An adapted distance between laws of SDEs

Benjamin A. Robinson

University of Vienna

June 7, 2023 — *Optimal Transport*, SIAM FME23

Joint work with

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Adapted Wasserstein distance between the laws of SDEs
(with J. Backhoff-Veraguas and S. Källblad) — arXiv:2209.03243, Sep 2022

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Adapted Wasserstein distance between the laws of SDEs
(with J. Backhoff-Veraguas and S. Källblad) — arXiv:2209.03243, Sep 2022

Adapted Wasserstein distance for SDEs with irregular coefficients
(with M. Szölgyenyi) — in preparation

Distances between stochastic processes

$(X_n)_{n\in\{1,\dots N\}}$, $(Y_n)_{n\in\{1,\dots,N\}}$ real-valued stochastic processes

$\rightsquigarrow$ μ, ν probability measures on $\mathbb{R}^N$

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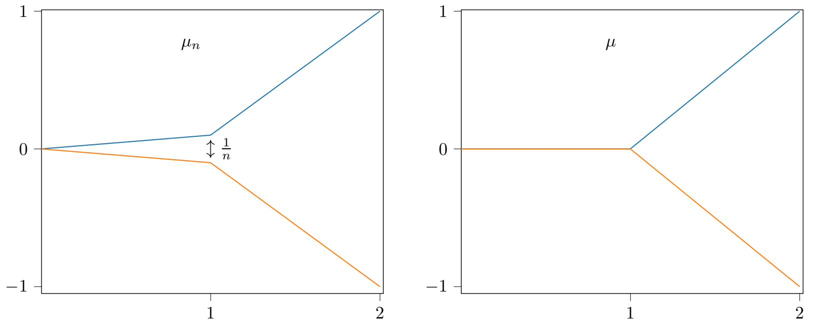
How to choose a "good" distance $d(\mu, \nu)$?

Distances between stochastic processes



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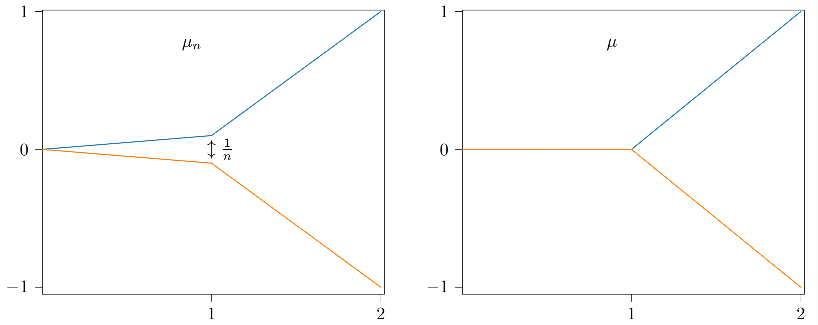

"Can get rich" "Cannot get rich"

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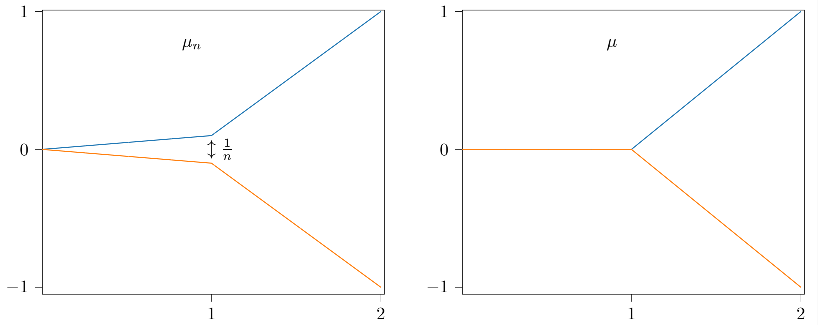

$$V_n := \sup_\tau \mathbb{E}^{\mu_n}[X_\tau] \approx \frac{1}{2}$$

$$V := \sup_\tau \mathbb{E}^{\mu}[X_\tau] = 0$$

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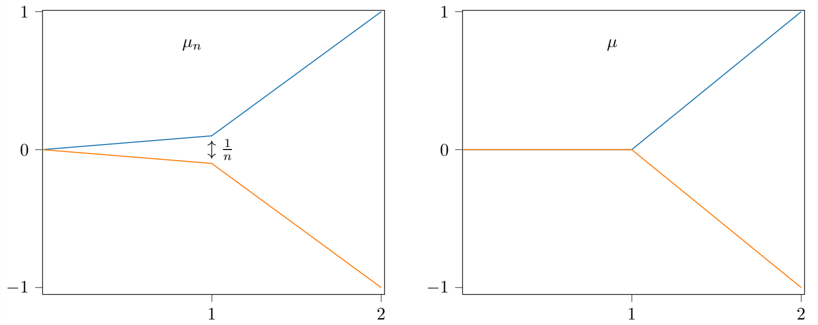

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$$V_n \not\to V$$

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E.g. Wasserstein distance $\mathcal{W}_2$ — from optimal transport

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Metrises weak convergence: $\mu_n \rightharpoonup \mu$ iff $\mathcal{W}_2(\mu_n,\mu) \to 0$.

Optimal transport

Probability measures μ, ν on $\mathbb{R}^N$

Find

$$\inf_{T\colon T_\# \mu = \nu} \mathbb{E}\left[\sum_{n=1}^{N} |T_n(X) - X_n|^2\right].$$

$$T(X) = (T_1(X_1, \ldots, X_N), \ldots, T_N(X_1, \ldots, X_N))$$

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Monge (1781), ...

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Kantorovich (1942), ... $\rightsquigarrow T$ random:

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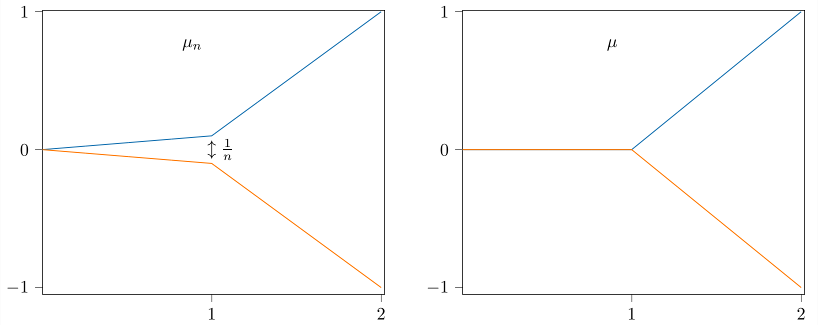
Monge (1781), ...

Kantorovich (1942), ... $\rightsquigarrow T$ random: replace $(X, T(X))$ with coupling (X, Y), $X \sim \mu$, $Y \sim \nu$.

Distances between stochastic processes



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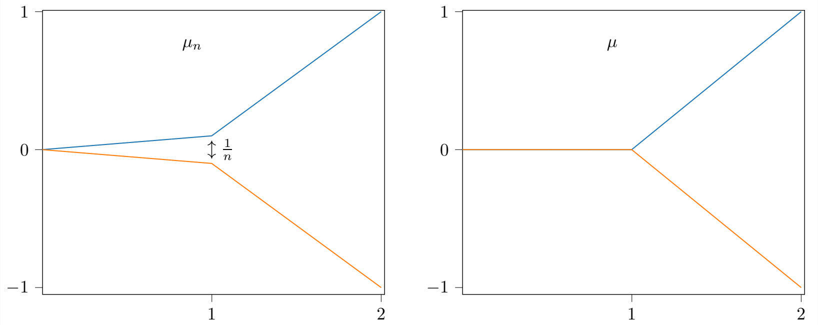

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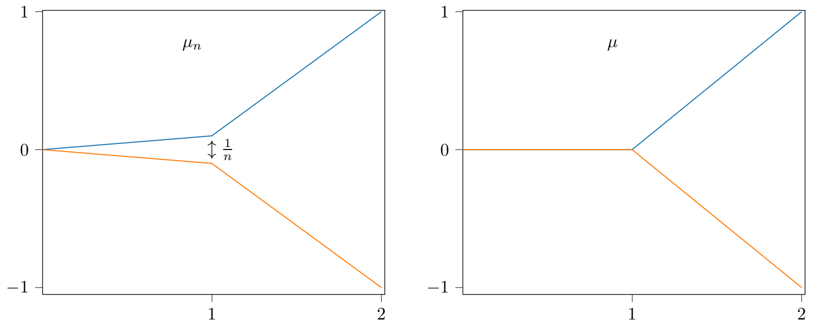

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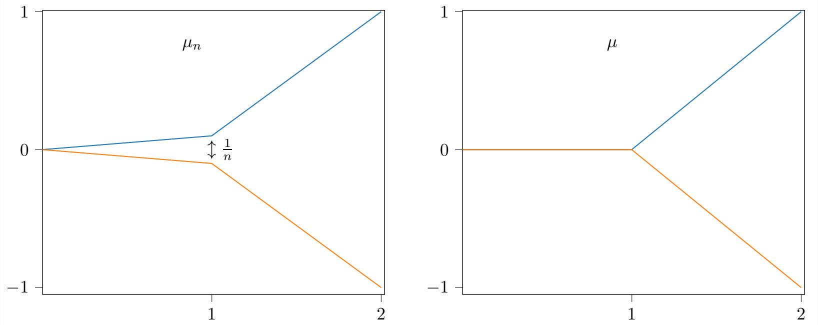

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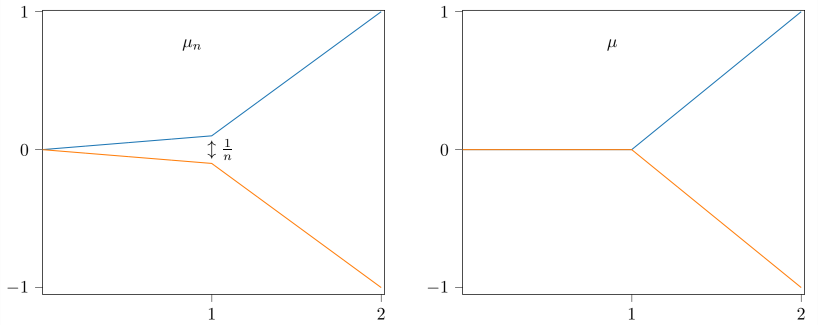

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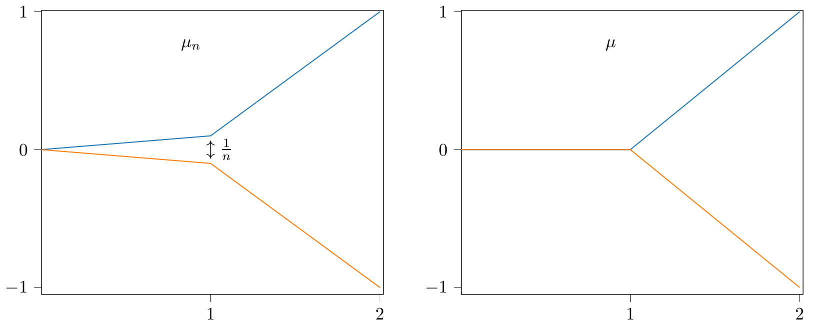
$V_n \not\to V$ but $\mu_n \rightharpoonup \mu$

Distances between stochastic processes



Want

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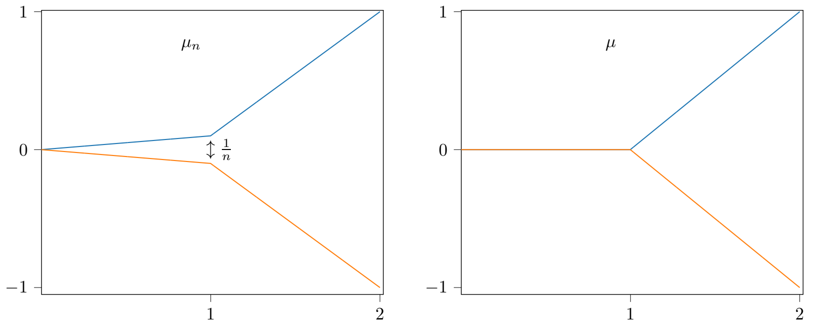
Distances between stochastic processes



Want

$$d(\mu_n, \mu) \not\to 0$$

E.g. Acciaio, Aldous, Backhoff-Veraguas, Bartl, Beiglböck, Bion-Nadal, Eder, Hellwig, Lassalle, Pammer, Pflug, Pichler, Posch, Talay, among others ...

Adapted topology

$$\mu, \nu \in \mathcal{P}(\mathbb{R}^N) \quad \rightsquigarrow \quad \mathcal{W}_2^2(\mu, \nu) := \inf_{T\colon T_\# \mu = \nu} \mathbb{E}\left[\sum_{n=1}^{N} |T_n(X) - X_n|^2\right].$$

$$T(X) = (T_1(X_1, \ldots, X_N), \ldots, T_N(X_1, \ldots, X_N))$$

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Knothe–Rosenblatt rearrangement

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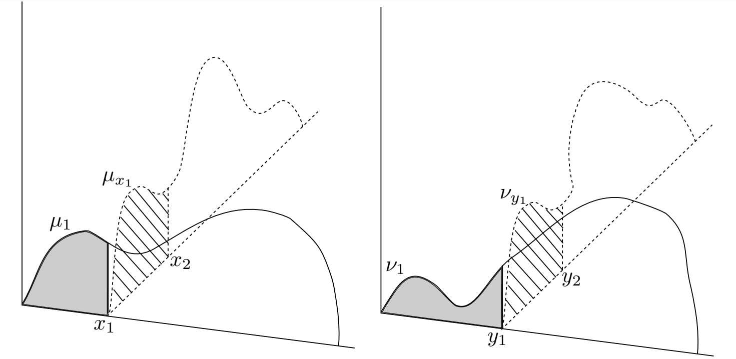
Knothe–Rosenblatt rearrangement

— generalisation of monotone rearrangement



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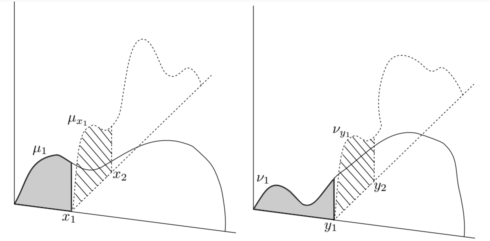
Knothe–Rosenblatt rearrangement

$$Y_k = T_k^{\mathrm{KR}}(X_1, \ldots, X_k) = F^{-1}_{\nu_{Y_1,\ldots,Y_{k-1}}} \circ F_{\mu_{X_1,\ldots,X_{k-1}}}(X_k),$$



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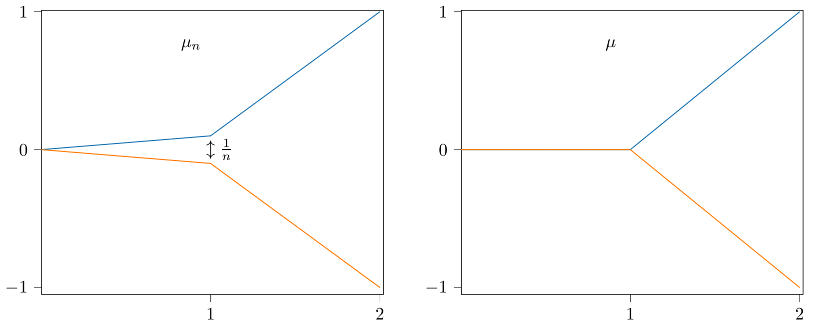
Theorem [Rüschendorf '85] [Posch '23+]

Under a monotonicity condition, the unique optimiser is the Knothe–Rosenblatt map T^{KR}. This induces the adapted weak toology.

Distances between stochastic processes



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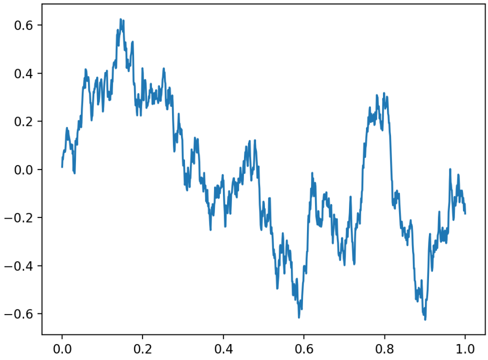
$$\mathcal{AW}(\mu_n, \mu) \not\to 0$$

Coupling SDEs

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$$

$$\rightsquigarrow$$

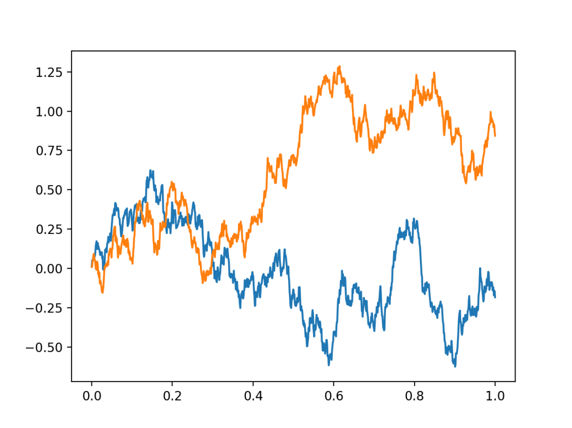
$$\mu \in \mathcal{P}(\Omega), \quad \Omega := C([0,1], \mathbb{R})$$



Coupling SDEs

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t \qquad \rightsquigarrow \qquad \mu$$
$$\mathrm{d}\bar{X}_t = \bar{b}(\bar{X}_t)\mathrm{d}t + \bar{\sigma}(\bar{X}_t)\mathrm{d}\bar{W}_t \qquad \rightsquigarrow \qquad \nu.$$

$b, \bar{b} : \mathbb{R} \to \mathbb{R}$, $\sigma, \bar{\sigma} : \mathbb{R} \to \mathbb{R}_+$



Coupling SDEs

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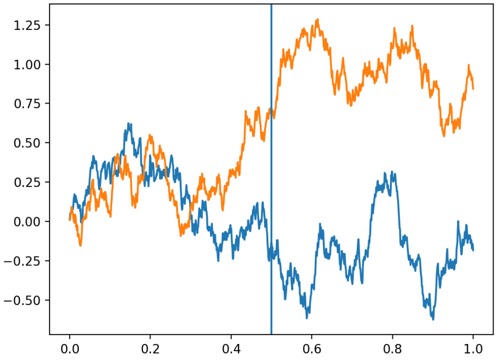

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Alternatives e.g. [Backhoff-Veraguas et al. '18], [Bartl et al. '23+]

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Applications to

- Stability in finance [Backhoff-Veraguas at al. '20]
- Martingale Optimal Transport [Backhoff-Veraguas et al. '18],
- Mimicking martingales **[Pammer, R. , Schachermayer '22]**

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Synchronous coupling

Continuous-time analogue of Knothe–Rosenblatt coupling

$$W = \bar{W}$$

Adapted topology

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Theorem 1 [Backhoff-Veraguas, Källblad, R. '22]

Optimising over adapted maps T

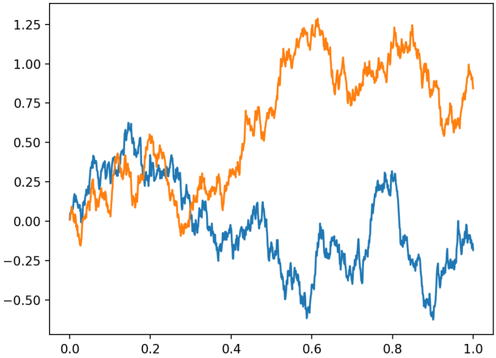
$$\Leftrightarrow$$

Optimising over correlations between $W, \bar{W}$.

Adapted topology

Example

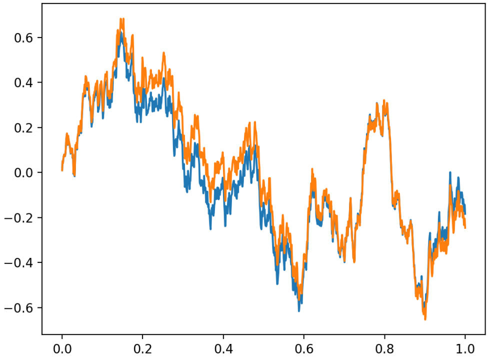
Product coupling — $W, \bar{W}$ independent



Adapted topology

Synchronous coupling

Choose the same driving Brownian motion $W = \bar{W}$.



Adapted topology

$$\begin{aligned} \mathrm{d}X_t &= b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t && \rightsquigarrow && \mu \\ \mathrm{d}\bar{X}_t &= \bar{b}(\bar{X}_t)\mathrm{d}t + \bar{\sigma}(\bar{X}_t)\mathrm{d}\bar{W}_t && \rightsquigarrow && \nu. \end{aligned}$$

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Theorem 2 [Backhoff-Veraguas, Källblad, R. '22]

Suppose that the coefficients are continuous with linear growth and that pathwise uniqueness holds the synchronous coupling is optimal.

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Theorem 2 [Backhoff-Veraguas, Källblad, R. '22]

Suppose that the coefficients are continuous with linear growth and that pathwise uniqueness holds the synchronous coupling is optimal.

Theorem 3 [R., Szölgyenyi '23+]

Under very mild conditions, the synchronous coupling is optimal, and we have an efficient method to compute $\mathcal{AW}_p(\mu, \nu)$.

A monotone numerical scheme

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t$$

Euler scheme

$$X_0^h = X_0,$$
$$X_t^h = X_{kh}^h + b(X_{kh})(t - kh), \quad t \in (kh, (k+1)h].$$

A monotone numerical scheme

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}W_t$$

Euler–Maruyama scheme

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Write $X_k^h := X_{kh}^h$ and $\mu^h = \mathrm{Law}((X_k^h)_k)$.

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Remark

$X_k^h \mapsto X_{(k+1)}^h$ is increasing if b is Lipschitz, $h \ll 1$

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Remark

$X_k^h \mapsto X_{(k+1)}^h$ is increasing if b is Lipschitz, $h \ll 1$

Corollary

The unique discrete-time bi-causal optimal coupling between μ^h, ν^h is the Knothe–Rosenblatt coupling.

A monotone numerical scheme

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Monotone Euler–Maruyama scheme

$X_0^h = X_0,$

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Corollary

The unique discrete-time bi-causal optimal coupling between μ^h, ν^h is the Knothe–Rosenblatt coupling.

Summary

- We prove optimality of the synchronous coupling;
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- Equivalence of topologies on a compact set;
- Introduce a numerical method to compute the value;

Summary

- Extension to SDEs with irregular drifts — *work in progress with Michaela Szölgyenyi*
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Theorem 3 [Backhoff-Veraguas, Källblad, R. '22]

Suppose that $(W, \bar{W})$ ρ-correlated induces an optimal coupling for $\mathcal{AW}_p(\mu^h, \nu^h)$, for all $h > 0$.

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Theorem 2 [Backhoff-Veraguas, Källblad, R. '22]

Suppose that the coefficients are continuous with linear growth and that pathwise uniqueness holds.

Then the synchronous coupling is optimal for $\mathcal{AW}_p(\mu, \nu)$.

Equality of topologies

Define the compact set

$$A_\Lambda := \{\phi : \mathbb{R} \to \mathbb{R} \colon\ |\phi(y) - \phi(x)| \leq \Lambda |y - x|, |\phi(0)| \leq \Lambda\}$$

and

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The following metrics induce the same topology on $\mathcal{P}_\Lambda$:

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