An SDE with no strong solution arising from a problem of stochastic control

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Introduction and Motivation

Problem

Minimise

$$\mathbb{E}\left[\int_0^{\tau_D} f(X_s)\,\mathrm{d}s + g(X_{\tau_D})\right]$$

over all continuous martingales X with fixed quadratic variation, defined on some bounded domain

$$D \subset \mathbb{R}^2.$$

Motivation

- Under minimal modelling assumptions, find best case models
- Connections to martingale optimal transport

Problem Formulation

Problem Statement

Fix a probability space on which a 1-dimensional Brownian motion B is defined, with natural filtration $\mathbb{F}$.

Let X^σ be a strong solution to

$$\mathrm{d}X_t^\sigma = \sigma_t \, \mathrm{d}B_t; \quad X_0^\sigma = x,$$

for processes $(\sigma_t)_{t\geq 0} \in \mathcal{U}$, where $\mathcal{U}$ is the set of $\mathbb{F}$-progressively measurable processes taking values in

$$U = \{\sigma \in \mathbb{R}^2 \colon \operatorname{Tr}(\sigma\sigma^\top) = 1\}.$$

Find the value function

$$v(x) := \inf_{\sigma\in\mathcal{U}} \mathbb{E}^x \left[\int_0^\tau f(X_s^\sigma) \, \mathrm{d}s + g(X_\tau^\sigma) \right].$$

Assumptions

$$v(x) := \inf_{\sigma \in \mathcal{U}} \mathbb{E}^x \left[\int_0^\tau f(X_s^\sigma)\, \mathrm{d}s + g(X_\tau^\sigma) \right].$$

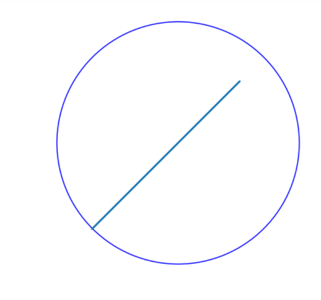
1. $D = B_R(0) \subset \mathbb{R}^2$
2. f radially symmetric; i.e. $f(x) = \tilde{f}(|x|)$
3. g constant
4. f continuous
5. $\tilde{f}'(r+)$ exists for all $r \geq 0$ with $\lim_{r \to 0} r \tilde{f}'(r) = 0$

Optimal Behaviour

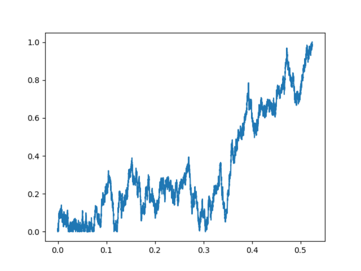
Radial Motion

Optimal behaviour for $\tilde{f}$ increasing

- Control: $$\sigma_t = \frac{1}{|x|}x$$
- Radius process: $$\mathrm{d}R_t = \mathrm{d}W_t$$



Sample path of X_t

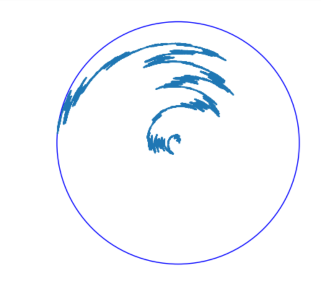


Sample path of R_t

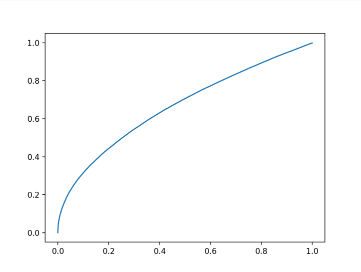
Tangential Motion

Optimal behaviour for $\tilde{f}$ decreasing

- Control:
$$\sigma_t = \frac{1}{|X_t|} X_t^{\perp}$$
- Radius process:
$$\mathrm{d}R_t = \frac{1}{2R_t}\,\mathrm{d}t \quad \Rightarrow \quad R_t = \sqrt{|x| + t}$$



Sample path of X_t



Sample path of R_t

Construction of solution

Under our assumptions the optimal strategy is to switch between radial and tangential motion.

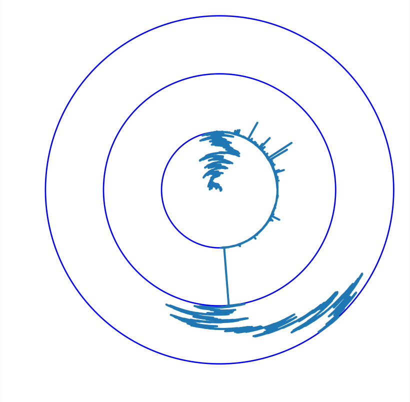


Figure 3: Possible trajectory

Proof of Optimality

Under the given assumptions, we use the theory of viscosity solutions to show optimality:

1. The value function v is continuous and M-convex
2. v satisfies a dynamic programming principle
3. v is the unique viscosity solution to

$$\begin{cases} \inf_{\sigma \in U} \mathrm{Tr}(D^2 v \sigma \sigma^\top) = -f & \text{in} \quad D \\ v = g & \text{on} \quad \partial D \end{cases} \tag{HJB}$$

4. Find switching points to construct candidate value function V
5. The candidate function V solves (HJB)

Hence $v = V$.

Behaviour at the Origin

Relaxing Assumptions

$$v(x) := \inf_{\sigma \in \mathcal{U}} \mathbb{E}^x \left[\int_0^\tau f(X_s^\sigma)\, \mathrm{d}s + g(X_\tau^\sigma) \right].$$

1. $D = B_R(0) \subset \mathbb{R}^2$
2. f radially symmetric; i.e. $f(x) = \tilde{f}(|x|)$
3. g constant
4. f continuous
5. $\tilde{f}'(r+)$ exists for all $r \geq 0$ with $\lim_{r \to 0} r\tilde{f}'(r) = 0$

Relaxing Assumptions

$$v(x) := \inf_{\sigma \in \mathcal{U}} \mathbb{E}^x \left[\int_0^\tau f(X_s^\sigma)\,\mathrm{d}s + g(X_\tau^\sigma) \right].$$

1. $D = B_R(0) \subset \mathbb{R}^2$
2. f radially symmetric; i.e. $f(x) = \tilde{f}(|x|)$
3. g constant
4. f continuous in $D \setminus \{0\}$

5.* $\tilde{f}'(r+)$ exists for all $r \geq 0$

Behaviour at the origin

If X moves tangentially at the origin, solving

$$\mathrm{d}X_t = \frac{1}{|X_t|} X_t^{\perp} \, \mathrm{d}B_t; \quad X_0 = 0,$$

then the cost up to leaving a ball $B_\varepsilon(0)$ is

$$\mathbb{E}^0 \left[\int_0^{\tau_\varepsilon} f(X_s) \, \mathrm{d}s \right] = 2 \int_0^{\varepsilon} \xi \tilde{f}(\xi) \, \mathrm{d}\xi.$$

Claim: For $\tilde{f}(r) \sim -\frac{1}{r^\beta}$, $\beta \in [1, 2)$,

all other admissible strategies have a strictly greater cost.

An SDE with No Strong Solution

An SDE with no strong solution

Theorem

The SDE

$$\mathrm{d}X_t = \frac{1}{|X_t|} X_t^{\perp} \, \mathrm{d}B_t; \quad X_0 = 0$$

has no strong solution.

Known SDEs with no strong solution

Tanaka's SDE

$$\mathrm{d}X_t = \operatorname{sign}(X_t)\,\mathrm{d}B_t$$

Key idea:

$$\mathcal{F}_t^B \subseteq \mathcal{F}_t^{|X|} \subsetneq \mathcal{F}_t^X$$

Tsirelson's SDE

$$\mathrm{d}X_t = b(t, (X_s)_{s\leq t})\,\mathrm{d}t + \mathrm{d}B_t$$

Key idea:

$b(t, (X_s)_{s\leq t})$ is uniform on $[0, 1)$ and independent of $\mathcal{F}_\infty^B$.

An SDE with no strong solution

Theorem
The SDE

$$\mathrm{d}X_t = \frac{1}{|X_t|} X_t^{\perp} \, \mathrm{d}B_t; \quad X_0 = 0$$

has no strong solution.

- The proof uses ideas from the study of Tsirelson's equation.
- We introduce Circular Brownian Motion, as in [Émery and Schachermayer, 1999].

Sketch of Proof

Write solutions $t \mapsto X_t \in \mathbb{R}^2$ to the SDE as

$$X_t = R_t \begin{bmatrix} \cos\theta_t \\ \sin\theta_t \end{bmatrix}.$$

Then

$$R_t = \sqrt{t}$$

and θ satisfies

$$\mathrm{d}\theta_t = t^{-\frac{1}{2}}\,\mathrm{d}B_t.$$

Then θ is a deterministic time change of a circular Brownian motion (CBM).

Sketch of Proof

Introduce the innovation filtration $\mathcal{H}$ of θ:

$$\mathcal{H}_t := \sigma\left(\{\theta_s - \theta_r \colon r < s \leq t\}\right).$$

By a result of [Émery and Schachermayer, 1999],

- θ_t is uniform on $[0, 2\pi)$;
- θ_t is independent of $\mathcal{H}_\infty$.

In particular,

$$\mathcal{H}_t \subsetneq \mathcal{F}^\theta_t.$$

Sketch of Proof

We have shown that

$$\mathcal{H}_t \subsetneq \mathcal{F}_t^{\theta}.$$

However,

$$B_t - B_s = \int_s^t r^{\frac{1}{2}}\,\mathrm{d}\theta_t \quad \text{is } \mathcal{H}_t\text{-measurable,}$$

and so, since $B_s \to 0$ as $s \to 0$,

$$\mathcal{F}_t^{B} \subseteq \mathcal{H}_t.$$

Hence θ is not adapted to the natural filtration of B.

Gap between weak and strong values

Theorem

The SDE

$$\mathrm{d}X_t = \frac{1}{|X_t|} X_t^{\perp} \, \mathrm{d}B_t; \quad X_0 = 0$$

has no strong solution.

However, there is a weak solution.

We can define a weak value function as in [El Karoui and Tan, 2013]

$$v^W(x) := \inf_{\mathbb{P} \in \mathcal{P}_x} \mathbb{E}^{\mathbb{P}} \left[\int_0^{\tau} f(X_s) \, \mathrm{d}s + g(X_\tau) \right].$$

Then, for $\tilde{f}(r) \sim -\frac{1}{r^\beta}$, $\beta \in [1, 2)$, the weak solution to the above SDE attains the weak value at the origin.

Gap between weak and strong values

$$v(x) := \inf_{\sigma \in \mathcal{U}} \mathbb{E}^x \left[\int_0^\tau f(X_s^\sigma)\,\mathrm{d}s + g(X_\tau^\sigma) \right]$$

$$v^W(x) := \inf_{\mathbb{P} \in \mathcal{P}_x} \mathbb{E}^{\mathbb{P}} \left[\int_0^\tau f(X_s)\,\mathrm{d}s + g(X_\tau) \right]$$

Conjecture

Suppose that there exists $\alpha \in (0, \infty)$ *and* $\beta^* \in [1, 2)$ *such that*

$$\lim_{r \to 0} r^\beta \tilde{f}(r) = \begin{cases} +\infty, & \beta < \beta^*, \\ \alpha, & \beta = \beta^*. \end{cases}$$

Then

$$v^W(0) < v(0).$$

References

El Karoui, N. and Tan, X. (2013).
Capacities, measurable selection and dynamic programming Part II: application in stochastic control problems.
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Émery, M. and Schachermayer, W. (1999).
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In Azéma, J., Émery, M., Ledoux, M., and Yor, M., editors, *Séminaire de Probabilités XXXIII*, volume 1709, pages 291–303. Springer Berlin Heidelberg, Berlin, Heidelberg.