \theta_R)h}^h|^{\frac{p}{2}}) \\ &\leq c_3(p, T) \left[\frac{1}{2} + \left(\frac{\alpha^{\frac{p}{2}}}{2} + \beta^{\frac{p}{4}} \right) |Z_{(k \wedge \theta_R)h}^h|^p \right], \end{aligned}$$

for all $k \in \{0, \dots, N\}$. Combining the previous estimates and setting $c_4(p, T) := 2^{-1} T^{\frac{p}{4}} c_2(p, T) c_3(p, T)$ and $c_5(p, T) := (2^{-1} \alpha^{\frac{p}{2}} + \beta^{\frac{p}{4}}) T^{\frac{p}{4} - 1} c_2(p, T) c_3(p, T)$, we arrive at

$$(4.22) \quad \mathbb{E} \left[\sup_{\ell \in \{0, \dots, M\}} \left| \sum_{k=0}^{\ell \wedge \theta_R - 1} Z_{kh}^h \tilde{\sigma}(Z_{kh}^h) \Delta W_{k+1}^h \right|^{\frac{p}{2}} \right] \leq c_4(p, T) + c_5(p, T) h \sum_{k=0}^{M-1} \mathbb{E} [|Z_{(k \wedge \theta_R)h}^h|^p].$$

For the final term in (4.20), note that

$$(4.23) \quad \begin{aligned} \mathbb{E} \left[\sup_{\ell \in \{0, \dots, M\}} \left(\sum_{k=0}^{\ell \wedge \theta_R - 1} |\tilde{\sigma}(Z_{kh}^h) \Delta W_{k+1}^h|^2 \right)^{\frac{p}{2}} \right] &\leq N^{\frac{p}{2} - 1} \mathbb{E} \left[\sup_{\ell \in \{0, \dots, M\}} \sum_{k=0}^{\ell \wedge \theta_R - 1} |\tilde{\sigma}(Z_{kh}^h) \Delta W_{k+1}^h|^p \right] \\ &= N^{\frac{p}{2} - 1} \sum_{k=0}^{M \wedge \theta_R - 1} \mathbb{E} [|\tilde{\sigma}(Z_{kh}^h)|^p |\Delta W_{k+1}^h|^p]. \end{aligned}$$

For each $k \in \{0, \dots, N-1\}$, by independence of increments, we have $\mathbb{E} [|\tilde{\sigma}(Z_{kh}^h)|^p |\Delta W_{k+1}^h|^p] = \mathbb{E} [|\tilde{\sigma}(Z_{kh}^h)|^p] \mathbb{E} [|\Delta W_{k+1}^h|^p]$. The monotonicity condition (4.9) implies that $\mathbb{E} [|\tilde{\sigma}(Z_{kh}^h)|^p] \leq c_6(p, T) (\alpha^{\frac{p}{2}} + \beta^{\frac{p}{2}} \mathbb{E} [|Z_{kh}^h|^p])$, for some constant $c_6(p, T) > 0$. By definition of the truncated Brownian motion W^h , there exists a constant $c_7(p, T) > 0$ such that $\mathbb{E} [|\Delta W_{k+1}^h|^p] \leq \mathbb{E} [|\Delta W_{k+1}|^p] \leq c_7(p, T) h^{\frac{p}{2}}$. Hence, by (4.23), we get

$$(4.24) \quad \mathbb{E} \left[\sup_{\ell \in \{0, \dots, M\}} \left(\sum_{k=0}^{\ell \wedge \theta_R - 1} |\tilde{\sigma}(Z_{kh}^h) \Delta W_{k+1}^h|^2 \right)^{\frac{p}{2}} \right] \leq c_8(p, T) + c_9(p, T) h \sum_{k=0}^{M-1} \mathbb{E} [|Z_{(k \wedge \theta_R)h}^h|^p],$$

where $c_8(p, T) := c_6(p, T) c_7(p, T) \alpha^{\frac{p}{2}} T^{\frac{p}{2}}$ and $c_9(p, T) := c_6(p, T) c_7(p, T) \beta^{\frac{p}{2}} T^{\frac{p}{2} - 1}$.

Combining (4.17) with (4.22) and (4.24) gives

$$\begin{aligned} \mathbb{E} \left[\sup_{\ell \in \{0, \dots, M\}} \left| \sum_{k=0}^{\ell \wedge \theta_R - 1} Z_{kh}^h \tilde{\sigma}(Z_{kh}^h) \Delta W_{k+1}^h \right|^{\frac{p}{2}} \right] &\leq \tilde{c}_4(p, T) + \tilde{c}_5(p, T) h \sum_{k=0}^{M-1} \mathbb{E} [|Y_{(k \wedge \theta_R)h}^h|^p], \\ \mathbb{E} \left[\sup_{\ell \in \{0, \dots, M\}} \left(\sum_{k=0}^{\ell \wedge \theta_R - 1} |\tilde{\sigma}(Z_{kh}^h) \Delta W_{k+1}^h|^2 \right)^{\frac{p}{2}} \right] &\leq \tilde{c}_8(p, T) + \tilde{c}_9(p, T) h \sum_{k=0}^{M-1} \mathbb{E} [|Y_{(k \wedge \theta_R)h}^h|^p], \end{aligned}$$

for constants $\tilde{c}_4(p, T) = c_4(p, T) + 2^{\frac{p}{2}-1}(2\alpha h K)^{\frac{p}{2}} T c_5(p, T)$, $\tilde{c}_5(p, T) = 2^{\frac{p}{2}-1} K^{\frac{p}{2}} c_5(p, T)$, $\tilde{c}_8(p, T) = c_8(p, T) + 2^{\frac{p}{2}-1}(2\alpha h K)^{\frac{p}{2}} T c_9(p, T)$, $\tilde{c}_9(p, T) = 2^{\frac{p}{2}-1} K^{\frac{p}{2}} c_9(p, T)$. Combining this with (4.20) and (4.21), we find that

$$\mathbb{E} \left[\sup_{\ell \in \{0, \dots, M\}} |Y_{(\ell \wedge \theta_R)h}^h|^p \right] \leq \tilde{c}_0(p, T) + \tilde{c}_1(p, T) h \sum_{k=0}^{M-1} \mathbb{E} \left[\sup_{j \in \{0, \dots, k\}} |Y_{(j \wedge \theta_R)h}^h|^p \right],$$

where $\tilde{c}_0(p, T) = c_0(p, T) + 2^{\frac{3p}{2}-2} \tilde{c}_4(p, T) + 2^{p-2} \tilde{c}_8(p, T)$ and $\tilde{c}_1(p, T) = c_1(p, T) + 2^{\frac{3p}{2}-2} \tilde{c}_5(p, T) + 2^{p-2} \tilde{c}_9(p, T)$.

In light of (4.19), we can apply the discrete Grönwall inequality (see, e.g. [45, Lemma 3.4]) to conclude that

$$\mathbb{E} \left[\sup_{\ell \in \{0, \dots, N\}} |Y_{(\ell \wedge \theta_R)h}^h|^p \right] \leq \tilde{c}_0(p, T) e^{\tilde{c}_1(p, T)T}.$$

We can now take the limit $R \rightarrow \infty$ and apply Fatou's lemma to find

$$\mathbb{E} \left[\sup_{\ell \in \{0, \dots, N\}} |Y_{\ell h}^h|^p \right] \leq \tilde{c}_0(p, T) e^{\tilde{c}_1(p, T)T}.$$

A further application of (4.17) yields

$$\mathbb{E} \left[\sup_{\ell \in \{0, \dots, N\}} |Z_{\ell h}^h|^p \right] \leq 2^{\frac{p}{2}-1} K^{\frac{p}{2}} \left(\tilde{c}_0(p, T) e^{\tilde{c}_1(p, T)T} + (2\alpha h)^{\frac{p}{2}} \right).$$

Set $C_0(p, T) := 2^{\frac{p}{2}-1} K^{\frac{p}{2}} (\tilde{c}_0(p, T) e^{\tilde{c}_1(p, T)T} + (\frac{\alpha}{\beta})^{\frac{p}{2}})$ and $C(p, T) := 2^{2(p-1)} L_{G^{-1}}^p (C_0(p, T) + |G(x_0)|^p) + 2^{p-1} |x_0|^p$. Applying the bound (4.16) we conclude that

$$\mathbb{E} \left[\sup_{k \in \{0, \dots, N\}} |X_{kh}^h|^p \right] \leq C(p, T).$$

The result for \tilde{X}^h is proved analogously. \square

We prove the following strong convergence rates.

Theorem 4.19. *Suppose that the coefficients of (1.1) satisfy Assumption 4.1, and let X be the unique strong solution of (1.1). For any $N \in \mathbb{N}$ sufficiently large and $h = T/N$, denote by X^h the associated transformed monotone semi-implicit Euler–Maruyama scheme defined by (4.14). Then, for any $p \geq 1$ there exists $C \in (0, \infty)$ such that*

$$\sup_{k \in \{1, \dots, N\}} \mathbb{E} [|X_{kh} - X_{kh}^h|^p]^{\frac{1}{p}} \vee \mathbb{E} \left[\sum_{k=0}^{N-1} \int_{kh}^{(k+1)h} |X_t - X_{kh}^h|^p dt \right]^{\frac{1}{p}} \leq \begin{cases} Ch^{\frac{1}{2}}, & p \in [1, 2], \\ Ch^{\frac{1}{p(p-1)}}, & p \in [2, \infty). \end{cases}$$

The same estimates hold for the scheme \tilde{X}^h defined via non-truncated Brownian increments.

Remark 4.20.

- (i) To our knowledge, this is the first result on strong convergence of an implicit scheme in the presence of *both* discontinuities *and* exponential growth in the drift coefficient.
- (ii) For $p \in [1, 2]$, we obtain the strong L^p convergence rate of 1/2 that we expect. For $p > 2$, we leave the question of improving the strong L^p convergence rate for future research.
- (iii) In the case that b is continuous, our result still strengthens Hu [30, Theorem 2.4], since we relax the continuous differentiability assumption on the coefficients and consider general $p \in [1, \infty)$, not only $p = 2$.

Proof of Theorem 4.19. As in the proof of Theorem 4.13, $X_t = G^{-1}(Z_t)$, for all $t \in [0, T]$, where Z is the unique strong solution of (1.1) with coefficients $(\tilde{b}, \tilde{\sigma})$, as defined in (4.8), and $Z_0 = G(x_0)$.

Now, for $k \in \{0, \dots, N\}$, define $Z_{kh}^h := G(X_{kh}^h)$, so that

$$(4.25) \quad \begin{aligned} \mathbb{E}[|X_{kh} - X_{kh}^h|^2] &\leq L_{G^{-1}}^2 \mathbb{E}[|Z_{kh} - Z_{kh}^h|^2], \\ \mathbb{E}\left[\sum_{k=0}^{N-1} \int_{kh}^{(k+1)h} |X_t - X_{kh}^h|^2 dt\right] &\leq L_{G^{-1}}^2 \mathbb{E}\left[\sum_{k=0}^{N-1} \int_{kh}^{(k+1)h} |Z_t - Z_{kh}^h|^2 dt\right], \end{aligned}$$

using the Lipschitz property of G^{-1} proved in Lemma 4.7. Therefore it suffices to analyse the L^2 error between Z^h and Z .

We have $Z_0^h = Z_0$ and, for $k \in \{0, \dots, N-1\}$,

$$(4.26) \quad Z_{(k+1)h}^h = Z_{kh}^h + h\tilde{b}(Z_{(k+1)h}^h) + \tilde{\sigma}(Z_{kh}^h)\Delta W_{k+1}^h.$$

For any $k \in \{0, \dots, N-1\}$, we can write

$$Z_{(k+1)h} = Z_{kh} + \tilde{b}(Z_{(k+1)h})h + \tilde{\sigma}(Z_{kh}^h)\Delta W_{k+1}^h + B_{k+1} + \Sigma_{k+1} + \Sigma_{k+1}^h + \mathcal{E}_{k+1},$$

where we define the error terms

$$\begin{aligned} B_{k+1} &:= \int_{kh}^{(k+1)h} (\tilde{b}(Z_s) - \tilde{b}(Z_{(k+1)h}))ds, & \Sigma_{k+1} &:= \int_{kh}^{(k+1)h} (\tilde{\sigma}(Z_s) - \tilde{\sigma}(Z_{kh}))dW_s, \\ \Sigma_{k+1}^h &:= (\tilde{\sigma}(Z_{kh}) - \tilde{\sigma}(Z_{kh}^h))\Delta W_{k+1}^h, & \mathcal{E}_{k+1} &:= \tilde{\sigma}(Z_{kh})[\Delta W_{k+1} - \Delta W_{k+1}^h]. \end{aligned}$$

Note that, if we define $\tilde{Z}_{kh}^h := G(\tilde{X}_{kh}^h)$, for $k \in \{0, \dots, N\}$, then \tilde{Z}^h satisfies the analogue estimates to (4.25) and also satisfies (4.26) with the truncated increment ΔW_{k+1}^h replaced by ΔW_{k+1} . Thus we can write

$$Z_{(k+1)h} = Z_{kh} + \tilde{b}(Z_{(k+1)h})h + \tilde{\sigma}(\tilde{Z}_{kh}^h)\Delta W_{k+1} + B_{k+1} + \Sigma_{k+1} + \tilde{\Sigma}_{k+1},$$

with $\tilde{\Sigma}_{k+1} := (\tilde{\sigma}(Z_{kh}) - \tilde{\sigma}(\tilde{Z}_{kh}^h))\Delta W_{k+1}$. We can bound the second moment of $\tilde{\Sigma}_{k+1}$ in exactly the same way as we do for Σ_{k+1}^h below. Thus the estimates that we prove for Z^h also hold for \tilde{Z}^h with only a minor modification in the proof.

Now we consider

$$Z_{(k+1)h} - Z_{(k+1)h}^h = Z_{kh} - Z_{kh}^h + (\tilde{b}(Z_{(k+1)h}) - \tilde{b}(Z_{(k+1)h}^h))h + B_{k+1} + \Sigma_{k+1} + \Sigma_{k+1}^h + \mathcal{E}_{k+1}.$$

By Lemma 4.11, \tilde{b} is absolutely continuous and one-sided Lipschitz with constant $L_{\tilde{b}}$. Therefore the Lebesgue-almost everywhere derivative \tilde{b}' exists and satisfies $\tilde{b}' \leq L_{\tilde{b}}$. The fundamental theorem of calculus shows that for all $x, y \in \mathbb{R}$,

$$\tilde{b}(x) - \tilde{b}(y) = \int_y^x \tilde{b}'(s)ds = \left(\int_0^1 \tilde{b}'(ux + (1-u)y)du\right)(x-y).$$

Hence

$$(4.27) \quad \begin{aligned} &\left(1 - h \int_0^1 \tilde{b}'(uZ_{(k+1)h} - (1-u)Z_{(k+1)h}^h)du\right)(Z_{(k+1)h} - Z_{(k+1)h}^h) \\ &= (Z_{kh} - Z_{kh}^h) + B_{k+1} + \Sigma_{k+1} + \Sigma_{k+1}^h + \mathcal{E}_{k+1}. \end{aligned}$$

Then, similarly to [30, Proof of Theorem 2.4], since $\tilde{b}' \leq L_{\tilde{b}}$ Lebesgue-almost everywhere and $h < 1/L_{\tilde{b}}$, the inverse

$$\mathcal{I}_{k+1} := \left(1 - h \int_0^1 \tilde{b}'(uZ_{(k+1)h} - (1-u)Z_{(k+1)h}^h)du\right)^{-1}$$

exists and $|\mathcal{I}_{k+1}| \leq (1 - L_{\tilde{b}}h)^{-1}$. Hence, by (4.27),

$$Z_{(k+1)h} - Z_{(k+1)h}^h = \mathcal{I}_{k+1}(Z_{kh} - Z_{kh}^h) + \mathcal{I}_{k+1}B_{k+1} + \mathcal{I}_{k+1}\Sigma_{k+1} + \mathcal{I}_{k+1}\Sigma_{k+1}^h + \mathcal{I}_{k+1}\mathcal{E}_{k+1},$$

and iterating gives

$$(4.28) \quad \begin{aligned} Z_{(k+1)h} - Z_{(k+1)h}^h &= \sum_{j=1}^{k+1} \mathcal{I}_{k+1} \dots \mathcal{I}_j B_j + \sum_{j=1}^{k+1} \mathcal{I}_{k+1} \dots \mathcal{I}_j \Sigma_j \\ &+ \sum_{j=1}^{k+1} \mathcal{I}_{k+1} \dots \mathcal{I}_j \Sigma_j^h + \sum_{j=1}^{k+1} \mathcal{I}_{k+1} \dots \mathcal{I}_j \mathcal{E}_j. \end{aligned}$$

We bound the second moment of the first term, using the Cauchy–Schwarz inequality, by

$$\mathbb{E} \left[\left| \sum_{j=1}^{k+1} \mathcal{I}_{k+1} \dots \mathcal{I}_j B_j \right|^2 \right] \leq \sum_{j=1}^{k+1} \mathbb{E} |\mathcal{I}_{k+1} \dots \mathcal{I}_j|^2 \cdot \sum_{j=1}^{k+1} \mathbb{E} |B_j|^2 \leq (k+1)(1 - L_{\bar{b}}h)^{-2(k+1)} \sum_{j=1}^{k+1} \mathbb{E} |B_j|^2.$$

Since $L_{\bar{b}}h \leq \frac{1}{2}$, setting $c_1 := 4L_{\bar{b}}$ gives

$$(1 - L_{\bar{b}}h)^{-2(k+1)} \leq \exp \left\{ 2(k+1) \frac{L_{\bar{b}}h}{1 - L_{\bar{b}}h} \right\} \leq \exp\{c_1 T\}.$$

We can also apply the Cauchy–Schwarz inequality to the bound (4.10) from Lemma 4.11 to find a constant $c_2 \in (0, \infty)$ such that $\mathbb{E}[|B_j|^2] \leq c_2 h^3$, for all $j \in \{1, \dots, N\}$. Therefore

$$(4.29) \quad \mathbb{E} \left[\left| \sum_{j=1}^{k+1} \mathcal{I}_{k+1} \dots \mathcal{I}_j B_j \right|^2 \right] \leq c_2 T^2 \exp(c_1 T) h.$$

To estimate the remaining terms in (4.28), we set $\mathcal{S}_0 = 0$ and, for all $k \in \{1, \dots, N\}$, denote

$$\mathcal{S}_k := \sum_{j=1}^k \mathcal{I}_k \dots \mathcal{I}_j \Sigma_j + \sum_{j=1}^k \mathcal{I}_k \dots \mathcal{I}_j \Sigma_j^h + \sum_{j=1}^k \mathcal{I}_k \dots \mathcal{I}_j \mathcal{E}_j.$$

Observe that, for all $k \in \{0, \dots, N-1\}$,

$$\mathcal{S}_{k+1} = \mathcal{I}_k (\mathcal{S}_k + \Sigma_k + \Sigma_k^h + \mathcal{E}_k).$$

By independence of Brownian increments, we have that \mathcal{S}_k is independent of Σ_k , Σ_k^h , and \mathcal{E}_k . Using Pythagoras' theorem and then the Cauchy–Schwarz inequality, we see that the second moment is bounded by

$$(4.30) \quad \mathbb{E}[|\mathcal{S}_{k+1}|^2] \leq (1 - L_{\bar{b}}h)^{-2} (\mathbb{E}[|\mathcal{S}_k|^2] + 4\mathbb{E}[|\Sigma_k|^2] + 4\mathbb{E}[|\Sigma_k^h|^2] + 4\mathbb{E}[|\mathcal{E}_k|^2]).$$

We note that, in order to iterate this bound, it is essential that no constant appears in front of the term $\mathbb{E}[|\mathcal{S}_k|^2]$. Thus the above argument using independence does not directly generalise to p^{th} moments for $p \neq 2$.

Now, by Lemma 4.11, $\tilde{\sigma}$ is Lipschitz with constant $L_{\tilde{\sigma}}$. Therefore, by the Itô isometry,

$$\mathbb{E}[|\Sigma_{k+1}|^2] = \mathbb{E} \left[\int_{kh}^{(k+1)h} |\tilde{\sigma}(Z_s) - \tilde{\sigma}(Z_{kh})|^2 ds \right] \leq L_{\tilde{\sigma}}^2 \mathbb{E} \left[\int_{kh}^{(k+1)h} |Z_s - Z_{kh}|^2 ds \right].$$

Then the bound (4.12) gives a constant $c_3 \in (0, \infty)$ such that

$$(4.31) \quad \mathbb{E}[|\Sigma_{k+1}|^2] \leq c_3 L_{\tilde{\sigma}}^2 \int_{kh}^{(k+1)h} |s - kh| ds \leq c_3 L_{\tilde{\sigma}}^2 h^2.$$

By the independence of Z_{kh} from the increments ΔW_{k+1} and ΔW_{k+1}^h , together with the linear growth of $\tilde{\sigma}$ and the second moment bound from Proposition 4.18, there exists $c_4 \in (0, \infty)$ such that

$$\mathbb{E}[|\mathcal{E}_{k+1}|^2] = \mathbb{E}[|\tilde{\sigma}(Z_{kh})|^2] \cdot \mathbb{E}[|\Delta W_{k+1} - \Delta W_{k+1}^h|^2] \leq c_4 \mathbb{E}[|\Delta W_{k+1} - \Delta W_{k+1}^h|^2].$$

Adapting the proof of [7, Lemma 3.13] gives a bound on the second moment $\mathbb{E}[|\Delta W_{k+1} - \Delta W_{k+1}^h|^2]$, which implies that there exists $c_5 \in (0, \infty)$ such that

$$(4.32) \quad \mathbb{E}[|\mathcal{E}_{k+1}|^2] \leq c_5 h^2.$$

An application of the triangle inequality then yields the existence of a constant $c_6 \in (0, \infty)$ such that $\mathbb{E}[|\Delta W_{k+1}^h|^2] \leq 2(\mathbb{E}[|\mathcal{E}_{k+1}|^2] + \mathbb{E}[|\Delta W_{k+1}|^2]) \leq c_6 h$. This, the independence of Z_{kh} and Z_{kh}^h from the increment ΔW_{k+1}^h , and the Lipschitz continuity of $\tilde{\sigma}$ yield the bound

$$(4.33) \quad \begin{aligned} \mathbb{E}[|\Sigma_{k+1}^h|^2] &= \mathbb{E}[|\tilde{\sigma}(Z_{kh}) - \tilde{\sigma}(Z_{kh}^h)|^2] \cdot \mathbb{E}[|\Delta W_{k+1}^h|^2] \\ &\leq c_6 L_{\tilde{\sigma}}^2 \mathbb{E}[|Z_{kh} - Z_{kh}^h|^2] h. \end{aligned}$$

Combining (4.28) with the definition of \mathcal{S}_k and the bound (4.29) gives

$$(4.34) \quad \mathbb{E}[|Z_{kh} - Z_{kh}^h|^2] \leq 2c_2 T^2 \exp(c_1 T) h + 2\mathbb{E}[|\mathcal{S}_{k+1}|^2].$$

We now substitute the bounds (4.31), (4.32), and (4.33) into (4.30), to get

$$\mathbb{E}[|\mathcal{S}_{k+1}|^2] \leq (1 - L_{\tilde{b}} h)^{-2} (\mathbb{E}[|\mathcal{S}_k|^2] + 4c_3 L_{\tilde{\sigma}}^2 h^2 + 4c_6 L_{\tilde{\sigma}}^2 \mathbb{E}[|Z_{(k-1)h} - Z_{(k-1)h}^h|^2] h + 4c_5 h^2).$$

Then (4.34) gives

$$\mathbb{E}[|\mathcal{S}_{k+1}|^2] \leq (1 - L_{\tilde{b}} h)^{-2} (1 + 8c_6 L_{\tilde{\sigma}}^2 h) \mathbb{E}[|\mathcal{S}_k|^2] + c_7 h^2 (1 - L_{\tilde{b}} h)^{-2},$$

where $c_7 := 4(c_3 L_{\tilde{\sigma}}^2 + 2c_2 c_6 L_{\tilde{\sigma}}^2 T^2 \exp(c_1 T) + c_5)$.

Since $L_{\tilde{b}} h \leq \frac{1}{2}$, we can check that $1 \leq (1 - L_{\tilde{b}} h)^{-2} (1 + 8c_6 L_{\tilde{\sigma}}^2 h) \leq 1 + c_8 h$, where $c_8 := 32c_6 L_{\tilde{\sigma}}^2$, and we have

$$\mathbb{E}[|\mathcal{S}_{k+1}|^2] \leq (1 + c_8 h) \mathbb{E}[|\mathcal{S}_k|^2] + 4c_7 h^2.$$

Iterating, we obtain

$$\mathbb{E}[|\mathcal{S}_{k+1}|^2] \leq 4c_7 h^2 \sum_{j=0}^k (1 + c_8 h)^j \leq 4c_7 h^2 \sum_{j=0}^k \exp(c_8 h N) \leq 4c_7 T \exp(c_8 T) h.$$

Finally, combining this with (4.34) gives a constant $c_9 \in (0, \infty)$ such that

$$(4.35) \quad \mathbb{E}[|Z_{kh} - Z_{kh}^h|^2] \leq c_9 h.$$

Now by (4.12), for $k \in \{0, \dots, N-1\}$ and $t \in [kh, (k+1)h)$, there exists a constant $c_{10} \in (0, \infty)$ such that $\mathbb{E}[|Z_t - Z_{kh}|^2] \leq c_{10} h$. Combining this with (4.35) and the triangle inequality, we have

$$(4.36) \quad \begin{aligned} \mathbb{E} \left[\sum_{k=0}^{N-1} \int_{kh}^{(k+1)h} |Z_t - Z_{kh}^h|^2 dt \right] &\leq 2 \sum_{k=0}^{N-1} \int_{kh}^{(k+1)h} (\mathbb{E}[|Z_t - Z_{kh}|^2] + \mathbb{E}[|Z_{kh}^h - Z_{kh}|^2]) dt \\ &\leq 2Nh(c_9 h + c_{10} h) = \tilde{C} h, \end{aligned}$$

for $\tilde{C} := 2T(c_9 + c_{10})$.

The estimates (4.25) applied to (4.35) and (4.36) yield

$$\sup_{k \in \{1, \dots, N\}} \mathbb{E}[|X_{kh} - X_{kh}^h|^2]^{\frac{1}{2}} \leq L_{G^{-1}} c_9^{\frac{1}{2}} h^{\frac{1}{2}},$$

and

$$(4.37) \quad \mathbb{E} \left[\sum_{k=0}^{N-1} \int_{kh}^{(k+1)h} |X_t - X_{kh}^h|^2 dt \right]^{\frac{1}{2}} \leq L_{G^{-1}} \tilde{C}^{\frac{1}{2}} h^{\frac{1}{2}}$$

Applying Jensen's inequality, we obtain the same rate of convergence for $p \in [1, 2)$. For $p \in (2, \infty)$, we apply Hölder's inequality to find

$$\begin{aligned} \mathbb{E} \left[\sum_{k=0}^{N-1} \int_{kh}^{(k+1)h} |X_t - X_{kh}^h|^p dt \right]^{\frac{1}{p}} &= \mathbb{E} \left[\sum_{k=0}^{N-1} \int_{kh}^{(k+1)h} |X_t - X_{kh}^h|^{\frac{2}{p-1}} \cdot |X_t - X_{kh}^h|^{\frac{(p-2)(p+1)}{p-1}} dt \right]^{\frac{1}{p}} \\ &\leq \mathbb{E} \left[\sum_{k=0}^{N-1} \int_{kh}^{(k+1)h} |X_t - X_{kh}^h|^2 dt \right]^{\frac{1}{p(p-1)}} \mathbb{E} \left[\sum_{k=0}^{N-1} \int_{kh}^{(k+1)h} |X_t - X_{kh}^h|^{p+1} dt \right]^{\frac{p-2}{p(p-1)}} \\ &\leq Ch^{\frac{1}{p(p-1)}}, \end{aligned}$$

for some constant $C \in (0, \infty)$, using the L^2 bound (4.37), combined with the bound on the $(p+1)$ -moment of the scheme shown in Proposition 4.18, and the bound on the $(p+1)$ -moment of the solution of the SDE shown in Proposition 4.12. In the same way, we show that

$$\sup_{k \in \{1, \dots, N\}} \mathbb{E}[|X_{kh} - X_{kh}^h|^p] \leq Ch^{\frac{1}{p(p-1)}},$$

which concludes the proof. \square

Remark 4.21 (additive noise). Under the additional assumption that $\sigma \equiv 1$ we only require estimates for the error terms B_k , but not for $\Sigma_k, \Sigma_k^h, \mathcal{E}_k$, in the proof of Theorem 4.19. Repeated applications of Hölder's inequality yield p^{th} moment estimates analogous to (4.29) with order $h^{\frac{p}{2}}$. Proceeding as in the proof of Theorem 4.19, we find the strong L^p convergence rate $1/2$ for all $p \geq 1$ in the case of additive noise.

4.3. Application to optimal transport. We next apply the transformed monotone semi-implicit Euler–Maruyama scheme from Definition 4.14 to finding the optimiser of (BCOT) between the laws of SDEs satisfying Assumption 4.1. To this end, we will verify the following properties of the scheme.

Lemma 4.22. *Suppose that the coefficients of (1.1) satisfy Assumption 4.1. Then, for all h sufficiently small, the transformed monotone semi-implicit Euler–Maruyama scheme (4.14) is stochastically increasing and, for $k \in \{0, \dots, N-1\}$, the map $\Delta W_{k+1}^h \mapsto X_{(k+1)h}^h$ is increasing.*

Proof. By Lemma 4.11, $\tilde{\sigma}$ is Lipschitz with coefficient $L_{\tilde{\sigma}} \in (0, \infty)$. Fix $k \in \{0, \dots, N-1\}$ and $x, y \in \mathbb{R}$ with $x < y$. Then, using Definition 2.15 of ΔW^h ,

$$\begin{aligned} x + \tilde{\sigma}(x)\Delta W_{k+1}^h - (y + \tilde{\sigma}(y)\Delta W_{k+1}^h) &= x - y + (\tilde{\sigma}(x) - \tilde{\sigma}(y))\Delta W_{k+1}^h \\ &\leq (1 + L_{\tilde{\sigma}}\Delta W_{k+1}^h)(x - y) \\ &\leq (1 - L_{\tilde{\sigma}}A_h)(x - y). \end{aligned}$$

For $h > 0$ sufficiently small that $1 - L_{\tilde{\sigma}}A_h > 0$, we have that the map $(\text{id} + \Delta W_{k+1}^h\tilde{\sigma})$ is strictly increasing with strictly increasing inverse. Let us also take h sufficiently small that $(\text{id} - h\tilde{b})^{-1}$ is well-defined and strictly increasing by Lemma 4.16. Then we can rewrite (4.14) as

$$(4.38) \quad X_{(k+1)h}^h = G^{-1} \circ (\text{id} - h\tilde{b})^{-1} \circ (\text{id} + \Delta W_{k+1}^h\tilde{\sigma}) \circ G(X_{kh}^h).$$

Recall that G and G^{-1} are strictly increasing by Lemma 4.7. Thus $X_{(k+1)h}^h$ is obtained from X_{kh}^h by the concatenation of four strictly increasing maps. This shows that $(X_{kh}^h)_{k \in \{0, \dots, N\}}$ is stochastically increasing. Noting that $\tilde{\sigma}$ is non-negative, the second claim also follows from (4.38). \square

We can now state and prove our optimality result.

Theorem 4.23. *Suppose that the coefficients (b, σ) and $(\bar{b}, \bar{\sigma})$ of (2.6) satisfy Assumption 4.1 and write μ, ν for the laws of the respective strong solutions X, \bar{X} . Let $c: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function for which there exists $K \geq 0$ and $p \geq 1$ such that, for each fixed $t \in [0, T]$, $c_t(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies (2.3) and (2.4). Then the synchronous coupling attains the value $V_c(\mu, \nu)$ of the bicausal optimal transport problem (BCOT).*

Proof. We proved strong existence and pathwise uniqueness in Theorem 4.13. For $N \in \mathbb{N}$, $h = T/N$, denote by $(X_{kh}^h)_{k \in \{1, \dots, N\}}$ and $(\bar{X}_{kh}^h)_{k \in \{1, \dots, N\}}$ the transformed monotone semi-implicit Euler–Maruyama scheme for the solutions X and \bar{X} of (1.1), respectively, as defined in Definition 4.14. By Lemma 4.22, (X^h, \bar{X}^h) and (X, \bar{X}) satisfy Assumptions 3.1 (i) and (ii). By Theorem 4.19, Assumption 3.1 (iii) is also satisfied. We conclude by applying Proposition 3.2. \square

5. SDES WITH BOUNDED MEASURABLE DRIFT

We now consider a second class of irregular coefficients. We only require the drift coefficient to be bounded and measurable, and we allow the diffusion coefficient to be Hölder continuous in space rather than Lipschitz, but we do require a uniform non-degeneracy condition and boundedness of the diffusion, which was not needed in Section 4. The coefficients may be time-dependent. In this setting, we again obtain optimality of the synchronous coupling for (BCOT) between the laws of SDEs. Similarly to Section 4, the proof relies on a transformation of the SDEs. Here we use the drift-removing transformation that is defined by Zvonkin [81, Theorem 1]. We solve a transformed bicausal optimal transport problem, which we show to be equivalent to (BCOT). This is a different approach from Section 4; in Section 5.1 we give a counterexample to show that the former approach does not yield such a general result.

We make the following assumptions on the coefficients (b, σ) of the SDE (1.1), following Zvonkin [81].

Assumption 5.1. Suppose that the coefficients (b, σ) satisfy the following:

- (i) b is bounded and measurable;
- (ii) σ is bounded, continuous in time, and α -Hölder continuous in space, for some $\alpha \in [\frac{1}{2}, 1]$;
- (iii) σ is uniformly non-degenerate; that is, for all $x \in \mathbb{R}$, $\sigma^2(x) \geq C > 0$.

Remark 5.2.

- (1) Under Assumption 5.1, Zvonkin [81, Theorem 4] proves existence and uniqueness of strong solutions of the SDE (1.1) via a transformation that removes the drift.
- (2) A similar transformation to that of [81] was used by Talay [76] to analyse the numerical approximation of SDEs. More recently, Backhoff-Veraguas et al. [7], Gerencsér, Lampl, and Ling [21], Neuenkirch and Szölgyenyi [54], Ngo and Taguchi [56, 57, 58] use a similar drift-removing transformation in the numerical approximation of SDEs with irregular coefficients, under stronger conditions than Assumption 5.1.
- (3) For the SDE (1.1) with coefficients satisfying Assumption 5.1, strong convergence rates for the Euler–Maruyama scheme have been obtained by Gyöngy and Rásonyi [26] under additional conditions on the drift, and by Dareiotis and Gerencsér [17], Dareiotis et al. [18] for time-homogeneous coefficients under additional conditions on the diffusion.

Example 5.3. Consider the SDE

$$dX_t = \text{sign}(\sin(X_t)) + (1 + \sqrt{|X_t|} \mathbf{1}_{\{|X_t| \leq 4\}} + 2 \mathbf{1}_{\{|X_t| > 4\}}) dW_t; \quad X_0 = x_0 \in \mathbb{R}.$$

The coefficients of this SDE satisfy Assumption 5.1.

Suppose that (b, σ) satisfies Assumption 5.1, and let X be the unique strong solution of (1.1) with coefficients (b, σ) . Zvonkin [81, Theorem 1] gives the existence of a function $u: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ that is in the Sobolev space $W_p^{1,2}([0, T] \times D)$, for any bounded domain $D \subset \mathbb{R}$ and $p > \frac{3}{2}$, and solves the PDE problem

$$\begin{aligned} \partial_t u_t(x) + \partial_x u_t(x) b_t(x) + \frac{1}{2} \partial_{xx} u_t(x) \sigma_t^2(x) &= 0, \quad (t, x) \in [0, T] \times \mathbb{R}, \\ u_T(x) &= x, \quad x \in \mathbb{R}. \end{aligned}$$

By [81, Theorem 2], there exists an inverse function $v: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying $(v_t \circ u_t)(x) = x$ for all $t \in [0, T]$, $x \in \mathbb{R}$.

Define $Z_t := u_t(X_t)$ for all $t \in [0, T]$. Then [81, Theorem 3] shows that we can apply Itô's formula to obtain

$$(5.1) \quad dZ_t = \tilde{\sigma}_t(Z_t) dW_t, \quad Z_0 = u_0(x_0),$$

where $\tilde{\sigma}_t := (\partial_x u_t \cdot \sigma_t) \circ v_t$, for all $t \in [0, T]$.

We recall the following properties of u and v from [81, Theorem 2] without proof.

Lemma 5.4. *Suppose that (b, σ) satisfy Assumption 5.1. Then, uniformly in $t \in [0, T]$,*

- (i) $x \mapsto u_t(x)$ and $x \mapsto v_t(x)$ are strictly increasing and Lipschitz continuous;
- (ii) $x \mapsto \partial_x u_t(x)$ is bounded from above, bounded away from zero, and β -Hölder continuous, for any $\beta \in (0, 1)$;

and, uniformly in $x \in D$, for any bounded domain $D \subset \mathbb{R}$,

- (iii) $t \mapsto u_t(x)$, $t \mapsto v_t(x)$, and $t \mapsto \partial_x u_t(x)$ are β -Hölder continuous for any $\beta \in (0, 1)$.

The following is an immediate consequence of Lemma 5.4.

Corollary 5.5. *Suppose that (b, σ) satisfies Assumption 5.1. Then the diffusion coefficient $\tilde{\sigma}$ of the transformed SDE (5.1) is bounded from above and bounded away from zero. Moreover, for any $t \in [0, T]$, $x \mapsto \tilde{\sigma}_t(x)$ is α -Hölder continuous and, for any $x \in \mathbb{R}$, $t \mapsto \tilde{\sigma}_t(x)$ is β -Hölder continuous for any $\beta \in (0, \alpha)$.*

Let $c: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function for which there exists $K \geq 0$ and $p \geq 1$ such that, for each fixed $t \in [0, T]$, $c_t(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies the polynomial growth condition (2.3) and the quasi-monotonicity condition (2.4). Suppose that (b, σ) and $(\bar{b}, \bar{\sigma})$ satisfy Assumption 5.1, and write μ, ν for the laws of the solutions X, \bar{X} of (2.6). For $(\bar{b}, \bar{\sigma})$, define functions \bar{u}, \bar{v} analogously to u, v and define $\bar{Z}_t = \bar{u}_t(\bar{X}_t)$ for all $t \in [0, T]$.

In order to find the value $V_c(\mu, \nu)$ of (BCOT), we consider the following auxiliary problem. Write $\tilde{\mu}, \tilde{\nu}$ for the laws of Z, \bar{Z} , respectively, and define a cost function $\tilde{c}: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$(5.2) \quad \tilde{c}_t(z, \bar{z}) := c_t(v_t(z), \bar{v}_t(\bar{z})).$$

Consider the problem

$$(5.3) \quad V_{\tilde{c}}(\tilde{\mu}, \tilde{\nu}) := \inf_{\pi \in \text{Cpl}_{bc}(\tilde{\mu}, \tilde{\nu})} \mathbb{E}^\pi \left[\int_0^T \tilde{c}_t(\omega_t, \bar{\omega}_t) dt \right].$$

We now verify that \tilde{c} inherits the polynomial growth and quasi-monotonicity of properties from c .

Lemma 5.6. *Suppose that (b, σ) and $(\bar{b}, \bar{\sigma})$ satisfy Assumption 5.1. Let $c: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function for which there exists $K \geq 0$ and $p \geq 1$ such that, for each fixed $t \in [0, T]$, $c_t(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies (2.3) and (2.4). Then, there exists $\tilde{K} \geq 0$ such that, for each $t \in [0, T]$, the function $\tilde{c}_t: \mathbb{R} \times \mathbb{R}$ defined by (5.2) is also continuous, satisfies (2.3) with constant $\tilde{K} \geq 0$ and power p , and satisfies (2.4).*

Proof. For any $t \in [0, T]$, \tilde{c}_t is continuous, since c_t is continuous and v_t, \bar{v}_t are Lipschitz by Lemma 5.4. By assumption, there exist $p \geq 1$ and $K \geq 0$ such that, for all $t \in [0, T]$, c_t satisfies the growth bound (2.3). Now fix $t \in [0, T]$. Then

$$|\tilde{c}_t(z, \bar{z})| = |c_t(v_t(z), \bar{v}_t(\bar{z}))| \leq K(1 + |v_t(z)|^p + |\bar{v}_t(\bar{z})|^p).$$

Writing L for the Lipschitz constant of v_t , we have $|v_t(z)| \leq L|z| + L|u_t(x_0)| + |x_0|$, and similarly for $\bar{v}_t(\bar{z})$. Thus the growth bound (2.3) holds for \tilde{c}_t with a constant \tilde{K} independent of t , and with power p . It remains to prove that \tilde{c}_t satisfies (2.4). Since v_t, \bar{v}_t are increasing by Lemma 5.4, this follows immediately from the fact that c_t satisfies (2.4). \square

Now we show that (5.3) is equivalent to (BCOT).

Lemma 5.7. *Suppose that (b, σ) and $(\bar{b}, \bar{\sigma})$ satisfy Assumption 5.1. Let $c: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function. Then $V_c(\mu, \nu) = V_{\tilde{c}}(\tilde{\mu}, \tilde{\nu})$.*

Proof. Let $\pi \in \text{Cpl}_{\text{bc}}(\mu, \nu)$. By [7, Proposition 2.2], $\pi = \text{Law}(X, \bar{X})$, where (X, \bar{X}) is the unique strong solution of the system of SDEs (2.6) driven by some correlated Brownian motion (W, \bar{W}) . Now define (Z, \bar{Z}) by $Z_t = u_t(X_t)$, $\bar{Z}_t = \bar{u}_t(\bar{X}_t)$, for all $t \in [0, T]$. Then Z is the unique strong solution of the SDE (5.1) driven by W , and \bar{Z} of the analogous transformed SDE for $(\bar{b}, \bar{\sigma})$ driven by \bar{W} . Hence, by [7, Proposition 2.2] again, we can define a bicausal coupling by $\tilde{\pi} := \text{Law}(Z, \bar{Z}) \in \text{Cpl}_{\text{bc}}(\tilde{\mu}, \tilde{\nu})$. By the definition of \tilde{c} , we can write

$$(5.4) \quad \mathbb{E}^{\tilde{\pi}} \left[\int_0^T \tilde{c}_t(\omega_t, \bar{\omega}_t) dt \right] = \mathbb{E} \left[\int_0^T c_t(v_t(Z_t), \bar{v}_t(\bar{Z}_t)) dt \right] = \mathbb{E}^{\pi} \left[\int_0^T c_t(\omega_t, \bar{\omega}_t) dt \right].$$

Starting from an arbitrary $\tilde{\pi} \in \text{Cpl}_{\text{bc}}(\tilde{\mu}, \tilde{\nu})$, we can construct $\pi \in \text{Cpl}_{\text{bc}}(\mu, \nu)$ in a similar manner, via the inverse transform v . Thus, taking the infimum on both sides of (5.4), we obtain $V_c(\mu, \nu) = V_{\tilde{c}}(\tilde{\mu}, \tilde{\nu})$. \square

Combining the preceding results, we obtain optimality of the synchronous coupling for (BCOT).

Theorem 5.8. *Suppose that (b, σ) , $(\bar{b}, \bar{\sigma})$ satisfy Assumption 5.1, and write μ, ν for the laws of X, \bar{X} , respectively. Let $c: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function for which there exists $K \geq 0$ and $p \geq 1$ such that, for each fixed $t \in [0, T]$, $c_t(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies (2.3) and (2.4). Then the synchronous coupling attains the value $V_c(\mu, \nu)$ of the bicausal optimal transport problem (BCOT).*

Proof. Thanks to Lemma 5.7, it suffices to prove that the synchronous coupling $\pi_{\tilde{\mu}, \tilde{\nu}}^{\text{sync}}$ attains the value $V_{\tilde{c}}(\tilde{\mu}, \tilde{\nu})$, where $\tilde{\mu}, \tilde{\nu}$ are the laws of solutions of transformed SDEs of the form (5.1). By Corollary 5.5, the coefficients of these SDEs are bounded and continuous, and pathwise uniqueness holds by Yamada and Watanabe [79, Theorem 1]. Thus the coefficients of each of the transformed SDEs satisfy Assumption 3.6. By Lemma 5.6, \tilde{c} satisfies the conditions of Corollary 3.7, and applying this result completes the proof. \square

5.1. Lack of monotonicity and counterexamples. In the proof of Theorem 5.8, we consider transformed SDEs with Hölder continuous diffusion coefficients that may not be Lipschitz. Recall the monotone Euler–Maruyama scheme from Section 2.3 that was introduced in [7, Definition 3.11]. We show that this scheme is not stochastically monotone for SDEs whose diffusion coefficient is not Lipschitz. Thus we could not apply Proposition 3.2 with the monotone Euler–Maruyama scheme to prove optimality in this case.

Proposition 5.9. *Let $\sigma: \mathbb{R} \rightarrow (0, \infty)$ be α -Hölder continuous for some $\alpha \in [\frac{1}{2}, 1)$, bounded from above, and bounded away from zero. Consider the SDE $dZ_t = \sigma(Z_t) dW_t$, with $Z_0 = z_0 \in \mathbb{R}$. If σ is not Lipschitz continuous, then the monotone Euler–Maruyama scheme for Z is not stochastically monotone for any step size.*

Remark 5.10. In the setting of Proposition 5.9, the monotone Euler–Maruyama scheme coincides with the transformed monotone semi-implicit Euler–Maruyama scheme, since there is no drift term.

Proof of Proposition 5.9. The conditions of the proposition are sufficient to guarantee the existence of a unique strong solution Z of the SDE by, e.g. [81, Theorem 4]. For any $N \in \mathbb{N}$ and $h = \frac{T}{N}$, let Z^h denote the monotone Euler–Maruyama scheme for Z with step size h , and let A_h be the truncation level given in Definition 2.15. Note that, since σ is not Lipschitz continuous, there exist $z, \bar{z} \in \mathbb{R}$ with $z < \bar{z}$ such that

$$(5.5) \quad |\sigma(\bar{z}) - \sigma(z)| > A_h^{-1}(\bar{z} - z) > 0.$$

Now consider the random variables $Y := z + \sigma(z)\xi^h$, $\bar{Y} := \bar{z} + \sigma(\bar{z})\xi^h$, where ξ^h is a random variable with the same law as an increment of the truncated Brownian motion W^h . For all $a \in \mathbb{R}$, we have

$$\mathbb{P}[Y \leq a] = \mathbb{P}\left[\xi^h \leq \frac{a-z}{\sigma(z)}\right] = \begin{cases} 0, & a \leq z - A_h\sigma(z), \\ 2\Phi\left(h^{-\frac{1}{2}}\frac{a-z}{\sigma(z)}\right), & a \in (z - A_h\sigma(z), z + A_h\sigma(z)), \\ 1, & a \geq z + A_h\sigma(z), \end{cases}$$

where $\Phi: \mathbb{R} \rightarrow [0, 1]$ is the distribution function of a standard Gaussian, which is strictly increasing. The same holds for \bar{Y} with z replaced by \bar{z} . We aim to show that there exist $a_*, a^* \in \mathbb{R}$ such that $\mathbb{P}[\bar{Y} \leq a_*] < \mathbb{P}[Y \leq a_*]$ and $\mathbb{P}[Y \leq a^*] < \mathbb{P}[\bar{Y} \leq a^*]$. This would imply that Z^h is neither stochastically decreasing nor increasing. Suppose first that $\sigma(\bar{z}) \geq \sigma(z)$. Then rearranging (5.5) gives

$$\bar{z} - A_h\sigma(\bar{z}) < z - A_h\sigma(z).$$

So there exists $a^* \in (\bar{z} - A_h\sigma(\bar{z}), z - A_h\sigma(z))$, and we see that

$$\mathbb{P}[Y \leq a^*] = 0, \text{ while } \mathbb{P}[\bar{Y} \leq a^*] > 0.$$

Set $a_* = z + A_h\sigma(z)$ and note that $a_* < \bar{z} + A_h\sigma(\bar{z})$. Therefore $\mathbb{P}[Y \leq a_*] = 1$ and $\mathbb{P}[\bar{Y} \leq a_*] < 1$.

On the other hand, if $\sigma(\bar{z}) < \sigma(z)$, then $\bar{z} + A_h\sigma(\bar{z}) < z + A_h\sigma(z)$, and so there exists a^* such that $\mathbb{P}[\bar{Y} \leq a^*] = 1$ and $\mathbb{P}[Y \leq a^*] < 1$. Now setting $a_* = \bar{z} - A_h\sigma(\bar{z}) > z - A_h\sigma(z)$ gives $\mathbb{P}[\bar{Y} \leq a_*] = 0$ and $\mathbb{P}[Y \leq a_*] > 0$. Hence Z^h is not stochastically monotone. \square

A natural generalisation of stochastic monotonicity is monotonicity with respect to second order stochastic dominance. We now provide an example to show that, for two marginals that are increasing in second order stochastic dominance, the Knothe–Rosenblatt rearrangement may fail to be optimal for the bicausal transport problem, if one of the marginals is not stochastically increasing.

Example 5.11 (second order stochastic dominance is insufficient for optimality). Consider the discrete-time Markov processes $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$, with first marginals given by $\mathbb{P}[X_1 = -1/2] = \mathbb{P}[X_1 = 1/2] = \mathbb{P}[Y_1 = -1/2] = \mathbb{P}[Y_1 = 1/2] = 1/2$, and conditional second marginals by $\mathbb{P}[X_2 = 2 | X_1 = -1/2] = \mathbb{P}[X_2 = -2 | X_1 = -1/2] = 1/2$, $\mathbb{P}[X_2 = 0 | X_1 = 1/2] = 1$, and $\mathbb{P}[Y_2 = -2 | Y_1 = -1/2] = \mathbb{P}[Y_2 = 0 | Y_1 = -1/2] = \mathbb{P}[Y_2 = -2 | Y_1 = 1/2] = \mathbb{P}[Y_2 = 2 | Y_1 = 1/2] = 1/2$.

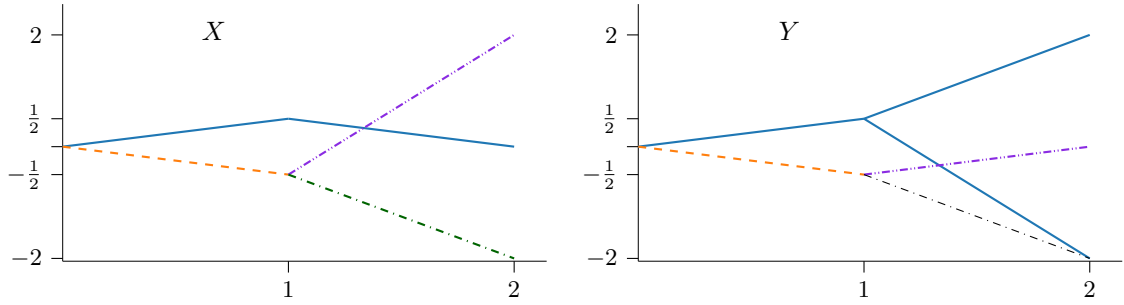
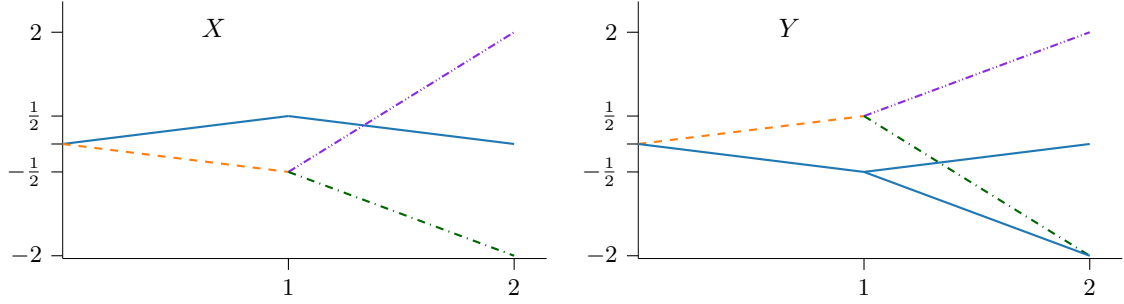
We claim that X is increasing with respect to second order stochastic dominance, but not with respect to first order stochastic dominance. Denote by $F_-, F_+ : \mathbb{R} \rightarrow [0, 1]$ the distribution functions of the second marginal of X , conditional on $X_1 = -\frac{1}{2}$ and $X_1 = \frac{1}{2}$, respectively; that is

$$F_-(x) = \begin{cases} 0, & x < -2, \\ \frac{1}{2}, & x \in [-2, 2), \\ 1, & x \geq 2, \end{cases} \quad F_+(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

We see that, for any $x \in [0, 2)$, $F_-(x) < F_+(x)$, and so X is not stochastically increasing. However, integrating, we find that

$$\int_{-\infty}^x (F_-(u) - F_+(u)) du = \begin{cases} 0, & x < -2, \\ \frac{x+2}{2} \geq 0, & x \in [-2, 0), \\ \frac{2-x}{2} \geq 0, & x \in [0, 2), \\ 0, & x \geq 2. \end{cases}$$

Hence X is increasing with respect to second order stochastic dominance. By inspection of the conditional distribution functions for Y , we see that Y is stochastically monotone, and therefore also increasing with respect to second order stochastic dominance. Thus X and Y are co-monotone

(A) The laws of X and Y coupled with the Knothe–Rosenblatt rearrangement π_{KR} .(B) The laws of X and Y coupled with an alternative bicausal coupling π_{AT} .FIGURE 3. Two couplings of the laws of X and Y . In each case, paths with the same colour and line style are coupled with each other.

with respect to *second order* stochastic dominance, but they are *not* stochastically co-monotone (i.e. co-monotone with respect to first order stochastic dominance).

We now show that second order monotonicity is not sufficient for optimality of the Knothe–Rosenblatt rearrangement. We aim to solve (BCOT) for the quadratic cost; that is we want to find

$$\mathcal{AW}_2^2(\mu, \nu) = \inf_{\pi \in \text{Cpl}_{\text{bc}}(\mu, \nu)} \mathbb{E}^\pi [|X_1 - Y_1|^2 + |X_2 - Y_2|^2].$$

We first compute the cost induced by the Knothe–Rosenblatt rearrangement $\pi_{\text{KR}}(\mu, \nu)$, shown in Figure 3a. Under this coupling, the first marginals are coupled monotonically, and so we have $\mathbb{E}[|X_1 - Y_1|^2] = 0$. Now, conditional on the event that $(X_1, Y_1) = (\frac{1}{2}, \frac{1}{2})$, we couple the second marginals monotonically and compute $\mathbb{E}[|X_2 - Y_2|^2] = \frac{1}{2}(|0 - 2|^2 + |0 + 2|^2) = 4$. Conditional on the complementary event $(X_1, Y_1) = (-\frac{1}{2}, -\frac{1}{2})$, we find $\mathbb{E}[|X_2 - Y_2|^2] = \frac{1}{2}(|2 - 0|^2 + |-2 + 2|^2) = 2$. We conclude that

$$\mathbb{E}^{\pi_{\text{KR}}(\mu, \nu)} [|X_1 - Y_1|^2 + |X_2 - Y_2|^2] = 0 + \frac{1}{2}(4 + 2) = 3.$$

Now consider an alternative bicausal coupling $\pi_{\text{AT}}(\mu, \nu)$, defined by taking the *antitone* coupling between the first marginals (i.e. the opposite of the monotone coupling) and then proceeding as in the Knothe–Rosenblatt rearrangement by coupling the conditional second marginals monotonically, as shown in Figure 3b. Under this coupling there is a non-zero contribution to the cost from the first marginals: $\mathbb{E}[|X_1 - Y_1|^2] = 1$. Conditional on $(X_1, Y_1) = (\frac{1}{2}, -\frac{1}{2})$, we have $\mathbb{E}[|X_2 - Y_2|^2] = \frac{1}{2}(|0 - 0|^2 + |0 + 2|^2) = 2$, and conditional on the complement, $\mathbb{E}[|X_2 - Y_2|^2] = \frac{1}{2}(|2 - 2|^2 + |-2 + 2|^2) = 0$. This gives a total cost of

$$\mathbb{E}^{\pi_{\text{AT}}(\mu, \nu)} [|X_1 - Y_1|^2 + |X_2 - Y_2|^2] = 1 + \frac{1}{2}(2 + 0) = 2 < 3.$$

Hence the Knothe–Rosenblatt rearrangement is not optimal.

6. GENERAL OPTIMALITY OF THE SYNCHRONOUS COUPLING

In the previous sections we have identified different classes of coefficients of the SDEs (2.6) for which the synchronous coupling is optimal for (BCOT). In fact, the same optimality result holds for SDEs whose coefficients belong to two different classes; that is they satisfy one of Assumptions 3.6, 4.1, and 5.1.

Theorem 6.1. *Suppose that (b, σ) and $(\bar{b}, \bar{\sigma})$ each satisfy any one of Assumptions 3.6, 4.1, and 5.1, where the two pairs of coefficients may each satisfy a different assumption. Write μ, ν for the laws of X, \bar{X} , respectively. Let $c: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function for which there exists $K \geq 0$ and $p \geq 1$ such that, for each fixed $t \in [0, T]$, $c_t(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies (2.3) and (2.4). Then the synchronous coupling attains the value $V_c(\mu, \nu)$ of the bicausal optimal transport problem (BCOT).*

Proof. First suppose that (b, σ) satisfies either the global Lipschitz condition Assumption 3.4 (this implies Assumption 3.6) or Assumption 4.1. Then take X^h to be the monotone Euler–Maruyama scheme defined in [7, Definition 3.11], or the transformed monotone semi-implicit Euler–Maruyama scheme defined in Definition 4.14, respectively. If $(\bar{b}, \bar{\sigma})$ also satisfies one of the above conditions, then taking \bar{X}^h to be the respective numerical scheme, Assumption 3.1 is satisfied by (X, \bar{X}) and (X^h, \bar{X}^h) . Then we can conclude by Proposition 3.2. If $(\bar{b}, \bar{\sigma})$ satisfies Assumption 3.6, but not necessarily a global Lipschitz condition, then we can approximate $(\bar{b}, \bar{\sigma})$ locally uniformly in space by a sequence of Lipschitz functions that have uniform linear growth. Then we conclude by applying the stability result Lemma 3.3 to the previous case.

Now suppose that (b, σ) satisfies either Assumption 3.6 or Assumption 4.1, and $(\bar{b}, \bar{\sigma})$ satisfies Assumption 5.1. Then we adapt the proof of Theorem 5.8 as follows. Take the transformation \bar{u} to be as in the proof of Theorem 5.8, and replace the transformation u with the identity. Then, following the proof of Theorem 5.8, we show that (BCOT) is equivalent to a bicausal transport problem between the laws of transformed SDEs, whose coefficients satisfy either Assumption 3.6 or Assumption 4.1. From the cases considered above, we know that the synchronous coupling is optimal for the transformed problem, and by equivalence of the problems, the synchronous coupling is also optimal for (BCOT).

Noting that the roles of μ and ν are interchangeable, all remaining cases are covered by Corollary 3.7 and Theorems 4.23 and 5.8. □

7. COMPUTATION OF THE ADAPTED WASSERSTEIN DISTANCE

Our results immediately suggest an efficient method for computing the value of a bicausal optimal transport problem. This is in sharp contrast to the general situation in classical and constrained optimal transport problems, when the optimiser is not known. Therefore our results contribute to the efficiency of computations in applications, such as in the calculation of robust bounds in mathematical finance, as discussed in Section 8.

In classical optimal transport, entropic regularisation is used in computation. Sinkhorn’s algorithm can be used to efficiently compute the value of the regularised problem, but not of the original unregularised problem; see Cuturi [15], Peyré and Cuturi [59]. For bicausal optimal transport in discrete time, Eckstein and Pammer [19], Pichler and Weinhardt [62] introduce two different adapted analogues of Sinkhorn’s algorithm. However, each of these algorithms either include a discretisation of the marginal measures or work only on a discrete state space, and generalising them to both continuous time and space would be prohibitively costly. Under rather strong assumptions on the coefficients, Bion-Nadal and Talay [10] consider a stochastic control problem for SDEs, which is shown to be equivalent to finding the adapted 2-Wasserstein distance in [7, Proposition

2.2]. This control formulation gives rise to a Hamilton-Jacobi-Bellman equation, which one could solve numerically.

7.1. Computation via the synchronous coupling. Let $c: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function for which there exists $K \geq 0$ and $p \geq 1$ such that, for each fixed $t \in [0, T]$, $c_t(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies (2.3) and (2.4). Suppose that (b, σ) satisfies one of Assumptions 3.6, 4.1, and 5.1, and that $(\bar{b}, \bar{\sigma})$ also satisfies one of these assumptions. Then, by Theorem 6.1, the synchronous coupling is optimal for $V_c(\mu, \nu)$. Writing (X, \bar{X}) for the solution of the SDEs (2.6) driven by a common Brownian motion W , we thus have

$$V_c(\mu, \nu) = \mathbb{E} \left[\int_0^T c_t(X_t, \bar{X}_t) dt \right].$$

In particular, taking $c_t(x, y) = |x - y|^p$, for all $t \in [0, T]$, $x, y \in \mathbb{R}$, the adapted Wasserstein distance is given by

$$\mathcal{AW}_p^p(\mu, \nu) = \mathbb{E} \left[\int_0^T |X_t - \bar{X}_t|^p dt \right].$$

For $N \in \mathbb{N}$, set $h = T/N$ and take X^h, \bar{X}^h to be càdlàg processes adapted to the natural filtration of W such that X^h (resp. \bar{X}^h) approximates X (resp. \bar{X}) strongly in L^p . Define $S^h := \mathbb{E}[\int_0^T |X_t^h - \bar{X}_t^h|^p dt]^{\frac{1}{p}}$. Then

$$(7.1) \quad |\mathcal{AW}_p(\mu, \nu) - S^h| \leq \mathbb{E} \left[\int_0^T |X_t - X_t^h|^p dt \right]^{\frac{1}{p}} + \mathbb{E} \left[\int_0^T |\bar{X}_t - \bar{X}_t^h|^p dt \right]^{\frac{1}{p}} \xrightarrow{h \rightarrow 0} 0.$$

Note that each term in the sum on the right hand side of (7.1) depends on only one of the processes X, \bar{X} and so these terms can be estimated separately. Thus the problem of computing the adapted Wasserstein distance is reduced to the well-studied problem of numerically solving two one-dimensional SDEs. Monte Carlo methods can then be applied to compute these approximations.

There are now two error sources to consider in the numerical computation of $\mathcal{AW}_p(\mu, \nu)$: the convergence rate of the schemes X^h, \bar{X}^h , and the computational cost of Monte Carlo estimation for S^h . Given numerical schemes for which we have strong convergence rates, one can efficiently implement a multi-level Monte Carlo (MLMC) method to reduce the overall computational cost; see, for example, Giles and Szpruch [23].

Remark 7.1. The bound (7.1) shows that, for each SDE, we may take any numerical scheme that converges strongly in L^p . For example, for any SDE with Lipschitz coefficients, we can apply the Euler–Maruyama scheme defined in Section 2.3. In the case of discontinuous and exponentially growing drift, we proved strong L^p convergence of the transformed (monotone) semi-implicit Euler–Maruyama scheme in Theorem 4.19, which allows us to compute the adapted Wasserstein distance when one of the SDEs has coefficients satisfying Assumption 4.1. When additional conditions hold, however, we may exploit the existing convergence results that are discussed in Remark 4.4 to improve computational efficiency. While stochastic monotonicity of the scheme is essential to the proof of optimality in Theorem 4.23, this condition is not needed for the computation of the adapted Wasserstein distance. Under Assumption 5.1, suitable numerical schemes are discussed in Remark 5.2.

Example 7.2. As a toy example, consider the situation that b, \bar{b} are Lipschitz and $\sigma, \bar{\sigma}$ are strictly positive constants. In this case, taking X^h, \bar{X}^h as the standard Euler–Maruyama schemes for X, \bar{X} , respectively, (7.1) implies that $|\mathcal{AW}_p(\mu, \nu) - S^h| \leq C(p, T)N^{-1}$. This follows from the order 1 strong convergence of the Euler–Maruyama scheme in the case of Lipschitz drift and additive noise, where it is equivalent to the Milstein scheme [47].

7.2. Numerical examples. In each of the examples below, we consider an SDE with solution X and, for $k \in \mathbb{N}$, the solution $\bar{X}^{(k)}$ of an SDE obtained by some perturbation of the coefficients of the original SDE, where the magnitude of the perturbation increases with the parameter k . We fix the terminal time $T = 1$ and consider the adapted Wasserstein distance $\mathcal{AW}_2(\mu, \nu^{(k)})$, where μ and $\nu^{(k)}$ denote the laws of X and $\bar{X}^{(k)}$, respectively. Supposing that the coefficients of the SDEs for X and $X^{(k)}$ satisfy the assumptions of Theorem 6.1, this distance is attained by the synchronous coupling. For $h > 0$, let X^h (resp. $\bar{X}^{(k),h}$) denote numerical approximations of X (resp. $\bar{X}^{(k)}$) that converge strongly in L^2 with known rates and are each driven by a common one-dimensional Brownian motion. We fix the step size $h = 2^{-12}$ for the numerical approximations and average over 2^{12} sample paths. In light of the bound (7.1), we can then approximate the squared adapted Wasserstein distance $\mathcal{AW}_2^2(\mu, \nu^{(k)})$ by the Monte Carlo estimate

$$(7.2) \quad \widehat{\mathcal{AW}}_2^2(\mu, \nu^{(k)}) = \frac{1}{2^{12}(2^{12} + 1)} \sum_{i=1}^{2^{12}} \sum_{j=0}^{2^{12}} |X_{jh}^h(\omega_i) - \bar{X}_{jh}^{(k),h}(\omega_i)|^2.$$

In each example below, we plot $\widehat{\mathcal{AW}}_2^2(\mu, \nu^{(k)})$ against the size of the perturbation k .

Example 7.3 (An SDE with discontinuous drift). Consider SDEs

$$(7.3) \quad \begin{aligned} dX_t &= dW_t, & X_0 &= x_0, \\ d\bar{X}_t^{(k)} &= \frac{k}{10} \text{sign}(\bar{X}_t^{(k)}) dt + dW_t, & \bar{X}_0^{(k)} &= x_0, \quad k \in \{1, \dots, 10\}. \end{aligned}$$

The solution X of the first SDE in (7.3) can be approximated by the Euler–Maruyama scheme with strong convergence order 1. For the SDEs in the second line of (7.3), however, Ellinger, Müller-Gronbach, and Yaroslavtseva [20], Müller-Gronbach and Yaroslavtseva [52], Przybyłowicz, Schwarz, and Szölgyenyi [65] show an upper bound of 3/4 for the strong convergence rate of any numerical scheme that uses finitely many approximations of the driving Brownian motion (such as the Euler–Maruyama scheme). This is one of the structural properties that distinguishes the class of SDEs with discontinuous drift. For the adapted Wasserstein distance between the induced measures, however, we observe no gap comparable to the gap between the strong convergence rates. Each of the SDEs in (7.3) satisfies Assumption 4.1, and so Theorem 6.1 implies that the synchronous coupling attains the adapted Wasserstein distance between the laws of the solutions. Thus, we can estimate this distance by (7.2), taking $X^h, \bar{X}^{(k),h}$ to be the respective Euler–Maruyama schemes, each driven by a common Brownian motion. The relationship between this approximation of the adapted Wasserstein distance and the size of the jump in the drift is shown in Figure 4.

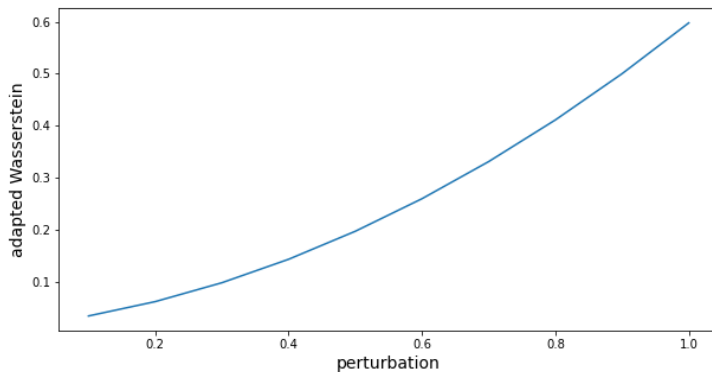


FIGURE 4. Approximation of the adapted Wasserstein distance for the SDEs (7.3).

Example 7.4 (CIR model). Consider a CIR process $(X_t)_{t \in [0, T]}$, solving the SDE

$$(7.4) \quad dX_t = \kappa(\eta - X_t) dt + \gamma\sqrt{X_t} dW_t, \quad X_0 = x_0,$$

for some $\kappa \in \mathbb{R}$, $\eta \in [0, \infty)$, $\gamma, x_0 \in (0, \infty)$. We suppose that $2\kappa\eta \geq \gamma^2$, so that $X_t > 0$ for all $t \in [0, T]$ by Feller's test; see, for example, [36, Theorem 5.29]. We call the parameter η the mean reversion level, κ the mean reversion speed, and γ the diffusion parameter. Such a process is commonly used to model interest rates.

Here we have a Lipschitz-continuous drift coefficient and a $1/2$ -Hölder continuous diffusion coefficient. The classical result of Yamada and Watanabe [79, Theorem 1] gives pathwise uniqueness for (7.4). Thus the coefficients of (7.4) satisfy Assumption 3.6. Consider also the process $\bar{X}^{(k)}$ solving the same SDE (7.4) with perturbed parameters $\bar{\kappa}, \bar{\eta}, \bar{\gamma}$ and $\bar{X}_0^{(k)} = x_0$. Since the coefficients of this SDE also satisfy Assumption 3.6, we have by Theorem 6.1 that the synchronous coupling attains the adapted Wasserstein distance between their laws.

Since the CIR process remains positive, its approximation requires a modification to the standard Euler–Maruyama scheme. For example, we can take the symmetrised Euler–Maruyama scheme that is defined in Berkaoui, Bossy, and Diop [9] by taking the absolute value at each step of the Euler–Maruyama scheme. Under the condition that $2\kappa\eta \geq \gamma^2$, [9, Theorem 2.2] shows that the symmetrised Euler–Maruyama scheme converges strongly in L^2 at rate $1/2$. Note that [27, Theorem 1.1] shows that this convergence rate does not hold when $2\kappa\eta < \gamma^2$. In this example we fix the parameters $\kappa = \eta = \gamma = 1$ for the process X , and for the process $\bar{X}^{(k)}$ we perturb each of these parameters in turn, fixing the other two equal to those of the process X . For each perturbation, we approximate $X, \bar{X}^{(k)}$ by the numerical scheme described above, taking the same Brownian increments in both X^h and $\bar{X}^{(k),h}$. We then estimate the adapted Wasserstein distance between the laws of X and $\bar{X}^{(k)}$ by (7.2).

Figure 5 shows an approximation of the adapted Wasserstein distance plotted against the absolute size of the perturbation for each parameter on the left-hand side, and the same on a log-scale on the right-hand side. As expected, we observe clearly in Figure 5 that increasing the perturbation in each of the parameters increases the adapted Wasserstein distance between the reference measure and the measure induced by the process with perturbed parameters. We observe that a perturbation of the diffusion parameter has a bigger effect than that of the mean reversion level, which has a much bigger effect than a perturbation of the mean reversion speed. This behaviour depends, however, on the choice of the parameters for the process X . As the log-plot confirms, the adapted Wasserstein distance grows exponentially in the size of the perturbation. This observation confirms that the adapted Wasserstein distance successfully captures the difference between CIR models with different parameters.

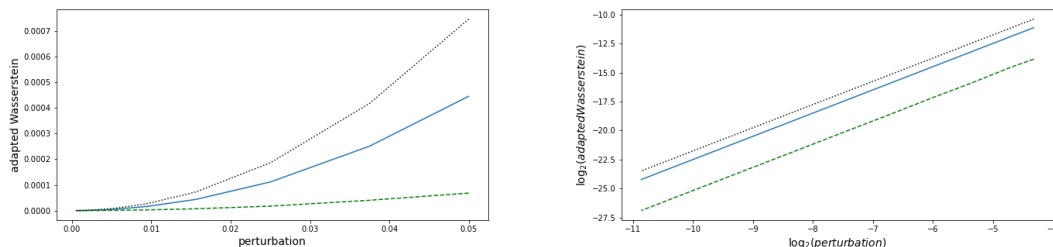


FIGURE 5. Approximation of the adapted Wasserstein distance for the CIR process; the dotted black line shows the effect of a perturbed diffusion parameter γ , the solid blue line shows the effect of a perturbed mean reversion level η , and the dashed green line shows the effect of a perturbed mean reversion speed κ .

8. APPLICATION TO ROBUST OPTIMISATION

We finally turn to applications in the robust approach to stochastic optimisation. For the discrete-time adapted Wasserstein distance, several results have been proven on stability of various stochastic optimisation problems [5], continuity of optimal stopping problems [1, 6], and model-sensitivity of multi-period optimisation problems [3, 8]. Fully extending such results to continuous time remains open.

In this section, we equip the space $\Omega = C([0, T], \mathbb{R})$ with the L^∞ norm $\|\omega\|_\infty := \sup_{t \in [0, T]} |\omega_t|$, instead of the L^p norm, and we define the adapted Wasserstein distance with respect to the L^∞ norm, analogously to Definition 2.11. We prove Lipschitz continuity of optimal stopping problems with respect to the adapted 1-Wasserstein distance on $\mathcal{P}_2(\Omega)$. We thus extend [6, Lemma 7.1] from discrete-time processes to one-dimensional continuous-time processes with continuous paths.

8.1. Optimal stopping. For any measurable function $f: \mathbb{R} \rightarrow \mathbb{R}$, define the value $v^f: \mathcal{P}_2(\Omega) \rightarrow \mathbb{R}$ of the optimal stopping problem for f by

$$v^f(\mu) := \inf_{\tau} \mathbb{E}^\mu[f(\omega_\tau)],$$

for all $\mu \in \mathcal{P}_2(\Omega)$, where ω is again the canonical process on Ω , and the infimum is taken over all stopping times with respect to the completion of the canonical filtration under μ . We restrict ourselves to such Markovian optimal stopping problems for simplicity of the presentation, but note that one can prove the following result in more generality, following the same strategy.

Theorem 8.1. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be Lipschitz with Lipschitz constant $L \in (0, \infty)$. Then, for any $\mu, \nu \in \mathcal{P}_2(\Omega)$, we have the bound*

$$(8.1) \quad |v^f(\mu) - v^f(\nu)| \leq L \cdot \inf_{\pi \in \text{Cpl}_{\text{bc}}(\mu, \nu)} \mathbb{E}^\pi \left[\sup_{0 \leq t \leq T} |\omega_t - \bar{\omega}_t| \right].$$

We have the following immediate corollary, using only the fact that the synchronous coupling is bicausal by [7, Proposition 2.2].

Corollary 8.2. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be Lipschitz with Lipschitz constant $L \in (0, \infty)$. Then, for any $\mu, \nu \in \mathcal{P}_2(\Omega)$, we have the bound*

$$|v^f(\mu) - v^f(\nu)| \leq L \cdot \mathbb{E}^{\pi^{\text{sync}}} \left[\sup_{0 \leq t \leq T} |\omega_t - \bar{\omega}_t| \right].$$

Proof of Theorem 8.1. Let $\pi \in \text{Cpl}_c(\mu, \nu)$ and let (X, Y) be a pair of random variables on $\Omega \times \Omega$ with joint law π . As before, let \mathcal{F}^X and \mathcal{F}^Y denote the completions of the natural filtrations of X and Y under the measures μ and ν , respectively. Fix $\varepsilon > 0$, and take an ε -optimal \mathcal{F}^Y -stopping time τ^Y ; i.e. $\mathbb{E}[f(Y_{\tau^Y})] \leq v^f(\nu) + \varepsilon$.

For each $u \in [0, 1]$, define $\theta_u := \inf\{t \geq 0: \pi(\tau^Y \leq t \mid \mathcal{F}_t^X) \geq u\}$. We claim that θ_u is an \mathcal{F}^X -stopping time. By continuity of the paths of processes X and Y , we know that, for any $u \in [0, 1]$ and $t \in [0, T]$,

$$\{\theta_u \leq t\} = \{\pi(\tau^Y \leq t \mid \mathcal{F}_t^X) \geq u\}.$$

So it suffices to show that, for each $t \in [0, T]$, the conditional probability $\pi(\tau^Y \leq t \mid \mathcal{F}_t^X)$ is \mathcal{F}_t^X -measurable. Since $\pi \in \text{Cpl}_c(\mu, \nu)$, the causality condition (2.2) states that, under π , \mathcal{F}_t^Y is conditionally independent of \mathcal{F}_t^X , given \mathcal{F}_t^X . An equivalent formulation of this conditional independence condition (see, e.g. Kallenberg [34, Proposition 5.6]) is that for any $F \in \mathcal{F}_t^Y$,

$$(8.2) \quad \pi(F \mid \mathcal{F}_t^X \vee \mathcal{F}_t^X) = \pi(F \mid \mathcal{F}_t^X).$$

In fact, $\mathcal{F}_t^X \subset \mathcal{F}_t^X$, and so $\mathcal{F}_t^X \vee \mathcal{F}_t^X = \mathcal{F}_t^X$. By the definition of a stopping time, $\{\tau^Y \leq t\} \in \mathcal{F}_t^Y$. Therefore (8.2) gives us

$$\pi(\tau^Y \leq t \mid \mathcal{F}_t^X) = \pi(\tau^Y \leq t \mid \mathcal{F}_t^X),$$

which is then \mathcal{F}_t^X -measurable. This proves our claim that θ_u is an \mathcal{F}^X -stopping time for each $u \in [0, 1]$. Thus

$$v^f(\mu) \leq \inf_{u \in [0, 1]} \mathbb{E}^\pi[f(X_{\theta_u})] \leq \int_0^1 \mathbb{E}^\pi[f(X_{\theta_u})] du.$$

Applying the tower property of conditional expectation and Fubini's theorem to exchange the order of integration, we arrive at

$$(8.3) \quad v^f(\mu) \leq \mathbb{E}^\pi \left[\int_0^1 \mathbb{E}^\pi[f(X_{\theta_u}) \mid \mathcal{F}_T^X] du \right].$$

Now fix a path $\omega \in \Omega$ of the process X . We can define a probability measure \mathbb{P}_ω on $[0, T]$ by $\mathbb{P}_\omega([0, t]) = \pi(\tau^Y \leq t \mid \mathcal{F}_T^X)(\omega)$, for all $t \in [0, T]$. Then $u \mapsto \theta_u(\omega)$ is the quantile function of \mathbb{P}_ω , and we see that

$$\mathbb{E}^\pi[f(X_{\tau^Y}) \mid \mathcal{F}_T^X](\omega) = \int_0^T f(\omega_t) d\mathbb{P}_\omega(t) = \int_0^1 f(\omega_{\theta_u(\omega)}) du = \int_0^1 \mathbb{E}^\pi[f(X_{\theta_u}) \mid \mathcal{F}_T^X](\omega) du.$$

Taking the expectation with respect to π and substituting into (8.3), we find

$$v^f(\mu) \leq \mathbb{E}^\pi [\mathbb{E}^\pi[f(X_{\tau^Y}) \mid \mathcal{F}_T^X]] = \mathbb{E}^\pi[f(X_{\tau^Y})].$$

Hence $v^f(\mu) - v^f(\nu) \leq \mathbb{E}^\pi[f(X_{\tau^Y}) - f(Y_{\tau^Y})] + \varepsilon$. Since f is Lipschitz with Lipschitz constant L , we have

$$v^f(\mu) - v^f(\nu) \leq L \cdot \mathbb{E}^\pi[|X_{\tau^Y} - Y_{\tau^Y}|] + \varepsilon \leq L \cdot \mathbb{E}^\pi[\|X - Y\|_\infty] + \varepsilon.$$

Exchanging the roles of μ and ν and noting that $\varepsilon > 0$ and $\pi \in \text{Cpl}_c(\mu, \nu)$ were chosen arbitrarily completes the proof. \square

Remark 8.3. The quantity on the right-hand side of (8.1) is the adapted 1-Wasserstein distance with respect to the L^∞ norm on Ω . In general, such a bound does not hold for \mathcal{AW}_1 defined with respect to the L^1 norm on Ω , as in Definition 2.11. For example, take f to be the identity and ν concentrated on the constant zero function. Choose μ such that, for some $\varepsilon \in (0, 1)$, $\mathbb{E}^\mu[\int_0^T |\omega_t| dt] \leq \varepsilon$ and, for some $\delta \in [0, T]$, $\omega_\delta = 1$ μ -almost surely. Then $|v^f(\mu) - v^f(\nu)| = |v^f(\mu)| \geq 1$, while $\mathcal{AW}_1(\mu, \nu) \leq \varepsilon$. Therefore the bound (8.1) does not hold when the right-hand side is replaced by \mathcal{AW}_1 defined as in Definition 2.11.

In the case of SDEs with sufficiently regular coefficients, however, continuity of optimal stopping does hold for the adapted Wasserstein distance \mathcal{AW}_p defined with respect to the L^p norm on Ω for any $p \geq 1$. For $\Lambda > 0$, define $\mathcal{P}^\Lambda := \{\text{Law}(X) : X \text{ solves (1.1) with some } \Lambda\text{-Lipschitz } (b, \sigma)\}$.

Corollary 8.4. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be Lipschitz and let $\Lambda > 0$. Then v^f is continuous on \mathcal{P}^Λ with respect to \mathcal{AW}_p for any $p \geq 1$.*

Proof. By [7, Theorem 1.5], the adapted Wasserstein distance defined with respect to the L^∞ norm on Ω induces the same topology as \mathcal{AW}_p for all $p \geq 1$. Thus the result follows from Theorem 8.1. \square

Establishing conditions for optimality of the synchronous coupling for the adapted Wasserstein distance defined with respect to the L^∞ norm on Ω is an open problem. Thus, the right-hand side of (8.1) is not computationally tractable. In contrast to that, the bound in Corollary 8.2 does admit an efficient numerical approximation, following the approach of Section 7.1. Hence, our methods provide computationally feasible bounds on the values of optimisation problems, making such problems tractable for applications.

APPENDIX A. PROOFS FROM SECTION 3

Proof of Proposition 3.2. For $h > 0$, Assumption 3.1 (i) implies that the Knothe–Rosenblatt rearrangement is optimal for (2.5) between μ^h and ν^h , by Proposition 2.9. Supposing that the

discretised processes (X^h, \bar{X}^h) are driven by a common Brownian motion, Assumption 3.1 (ii) implies that the Knothe–Rosenblatt rearrangement is equal to $\text{Law}(X^h, \bar{X}^h)$, by Proposition 2.8. This together with Assumption 3.1 (iii) allows us to conclude that the limit (3.1) holds and the synchronous coupling attains the value (BCOT) of the continuous-time bicausal transport problem. This final step follows along the same lines as in the proof of [7, Proposition 3.24], and we omit the details here. \square

Proof of Lemma 3.3. Let $\pi \in \text{Cpl}_{\text{bc}}(\mu, \nu)$. Then by [7, Proposition 2.2] on the relation between bicausal couplings and correlated Brownian motions, there exists a correlated Brownian motion (W, \bar{W}) such that $\pi = \text{Law}(X, \bar{X})$ when (2.6) is driven by (W, \bar{W}) . Taking the same driving Brownian motion, we write $\pi^N := \text{Law}(X^N, \bar{X}^N) \in \text{Cpl}_{\text{bc}}(\mu^N, \nu^N)$, for each $N \in \mathbb{N}$. Interpreting each such system of SDEs as a two-dimensional SDE, the stability result of Stroock and Varadhan [74, Theorem 11.1.4] implies the weak convergence $\pi^N \rightarrow \pi$ as $N \rightarrow \infty$. Moreover, the uniform linear growth bound implies that, for any $p \geq 1$, there exists a constant $C > 0$ such that $\mathbb{E}^{\pi^N}[\sup_{t \in [0, T]} |(\omega_t, \bar{\omega}_t)|^p] \leq C$ for all $N \in \mathbb{N}$; see, e.g. [22, Lemma 3.8]. Thus $\mathbb{E}^{\pi^N}[\phi(\omega, \bar{\omega})] \rightarrow \mathbb{E}^\pi[\phi(\omega, \bar{\omega})]$, for any functional $\phi: \Omega \times \Omega \rightarrow \mathbb{R}$ that is continuous with order p polynomial growth with respect to the uniform norm on $\Omega \times \Omega$; see, e.g. [78, Definition 6.8]. We claim that $(\omega, \bar{\omega}) \mapsto \int_0^T c_t(\omega_t, \bar{\omega}_t) dt$ satisfies these conditions. Indeed the polynomial growth of order p follows immediately from the bound (2.3), which holds uniformly in t . Continuity follows from (2.3) and the continuity of the maps $(x, y) \mapsto c_t(x, y)$ for each $t \in [0, T]$ via the Lebesgue–Vitali dominated convergence theorem. Hence

$$(A.1) \quad \mathbb{E}^{\pi^N} \left[\int_0^T c_t(\omega_t, \bar{\omega}_t) dt \right] \xrightarrow{N \rightarrow \infty} \mathbb{E}^\pi \left[\int_0^T c_t(\omega_t, \bar{\omega}_t) dt \right].$$

Now take a common one-dimensional Brownian motion driving (2.6) for each set of coefficients. Then we have

$$\pi_{\mu^N, \nu^N}^{\text{sync}} = \text{Law}(X^N, \bar{X}^N) \xrightarrow{N \rightarrow \infty} \text{Law}(X, \bar{X}) = \pi_{\mu, \nu}^{\text{sync}}.$$

Given that $\pi_{\mu^N, \nu^N}^{\text{sync}}$ minimises the left-hand side of (A.1) over $\text{Cpl}_{\text{bc}}(\mu^N, \nu^N)$ for each $N \in \mathbb{N}$, we have that $\pi_{\mu, \nu}^{\text{sync}}$ minimises the right-hand side of (A.1) over $\text{Cpl}_{\text{bc}}(\mu, \nu)$ and the required convergence holds. \square

Proof of Proposition 3.5. Under Assumption 3.4, we can show that the monotone Euler–Maruyama scheme satisfies Assumption 3.1, following the proofs of [7, Lemma 3.15, 3.16, 3.17]. The result then follows from Proposition 3.2. \square

Proof of Corollary 3.7. For each SDE, existence of a weak solution is guaranteed by Skorokhod [72, Chapter 3, Section 3], and given that we assumed pathwise uniqueness, Yamada and Watanabe [79, Corollary 1] implies that there exists a unique strong solution. We can approximate the coefficients (b, σ) and $(\bar{b}, \bar{\sigma})$ locally uniformly in space by coefficients satisfying Assumption 3.4 and the conditions of Lemma 3.3. We conclude by combining Proposition 3.5 and Lemma 3.3. \square

APPENDIX B. PROOFS FROM SECTION 4.1

Proof of Lemma 4.6. Direct computation of the first and second derivatives of $\bar{\phi}_k$ yield the first three points. We also observe that the first derivative is continuous and bounded, and hence $\bar{\phi}_k$ is Lipschitz. The second derivative is defined almost everywhere and is continuous and bounded on $(-\infty, \xi_k) \cup (\xi_k, \infty)$. Combined with the fact that the one-sided second derivatives at ξ_k exist and are finite, this shows that $\bar{\phi}'_k$ is Lipschitz. Finally, differentiating $\bar{\phi}''_k$ once again on $(-\infty, \xi_k) \cup (\xi_k, \infty)$ gives a bounded third derivative, and hence $\bar{\phi}''_k$ is piecewise Lipschitz. \square

Proof of Lemma 4.7. By Lemma 4.6, $\bar{\phi}_k(x) = 0$ for $|x - \xi_k| \geq c_0$ for each $k \in \{1, \dots, m\}$, and so we have $G(x) = x$ for $x \in \mathbb{R} \setminus \bigcup_{k \in \{1, \dots, m\}} (\xi_k - c_0, \xi_k + c_0)$. Moreover, for any $k \in \{1, \dots, m\}$ and $x \in [\xi_k - c_0, \xi_k + c_0]$, $G(x) = x + \alpha_k \bar{\phi}_k(x)$ and $G'(x) = 1 + \alpha_k \bar{\phi}'_k(x)$, implying that $G'(x) \in [1 - 6c_0|\alpha_k|, 1 + 6c_0|\alpha_k|]$ by Lemma 4.6. Thus, as in [42, Lemma 2.2], we observe that our choice of c_0 in (4.6) guarantees that G is strictly increasing on \mathbb{R} and therefore admits a strictly increasing inverse $G^{-1}: \mathbb{R} \rightarrow \mathbb{R}$. From the definition of G and the properties proved in Lemma 4.6, we immediately deduce that G is bounded and Lipschitz with bounded Lipschitz first derivative G' and bounded piecewise Lipschitz almost-everywhere second derivative G'' . By the inverse function theorem, G^{-1} has the same properties. \square

Proof of Lemma 4.9. As in Assumption 4.1, let $L_f, K_f, \gamma, \eta \in (0, \infty)$ denote the constants that appear in the one-sided Lipschitz condition, local Lipschitz condition and exponential growth condition for f . Write $L_g \in (0, \infty)$ for a Lipschitz constant of g , and $M_g := \sup_{x \in \mathbb{R}} |g(x)| < \infty$. Let $k \in \{0, \dots, m\}$. By Assumption 4.1 (A1) (i), f is absolutely continuous on (ξ_k, ξ_{k+1}) and, as noted in Remark 4.5, the one-sided limits $f(\xi_k-), f(\xi_k+)$ exist and are finite, and so $M_f := \sup_{x \in (\xi_k, \xi_{k+1})} |f(x)| < \infty$. Therefore, $(f \cdot g): (\xi_k, \xi_{k+1}) \rightarrow \mathbb{R}$ is absolutely continuous, as the product of bounded absolutely continuous functions. Now, for any $x, y \in (\xi_k, \xi_{k+1})$,

$$\begin{aligned} (x - y)(f(x)g(x) - f(y)g(y)) &= (x - y)(f(x) - f(y))g(x) + (x - y)(g(x) - g(y))f(y) \\ &\leq (L_f M_g + L_g M_f) |x - y|^2, \end{aligned}$$

and also

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &\leq |f(x) - f(y)||g(x)| + |f(y)||g(x) - g(y)| \\ &\leq K_f M_g (\exp\{\gamma|x|^\eta\} + \exp\{\gamma|y|^\eta\}) |x - y| + L_g M_f |x - y| \\ &\leq (K_f M_g + L_g M_f) (\exp\{\gamma|x|^\eta\} + \exp\{\gamma|y|^\eta\}) |x - y|. \end{aligned}$$

Thus, $(f \cdot g): \mathbb{R} \rightarrow \mathbb{R}$ satisfies Assumption 4.1 (A1). \square

Proof of Lemma 4.10. By Assumption 4.1 (A1) and the continuity of f , Assumption 4.1 (A1) (i)–(iii) also hold on the closed intervals $[\xi_k, \xi_{k+1}]$, for all $k \in \{1, \dots, m-1\}$. By absolute continuity on the closed intervals, we can define $f' \in L^1(\mathbb{R})$ that agrees with the almost-everywhere derivative of f on each open interval. Linearity of the integral then implies that f' is the almost-everywhere derivative of f on \mathbb{R} . Therefore f is absolutely continuous on \mathbb{R} . In this spirit, it is straightforward to show that Assumption 4.1 (A1) (ii) and (iii) also extend to the whole of \mathbb{R} . \square

Proof of Lemma 4.11. First, as in [42], we note that the constants $\alpha_k, k \in \{1, \dots, m\}$ are chosen in (4.5) such that $bG' + 1/2\sigma^2 G''$ is continuous on \mathbb{R} . Indeed by Lemma 4.6, for each $k \in \{1, \dots, m\}$, $G'(\xi_k) = 1 + \alpha_k \bar{\phi}'_k(\xi_k) = 1$, $G''(\xi_k+) = \alpha_k \bar{\phi}''_k(\xi_k+) = 2\alpha_k$, $G''(\xi_k-) = -2\alpha_k$, and so

$$(bG' + \frac{1}{2}\sigma^2 G'')(\xi_k+) - (bG' + \frac{1}{2}\sigma^2 G'')(\xi_k-) = b(\xi_k+) - b(\xi_k-) + 2\alpha_k \sigma^2(\xi_k) = 0.$$

Continuity on \mathbb{R} then follows from Lemma 4.7. Also by Lemma 4.7, G' is bounded and Lipschitz. Therefore, by Lemma 4.9, bG' satisfies Assumption 4.1 (A1). The product $\sigma^2 G''$ is zero on $\mathbb{R} \setminus \bigcup_{k \in \{1, \dots, m\}} (\xi_k - c_0, \xi_k + c_0)$, and σ^2 and G'' , and hence their product, are bounded and Lipschitz on each interval $[\xi_k - c_0, \xi_k + c_0]$, $k \in \{1, \dots, m\}$. Thus $\sigma^2 G''$ satisfies Assumption 4.1 (A1). Taking the sum, $bG' + 1/2\sigma^2 G''$ satisfies Assumption 4.1 (A1), is continuous on \mathbb{R} , and hence satisfies Assumption 4.1 (A1) (i)–(iii) globally on \mathbb{R} , by Lemma 4.10. Taking the composition with G^{-1} , which is also Lipschitz by Lemma 4.7, preserves these properties, and we conclude that \tilde{b} satisfies statements (i)–(iii) of the lemma.

Now, $\sigma G'$ is globally Lipschitz, as the product of the Lipschitz function σ and the bounded Lipschitz function G' that is zero on $\mathbb{R} \setminus \bigcup_{k \in \{1, \dots, m\}} (\xi_k - c_0, \xi_k + c_0)$. Taking the composition

with the Lipschitz function G^{-1} preserves the Lipschitz property, and this concludes the proof of statement (iv).

Finally, statement (v) of the lemma follows from statements (ii) and (iv). \square

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