Optimal control of martingales in a radially symmetric environment and an SDE with no strong solution

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Problem statement

Minimise

$$\mathbb{E}\left[\int_0^{\tau_D} f(X_s)\,\mathrm{d}s + g(X_{\tau_D})\right]$$

over all continuous martingales X with unit quadratic variation, defined on some bounded domain

$$D \subset \mathbb{R}^d.$$

Motivation

Problem formulation

Optimal behaviour

Construction of explicit solution

Extension of results

An SDE with no strong solution

Motivation

Motivation

An example of a problem in mathematical finance is to find

$$v(\mathbb{P}) = \sup_{\tau} \mathbb{E}^{\mathbb{P}} \left[e^{-r\tau} \left(K - |X_\tau| \right)_+ \right].$$

This value may be very sensitive to the choice of the measure $\mathbb{P}$.

Robust finance is concerned with finding model-independent bounds such as

$$\inf_{\mathbb{P}} v(\mathbb{P}) = \sup_{\tau} \inf_{\mathbb{P}} \mathbb{E}^{\mathbb{P}} \left[e^{-r\tau} \left(K - |X_\tau| \right)_+ \right].$$

These bounds can be found using techniques from Skorokhod embedding and martingale optimal transport.

Problem formulation

Strong formulation

Control set: Define $U := \left\{\sigma \in \mathbb{R}^{d,d} \colon\ \mathrm{Tr}(\sigma\sigma^\top) = 1\right\}$

Fix a probability space on which a d-dimensional Brownian motion B is defined, with natural filtration $\mathbb{F}$.

Let $\mathcal{U}$ be the set of U-valued $\mathbb{F}$-progressively measurable processes.

Dynamics: For $x \in D$ and $\sigma \in \mathcal{U}$, let X^σ be a strong solution to

$$\mathrm{d}X_t = \sigma_t \,\mathrm{d}B_t; \quad X_0 = x.$$

Value function: Find the strong value function $v^S : D \to \mathbb{R}$,

$$v^S(x) := \inf_{\sigma \in \mathcal{U}} \mathbb{E}^x \left[\int_0^\tau f(X_s^\sigma)\,\mathrm{d}s + g(X_\tau^\sigma)\right]$$

Weak formulation

Control set: Define $U := \left\{\sigma \in \mathbb{R}^{d,d} \colon \operatorname{Tr}(\sigma\sigma^\top) = 1\right\}$

Dynamics: Fix $x \in D$ and let $\mathcal{P}_x$ be the set of probability measures on $\Omega \times \mathcal{B}(\mathbb{R}_+, U)$ such that, for any $\mathbb{P} \in \mathcal{P}_x$,

- The process

  $$t \mapsto \phi(X_t) - \phi(X_0) - \frac{1}{2}\int_0^t \operatorname{Tr}(D^2\phi(X_s)\nu_s\nu_s^\top)\,\mathrm{d}s$$

  is a martingale for any $\phi \in C^2(D)$, and
- $\mathbb{P}(X_0 = x) = 1$.

Value function: Find the weak value function $v^W : D \to \mathbb{R}$,

$$v^W(x) := \inf_{\mathbb{P} \in \mathcal{P}_x} \mathbb{E}^{\mathbb{P}}\left[\int_0^\tau f(X_s)\,\mathrm{d}s + g(X_\tau)\right]$$

Assumptions

$$v^W(x) := \inf_{\mathbb{P}\in\mathcal{P}_x} \mathbb{E}^{\mathbb{P}}\left[\int_0^\tau f(X_s)\,\mathrm{d}s + g(X_\tau)\right],$$

1. $D = B_R(0) \subset \mathbb{R}^d$
2. f is radially symmetric; i.e. $f(x) = \tilde{f}(|x|)$
3. g is constant
4. f is continuous
5. $\tilde{f}'_+(r)$ exists for all $r \geq 0$
6. $\tilde{f}$ is monotone and sufficiently smooth near the origin

Theorem [El Karoui and Tan, 2013]

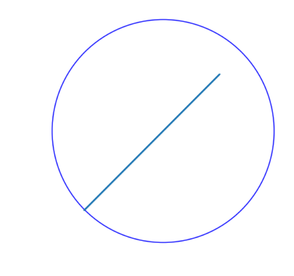
The weak and strong formulations are equivalent:

$$v := v^W = v^S.$$

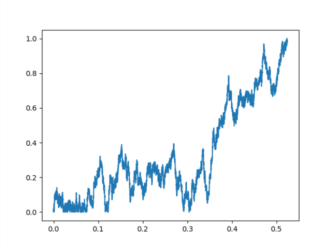
Optimal behaviour

Radial motion

Optimal behaviour for $\tilde{f}$ monotonically increasing



Sample path of X_t

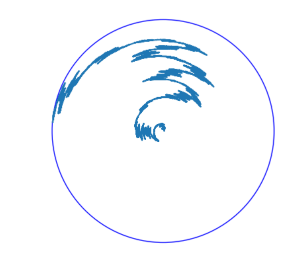


Sample path of R_t

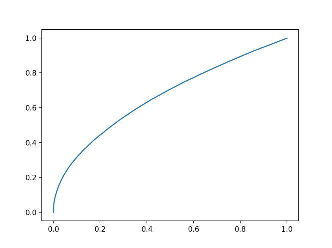
- Control:
$$\sigma_t = \frac{1}{|x|}[x; 0; \ldots ; 0]$$
- Radius process:
$$\mathrm{d}R_t = \mathrm{d}W_t$$

Tangential motion

Optimal behaviour for $\tilde{f}$ monotonically decreasing



Sample path of X_t



Sample path of R_t

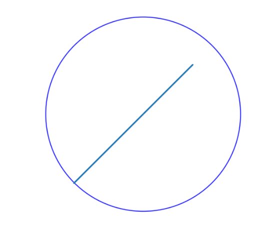
- Control:

$$\sigma_t = \frac{1}{|X_t|}[X_t^{\perp}; 0; \ldots ; 0]$$

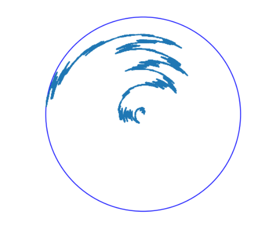
- Radius process:

$$\mathrm{d}R_t = \frac{1}{2R_t}\,\mathrm{d}t \quad \Rightarrow \quad R_t = \sqrt{|x| + t}$$

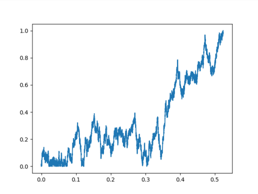
Two optimal behaviour regimes



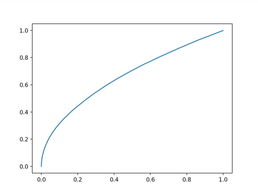
(a) Sample path of radial motion



(b) Sample path of tangential motion



(c) Sample path of radius process for (a)



(d) Sample path of radius process for (b)

Two optimal behaviour regimes

Claim

For any f satisfying Assumptions 1–6, an optimal strategy is to switch between radial and tangential motion.

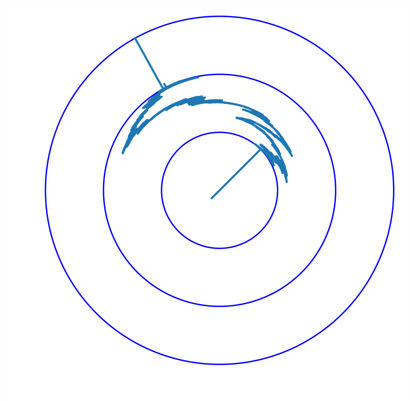


Figure 4: A possible optimal trajectory

Construction of explicit solution

Method of solution

1. Prove that the value function v is the unique viscosity solution to

$$\begin{cases} -\frac{1}{2}\inf_{\sigma\in U}\mathrm{Tr}(D^2 v\sigma\sigma^\top) = f & \text{in} \quad D \\ v = g & \text{on} \quad \partial D \end{cases} \tag{HJB}$$

2. Find switching points to construct candidate value function V
3. Show that the candidate function V solves (HJB)

Construction of value function

Claim that the optimal strategy is to switch between radial and tangential motion.

Then $v(x) = \tilde{v}(|x|)$, where

- $\tilde{v}(R) = g$
- and, for $r \in (0, R)$, either

$$-\frac{1}{2}\tilde{v}''(r) = \tilde{f}(r), \quad \text{or}$$
$$-\frac{1}{2r}\tilde{v}'(r) = \tilde{f}(r).$$

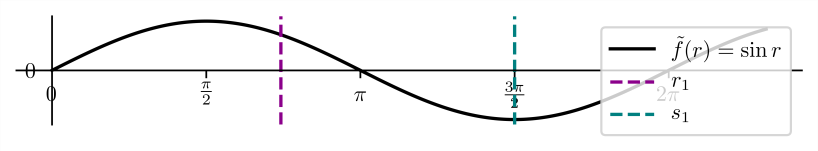
To minimise

$$\tilde{v}(r) = g - \int_r^R \tilde{v}'(s)\,\mathrm{d}s,$$

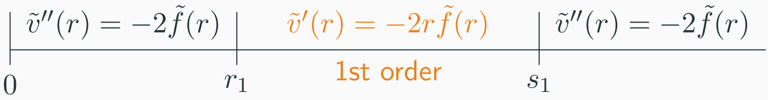
we seek to maximise $\tilde{v}'(r)$.

An example

Consider the cost function $f(x) = \sin(|x|)$

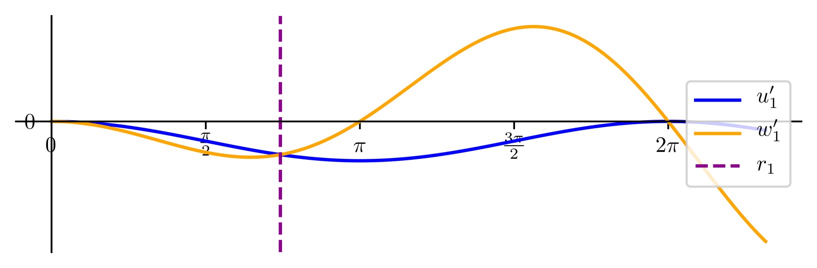


Switching points



Return to the example

$$u_1''(r) = -2\tilde{f}(r), \quad (u_1)'_+(0) = 0$$
$$w_1'(r) = -2r\tilde{f}(r)$$



Switching points

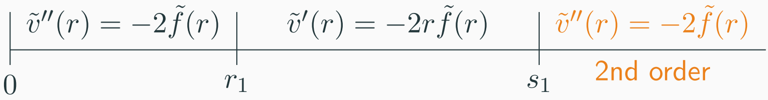
$\tilde{v}''(r) = -2\tilde{f}(r)$	$\tilde{v}'(r) = -2r\tilde{f}(r)$	$\tilde{v}''(r) = -2\tilde{f}(r)$
0	r_1 — 1st order	s_1

Switching point is determined by

$$\begin{aligned} r_1 &= \inf\{r > s_0 \colon \tilde{v}'(r) < -2r\tilde{f}(r)\} \\ &= \inf\{r > s_0 \colon \int_0^r \tilde{f}(s)\,\mathrm{d}s > r\tilde{f}(r)\}. \end{aligned}$$

By continuity of f, we have smooth fit at r_1, even though the local time is zero.

Switching points

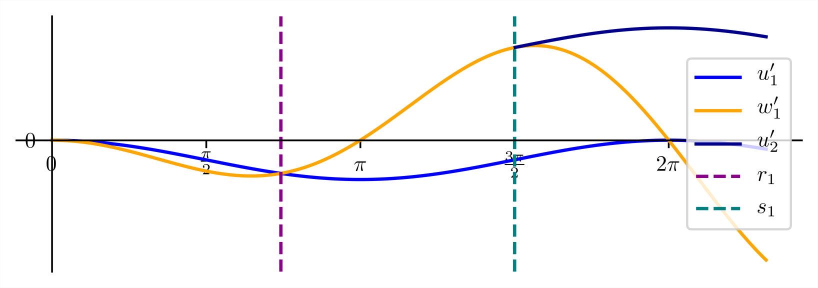


We need to enforce smooth fit at s_1

Return to the example

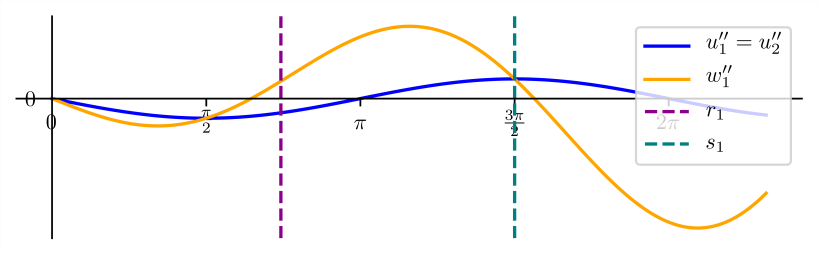
$$w_1'(r) = -2r\tilde{f}(r)$$

$$u_2''(r) = -2\tilde{f}(r), \quad (u_2)'_+(s_1) = w_1'(s_1)$$



Return to the example

$$w_1'(r) = -2r\tilde{f}(r)$$
$$u_2''(r) = -2\tilde{f}(r), \quad (u_2)'_+(s_1) = w_1'(s_1)$$



Switching points

$\tilde{v}''(r) = -2\tilde{f}(r)$ | $\tilde{v}'(r) = -2r\tilde{f}(r)$ | $\tilde{v}''(r) = -2\tilde{f}(r)$

0 | r_1 | s_1 | 2nd order

We need to enforce smooth fit at s_1, and we need a 2nd order condition to determine the switching point:

$$\begin{aligned} s_1 &= \inf\{r > s_0 \colon \tilde{v}''_+(r) < -2\tilde{f}(r)\} \\ &= \inf\{r > s_0 \colon \tilde{f}'_+(r) > 0\}. \end{aligned}$$

Switching points

Continue in this way to construct a sequence of switching points

$$s_0 < r_1 < \ldots < r_i < s_i < \ldots,$$

with

$$r_i := \inf \left\{ r > s_{i-1} \colon \int_{s_{i-1}}^{r} \tilde{f}(s) \, \mathrm{d}s > r \tilde{f}(r) \right\},$$

and

$$s_i := \inf \left\{ r > r_i \colon \tilde{f}'_+(r) > 0 \right\}.$$

We arrive at the following candidate value function $V : D \to \mathbb{R}$.

Candidate value function

Case 1: If $\tilde{f}$ is increasing in $(0,\eta)$, then set $s_0 = 0$ and let $K \in \mathbb{N}$ be such that $R \in (s_{K-1}, s_K]$. For $x \in D$, define

$$\begin{aligned} V(x) &= g - 2\int_{R\vee r_K}^{s_K} s\tilde{f}(s)\,\mathrm{d}s \\ &\quad - 2(r_K - R\wedge r_K)s_{K-1}\tilde{f}(s_{K-1}) - 2\int_{R\wedge r_K}^{r_K}\int_{s_{K-1}}^{s}\tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s \\ &\quad + 2\sum_{i=1}^{K}\mathbb{1}_{\{(s_{i-1},s_i]\}}(|x|)\Big[(r_i - |x|\wedge r_i)s_{i-1}\tilde{f}(s_{-1}) \\ &\qquad\qquad + \int_{|x|\wedge r_i}^{r_i}\int_{s_{i-1}}^{s}\tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s + \int_{|x|\vee r_i}^{s_i} s\tilde{f}(s)\,\mathrm{d}s + \mathfrak{F}_i^K\Big]. \end{aligned}$$

Candidate value function

Case 2: If $\tilde{f}$ is decreasing in $(0,\eta)$, then set $r_0 = 0$ and let $L \in \mathbb{N}$ be such that $R \in (r_L, r_{L+1}]$. For $x \in D$, define

$$\begin{aligned} V(x) &= g - 2\int_{R\wedge s_L}^{s_L} s\tilde{f}(s)\,\mathrm{d}s \\ &\quad + 2(R\vee s_L - s_L)s_L\tilde{f}(s_L) + 2\int_{s_L}^{R\vee s_L}\int_{s_L}^{s}\tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s \\ &\quad + 2\sum_{i=0}^{L}\mathbb{1}_{\{(r_i,r_{i+1}]\}}(|x|)\Bigg[\int_{|x|\wedge s_i}^{s_i} s\tilde{f}(s)\,\mathrm{d}s - (|x|\vee s_i - s_i)s_i\tilde{f}(s_i) \\ &\qquad\qquad - \int_{s_i}^{|x|\vee s_i}\int_{s_i}^{s}\tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s + \mathfrak{F}_i^L\Bigg]. \end{aligned}$$

Candidate value function

There exist constants C_i, $\tilde{C}_i$ such that

$$V(x) = \begin{cases} -2\int_{s_{i-1}}^{|x|}\int_{s_{i-1}}^{s}\tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s - 2\,|x|\,s_{i-1}\tilde{f}(s_{i-1}) + C_i, & |x| \in [s_{i-1}, r_i], \\ -2\int_{r_i}^{|x|} s\tilde{f}(s)\,\mathrm{d}s + \tilde{C}_i, & |x| \in [r_i, s_i]. \end{cases}$$

Theorem [Cox and R. 2020+]

Under Assumptions 1–6, the value function is given by

$$v = V.$$

Idea of the proof

1. Prove that the value function v
   - is continuous and semi-convex
   - satisfies a dynamic programming principle
   - is the unique viscosity solution to

$$\begin{cases} -\frac{1}{2}\inf_{\sigma \in U} \mathrm{Tr}(D^2 v \sigma \sigma^\top) = f & \text{in} \quad D \\ v = g & \text{on} \quad \partial D \end{cases} \tag{HJB}$$

2. Verify that V solves (HJB)
3. Conclude that $v = V$

Extension of results

Original assumptions

We now relax the assumptions:

$$v(x) := \inf_{\mathbb{P}\in\mathcal{P}_x} \mathbb{E}^{\mathbb{P}}\left[\int_0^\tau f(X_s)\,\mathrm{d}s + g(X_\tau)\right],$$

1. $D = B_R(0)$
2. f radially symmetric; i.e. $f(x) = \tilde{f}(|x|)$
3. g constant
4. f continuous
5. $\tilde{f}'(r+)$ exists for all $r \geq 0$
6. $\tilde{f}$ is monotone and sufficiently smooth near the origin

Relaxed assumptions

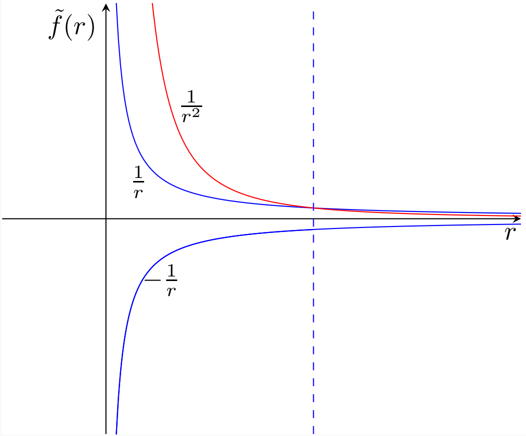
We now relax the assumptions:

$$v(x) := \inf_{\mathbb{P} \in \mathcal{P}_x} \mathbb{E}^{\mathbb{P}} \left[\int_0^\tau f(X_s) \, \mathrm{d}s + g(X_\tau) \right],$$

1. $D = B_R(0)$
2. f radially symmetric; i.e. $f(x) = \tilde{f}(|x|)$
3. g constant
4. f continuous in $D \setminus \{0\}$
5. $\tilde{f}'(r+)$ exists for all $r \geq 0$
6. $\tilde{f}$ is monotone near the origin

Extension of results

We can extend our results to allow the cost function to explode at the origin.



Extension of results

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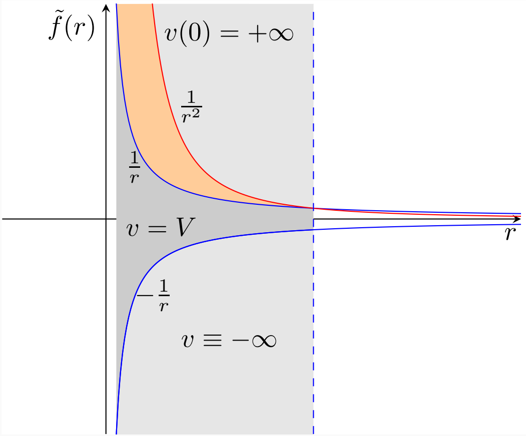


Gap between weak and strong values

Conjecture

Suppose that there exists $\alpha \in (0, \infty)$ and $\beta^* \in [1, 2)$ such that

$$\lim_{r \to 0} r^\beta \tilde{f}(r) = \begin{cases} +\infty, & \beta < \beta^*, \\ \alpha, & \beta = \beta^*. \end{cases}$$

Then

$$v^W(0) < v^S(0).$$

An SDE with no strong solution

An SDE with no strong solution

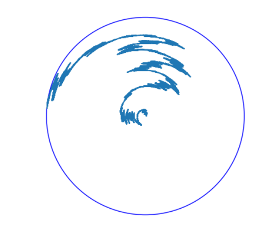
Fix $d = 2$ and let B be a one-dimensional Brownian motion.

Theorem [Larsson and Ruf, 2020]

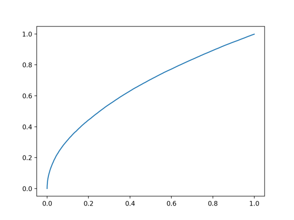
The SDE

$$\mathrm{d}X_t = \frac{1}{|X_t|} X_t^{\perp} \, \mathrm{d}B_t; \quad X_0 = 0$$

has a weak solution.



Sample path of X_t



Sample path of R_t

An SDE with no strong solution

Fix $d = 2$ and let B be a one-dimensional Brownian motion.

Theorem [Larsson and Ruf, 2020]

The SDE

$$\mathrm{d}X_t = \frac{1}{|X_t|} X_t^{\perp} \, \mathrm{d}B_t; \quad X_0 = 0$$

has a weak solution.

Therefore $v^W(0) = V(0)$.

Theorem [Cox and R. 2020+]

The SDE

$$\mathrm{d}X_t = \frac{1}{|X_t|} X_t^{\perp} \, \mathrm{d}B_t; \quad X_0 = 0$$

has no strong solution.

Gap between weak and strong values

Theorem [Cox and R. 2020+]

The SDE

$$\mathrm{d}X_t = \frac{1}{|X_t|} X_t^{\perp} \gamma_t \, \mathrm{d}B_t; \quad X_0 = 0$$

has a weak solution but no strong solution, for any ± 1-valued process γ depending only on the increments of X.

- The proof uses ideas from the study of Tsirelson's equation.
- We use properties of Circular Brownian Motion, as proved in [Émery and Schachermayer, 1999].

Gap between weak and strong values

Theorem [Cox and R. 2020+]

The SDE

$$\mathrm{d}X_t = \frac{1}{|X_t|} X_t^\perp \gamma_t \, \mathrm{d}B_t; \quad X_0 = 0$$

has a weak solution but no strong solution, for any ± 1-valued process γ depending only on the increments of X.

Conjecture

Suppose that there exists $\alpha \in (0, \infty)$ and $\beta^* \in [1, 2)$ such that

$$\lim_{r \to 0} r^\beta \tilde{f}(r) = \begin{cases} +\infty, & \beta < \beta^*, \\ \alpha, & \beta = \beta^*. \end{cases}$$

Then

$$v^W(0) < v^S(0).$$

Summary

- Constructed the value function explicitly for continuous radially symmetric costs
- Extended this result to costs that explode at the origin under certain growth conditions
- Conjecture that there exists a gap between the weak and strong value functions for a regime of moderate growth at the origin:
  - Proved that an SDE describing tangential motion has a weak solution but no strong solution started from the origin
  - Proved that a possible approximating sequence of SDEs have no strong solution
  - Require to prove that a more general form of the SDE has no strong solution

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